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Mean value formulas on surfaces in Grushin spaces

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Abstract. We prove (sub)mean value formulas at the point $0 \in \Sigma$ for (sub)harmonic functions on a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ where the differentiable structure and the surface measure depend on the ambient Grushin structure.

Keskiarvokaavoja Grushinin avaruuden pinnoilla

Tiivistelmä. Todistamme (ali)keskiarvokaavoja pisteessä $0 \in \Sigma$ sellaisen hyperpinnan $\Sigma \subset \mathbb{R}^{n+1}$ (ali)harmonisille funktioille, jonka derivoituva rakenne ja pinta-alamitta riippuvat ympäröivän avaruuden Grushinin rakenteesta.

1. Introduction

For $n \in \mathbb{N}$ and $\alpha > 0$, we consider the vector fields on $\mathbb{R}^{n+1}$

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, \quad X_{n+1} = |x|^\alpha \frac{\partial}{\partial y}. \tag{1.1}$$

Here, a generic point in $\mathbb{R}^{n+1}$ is denoted by $\xi = (x, y) = (x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1}$. We also consider the second order partial differential operator on $\mathbb{R}^{n+1}$ given by

$$\mathcal{L}\varphi = \sum_{i=1}^{n+1} X_i^2 \varphi = \Delta_x \varphi + |x|^{2\alpha} \partial_y^2 \varphi, \tag{1.2}$$

where $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. The operator $\mathcal{L}$ in (1.2) is known as Baouendi–Grushin operator, see [9] for a historical account and see also [7].

When α is an even integer, this operator is hypoelliptic and admits a fundamental solution with pole at any point $\xi_0 \in \mathbb{R}^{n+1}$ (see [1] for an explicit representation). When $\xi_0 = 0$, an explicit formula for this fundamental solution is in fact known for any $\alpha > 0$ (see [8]) and, up to a normalization constant, it is the function $\Gamma(\xi) = \varrho(\xi)^{1-n-\alpha}$, $\xi \neq 0$, where $\varrho \colon \mathbb{R}^{n+1} \to \mathbb{R}$ is the gauge function

$$\varrho(\xi) = \left(|x|^{2(\alpha+1)} + (\alpha+1)^2 y^2\right)^{\frac{1}{2(\alpha+1)}}. \tag{1.3}$$

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface of class C^2 with $0 \in \Sigma$. We declare the vector-fields (1.1) orthonormal and we project them onto the tangent space to Σ, getting tangential operators $\delta_1, \ldots, \delta_{n+1}$. We fix on Σ the hypersurface measure σ associated with (1.1) according to the theory of sub-elliptic perimeters and then we define the adjoint operators $\delta_1^*, \ldots, \delta_{n+1}^*$ with respect to σ. The natural restriction of $\mathcal{L}$ to Σ is the differential operator

$$\mathcal{L}_\Sigma = -\sum_{i=1}^{n+1} \delta_i^* \delta_i.$$

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In this paper, we investigate the validity of mean value formulas (sub-mean value formulas) at the point $0 \in \Sigma$ for functions $f \in C^2(\Sigma)$ satisfying $\mathcal{L}_\Sigma f = 0$ ($\mathcal{L}_\Sigma f \geq 0$, respectively).

The operator $\mathcal{L}$ is an example of "sub-Laplacian" or "sum of squares of vector fields" satisfying the Hörmander condition [10]. When $\mathcal{L} = \sum_{i=1}^m X_i^2$ is such an operator in $\mathbb{R}^{n+1}$, the validity of mean-value formulas for $\mathcal{L}$-harmonic functions is established in [4, 3, 5]. Denoting by $\Gamma(\cdot, \xi_0)$ the fundamental solution for $\mathcal{L}$ with pole at ξ_0, if a function f satisfies $\mathcal{L} f = 0$ then for any $r > 0$ and $\xi_0 \in \mathbb{R}^{n+1}$

$$f(\xi_0) = \frac{1}{r} \int_{\Omega_r(\xi_0)} f(\xi) K(\xi, \xi_0) \, d\xi,$$

where $\Omega_r(\xi_0) = \{\xi \in \mathbb{R}^{n+1} \colon \Gamma(\xi, \xi_0) > 1/r\}$ and $K(\xi, \xi_0) = |X\Gamma(\xi, \xi_0)|^2 / \Gamma(\xi, \xi_0)^2$, with $|X\Gamma(\cdot, \xi_0)|^2 = \sum_{i=1}^m (X_i \Gamma(\cdot, \xi_0))^2$. The appearance of the kernel K is due to the different symmetry of Carnot–Carathéodory balls associated with the vector-fields building up $\mathcal{L}$ and level sets of $\Gamma(\cdot, \xi_0)$.

In the Riemannian case, the validity of mean-value formulas on metric balls for harmonic functions leads to the notion of "harmonic manifold". Starting probably with [11], there exists a huge literature on the problem of characterizing harmonic manifolds and it is not possible to give a full account, here. In fact, our hypersurface Σ embedded in $\mathbb{R}^{n+1}$ with the Grushin structure is not a Riemannian manifold but rather a weighted Riemannian manifold that becomes singular at the point $0 \in \Sigma$, see Remark 2.1.

In the Grushin space, the harmonicity at $0 \in \Sigma$ is governed by the following structural function $q_\Sigma \colon \Sigma \setminus \{0\} \to \mathbb{R}$

$$q_\Sigma(\xi) = \langle X\varrho, \nu \rangle \left[(n + 3\alpha) \langle X \log \varrho, \nu \rangle - 2\alpha \langle \nabla_x \log |x|, \bar{\nu} \rangle + n H_\Sigma \right]. \tag{1.4}$$

Above, $X\varrho = (X_1 \varrho, \ldots, X_{n+1} \varrho)$ is the X-gradient of the gauge function ϱ, $\nu = (\bar{\nu}, \nu_{n+1})$ is the α-normal to Σ, $\langle \cdot, \cdot \rangle$ are standard scalar products in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, and H_Σ is the mean curvature of Σ associated with the Grushin structure. We say that Σ is α-harmonic if $q_\Sigma = 0$. In particular, any homogeneous hypersurface, $\langle X\varrho, \nu \rangle = 0$, is α-harmonic, as we show in Section 5.1.

Theorem 1.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be an α-harmonic hypersurface of class C^2 with $0 \in \Sigma$. Any function $f \in C^2(\Sigma)$ such that $\mathcal{L}_\Sigma f = 0$ satisfies the mean-value formula at 0*

$$f(0) = \frac{C_{\Sigma,n,\alpha}}{r^{n+\alpha}} \int_{B_r \cap \Sigma} f(\xi) \, |\delta \varrho(\xi)|^2 \, d\sigma, \tag{1.5}$$

for all $r \in (0, r_0)$ and for some $r_0 > 0$ depending on Σ. The constant $0 < C_{\Sigma,n,\alpha} < \infty$ is defined by

$$\frac{1}{C_{\Sigma,n,\alpha}} = \frac{1}{r^{n+\alpha}} \int_{B_r \cap \Sigma} |\delta \varrho(\xi)|^2 \, d\sigma, \tag{1.6}$$

where the right hand-side does not depend on $r \in (0, r_0)$.

Above, the balls are

$$B_r = \{\xi \in \mathbb{R}^{n+1} \colon \varrho(\xi) < r\}.$$

and $|\delta \varrho| \leq 1$ is the length of the tangential gradient of ϱ. When Σ is homogeneous, the kernel is $|\delta \varrho|^2 = |x|^{2\alpha} / \varrho^{2\alpha}$. In the case of α-subharmonic hypersurfaces, $q_\Sigma \geq 0$, the statement is similar.

Theorem 1.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be an α-subharmonic hypersurface of class C^2 with $0 \in \Sigma$. Any function $f \in C^2(\Sigma)$ such that $\mathcal{L}_\Sigma f \geq 0$ satisfies the following sub-mean-value formula at 0*

$$f(0) \leq \frac{C_{\Sigma,n,\alpha}}{r^{n+\alpha}} \int_{B_r \cap \Sigma} f(\xi)\, |\delta\varrho(\xi)|^2 \, d\sigma,$$

for all $r \in (0, r_0)$ and for some $r_0 > 0$ depending on Σ. The constant $0 < C_{\Sigma,n,\alpha} < \infty$ is defined by

$$\frac{1}{C_{\Sigma,n,\alpha}} = \lim_{r\to 0^+} \frac{1}{r^{n+\alpha}} \int_{B_r\cap\Sigma} |\delta\varrho(\xi)|^2 \, d\sigma. \tag{1.7}$$

The operator $\mathcal{L}$ in (1.2) and the hyper-surface measure σ are invariant with respect to the vertical translations $(x, y) \mapsto (x, y + y_0)$, for any fixed $y_0 \in \mathbb{R}$. Theorem 1.2 can be therefore extended to get mean value formulas at points $(0, y_0) \in \Sigma$ also with $y_0 \neq 0$. Obtaining mean value formulas at points $(x_0, y_0) \in \Sigma$ with $x_0 \neq 0$ is, instead, difficult because our knowledge of the fundamental solution of $\mathcal{L}$ with pole at (x_0, y_0) with $x_0 \neq 0$ is not explicit enough.

Our interest in sub-mean value formulas on hypersurfaces of $\mathbb{R}^{n+1}$ endowed with a system of Hörmander vector fields comes from the theory of minimal surfaces. One of the key tools in Bombieri–De Giorgi–Miranda's proof of the gradient estimate is the sub-mean value property for sub-harmonic functions on minimal surfaces of the Euclidean space, see [2]. In our setting, a minimal surface is defined by $H_\Sigma = 0$. This condition simplifies the structural function q_Σ, however, this is not sufficient to have $q_\Sigma \geq 0$.

The paper is organized as follows. In Section 2, we recall the basic definitions of the measure σ, of the α-normal ν of Σ, and of mean curvature H_Σ. In Section 3, we introduce the various differential operators and we develop a calculus on radial functions. The explicit computations of second order derivatives of ϱ is crucial, here. In Section 4, we prove Theorems 1.1 and 1.2. Finally, in Section 5 we study the structural function q_Σ.

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2. Perimeter and mean curvature of hypersurfaces

The *α-perimeter* of a Lebesgue measurable set $E \subset \mathbb{R}^{n+1}$ in an open set $A \subset \mathbb{R}^{n+1}$ is

$$P_\alpha(E; A) = \sup\left\{ \int_E \sum_{i=1}^{n+1} X_i\varphi_i(\xi)\, d\xi \colon \varphi \in C^1_c(A; \mathbb{R}^{n+1}),\ \max_{\xi\in A} |\varphi(\xi)| \leq 1 \right\}.$$

We are using the Lebesgue measure $d\xi = d\mathcal{L}^{n+1}$ in $\mathbb{R}^{n+1}$. When the boundary of E is locally the graph of a Lipschitz function, its α-perimeter has the following integral representation (see [6, Proposition 2.1])

$$P_\alpha(E; A) = \int_{\partial E\cap A} \sqrt{|\bar{N}|^2 + |x|^{2\alpha}|N'|^2}\, d\mathcal{H}^n, \tag{2.1}$$

where $N(\xi) = (\bar{N}(\xi), N'(\xi)) \in \mathbb{R}^n \times \mathbb{R}$ is the Euclidean outer unit normal to ∂E at the point ξ, and $\mathcal{H}^n$ is the standard n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. On top of its appearance as a sub-Riemannian perimeter, the relevance of the perimeter P_α is due to its relation with the Heisenberg perimeter. When $\alpha = 1$ and n is even, then the Heisenberg perimeter of a set with cylindrical symmetry coincides with its α-perimeter, see e.g., [6, Proposition 2.3].

Motivated by (2.1), when $\Sigma \subset \mathbb{R}^{n+1}$ is an orientable hypersurface that is locally a Lipschitz graph and $N = (\bar N, N')$ is its Euclidean normal, we call the Borel measure on Σ

$$\sigma = \sqrt{|\bar N|^2 + |x|^{2\alpha}|N'|^2}\,\mathcal{H}^n \llcorner \Sigma. \tag{2.2}$$

the α-perimeter measure of Σ.

The regular part of Σ is the set $\Sigma^* = \{\xi = (x,y) \in \Sigma \colon x \neq 0\}$. At $\mathcal{H}^n$-a.e. point $\xi \in \Sigma^*$ we can define the *α-normal* of Σ as the vector field $\nu = \sum_{i=1}^{n+1} \nu_i X_i$ with

$$\begin{aligned} \nu_i &= \frac{N_i}{\sqrt{|\bar N|^2 + |x|^{2\alpha}|N'|^2}}, \quad \text{for } i = 1,\dots,n,\\ \nu_{n+1} &= \frac{a N_{n+1}}{\sqrt{|\bar N|^2 + |x|^{2\alpha}|N'|^2}}. \end{aligned}$$

With abuse of notation, we identify ν with the mapping $\nu\colon \Sigma^* \to \mathbb{R}^{n+1}$ given by the vector of its coordinates $\nu(\xi) = (\nu_1(\xi),\dots,\nu_{n+1}(\xi)) \in \mathbb{R}^{n+1}$ for $\xi \in \Sigma^*$.

Remark 2.1. (Comparison with the Riemannian structure) The hypersurface measure σ and the α-normal ν can be interpreted in the following Riemannian terms. The tensor metric in $\mathbb{R}^{n+1} \setminus \{x = 0\}$ making $X_1,\dots,X_{n+1}$ orthonormal is

$$g_\alpha(\xi) = \begin{pmatrix} I_n & 0 \\ 0 & |x|^{-2\alpha} \end{pmatrix}, \quad x \neq 0.$$

When $x = 0$ the metric is not defined. The Riemannian volume associated with g_α is the measure $\mu = |x|^{-\alpha}\mathcal{L}^{n+1}$ and is singular at $x = 0$. The Riemannian surface area associated with g_α of a hypersurface Σ is the measure

$$\mu_\Sigma = |x|^{-\alpha}\sqrt{|\bar N|^2 + |x|^{2\alpha}|N'|^2}\,\mathcal{H}^n \llcorner \Sigma,$$

where $N = (\bar N, N')$ is the Euclidean unit normal. We deduce that Lebesgue measure and α-perimeter are weighted Riemannian volume and hypersurface measures with the same weight:

$$\mathcal{L}^{n+1} = |x|^\alpha \mu \quad \text{and} \quad \sigma = |x|^\alpha \mu_\Sigma.$$

We now focus on the case of graphs. Let $\Sigma = \Sigma_u = \{\xi = (x, u(x)) \in \mathbb{R}^{n+1} \colon x \in \Omega\}$ be the y-graph of a function $u \in C^1(\Omega)$ for some open set $\Omega \subset \mathbb{R}^n$. We shall assume that $0 \in \Omega$ and let $\Omega^* = \Omega \setminus \{0\}$. The *$\alpha$-unit normal* to Σ_u at points in Σ_u^* is the mapping $\nu = (\bar\nu, \nu_{n+1})\colon \Sigma^* \to \mathbb{R}^{n+1}$

$$\bar\nu = \frac{-\nabla u}{\sqrt{|\nabla u|^2 + |x|^{2\alpha}}}, \quad \nu_{n+1} = \frac{|x|^\alpha}{\sqrt{|\nabla u|^2 + |x|^{2\alpha}}}. \tag{2.3}$$

This normal is pointing upwards. Notice that $\nu = \nu(\xi)$ only depends on x and not on $y = u(x)$.

From (2.2) and from the area-formula, we deduce that the σ-area of Σ has the integral representation

$$\sigma(\Sigma_u) = \int_\Omega \sqrt{|\nabla u|^2 + |x|^{2\alpha}}\,dx = \int_\Omega v(x)\,dx, \tag{2.4}$$

where v is the σ-area element

$$v(x) = \sqrt{|\nabla u|^2 + a^2}, \quad a = |x|^\alpha. \tag{2.5}$$

If Σ_u minimizes the σ-area with respect to compact perturbations in Ω and $u \in C^2(\Omega)$, then u satisfies the partial differential equation of the minimal surface-type

$$\operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{|\nabla u(x)|^2 + |x|^{2\alpha}}}\right) = 0, \quad x \in \Omega^*.$$

This follows by a standard first variation procedure applied to (2.4). This suggests the following definition.

Definition 2.2. (α-mean curvature) Let Σ be the y-graph of a function $u \in C^2(\Omega)$. We define the *α-mean curvature* of Σ at the point $\xi = (x, u(x)) \in \Sigma^*$ as

$$H_\Sigma(\xi) = \frac{1}{n} \operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{|\nabla u(x)|^2 + |x|^{2\alpha}}}\right). \tag{2.6}$$

We say that Σ is an α-minimal hypersurface if $H_\Sigma = 0$ on Σ^*.

A more geometric definition of α-mean curvature will be presented in the next section.

3. Tangential operators and Laplacians

We introduce tangential differential operators on hypersurfaces in $\mathbb{R}^{n+1}$ endowed with the Grushin structure. Let $\Sigma \subset \mathbb{R}^{n+1}$ be an embedded hypersurface of class C^2 with α-normal $\nu\colon \Sigma^* \to \mathbb{R}^{n+1}$.

The X-gradient of a function $\varphi \in C^1(\mathbb{R}^{n+1})$ is the vector-field $X\varphi = \sum_{i=1}^{n+1} X_i\varphi\, X_i$ that we identify, with abuse of notation, with the vector of its coordinates $X\varphi = (X_1\varphi, \ldots, X_{n+1}\varphi)$. We denote the standard scalar product on $\mathbb{R}^{n+1}$ by $\langle \cdot, \cdot \rangle$.

Definition 3.1. (Tangential gradient) Let $\nu\colon \Sigma^* \to \mathbb{R}^{n+1}$ be the α-normal of Σ. The tangential gradient on Σ is the mapping $\delta\colon C^1(\Sigma) \to C(\Sigma^*; \mathbb{R}^{n+1})$

$$\delta\varphi = X\varphi - \langle X\varphi, \nu \rangle \nu. \tag{3.1}$$

For any $i = 1, \ldots, n+1$ we also let $\delta_i\varphi = X_i\varphi - \langle X\varphi, \nu \rangle \nu_i$.

In (3.1), φ is extended outside Σ in a C^1 way, and the definition will be independent of this extension. We are assuming that Σ is oriented and we are fixing a choice of α-normal. The definition does not depend on this choice. When Σ is a y-graph, we agree that ν is pointing upwards. In this case, the α-normal ν can be extended outside Σ in a way that is independent of the variable y. In the rest of the paper the surface Σ will be always assumed to be a y-graph.

The definition of the tangential operator δ in (3.1) is extrinsic. A different possibility could be to define the tangential gradient of functions on Σ using the Riemannian metric g_α. However, the choice in (3.1) is the correct one in order to recover the definition in (2.6) of α-mean curvature. Indeed, this definition reads

$$H_\Sigma = -\frac{1}{n} \sum_{i=1}^{n} X_i \nu_i, \tag{3.2}$$

and it can be rephrased in the following way.

Lemma 3.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a y-graph of class C^2. Then on Σ^* we have the identity*

$$H_\Sigma = -\frac{1}{n}\sum_{i=1}^{n+1} \delta_i \nu_i.$$

Proof. We use the definition of δ and observe that $\sum_{i=1}^{n+1} \nu_i X_k \nu_i = X_k(|\nu|^2/2) = 0$, and $X_{n+1}\nu_{n+1} = 0$, so that

$$\begin{aligned}
\sum_{i=1}^{n+1} \delta_i \nu_i &= \sum_{i=1}^{n} \delta_i \nu_i + \delta_{n+1}\nu_{n+1} \\
&= \sum_{i=1}^{n} X_i \nu_i - \sum_{i=1}^{n}\sum_{k=1}^{n+1} \nu_i \nu_k X_k \nu_i + X_{n+1}\nu_{n+1} - \nu_{n+1}\nu_k X_k \nu_{n+1} \\
&= \sum_{i=1}^{n} X_i \nu_i - \sum_{k=1}^{n+1} \left(\nu_k \sum_{i=1}^{n+1} \nu_i X_k \nu_i \right) + X_{n+1}\nu_{n+1} \\
&= \sum_{i=1}^{n} X_i \nu_i = -nH_\Sigma.
\end{aligned}$$

□

Next we introduce the adjoint operators δ_i^*, integrating by parts with respect to the measure σ.

Definition 3.3. (Adjoint tangential operators) For each $i = 1, \dots, n+1$, we define the *adjoint tangential operator* $\delta_i^* \colon C^1(\Sigma) \to C(\Sigma^*)$ through the identity

$$\int_\Sigma \psi\, \delta_i \varphi \, d\sigma = -\int_\Sigma \varphi\, \delta_i^* \psi \, d\sigma, \quad \varphi, \psi \in C_c^1(\Sigma). \tag{3.3}$$

The explicit formula for adjoint operators is given in the next lemma.

Lemma 3.4. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface with α-mean curvature H_Σ. For every $\psi \in C_c^1(\Sigma)$ and $i = 1, \dots, n+1$, we have on Σ^**

$$\delta_i^* \psi = -\delta_i \psi - \psi \left[\delta_i \log |x|^\alpha + \frac{1}{\nu_{n+1}} (\delta_{n+1}\nu_i - \delta_i \nu_{n+1}) + nH_\Sigma \nu_i \right]. \tag{3.4}$$

Proof. Let $\varphi, \psi \in C_c^1(\Sigma^*)$. Then by the area formula (2.4) with v as in (2.5)

$$\begin{aligned}
\int_\Sigma \varphi\, \delta_i^* \psi \, d\sigma &= \int_\Sigma \psi\, \delta_i \varphi \, d\sigma = \int_\Omega \psi (X_i \varphi - \langle X\varphi, \nu\rangle \nu_i) v \, dx \\
&= -\int_\Omega \varphi \left[X_i(v\psi) - \sum_{k=1}^{n} X_k(\nu_i \nu_k v \psi) \right] dx = -\int_\Sigma \varphi \frac{A}{v}\, d\sigma,
\end{aligned}$$

where in the last identity we set $A = X_i(v\psi) - \sum_{k=1}^{n} X_k(\nu_i \nu_k v\psi)$. We are left to prove that A/v is equal to the right-hand side in (3.4).

We have

$$\begin{aligned}
-\frac{A}{v} &= -\frac{1}{v}\left[\psi X_i v + v X_i \psi - v\nu_i \langle X\psi, \nu\rangle - \psi \sum_{k=1}^{n} X_k(\nu_i \nu_k v) \right] \\
&= -\delta_i \psi - \frac{\psi}{v}\left[X_i v - \nu_i \langle Xv, \nu\rangle - v \sum_{k=1}^{n} X_k(\nu_i \nu_k) \right].
\end{aligned}$$

In the term within brackets above, we easily recognize $X_i v - \nu_i \langle Xv, \nu\rangle = \delta v$. On the other hand, using $X_{n+1}\nu = 0$ and (3.2), we get

$$\sum_{k=1}^{n} X_k(\nu_i\nu_k) = -n\nu_i H_\Sigma - \frac{1}{\nu_{n+1}}\left[X_{n+1}\nu_i - \nu_{n+1}\langle X\nu_i, \nu\rangle\right] = -n\nu_i H_\Sigma - \frac{1}{\nu_{n+1}}\delta_{n+1}\nu_i.$$

Summarizing, we have

$$\frac{A}{v} = \delta_i\psi + \psi\left[\frac{\delta_i v}{v} + \frac{\delta_{n+1}\nu_i}{\nu_{n+1}} + nH_\Sigma\nu_i\right].$$

To prove our claim it then remains to check the following identity

$$\frac{\delta_i v}{v} = \frac{\delta_i a}{a} - \frac{\delta_i\nu_{n+1}}{\nu_{n+1}} = \delta_i(\log a) - \frac{\delta_i\nu_{n+1}}{\nu_{n+1}}.$$

Indeed, since $\nu_{n+1} = a/v$ we have

$$\frac{\delta_i a}{a} - \frac{\delta_i\nu_{n+1}}{\nu_{n+1}} = \frac{\delta_i a}{a} - \frac{v\delta_i(\frac{a}{v})}{a} = \frac{\delta_i a}{a} - \frac{v}{a}\left(\frac{\delta_i a}{v} - \frac{a}{v^2}\delta_i v\right) = \frac{\delta_i v}{v}.$$ □

Definition 3.5. (Tangential Laplacians) Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface of class C^2. The *tangential Laplacian* of Σ is the operator $\mathcal{L}_\Sigma \colon C^2(\Sigma) \to C(\Sigma^*)$

$$\mathcal{L}_\Sigma\varphi = -\sum_{i=1}^{n+1}\delta_i^*\delta_i\varphi.$$

The relation between $\mathcal{L}_\Sigma$ and the non-adjoint Laplacian

$$\Delta_\Sigma\varphi = \sum_{i=1}^{n+1}\delta_i^2\varphi. \tag{3.5}$$

is described in the following proposition.

Lemma 3.6. *For any $\varphi \in C^2(\Sigma)$ we have the identity*

$$\mathcal{L}_\Sigma\varphi = \Delta_\Sigma\varphi + \nu_{n+1}^2\langle\delta\varphi, \delta\log a\rangle + \delta_{n+1}\varphi\,\delta_{n+1}\log a \tag{3.6}$$

Proof. We have

$$\mathcal{L}_\Sigma\varphi = -\sum_{i=1}^{n+1}\delta_i^*\delta_i = \Delta_\Sigma\varphi + \langle\delta\varphi, \delta\log a\rangle - \frac{1}{\nu_{n+1}}\langle\delta\varphi, \delta\nu_{n+1} - \delta_{n+1}\nu\rangle.$$

By formula (3.4), for $i = 1, \dots, n+1$ we have

$$\delta\nu_{n+1} - \delta_{n+1}\nu = \nu_{n+1}\big[\delta\log a - \big(\bar{0}, \delta_{n+1}\log a\big)\big] + \nu_{n+1}^2\big[\nu\delta_{n+1}\log a - \nu_{n+1}\delta\log a\big].$$

Since $\langle\delta(\cdot), \nu\rangle = 0$, we deduce

$$-\frac{1}{\nu_{n+1}}\langle\delta\varphi, \delta\nu_{n+1} - \delta_{n+1}\nu\rangle = -\langle\delta\varphi, \delta\log a\rangle + \delta_{n+1}\varphi\delta_{n+1}\log a + \nu_{n+1}^2\langle\delta\varphi, \delta\log a\rangle,$$

proving the result. □

The formal Hessian of φ with respect to the vector-fields $X_1, \dots, X_{n+1}$ is the $(n+1)\times(n+1)$ matrix

$$X^2\varphi = \big(X_iX_j\varphi\big)_{i,j=1,\dots,n+1}.$$

The non-adjoint Laplacian Δ_Σ has a clear representation in terms of the Grushin operator $\mathcal{L}$ in (1.2), X^2 and α-mean curvature of Σ.

Lemma 3.7. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface of class C^2 with α-mean curvature H_Σ. For any $\varphi \in C^2(\mathbb{R}^{n+1})$ we have the identity on Σ^**

$$\Delta_\Sigma \varphi = \mathcal{L}\varphi - \langle (X^2\varphi)\nu, \nu\rangle + nH_\Sigma \langle X\varphi, \nu\rangle. \tag{3.7}$$

Proof. We first compute $\delta_i^2\varphi$, for $i \geq 1$. We have

$$\delta_i^2\varphi = \delta_i(X_i\varphi - \langle X\varphi, \nu\rangle \nu_i) = X_iX_i\varphi - \langle XX_i\varphi, \nu\rangle\nu_i - \langle X\varphi, \nu\rangle \delta_i\nu_i - \delta_i(\langle X\varphi, \nu\rangle)\nu_i.$$

Summing over i, we obtain the identities

$$\sum_{i=1}^{n+1} \langle XX_i\varphi, \nu\rangle \nu_i = \langle (X^2\varphi)\nu, \nu\rangle, \quad \sum_{i=1}^{n+1} \delta_i\nu_i = -nH_\Sigma,$$
$$\sum_{i=1}^{n+1} \delta_i(\langle X\varphi, \nu\rangle)\nu_i = \langle \delta(\langle X\varphi, \nu\rangle), \nu\rangle = 0,$$

and this completes the proof. □

We specialize the previous formulas to the case when φ is a radial function around $0 \in \mathbb{R}^{n+1}$. The symmetry is governed by the gauge function ϱ in (1.3). Below, we collect the differential identities concerning first and second order derivatives of ϱ. With the notation $\xi = (x, y)$ and $\varrho = \varrho(\xi)$ we have

$$\nabla_x\varrho = x|x|^{2\alpha}\varrho^{-(2\alpha+1)} \quad \text{and} \quad \partial_y\varrho = (\alpha+1)y\varrho^{-(2\alpha+1)}.$$

Then the squared norm of the X-gradient of ϱ is

$$|X\varrho|^2 = |\nabla_x\varrho|^2 + |x|^{2\alpha}|\partial_y\varrho|^2 = |x|^{2\alpha}\varrho^{-2\alpha}. \tag{3.8}$$

The second derivatives of ϱ are, with $i, j = 1, \dots, n$ and denoting by ε_{ij} the Kronecker symbol,

$$\begin{aligned}
X_iX_j\varrho &= |x|^{2\alpha}\left[\varepsilon_{ij} + 2\alpha\frac{x_ix_j}{|x|^2} - (2\alpha+1)\frac{x_ix_j|x|^{2\alpha}}{\varrho^{2(\alpha+1)}}\right],\\
X_iX_{n+1}\varrho &= (\alpha+1)|x|^{2\alpha}x_iy\varrho^{-2\alpha-1}\left[\frac{\alpha}{|x|^{\alpha+2}} - (2\alpha+1)\frac{|x|^{2\alpha}}{\varrho^{2(\alpha+1)}}\right],\\
X_{n+1}X_j\varrho &= -(2\alpha+1)(\alpha+1)|x|^{3\alpha}x_jy\varrho^{-4\alpha-3},\\
X_{n+1}^2\varrho &= (\alpha+1)|x|^{2\alpha}\varrho(x)^{-2\alpha-1}\left[1 - (2\alpha+1)(\alpha+1)\frac{y^2}{\varrho^{2(\alpha+1)}}\right].
\end{aligned} \tag{3.9}$$

From (3.9), we get the following formulas for the Laplacian $\mathcal{L}$ and for the quadratic form $\langle (X^2\varrho)\nu, \nu\rangle$:

$$\mathcal{L}\varrho = (n+\alpha)\frac{|X\varrho|^2}{\varrho}, \tag{3.10}$$

and

$$\begin{aligned}
\langle (X^2\varrho)\nu, \nu\rangle &= |\bar\nu|^2 + 2\alpha\frac{\langle x, \bar\nu\rangle^2}{|x|^2} - (2\alpha+1)\frac{|x|^{2\alpha}\langle x, \bar\nu\rangle^2}{\varrho^{2(\alpha+1)}}\\
&\quad + (\alpha+1)\langle x, \bar\nu\rangle^2\nu_{n+1}y\left(\frac{\alpha}{|x|^{\alpha+2}} - 2(2\alpha+1)\frac{|x|^{2\alpha}\langle x, \bar\nu\rangle^2}{\varrho^{2(\alpha+1)}}\right)\\
&\quad + (\alpha+1)\nu_{n+1}^2\left(1 - (\alpha+1)(2\alpha+1)\frac{y^2}{\varrho^{2(\alpha+1)}}\right).
\end{aligned} \tag{3.11}$$

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface with α-normal ν and α-mean curvature H_Σ. The structural function $q_\Sigma \colon \Sigma^* \to \mathbb{R}$ introduced in (1.4) governs the harmonicity of Σ at $0 \in \Sigma$ and appears in the following formula.

Theorem 3.8. (Tangential Laplacian of radial functions) *For any $\varphi \in C^2(\mathbb{R}^+)$, the function $\psi \in C^2(\mathbb{R}^{n+1} \setminus \{0\})$, $\psi = \varphi \circ \varrho$, satisfies the identity*

$$\mathcal{L}_\Sigma \psi = \left\{ \varphi''(\varrho) + \varphi'(\varrho) \frac{n+\alpha-1}{\varrho} \right\} |\delta\varrho|^2 + q_\Sigma \varphi'(\varrho). \tag{3.12}$$

Proof. In a first step, we prove that formula (3.12) holds with the following expression for q_Σ:

$$\begin{aligned} q_\Sigma = & -\frac{n+\alpha-1}{\varrho}|\delta\varrho|^2 + (n+\alpha)\frac{|X\varrho|^2}{\varrho} - \langle (X^2\varrho)\nu, \nu\rangle + nH_\Sigma \langle X\varrho, \nu\rangle \\ & + \alpha\nu_{n+1}^2 \langle \delta\varrho, \delta \log|x|\rangle + \alpha\delta_{n+1}\varrho\delta_{n+1}\log|x|. \end{aligned} \tag{3.13}$$

The proof combines formula (3.6) of Lemma 3.6 and formula (3.7) in Lemma 3.7:

$$\begin{aligned} \Delta_\Sigma \psi &= \mathcal{L}(\psi \circ \varrho) - \langle X^2(\psi\circ\varrho)\nu, \nu\rangle + nH_\Sigma \langle X(\psi\circ\varrho), \nu\rangle \\ &= \varphi''(\varrho)|\delta\varrho|^2 + \varphi'(\varrho)\big(\mathcal{L}\varrho - \langle (X^2\varrho)\nu,\nu\rangle + nH_\Sigma\langle X\varrho, \nu\rangle\big), \end{aligned}$$

where the last identity is a simple computation with the chain rule. On the other hand, we have

$$\mathcal{L}_\Sigma \psi = \Delta_\Sigma \psi + \alpha\nu_{n+1}^2 \langle \delta\psi, \delta(\log|x|)\rangle + \alpha\delta_{n+1}\psi\delta_{n+1}(\log|x|).$$

Since $\delta\psi = \varphi'(\varrho)\delta\varrho$, we deduce

$$\begin{aligned} \mathcal{L}_\Sigma \psi = \varphi''(\varrho)|\delta\varrho|^2 + \varphi'(\varrho)\big[&\mathcal{L}\varrho - \langle (X^2\varrho)\nu,\nu\rangle + nH_\Sigma\langle X\varrho,\nu\rangle \\ &+ \alpha\nu_{n+1}^2\langle \delta\varrho, \delta(\log|x|)\rangle + \alpha\delta_{n+1}\varrho\delta_{n+1}(\log|x|)\big] \end{aligned}$$

The proof of (3.12) with q_Σ as in (3.13) is then concluded by adding and subtracting the quantity $\frac{n+\alpha-1}{\varrho}|\delta\varrho|^2$ within squared brackets and using (3.10).

In the next step, we check that q_Σ in (3.13) is as in (1.4). We start by observing that an elementary computation gives

$$\nu_{n+1}^2\langle \delta\varrho, \delta\log|x|\rangle + \delta_{n+1}\varrho\delta_{n+1}\log|x| = \frac{|x|^{2\alpha}}{\varrho^{2\alpha+1}}\Big(\nu_{n+1}^2 - (\alpha+1)\nu_{n+1}|x|^{-2-\alpha}\langle x, \bar\nu\rangle y\Big),$$

and

$$-\frac{n+\alpha-1}{\varrho}|\delta\varrho|^2 + (n+\alpha)\frac{|X\varrho|^2}{\varrho} = \frac{|X\varrho|^2}{\varrho} + \frac{n+\alpha-1}{\varrho}\langle X\varrho, \nu\rangle^2.$$

Inserting formulas (3.9)–(3.11) into (3.13), we obtain

$$\begin{aligned} q_\Sigma = \frac{n+\alpha-1}{\varrho}&\langle X\varrho,\nu\rangle^2 + nH_\Sigma\langle X\varrho,\nu\rangle - \frac{|X\varrho|^2}{\varrho}\bigg[2\alpha\frac{\langle x,\bar\nu\rangle^2}{|x|^2} \\ &- (2\alpha+1)\frac{|x|^{2\alpha}}{\varrho^{2\alpha+2}}\langle x,\bar\nu\rangle^2 + 2\alpha(\alpha+1)\langle x,\bar\nu\rangle y|x|^{-2-\alpha}\nu_{n+1} \\ &- 2(\alpha+1)(2\alpha+1)\frac{|x|^\alpha}{\varrho^{2\alpha+2}}\langle x,\bar\nu\rangle y\nu_{n+1} \\ &- (\alpha+1)^2(2\alpha+1)y^2\nu_{n+1}^2\frac{1}{\varrho^{2\alpha+2}}\bigg]. \end{aligned} \tag{3.14}$$

Now we observe that

$$\langle X\varrho, \nu\rangle^2 = \frac{|x|^{2\alpha}}{\varrho^{4\alpha+2}} \left(|x|^\alpha \langle x, \bar\nu\rangle + (\alpha+1) y \nu_{n+1}\right)^2$$

and so, since $|X\varrho| = (|x|/\varrho)^\alpha$,

$$\langle X\varrho, \nu\rangle = \frac{|X\varrho|}{\varrho^{\alpha+1}} \left(|x|^\alpha \langle x, \bar\nu\rangle + (\alpha+1) y \nu_{n+1}\right). \tag{3.15}$$

Replacing last identity into (3.14) and after some computations that are omitted, we obtain (1.4). □

4. Mean value formulas

We are ready to prove Theorems 1.1 and 1.2. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface of class C^2 with $0 \in \Sigma$. We say that:

i) Σ is α-harmonic at 0 if $q_\Sigma = 0$;
ii) Σ is α-subharmonic at 0 if $q_\Sigma \geq 0$;
iii) Σ is α-superharmonic at 0 if $q_\Sigma \leq 0$.

Proof of Theorem 1.1. For any $\psi \in C_c^\infty(\Sigma)$, by the integration by parts formula (3.3) we have

$$0 = \int_\Sigma \mathcal{L}_\Sigma f\, \psi\, d\sigma = -\int_\Sigma \langle \delta f\, \delta\psi\rangle d\sigma = \int_\Sigma f\, \mathcal{L}_\Sigma \psi\, d\sigma. \tag{4.1}$$

We shall use this identity for functions ψ with radial structure around 0. Let $\chi \in C^\infty(\mathbb{R}^+)$ be a function such that $\chi(r) = 1$ for $0 < r < 1/2$ and $\chi(r) = 0$ for $r > 1$. We may also assume that $\chi' \leq 0$. With the notation $m(r) = r^{n+\alpha}$, for $0 < s < r$ we define the function $\vartheta_s \in C^\infty(\mathbb{R}^+)$

$$\vartheta_s(\varrho) = \frac{\partial}{\partial s}\left(\frac{1}{m(s)}\chi\left(\frac{m(\varrho)}{m(s)}\right)\right), \quad \varrho > 0. \tag{4.2}$$

Assume there exists a solution $\varphi_s \in C^\infty(\mathbb{R}^+)$ to the differential problem

$$\begin{cases} \varphi_s''(\varrho) + \frac{m''(\varrho)}{m'(\varrho)}\varphi_s'(\varrho) = \vartheta_s(\varrho), & \varrho > 0 \\ \varphi_s(\varrho) = 0, & \varrho > s. \end{cases} \tag{4.3}$$

Then we may consider the function $\psi_s(\xi) = \varphi_s(\varrho(\xi))$ for $\xi \in \Sigma$. By formula (3.12) with $q_\Sigma = 0$ and (4.3) we have

$$\mathcal{L}_\Sigma \psi_s = \left\{\varphi_s''(\varrho) + \varphi_s'(\varrho)\frac{n+\alpha-1}{\varrho}\right\}|\delta\varrho|^2 = \frac{\partial}{\partial s}\left(\frac{1}{m(s)}\chi\left(\frac{m(\varrho)}{m(s)}\right)\right)|\delta\varrho|^2,$$

and from (4.1) we deduce that

$$0 = \int_\Sigma f\, \mathcal{L}_\Sigma \psi_s\, d\sigma = \frac{\partial}{\partial s}\int_\Sigma f(\xi)\, \frac{1}{m(s)}\chi\left(\frac{m(\varrho)}{m(s)}\right)|\delta\varrho|^2\, d\sigma,$$

and thus for any $0 < s < r$

$$\frac{1}{m(s)}\int_\Sigma f(\xi)\,\chi\left(\frac{m(\varrho)}{m(s)}\right)|\delta\varrho|^2\, d\sigma = \frac{1}{m(r)}\int_\Sigma f(\xi)\,\chi\left(\frac{m(\varrho)}{m(r)}\right)|\delta\varrho|^2\, d\sigma.$$

We may approximate the characteristic function of the interval $(0,1) \subset \mathbb{R}$ by a sequence of functions χ as above. Passing to the limit in the previous identity, we get

$$\frac{1}{m(s)} \int_{B_s \cap \Sigma} f(\xi)\, |\delta \varrho|^2 \, d\sigma = \frac{1}{m(r)} \int_{B_r \cap \Sigma} f(\xi)\, |\delta \varrho|^2 \, d\sigma.$$

This formula holds for $f = 1$, proving that the right hand-side of (1.6) does not depend on $r > 0$. By continuity of f at 0, we get (1.5) with $C_{\Sigma,n,\alpha}$ as in (1.6).

We are left to show that problem (4.3) has a solution. A straightforward computation shows that the function ϑ_s in (4.2) reads

$$\vartheta_s(\varrho) = -\frac{m'(s)}{m(s)^2} \frac{\partial}{\partial \varrho} \left(m(\varrho) \chi \left(\frac{m(\varrho)}{m(s)} \right) \right),$$

and so the differential equation in (4.3) is equivalent to

$$\frac{\partial}{\partial \varrho} \big(m'(\varrho) \varphi_s'(\varrho) \big) = -\frac{m'(s)}{m(s)^2} \frac{\partial}{\partial \varrho} \left(m(\varrho) \chi \left(\frac{m(\varrho)}{m(s)} \right) \right).$$

Integrating with $\varphi_s'(\varrho) = 0$ for $\varrho > s$ we obtain:

$$\varphi_s'(\varrho) = -\frac{m'(s) m(\varrho)}{m(s)^2 m'(\varrho)} \chi \left(\frac{m(\varrho)}{m(s)} \right) = -\frac{\varrho}{s^{n+\alpha+1}} \chi \left(\frac{\varrho^{n+\alpha}}{s^{n+\alpha}} \right). \tag{4.4}$$

A final integration with $\varphi_s(\varrho) = 0$ for $\varrho > s$ yields

$$\varphi_s(\varrho) = \int_\varrho^\infty \frac{r}{s^{n+\alpha+1}} \chi \left(\frac{r^{n+\alpha}}{s^{n+\alpha}} \right) dr, \tag{4.5}$$

showing that we find a function satisfying as a matter of fact $\varphi_s(\varrho) = 0$ for $\varrho > s$. □

Remark 4.1. Using the technique of Theorem 1.1, with $m(r) = r^{n+\alpha}$ replaced by $m(r) = r^{n+\alpha+1}$, one obtains a mean value formula for $\mathcal{L}$-harmonic functions at $0 \in \mathbb{R}^{n+1}$, where $|\delta \varrho|$ in (1.1) is replaced by $|X \varrho|$, and $\mathcal{L}$ is the Grushin Laplacian (1.2). The same technique works when $\mathcal{L} = \sum_{i=1}^m X_j^2$ is the sub-Laplacian of any family $X_1, \dots, X_m$ of smooth vector fields in $\mathbb{R}^{n+1}$ satisfying the Hörmander condition, with $2 \le m \le n+1$ and admitting a global fundamental solution. The resulting mean-value formulas coincide with the formulas obtained in [5].

We explain the relation between the two approaches in the case of a Carnot group of topological dimension $d > 2$. Let Γ be the fundamental solution of the corresponding Carnot sub-Laplacian $\mathcal{L}$ with pole at 0. For a harmonic function $\mathcal{L} f = 0$, the mean value formula (1.4) proved in [5] reads

$$f(0) = \frac{1}{r} \int_{\{\xi \in \mathbb{R}^d : \Gamma(\xi) > \frac{1}{r}\}} f(\xi)\, |X(\log \Gamma)|^2 \, \varphi \left(\frac{1}{r \Gamma(\xi)} \right) d\xi, \quad r > 0, \tag{4.6}$$

where φ is any continuous function on the interval $[0,1]$ with unit integral.

Let $\varrho(\xi) = \Gamma(\xi)^{\frac{1}{Q-2}}$, $\xi \neq 0$, where $Q \in \mathbb{N}$ is the homogeneous dimension of the group. The Lebesgue measure of the balls $B_s = \{\xi \in \mathbb{R}^d \colon \varrho(\xi) < s\}$ satisfies $m(s) = \mathcal{L}^d(B_s) = C s^Q$ for some constant $C > 0$ depending on n and Q. Using the technique of Theorem 1.1 we get the mean value formula

$$f(0) = \frac{C_{d,Q}}{m(s)} \int_{B_s} f(\xi)\, |X \varrho(\xi)|^2 \, d\xi, \tag{4.7}$$

with $C_{d,Q} > 0$ fixed on choosing $f = 1$. Formula (4.7) is precisely formula (4.6) with the choice $\varphi(t) = \frac{Q}{Q-2} t^{\frac{2}{Q-2}}$. In fact, in this case we have

$$|X(\log \Gamma)|^2 \varphi\left(\frac{1}{r\Gamma}\right) = Q(Q-2) r^{\frac{2}{2-Q}} |X\varrho|^2,$$

and we obtain equivalence with (4.7) by setting $r = s^{Q-2}$.

Proof of Theorem 1.2. The proof is identical to the proof of Theorem 1.1 with minor modifications that we list below. For any nonnegative $\psi \in C_c^\infty(\Sigma)$, we have

$$0 \le \int_\Sigma f \, \mathcal{L}_\Sigma \psi \, d\sigma.$$

The function φ_s is the solution to (4.3) defined in (4.5). Notice that $\varphi_s' \le 0$ if $\chi \ge 0$ in (4.4). Then we have $q_\Sigma(\xi)\varphi_s'(\varrho(\xi)) \le 0$ for $\xi \in \Sigma^*$. By formula (3.12), the function $\psi_s = \varphi_s \circ \varrho$ thus satisfies

$$\mathcal{L}_\Sigma \psi_s \le \frac{\partial}{\partial s}\left(\frac{1}{m(s)} \chi\left(\frac{m(\varrho)}{m(s)}\right)\right) |\delta\varrho|^2,$$

and we get, for $0 < s < r$,

$$\frac{1}{m(s)} \int_{B_s \cap \Sigma} f(\xi)\, |\delta\varrho|^2 \, d\sigma \le \frac{1}{m(r)} \int_{B_r \cap \Sigma} f(\xi)\, |\delta\varrho|^2 \, d\sigma.$$

The choice $f = 1$ shows the existence of the limit in (1.7). □

Remark 4.2. If Σ is α-superharmonic, $q_\Sigma \le 0$, then a function f with $\mathcal{L}_\Sigma f \le 0$ satisfies the super-mean-value formula at 0:

$$f(0) \ge \frac{C_{\Sigma,n,\alpha}}{r^{n+\alpha}} \int_{B_r \cap \Sigma} f(\xi)\, |\delta\varrho(\xi)|^2 \, d\sigma.$$

The proof is the same as in the sub-harmonic case.

5. Analysis of the structural function q_Σ

5.1. Homogeneous hypersurfaces are harmonic. In $\mathbb{R}^{n+1}$ with the Grushin structure, we introduce the anisotropic dilations $d_\lambda\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $\lambda > 0$,

$$d_\lambda(\xi) = d_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y), \quad \xi \in \mathbb{R}^{n+1}.$$

We say that a set $\Sigma \subset \mathbb{R}^{n+1}$ is d_λ-homogeneous if $d_\lambda(\Sigma) = \Sigma$ for any $\lambda > 0$.

Lemma 5.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface of class C^2 with $0 \in \Sigma$. If Σ is d_λ-homogeneous then Σ is α-harmonic at 0.*

Proof. We check the claim when Σ is a y-graph $\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} \colon x \in \mathbb{R}^n\}$ for some function $u \in C^2(\mathbb{R}^n)$ satisfying the identity $u(\lambda x) = \lambda^{\alpha+1} u(x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$. Differentiating this identity at $\lambda = 1$ we get

$$\langle \nabla u(x), x\rangle = (\alpha+1) u(x), \quad x \in \mathbb{R}^n. \tag{5.1}$$

Using formulas (2.3) for the α-normal $\nu = (\bar\nu, \nu_{n+1})$, (5.1) is equivalent to

$$|x|^\alpha \langle x, \bar\nu\rangle + (\alpha+1) y \nu_{n+1} = 0, \quad (x, y) \in \Sigma^*.$$

By formula (3.15) this is in turn equivalent to $\langle X\varrho, \nu\rangle = 0$ on Σ^*, and this implies $q_\Sigma = 0$, see (1.4). □

When Σ is d_λ-homogeneous we have $\langle X\varrho, \nu\rangle = 0$ and, using (3.8), the kernel $|\delta\varrho|^2$ appearing in the mean value formula (1.6) reduces to

$$|\delta\varrho|^2 = |X\varrho|^2 = \frac{|x|^{2\alpha}}{\varrho^{2\alpha}}.$$

This kernel is 0-homogeneous with respect to the dilations d_λ and satisfies $|\delta\varrho|^2 \le 1$.

5.2. Flat case. The hyperplane $\Sigma = \{(x, y) \in \mathbb{R}^{n+1} \colon y = 0\}$ is d_λ homogeneous and it is therefore α-harmonic. The α-normal is constant, $\nu = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$, and it follows that Σ is also α-minimal, $H_\Sigma = 0$.

The tangential gradient δ reduces to the standard gradient $\delta\varphi = (\nabla_x\varphi, 0)$ and the non-adjoint Laplacian Δ_Σ in (3.5) reduces to the standard Laplacian Δ_x in $\mathbb{R}^n$, see (3.7). From formula (3.6) we deduce that

$$\mathcal{L}_\Sigma\varphi = \Delta_x\varphi + \alpha\langle\nabla_x \log|x|, \nabla_x\varphi\rangle = \frac{1}{|x|^\alpha}\operatorname{div}\Big(|x|^\alpha\nabla_x\varphi\Big).$$

Theorem 1.1 states that a function $\varphi \in C^2(\mathbb{R}^n)$ satisfying $\mathcal{L}_\Sigma\varphi = 0$ has the mean value property at 0

$$\varphi(0) = \frac{n+\alpha}{n\omega_n r^{n+\alpha}} \int_{\{|x|<r\}} \varphi(x)|x|^\alpha\, dx,$$

with $\omega_n = \mathcal{L}^n(\{|x| < 1\})$.

5.3. $\boldsymbol{\alpha}$-subharmonic surfaces. In this section, we look for sufficient conditions for a hypersurface Σ to be α-subharmonic at $0 \in \Sigma$.

Definition 5.2. (η-flatness) Let $\eta > 0$. We say that a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is η-flat at $0 \in \Sigma$ if there exists $r > 0$ such that its α-normal $\nu = (\bar\nu, \nu_{n+1})$ satisfies

$$(\alpha+1)|y||\nu_{n+1}| \le \eta|x|^\alpha|\langle x, \bar\nu\rangle| \tag{5.2}$$

for all points $\xi = (x, y) \in \Sigma^* \cap B_r$.

When Σ is the y-graph of a function u, condition (5.2) reads

$$(\alpha+1)|u| \le \eta|\langle x, \nabla_x u\rangle| \tag{5.3}$$

holding for points a neighborhood of $0 \in \mathbb{R}^n$.

Lemma 5.3. *Let $u \in C^1(\{|x| \le 1\})$ be a function satisfying* (5.3). *Then for any point $|x| \le 1$ we have*

$$|u(x)| \le \Big(\max_{|x|=1}|u|\Big)\,|x|^{\frac{\alpha+1}{\eta}}. \tag{5.4}$$

Proof. Let $|x| = 1$ be fixed. We prove the claim along the segment tx, with $t \in [0, 1]$. Letting $\varphi : [0, 1] \to \mathbb{R}$, $\varphi(t) = u(tx)$, assumption (5.3) reads

$$(\alpha+1)|\varphi(t)| \le \eta t|\varphi'(t)|. \tag{5.5}$$

If $\varphi = 0$ on $[0, 1]$, the claim is trivial. Then we can assume that the open set $A = \{t \in (0, 1)\colon \varphi(t) \ne 0\}$ is nonempty. This set is a finite or countable disjoint union of intervals $(a, b) \subset A$. We always have $\varphi(a) = 0$ and from (5.5) it follows that $\varphi' \ne 0$ on (a, b), say $\varphi'(t) > 0$ for any $t \in (a, b)$. Then φ is strictly monotone increasing and thus $\varphi(b) > 0$, and so $b = 1$. It follows that $A = (a, 1)$, for some $a \in [0, 1)$, and $\varphi = 0$ on $[0, a]$.

We can without loss of generality assume that $a = 0$ and conclude the proof in the following way. We have $\varphi > 0$ and, say, $\varphi' > 0$ on $(0, 1)$. Then (5.5) reads

$$\frac{d}{dt}\log(t^{\alpha+1}) \le \eta \frac{d}{dt}\log\varphi(t),$$

and integrating this inequality from $t = s$ to $t = 1$, $0 < s < 1$, we get $\varphi(s) \le \varphi(1)s^{\frac{\alpha+1}{\eta}}$. This implies (5.4) and the proof is concluded. □

Theorem 5.4. *Let $\Sigma \subset \mathbb{R}^{n+1}$ by a hypersurface of class C^2 with α-normal $\nu = (\bar\nu, \nu_{n+1})$ and α-mean curvature H_Σ. Assume that:*

i) *Σ is η-flat at $0 \in \Sigma$ for some*

$$0 < \eta < \frac{n+\alpha}{n+3\alpha}. \tag{5.6}$$

ii) *We have, with limit restricted to $\xi = (x, y) \in \Sigma$,*

$$\lim_{\xi\to 0}\frac{|x|^2 H_\Sigma(\xi)}{\langle \bar\nu(\xi), x\rangle} = 0. \tag{5.7}$$

Then there exists a $\delta > 0$ such that $q_\Sigma \ge 0$ on $\Sigma^* \cap B_\delta$.

Proof. Without loss of generality we may assume that $\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} : |x| \le 1\}$ for some function $u \in C^2(\{|x| \le 1\})$, and that (5.2) holds with $r = 1$. Letting $C_1 = \max_{|x|=1}|u(x)|$, points $(x, y) \in \Sigma$ satisfy by (5.4)

$$|y| \le C_1|x|^{\frac{\alpha+1}{\eta}}. \tag{5.8}$$

Formula (1.4) for q_Σ reads

$$q_\Sigma(\xi) = \langle X\varrho, \nu\rangle\left[\frac{n+3\alpha}{\varrho}\langle X\varrho, \nu\rangle - \frac{2\alpha}{|x|^2}\langle x, \bar\nu\rangle + nH_\Sigma\right],$$

and the inequality $q_\Sigma \ge 0$ is thus implied by

$$\left|\frac{2\alpha}{|x|^2}\langle x, \bar\nu\rangle - nH_\Sigma\right| \le \frac{n+3\alpha}{\varrho}|\langle X\varrho, \nu\rangle|. \tag{5.9}$$

Inserting the identity (3.15) into (5.9) and then using (3.8), we see that $q_\Sigma \ge 0$ is implied by

$$\left|\frac{2\alpha}{|x|^2}\langle x, \bar\nu\rangle - nH_\Sigma\right| \le \frac{n+3\alpha}{\varrho^{2\alpha+2}}|x|^\alpha\big||x|^\alpha\langle x, \bar\nu\rangle + (\alpha+1)y\nu_{n+1}\big|. \tag{5.10}$$

By the η-flatness condition (5.2), (5.10) is in turn implied by

$$\left|\frac{2\alpha}{|x|^2}\langle x, \bar\nu\rangle - nH_\Sigma\right| \le (1-\eta)\frac{n+3\alpha}{\varrho^{2\alpha+2}}|x|^{2\alpha}|\langle x, \bar\nu\rangle|, \tag{5.11}$$

and finally, using assumption (5.7), (5.11) is implied by

$$\left(2\alpha + \frac{n|x|^2|H_\Sigma|}{|\langle x, \bar\nu\rangle|}\right) \le (1-\eta)(n+3\alpha)\frac{|x|^{2\alpha+2}}{\varrho^{2\alpha+2}}. \tag{5.12}$$

Now observe that, by (5.8),

$$\frac{\varrho^{2(\alpha+1)}}{|x|^{2(\alpha+1)}} = 1 + \frac{(\alpha+1)^2y^2}{|x|^{2(\alpha+1)}} \le 1 + C_1(\alpha+1)^2|x|^{2(\alpha+1)(1/\eta-1)}. \tag{5.13}$$

By (5.13) and (5.7), inequality (5.12) is satisfied for $x = 0$ as a strict inequality, by (5.6). Our claim follows by a limiting argument. □

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