Frostman lemma revisited

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Abstract. We study sharpness of various generalizations of Frostman's lemma. These generalizations provide better estimates for the lower Hausdorff dimension of measures. As a corollary, we prove that if a generalized anisotropic gradient $(\partial_1^{m_1} f, \partial_2^{m_2} f, \ldots, \partial_d^{m_d} f)$ of a function f in d variables is a measure of bounded variation, then this measure is absolutely continuous with respect to the Hausdorff $d-1$ dimensional measure.

Paluu Frostmanin lemman pariin

Tiivistelmä. Työssä tutkitaan Frostmanin lemman eri yleistysten tarkkuutta. Nämä yleistykset antavat parempia arvioita mittojen Hausdorffin alaulottuvuudelle. Seurauksena todistetaan, että jos d-muuttujaisen funktion f sekakertalukuinen osittaisderivaattavektori $(\partial_1^{m_1} f, \partial_2^{m_2} f, \ldots, \partial_d^{m_d} f)$ on rajallisesti heilahteleva mitta, on tämä mitta absoluuttisesti jatkuva $(d-1)$-ulotteisen Hausdorffin mitan suhteen.

1. Introducion

The classical Frostman lemma says the following (see [11], Theorem 8.8 for details). Let $B_r(x)$ be the open Euclidean ball of radius r centered at $x \in \mathbb{R}^d$. For any Borel set $F \subset \mathbb{R}^d$, its α-Hausdorff measure is not zero if and only if there exists a finite non-zero Borel measure μ supported on F such that the inequality

$$(1.1) \qquad \mu(B_r(x)) \leq r^\alpha$$

holds true for any r and x.

The main objective of our investigation is to understand the dimensional properties of measures by means of new covering theorems. To this end, let us recall the definition of lower Hausdorff dimension of a measure.

The Hausdorff dimension of a set is defined as follows:

$$(1.2) \qquad \dim_{\mathcal{H}} F = \inf\{\alpha \mid \mathcal{H}^\alpha(F) = 0\},$$

where $\mathcal{H}^\alpha(F)$ is the α-Hausdorff measure of F:

$$(1.3) \qquad \mathcal{H}^\alpha(F) = \lim_{\delta \to 0} \inf_{\substack{F \subset \cup B_j \\ \operatorname{diam}(B_j) < \delta}} \sum_j \operatorname{diam}(B_j)^\alpha.$$

Let μ be a possibly vector-valued (with values in a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$) locally finite Borel measure on $\mathbb{R}^d$. The said definition reads as follows:

$$(1.4) \qquad \begin{aligned} \dim_{\mathcal{H}} \mu = \inf\{\alpha \mid {}&\text{there exists a Borel set } F \text{ such that} \\ &\dim_{\mathcal{H}} F \leq \alpha \text{ and } \mu(F) \neq 0\}. \end{aligned}$$

For example, the dimension of a delta-measure δ_x, $x \in \mathbb{R}^d$, is zero, the dimension of the Lebesgue measure λ_d on $\mathbb{R}^d$ is d, and, finally, $\dim_{\mathcal{H}}(\lambda_d + \delta_x) = 0$.

Since we will be working with the Hausdorff dimension only, we suppress the index $\mathcal{H}$ in our notation. See Chapter 10 in [9] for more information on Hausdorff dimension of measures. Note that, if a measure μ satisfies the estimate

$$(1.5) \qquad |\mu(B_r(x))| \lesssim r^\alpha$$

for all $x \in \mathbb{R}^d$ and $r > 0$ uniformly, then the Frostman lemma implies $\dim \mu \geq \alpha$. We have used the simpler-to-prove implication in the Frostman lemma. Here and in what follows the notation $A \lesssim B$ expresses there exists a uniform constant C such that $A \leq CB$. In particular, the constant in (1.5) should depend neither on x nor r.

The inequality (1.5) is not equivalent to the assertion $\dim \mu \geq \alpha$ or any small perturbation of it. A good example is the measure $|x|^{-\beta}\, dx$, $\beta < 1$, on the line, which has dimension 1, but violates (1.5) for $\alpha > 1 - \beta$ (Lemma 5.1 below provides generalizations of this example). On the other hand, the assertion $\dim \mu \geq \alpha$ is equivalent to the statement that for μ almost every x and any $\varepsilon > 0$ the inequality

$$(1.6) \qquad |\mu|(B_r(x)) \leq r^{\alpha - \varepsilon}$$

holds true for all sufficiently small r (depending on ε and x), see Proposition 10.2 in [9]. Therefore, the latter local condition is quite difficult to handle, and it is desirable to provide a more uniform global one.

Let μ be a non-negative scalar measure. Consider the energy integral

$$(1.7) \qquad \iint_{\mathbb{R}^{2d}} \frac{d\mu(x)\, d\mu(y)}{|x - y|^\alpha}.$$

If the energy integral converges, then $\dim \mu \geq \alpha$. The energy method may be efficient for some problems, but it has certain crucial limitations (see [12], Section 3.5 for the details on the energy method and discussion in Section 3.6 for limitations). The following lemma provides another uniform condition sufficient for $\dim \mu \geq \alpha$.

Lemma 1.1. (Stolyarov and Wojciechowski [15, Lemma 1]) *Suppose that $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a radially decreasing non-negative function supported in the unit ball. Assume that φ decreases as the radius grows and $\varphi(x) = 1$ when $|x| \leq \frac{3}{4}$. Let μ be a measure such that for every collection $B_{r_j}(x_j)$ of d-dimensional balls such that $B_{3r_j}(x_j)$ are disjoint, the estimate*

$$(1.8) \qquad \sum_j \left| \int_{\mathbb{R}^d} \varphi_{3r_j}(x_j + y)\, d\mu(y) \right| \lesssim \left(\sum r_j^\alpha \right)^\beta$$

holds true for some positive α and β. Then $\dim \mu \geq \alpha$. Here $\varphi_t(z) = \varphi\left(\frac{z}{t}\right)$.

In the case $\beta = 1$, the above lemma reduces to the classical Frostman lemma, however, the case $\beta < 1$ is a priori stronger (Lemma 5.1 below provides examples). In the case $\beta > 1$, the result is considerably sub-optimal with respect to classical Frostman's lemma. In [15], the lemma served as a technical tool to prove that any vector-valued measure of the form $(\partial_1^{m_1} f, \partial_2^{m_2} f, \ldots, \partial_d^{m_d} f)$ has dimension at least $d - 1$. Here ∂_j is the operator of differentiation with respect to j-th coordinate and $m_1, m_2, \ldots, m_d$ are arbitrary natural numbers. Note that the case $m_1 = m_2 = \ldots = m_d = 1$ may be deduced from the co-area formula for BV functions (see [1, Theorem 3.40] for this formula).

In recent years, there is an increasing interest in the geometry of measures satisfying PDE or Fourier constraints, like gradients of functions of bounded variation,

or divergence free measures, or, for example, the generalized gradient measures as above. We refer the reader to the papers [2, 3, 5, 4, 8, 13], to mention a few (see [17] for limiting Sobolev inequalities for vectorial differential operators that are closely related to the topic). Of primary interest is the dimension problem: what is the lowest possible Hausdorff dimension of a measure solving a specific PDE? Lemma 1.1 appeared useful in this context, in particular, an analog of this lemma plays the pivotal role in [7] (see Lemma 4.2 in that paper), where a simpler discrete analog of the dimension problem for Fourier constrained measures is solved. Some techniques of [7] were later applied to the original dimension problem (see [14] and [5]). A similar lemma provides good (better than the ones given by the energy method) dimensional estimates for Riesz products, see [6] (the proof of Theorem 2.8). The purpose of this article is to sharpen and generalize Lemma 1.1. These considerations will also lead to a corollary about generalized gradients (see Theorem 2.2 below).

2. Statement of results

There are several objects playing a subtle role in the statement of Lemma 1.1. The function φ is, in a sense, auxiliary. It allows to replace the expression $\mu(B_r(x))$ with a smoother one. Namely, the integral

$$(2.1) \qquad \int_{\mathbb{R}^d} \varphi_{3r}(x + y) \, d\mu(y)$$

may be thought of as a smoothing of $\mu(B_r(-x))$. And, the expression on the left hand side in (1.8) may be thought of as the sum $\sum_j |\mu(B_{r_j}(-x_j))|$. The parameter β is quite mysterious since the estimate $\dim \mu \geq \alpha$ does not depend on β. We prefer to replace the number β with a function, i.e., to estimate the left hand side of (1.8) with $g\left(\sum_j r_j^\alpha\right)$, where g is a certain weight function; in Lemma 1.1, $g(r) = r^\beta$. This generalization seems reasonable the light of the entropy estimates in [7].

Definition 2.1. A continuous non-decreasing function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ is called regular provided $g(0) = 0$, and for some fixed $c > 1$ we have $g(x) \asymp g(cx)$.

Here and in what follows $A \asymp B$ is short for $A \lesssim B$ and $B \lesssim A$. We say that the functions φ and ψ are equivalent if $\varphi(x) \asymp \psi(x)$ for any $x \in \mathbb{R}^d$. For any $\theta > 0$, the function $g(x) = x^\theta$ is regular. An example of a non regular function is $g(x) = e^{-1/x}$.

Definition 2.2. A regular function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the Dini condition provided

$$(2.2) \qquad \int_0^1 \frac{g(t)}{t} \, dt < \infty.$$

Many regular functions satisfy the Dini condition. For example, the function $g(x) = x^\theta$, $\theta > 0$, does. An example of a regular function violating the Dini condition is $g(x) = 1/|\log x|$.

We are ready to formulate our first main result. By a disjoint family of balls we mean a collection of balls none of which intersect.

Theorem 2.1. *Let φ be a radially symmetric, radially non-increasing function supported in the unit ball. Assume also $\varphi(x) = 1$ when $|x| \leqslant \frac{3}{4}$. Let ψ be equivalent*

to φ. Let μ be an $\mathbb{R}$-valued signed Radon measure, let g be a regular function that satisfies the Dini condition. Assume that

$$(2.3) \qquad \sum_{B_{r_j}(x_j) \in \mathfrak{B}} \left| \int_{\mathbb{R}^d} \psi \left(\frac{y - x_j}{r_j} \right) d\mu(y) \right| \lesssim g \left(\sum_{B_{r_j}(x_j) \in \mathfrak{B}} r_j^\alpha \right)$$

for any disjoint family of balls $\mathfrak{B}$. Then,

$$(2.4) \qquad |\mu|(A) \lesssim h(\mathcal{H}^\alpha(A)),$$

where $h(x) = \int_0^x \frac{g(t)}{t} \, dt$ and $A \subset \mathbb{R}^d$ is an arbitrary Borel set.

Remark 2.1. The constant in (2.4) may depend on the constant in (2.3), the equivalence of φ and ψ, and the regularity of g.

It is easy to see that $g \lesssim h$ for regular g. In particular, μ is absolutely continuous with respect to $\mathcal{H}^\alpha$, and Lemma 1.1 is a corollary of Theorem 2.1 ($g(t) = t^\beta$ is regular and satisfies the Dini condition). We formulate our second main result.

Theorem 2.2. *Let φ be a bounded function supported in the unit ball and such that $\varphi(x) \geqslant 1$ when $|x| \leqslant \frac{3}{4}$. Let μ be an $\mathbb{R}$-valued signed Radon measure and let g be a regular function. Assume also that $\alpha > d - 1$. If*

$$(2.5) \qquad \sum_{B_{r_j}(x_j) \in \mathfrak{B}} \left| \int_{\mathbb{R}^d} \varphi \left(\frac{y - x_j}{r_j} \right) d\mu(y) \right| \lesssim g \left(\sum_{B_{r_j}(x_j) \in \mathfrak{B}} r_j^\alpha \right)$$

for any disjoint family of balls $\mathfrak{B}$, then

$$(2.6) \qquad |\mu|(A) \lesssim g(\mathcal{H}^\alpha(A))$$

for any Borel set $A \subset \mathbb{R}^d$.

Note that unlike Theorem 2.1, φ is not assumed to be radial in Theorem 2.2. We do not know whether this assumption is necessary for Theorem 2.1. Though Theorems 2.1 and 2.2 look similar, their proofs are quite different. Theorem 2.2 has a nice corollary about PDE constrained measures.

Corollary 2.2. *Let $m_1, m_2, \ldots, m_d$ be natural numbers. Assume f is a distribution such that for any $j = 1, 2, \ldots, d$ the distribution $\partial_j^{m_j} f$ is a signed measure of bounded variation. Then, these signed measures are absolutely continuous with respect to $\mathcal{H}^{d-1}$.*

We also provide a version of Lemma 1.1 where the function φ is not compactly supported. This version is surprisingly easier to prove.

Lemma 2.3. *Let φ be a bounded radially symmetric, radially non-increasing function such that $B_3(0) \subset \operatorname{supp} \varphi$. Let ψ be equivalent to φ. Let μ be a signed measure of bounded variation, let g be a regular function. Assume*

$$(2.7) \qquad \sum_{B_{r_j}(x_j) \in \mathfrak{B}} \left| \int_{\mathbb{R}^d} \psi \left(\frac{y - x_j}{r_j} \right) d\mu(y) \right| \lesssim g \left(\sum_{B_{r_j}(x_j) \in \mathfrak{B}} r_j^\alpha \right)$$

for any disjoint family of balls $\mathfrak{B}$. If

$$(2.8) \qquad |\varphi(x)| = O(|x|^{-\alpha}), \quad x \to \infty,$$

then μ is absolutely continuous with respect to $\mathcal{H}^\alpha$. If

(2.9) $$|\varphi(x)| = o(|x|^{-\alpha}), \quad x \to \infty,$$

then the inequality

(2.10) $$|\mu|(A) \lesssim g(\mathcal{H}^\alpha(A))$$

holds true for any Borel set A.

The proofs of Theorems 2.1 and 2.2 are based on new covering lemmas, which may be interesting in their own right. They deal with a simple notion that seems to be important for working with local properties of signed measures.

Definition 2.3. A family $\mathfrak{B}$ of open Euclidean balls is called a supercovering of a set $A \subset \mathbb{R}^d$, provided

(2.11) $$A \subset \bigcup_{B_{r_j}(x_j) \in \mathfrak{B}} B_{\frac{r_j}{3}}(x_j).$$

Lemma 2.4. Let $\alpha \in [0, d]$. There exist the constants $q(\alpha) \in (0, 1)$ and $C(d, \alpha) > 0$ such that for any compact set $K \subset \mathbb{R}^d$, $\mathcal{H}^\alpha(K) < a$, and any $\varepsilon > 0$ there exists a finite supercovering $\mathfrak{B}$ satisfying the requirements

1) the center of any ball of $\mathfrak{B}$ lies inside K and its radius does not exceed ε,
2) there exists a natural number N such that $\mathfrak{B}$ may be split into N disjoint subfamilies $\mathfrak{B}^j$ satisfying the bound

(2.12) $$\sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant Cq^j a, \quad j = 1, 2, \ldots, N.$$

Remark 2.5. Lemma 2.4 does not provide any control on N.

Lemma 2.6. Let $\alpha \in (d - 1, d]$. There exist the constants $C_1(d) > 0$ and $C_2(d, \alpha) > 0$ such that for any bounded set $A \subset \mathbb{R}^d$, $\mathcal{H}^\alpha(A) < a$, and any $\varepsilon > 0$, there exists a supercovering $\mathfrak{B}$, satisfying the requirements

1) the center of any ball of $\mathfrak{B}$ lies inside A and its radius does not exceed ε,
2) the family $\mathfrak{B}$ may be split into C_1 disjoint subfamilies $\mathfrak{B}^j$ such that

(2.13) $$\sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant C_2 a, \quad j = 1, 2, \ldots, C_1.$$

The constants in the lemmas admit explicit bounds. The proofs of Lemmas 2.4 and 2.6 are presented in Section 3. The proofs of Theorems 2.1 and 2.2 as well as Corollary 2.2 may be found in Section 4. The last section contains some generalizations of these results and some examples.

3. Constructions of fine coverings

We need some tools.

Lemma 3.1. For any $d \in \mathbb{N}$ there exists a constant $\theta(d)$ such that the following holds. Let A be a bounded subset of $\mathbb{R}^d$ and let $\mathfrak{B}$ be a family of balls in $\mathbb{R}^d$. Assume that

(3.1) $$\forall x \in A \quad \exists B_r(y) \in \mathfrak{B} \quad \text{such that } x \in B_{\frac{r}{3}}(y).$$

Then there exists a subfamily $\mathfrak{B}' \subset \mathfrak{B}$ satisfying the requirements

1) $A \subset \bigcup_{B_{r_i}(x_i) \in \mathfrak{B}'} B_{r_i}(x_i),$

2) *the family $\mathfrak{B}'$ may be split into θ disjoint subfamilies.*

Lemma 3.1 is a particular case of the Morse covering theorem (see [10, p. 6]).

Definition 3.1. A subset $A \subset \mathbb{R}^d$ is called ε-separated if the distance between any two points in A is not less than ε. An ε-separated subset of A is called maximal if it is maximal by inclusion.

Proposition 3.2. *An ε-separated set can be split into at most 100^d 6ε-separated subsets.*

Proof. Let E be an ε-separated set and let E_1 be a maximal 6ε-separated subset of E. Define the sets E_j inductively. Let E_j be a maximal 6ε-separated subset of $E \setminus \bigcup_{i<j} E_i$ for $2 \leqslant j \leqslant 100^d$. We will proof that

$$(3.2) \qquad E = \bigcup_{j=1}^{100^d} E_j.$$

Assume the contrary. Let x be a point in $E \setminus \bigcup_{i \leqslant 100^d} E_i$. It is clear that $\operatorname{dist}(x, E_j) < 6\varepsilon$ for all $j = 1, 2, \ldots, 100^d$. Consequently, $B_{6\varepsilon}(x)$ contains at least 100^d points of E. Since E is ε-separated, this is a contradiction. $\qquad\square$

We omit the proof of the following proposition.

Proposition 3.3. *Let E be a maximal ε-separated subset of A. Then the family $\mathfrak{B} = \{B_{3\varepsilon}(x) \mid x \in E\}$ is a supercovering of set A.*

Lemma 3.4. *Let $\alpha \in [0, d]$. There exist the constants $C(\alpha)$ and $\theta(d)$ such that for any bounded $A \subset \mathbb{R}^d$, $\mathcal{H}^\alpha(A) < a$ (for some constant a), and any $\varepsilon > 0$ there exists a family of balls $\mathfrak{B}$ such that*

1) $\mathfrak{B}$ *is a covering of A,*
2) *the center of any ball in $\mathfrak{B}$ lies in A and its radius does not exceed ε,*
3) *the family $\mathfrak{B}$ may be split into θ disjoint subfamilies $\mathfrak{B}^j$ such that*

$$(3.3) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant Ca.$$

Proof. Since $\mathcal{H}^\alpha(A) < a$ there exists a family $\mathfrak{A}$ such that $\mathfrak{A}$ is a covering A, for any $D \in \mathfrak{A}$ we have $\operatorname{diam}(D) < \frac{\varepsilon}{1000}$, and

$$(3.4) \qquad \sum_{D_i \in \mathfrak{A}} \operatorname{diam}(D_i)^\alpha < a.$$

The family $\mathfrak{B}_0$ is defined in the following way: for any $D \in \mathfrak{A}$ we choose a point $x \in D \cap A$ and put the ball $B_{3\operatorname{diam}(D)}(x)$ into $\mathfrak{B}_0$. The family $\mathfrak{B}_0$ and the set A satisfy the conditions of Lemma 3.1. Let $\mathfrak{B}$ be the subfamily of $\mathfrak{B}_0$ provided by Lemma 3.1. The family $\mathfrak{B}$ splits into $\theta(d)$ disjoint subfamilies $\mathfrak{B}^j$ that satisfy the estimate:

$$(3.5) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant \sum_{B_{r_i}(x_i) \in \mathfrak{B}_0} r_i^\alpha \leqslant \sum_{D_i \in \mathfrak{A}} (3\operatorname{diam}(D_i))^\alpha \leqslant 3^\alpha a. \qquad \square$$

Lemma 3.5. *Let $K \subset \mathbb{R}^d$ be a compact set such that $\mathcal{H}^\alpha(K) < a$ (for some constant a) and let $\varepsilon > 0$. There exists a constant M and a family of closed balls $\mathfrak{B}$ such that*

1) $\mathfrak{B}$ *is a covering of K,*
2) *the center of any ball in $\mathfrak{B}$ lies inside K and its radius does not exceed ε,*

3) $\#\mathfrak{B} = M$ *and*

$$(3.6) \qquad \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}} r_i^\alpha < a,$$

4) *for any family* $\mathfrak{B}'$ *that satisfies 1), 2), 3) the inequality*

$$(3.7) \qquad \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}} r_i^\alpha \leqslant \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}'} r_i^\alpha$$

is true.

The constant M depends on K and ε. The set $\bar{D}$ is the closure of D.

Proof. Since $\mathcal{H}^\alpha(A) < a$ there exists a family $\mathfrak{A}$ such that $\mathfrak{A}$ is a covering of A, for any $D \in \mathfrak{A}$ we have $\mathrm{diam}(D) < \frac{\varepsilon}{1000}$, and

$$(3.8) \qquad \sum_{D \in \mathfrak{A}} \mathrm{diam}(D)^\alpha < a.$$

The family $\mathfrak{B}_0$ is defined in the following way: for any $D \in \mathfrak{A}$ we choose a point $x \in D \cap A$ and put the ball $\bar{B}_{\mathrm{diam}(D)}(x)$ into $\mathfrak{B}_0$ ($\mathfrak{B}_0$ is the first step of the construction towards $\mathfrak{B}$). Note that

$$(3.9) \qquad \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}_0} r_i^\alpha \leqslant \sum_{D \in \mathfrak{A}} \mathrm{diam}(D)^\alpha < a.$$

There exists $\delta > 0$ such that

$$(3.10) \qquad \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}_0} ((1+\delta)r_i)^\alpha < a.$$

If we multiply the radii of the balls $\bar{B} \in \mathfrak{B}_0$ by $1 + \delta$ and make the balls open, then we will get an open covering of the compact set K. Let $\mathfrak{B}_1$ be a finite subcovering of this open covering. Note that we still have

$$(3.11) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}_1} r_i^\alpha \leqslant \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}_0} ((1+\delta)r_i)^\alpha < a.$$

Therefore if we close the balls in the family $\mathfrak{B}_1$ and take $M = \#\mathfrak{B}_1$, we will have a family that satisfies conditions 1), 2), 3). Now we will prove that there exists an optimal family of this kind. The set $S \subset (\mathbb{R}^d)^M \times \mathbb{R}^M$ is defined by the formula

$$(3.12) \qquad S = \big\{ (x_1, x_2, \ldots, x_M, r_1, \ldots, r_M) \mid x_j \in K, \ r_j \in [0, \varepsilon], \ K \subset \bigcup \bar{B}_{r_j}(x_j) \big\}.$$

It is easy to see that S is compact. Thus, there exists a point in S that minimizes the continuous function

$$(3.13) \qquad L(x_1, x_2, \ldots, x_M, r_1, \ldots, r_M) = \sum_{j=1}^M r_j^\alpha.$$

This point corresponds to the desired optimal family $\mathfrak{B}$. $\qquad\square$

Proof of Lemma 2.4. Let $\mathfrak{B}_0$ be a family of closed balls constructed in Lemma 3.5. The family $\mathfrak{B}$ will be some transformation of the family $\mathfrak{B}_0$. First, we split $\mathfrak{B}_0$ into families $\mathfrak{B}_1$ and $\mathfrak{B}_2$. The family $\mathfrak{B}_1$ consists of balls with radii not less than $\frac{\varepsilon}{9}$, and the family $\mathfrak{B}_2$ consists of balls with radii less than $\frac{\varepsilon}{9}$. We will transform the families

$\mathfrak{B}_1$ and $\mathfrak{B}_2$ into the families $\mathfrak{C}_1$ and $\mathfrak{C}_2$ and set $\mathfrak{B} = \mathfrak{C}_1 \cup \mathfrak{C}_2$. Let K_0 be defined by the following formula:

$$(3.14) \qquad K_0 = K \cap \bigcup_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}_1} \bar{B}_{r_i}(x_i).$$

Let E be a maximal $\frac{\varepsilon}{3}$-separated subset of K_0. Let $\mathfrak{C}_1$ be defined by the formula

$$(3.15) \qquad \mathfrak{C}_1 = \{B_\varepsilon(x) \mid x \in E\}.$$

The family $\mathfrak{C}_1$ is a supercovering of the set K_0 by Proposition 3.3. The family $\mathfrak{C}_2$ is defined by the formula

$$(3.16) \qquad \mathfrak{C}_2 = \{B_{3r}(x) \mid \bar{B}_r(x) \in \mathfrak{B}_2\}.$$

The family $\mathfrak{C}_2$ is a supercovering of the set $K \setminus K_0$ (by Proposition 3.3). We will split $\mathfrak{C}_1$ and $\mathfrak{C}_2$ into subfamilies $\mathfrak{B}^j$ and this will complete the proof. The family $\mathfrak{C}_1$ can be split into disjoint subfamilies $\mathfrak{B}^1, \ldots, \mathfrak{B}^{100^d}$ (by Proposition 3.2)

$$(3.17) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant \sum_{B_{r_i}(x_i) \in \mathfrak{C}_1} r_i^\alpha = (\#\mathfrak{C}_1)\varepsilon^\alpha = (\#E)\varepsilon^\alpha.$$

The set E is an $\frac{\varepsilon}{3}$-separated set, so $\#(E \cap \bar{B}_\varepsilon(x)) \leqslant 100^d$ for any $x \in \mathbb{R}^d$. Since E is a subset of $\bigcup_{B_r(x) \in \mathfrak{B}_1} \bar{B}_r(x)$, we also have $\#E \leqslant 100^d \#\mathfrak{B}_1$. With these inequalities, we may continue the estimate:

$$(3.18) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant \#E\varepsilon^\alpha \leqslant 100^d \#\mathfrak{B}_1 \varepsilon^\alpha \leqslant 100^d 9^d \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}_1} r_i^\alpha \leqslant 100^d 9^d a.$$

The family $\mathfrak{C}_2$ will be split into $\mathfrak{B}^j$ for $j > 100^d$. We will also define the families $\mathfrak{B}^{>j}$ inductively. Let $\mathfrak{B}^{>100^d} = \mathfrak{C}_2$. Assume we have defined family $\mathfrak{B}^{>j}$. Then by Vitali's Lemma we can find its disjoint subfamily $\mathfrak{B}^{j+1}$ (if we multiply all radii in the family $\mathfrak{B}^{j+1}$ by 3, it will cover the family $\mathfrak{B}^{>j}$) and define the family $\mathfrak{B}^{>j+1} = \mathfrak{B}^{>j} \setminus \mathfrak{B}^{j+1}$. Let $q = 1 - 9^{-\alpha}$. We will prove the inequality

$$(3.19) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j+1}} r_i^\alpha \leqslant q \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j}} r_i^\alpha.$$

If we enlarge all the radii of the balls in $\mathfrak{B}^{j+1}$ 3 times, the obtained family will cover all the balls in $\mathfrak{B}^{>j+1}$. The ball $B_r(x) \in \mathfrak{B}^{>j}$ has its analogue $\bar{B}_{\frac{r}{3}}(x)$ in $\mathfrak{B}_0$. Let the family $\mathfrak{B}'$ be defined by the formula

$$(3.20) \qquad \mathfrak{B}' = \{\bar{B}_r(x) \mid \bar{B}_r(x) \in \mathfrak{B}_0 \text{ and } B_{3r}(x) \notin \mathfrak{B}^{>j}\} \cup \{\bar{B}_{3r}(x) \mid B_r(x) \in \mathfrak{B}^j\}.$$

The family $\mathfrak{B}'$ satisfies conditions 1), 2), 3) of Lemma 3.5, so we have

$$(3.21) \qquad \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}_0} r_i^\alpha \overset{\text{Lemma } 3.5}{\leqslant} \sum_{\bar{B}_{r_i}(x_i) \in \mathfrak{B}'} r_i^\alpha.$$

Consequently,

$$(3.22) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j}} \left(\frac{r_i}{3}\right)^\alpha \leqslant \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} (3r_i)^\alpha,$$

$$(3.23) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j+1}} r_i^\alpha = \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j}} r_i^\alpha - \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant q \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j}} r_i^\alpha,$$

If we iterate the latter inequality, we will have

$$(3.24) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j}} r_i^\alpha \leqslant q^{j-100^d} \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>100^d}} r_i^\alpha.$$

So, for $j > 100^d$,

$$(3.25) \qquad \begin{aligned} \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha &\leqslant \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>j-1}} r_i^\alpha \leqslant q^{j-100^d-1} \sum_{B_{r_i}(x_i) \in \mathfrak{B}^{>100^d}} r_i^\alpha \\ &\leqslant q^{j-100^d-1} 3^\alpha \sum_{B_{r_i}(x_i) \in \mathfrak{B}_0} r_i^\alpha \leqslant q^{j-100^d-1} 3^\alpha a. \end{aligned}$$

Let $C = q^{-100^d-1} 3^\alpha 100^d 9^d$. Then

$$(3.26) \qquad \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha \leqslant C q^j a. \qquad \qquad \square$$

Definition 3.2. The set $B_R(x) \setminus B_{R-r}(x)$ will be called a ring with center x, radius r, and size R.

Let ω_{d-1} be the area of the unit sphere in $\mathbb{R}^d$ and let π_d be the volume of the unit ball in $\mathbb{R}^d$.

Proposition 3.6. Let F be a ring with size R and radius r. Then the volume of F does not exceed $\omega_{d-1} R^{d-1} r$.

Proof. We will integrate 1 over F and make the spherical change of coordinates,

$$(3.27) \qquad \int_F 1 \, dx = \int_{R-r}^R \omega_{d-1} t^{d-1} \, dt \leqslant \int_{R-r}^R \omega_{d-1} R^{d-1} \, dt = \omega_{d-1} R^{d-1} r. \qquad \square$$

Proof of Lemma 2.6. Let $\mathfrak{B}_0^j$ be the families provided by Lemma 3.1. Every ball in these families will be transformed into at most than countable number of balls by the following algorithm. Pick a number $q \in (0, 1)$. Assume we have a ball $B_r(x)$. We split the ring $B_r(x) \setminus B_{\frac{r}{3}}(x)$ into countable number of rings $F^j(B_r(x))$ whose radii are decreasing like geometric progression. Let $r_j = \frac{r}{3} + \frac{2r(1-q)}{3} \sum_{k=0}^j q^k$, then

$$(3.28) \qquad F^j(B_r(x)) = B_{r_j}(x) \setminus B_{r_{j-1}}(x),$$

$$(3.29) \qquad B_r(x) \setminus B_{\frac{r}{3}}(x) = \bigcup_{j=0}^\infty F^j(B_r(x)).$$

If $l = \frac{2(1-q)}{3}$, then the radius of $F^j(B_r(x))$ is rlq^j. Let $E^j(B_r(x))$ be a maximal $\frac{rlq^{j+1}}{3}$-separated subset of $F^j(B_r(x)) \cap A$. The balls with centers in $E^j(B_r(x))$ and radii $\frac{rlq^{j+1}}{6}$ form a disjoint family. All the balls in this family are contained in $\frac{rlq^{j+1}}{6}$-neighborhood of the ring $F^j(B_r(x))$. This neighborhood is inside the ring with center x radius $rlq^j + 2\frac{rlq^{j+1}}{6} < 2rlq^j$, and size $\frac{rlq^{j+1}}{6} + r_j < r$. The sum of the volumes of balls does not exceed the volume of the ring, so

$$(3.30) \qquad (\#E^j(B_r(x)) \pi_d \left(\frac{rlq^{j+1}}{6} \right)^d \leqslant 2\omega_{d-1} lq^j r^d,$$

$$(3.31) \qquad (\#E^j(B_r(x))) \leqslant C_0 q^{(1-d)j},$$

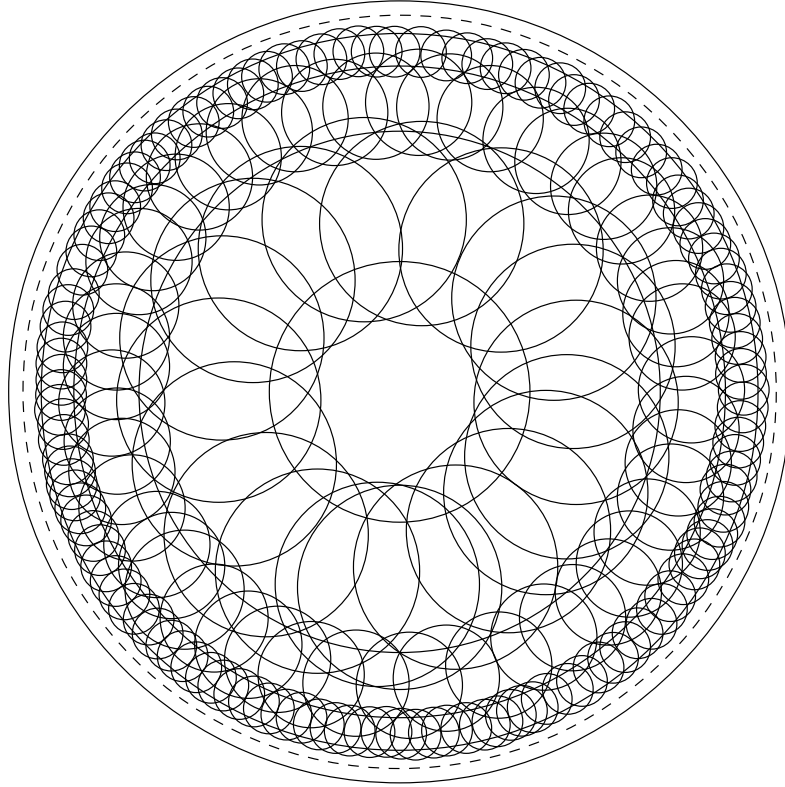where $C_0 = \frac{2\omega_{d-1}l}{\pi_d(\frac{lq}{6})^d}$.

Figure 1. Families $\mathfrak{C}^j(B_r(x))$.

The family $\mathfrak{C}^j(B_r(x))$ contains the balls with radii rlq^{j+1} and centers in the set $E^j(B_r(x))$ (see Figure 1). The family $\mathfrak{C}^j(B_r(x))$ is supercovering of $F^j(B_r(x)) \cap A$ by Proposition 3.3, and can be split into disjoint subfamilies $\mathfrak{C}_1^j(B_r(x)), \ldots, \mathfrak{C}_{100^d}^j(B_r(x))$ by Proposition 3.2. The balls in the family $\mathfrak{C}^j(B_r(x))$ does not intersect the rings $F^i(B_r(x))$ if $|i - j| \geqslant 2$, so the balls in family $\mathfrak{C}^j(B_r(x))$ do not intersect the balls in family $\mathfrak{C}^i(B_r(x))$ if $|i - j| \geqslant 3$. Let the family $\mathfrak{A}_{j,i}(B_r(x))$ for $0 \leqslant j \leqslant 2$ and $1 \leqslant i \leqslant 100^d$ be defined by the formula

$$(3.32) \qquad \mathfrak{A}_{j,i}(B_r(x)) = \bigcup_{k=0}^{\infty} \mathfrak{C}_i^{3k+j}(B_r(x)).$$

We can write the inequality

$$(3.33) \qquad \sum_{B_{\tau_l}(y_l) \in \mathfrak{A}_{j,i}(B_r(x))} \tau_l{}^\alpha \overset{(3.31)}{\leqslant} \sum_{k=0}^{\infty} C_0 q^{-(3k+j)(d-1)} (lrq^{(3k+j)+1})^\alpha$$

$$\leqslant \sum_{k=0}^{\infty} C_0 q^{-k(d-1)} (lrq^{k+1})^\alpha = Cr^\alpha,$$

where $C = C_0 l^\alpha q^\alpha \sum_{k=0}^{\infty} q^{k(\alpha-d+1)}$. Here we use that $\alpha \in (d-1, d]$. Now we are ready to define a supercovering $\mathfrak{B}$ of the set A:

$$(3.34) \qquad \mathfrak{B} = \left(\bigcup_m \bigcup_{B_{r_k}(x_k) \in \mathfrak{B}_0^m} \bigcup_{i,j} \mathfrak{A}_{j,i}(B_{r_k}(x_k)) \right) \bigcup \left(\bigcup_j \mathfrak{B}_0^j \right).$$

We split $\mathfrak{B}$ into subfamilies $\mathfrak{B}^{m,j,i}$ and $\mathfrak{B}_0^j$, where

$$(3.35) \qquad \mathfrak{B}^{m,j,i} = \bigcup_{B_{r_k}(x_k) \in \mathfrak{B}_0^m} \mathfrak{A}_{j,i}(B_{r_k}(x_k)),$$

and finish the proof with the estimate

$$(3.36) \qquad \sum_{B_{r_k}(x_k) \in \mathfrak{B}^{m,i,j}} r_k^\alpha = \sum_{B_{r_k}(x_k) \in \mathfrak{B}_0^m} \sum_{B_{\tau_l}(y_l) \in \mathfrak{A}_{j,i}(B_r(x))} \tau_l^\alpha \overset{(3.33)}{\leqslant} C \sum_{B_{r_k}(x_k) \in \mathfrak{B}_0^m} r_k^\alpha < C_2 a. \qquad \square$$

4. Proofs of theorems

Lemma 4.1. *Let μ be a signed measure, let A_+ and A_- be the sets of its Hahn decomposition, let μ_+ and μ_- be its positive and negative parts. Consider the set*

$$(4.1) \qquad P_{+,\varepsilon} = \{x \in A_+ \mid \exists \delta(x) \text{ such that } \forall r < \delta(x) \ \mu_-(B_r(x)) \leqslant \varepsilon \mu_+(B_r(x))\}.$$

Then $\mu_+(A) = \mu_+(P_{+,\varepsilon})$.

See the preprint [16] for the proof of a similar lemma (Lemma 4 of that paper). Consider the set $P_{+,\varepsilon}^{(N)}$ given by formula

$$(4.2) \qquad P_{+,\varepsilon}^{(N)} = \left\{ x \in A_+ \mid \forall r < \tfrac{1}{N} \ \mu_-(B_r(x)) \leqslant \varepsilon \mu_+(B_r(x)) \right\}.$$

Lemma 4.2. *Let $x \in P_{+,\varepsilon}^{(N)}$ and let $r < \tfrac{1}{N}$. Then*

$$(4.3) \qquad \int \varphi\left(\frac{y-x}{r}\right) d\mu_-(y) \leqslant \varepsilon \int \varphi\left(\frac{y-x}{r}\right) d\mu_+(y)$$

for any radial non-negative test-function φ supported in $B_1(0)$ that decreases as the radius grows.

Lemma 4.3. *Let $x \in P_{+,\varepsilon}^{(N)}$ and let $r < \tfrac{1}{N}$. Suppose that φ is a radial non-negative function supported in a unit ball that decreases as the radius grows and let ψ be a function such that $\varphi \leqslant \psi \leqslant \tfrac{1}{2\varepsilon}\varphi$. Then*

$$(4.4) \qquad \int \psi\left(\frac{y-x}{r}\right) d\mu_+(y) \leqslant 2 \int \psi\left(\frac{y-x}{r}\right) d\mu(y).$$

Proof. We write the estimates:

$$(4.5) \qquad \int \psi\left(\frac{y-x}{r}\right) d\mu_-(y) \leqslant \frac{1}{2\varepsilon} \int \varphi\left(\frac{y-x}{r}\right) d\mu_-(y) \leqslant \frac{1}{2} \int \psi\left(\frac{y-x}{r}\right) d\mu_+(y).$$

So we can write

$$(4.6) \qquad \int_{\mathbb{R}^d} \psi\left(\frac{y-x}{r}\right) d\mu(y) = \int_{\mathbb{R}^d} \psi\left(\frac{y-x}{r}\right) d\mu_+(y) - \int_{\mathbb{R}^d} \psi\left(\frac{y-x}{r}\right) d\mu_-(y)$$

$$\geqslant \frac{1}{2} \int_{\mathbb{R}^d} \psi\left(\frac{y-x}{r}\right) d\mu_+(y). \qquad \square$$

Proof of Theorem 2.1. Without loss of generality we may assume that $\varphi \leqslant \psi \leqslant C\varphi$. Suppose that A is a set such that $\mathcal{H}^\alpha(A) < a$, we will prove that $|\mu|(A) \lesssim h(a)$. Due to Lemma 4.1 it is enough to prove that $\mu_+(P_{+,\frac{1}{2C}}) \lesssim h(a)$, this provides the bound for $\mu_+(A)$; the bound for $\mu_-(A)$ is similar. Note that $P_{+,\frac{1}{2C}} = \bigcup P_{+,\frac{1}{2C}}^{(N)}$, so $\mu_+(P_{+,\frac{1}{2C}}) \leqslant 2\mu_+(P_{+,\frac{1}{2C}}^{(N)})$ for N large enough. Let K be a compact subset of $P_{+,\varepsilon}^{(N)}$

such that $\mu_+(P^{(N)}_{+,\varepsilon}) \leqslant 2\mu_+(K)$. We will prove that $\mu_+(K) \lesssim h(a)$. Let $\mathfrak{B}$ be a supercovering of set K provided by Lemma 2.4, and let $\mathfrak{B}^j$ be the corresponding subfamilies of $\mathfrak{B}$. We can write

$$\mu_+(K) \lesssim \sum_{B_{r_i}(x_i) \in \mathfrak{B}} \int \psi\left(\frac{y-x_i}{r_i}\right) d\mu_+(y) \overset{\text{Lem 4.3}}{\lesssim} \sum_{B_{r_i}(x_i) \in \mathfrak{B}} \int \psi\left(\frac{y-x_i}{r_i}\right) d\mu(y)$$

$$(4.7) \qquad = \sum_{j=1}^M \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} \int \psi\left(\frac{y-x_i}{r_i}\right) d\mu(y) \lesssim \sum_{j=1}^M g\left(\sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha\right)$$

$$\leqslant \sum_{j=1}^M g(Cq^j a) \lesssim \sum_{j=1}^M g(q^j a) \lesssim \int_0^a \frac{g(t)}{t} dt = h(a). \qquad \square$$

Proof of Theorem 2.2. Let A be the same set as in the previous proof and let A_+ be the positive part of it. Consider the compact set K such that $K \subset A_+$ and $\mu_+(A_+) \leqslant 2\mu_+(K)$. Let V be a open set such that $A_+ \subset V$ and $\mu_-(V) \leqslant \varepsilon$. We will prove that $\mu_+(K) \lesssim g(a)$. Let $\mathfrak{B}$ be a supercovering of K provided by Lemma 2.6. We may assume that any ball from $\mathfrak{B}$ lies in V. We can write

$$\mu_+(K) \lesssim \sum_{B_{r_i}(x_i) \in \mathfrak{B}} \int \varphi\left(\frac{y-x_i}{r_i}\right) d\mu_+(y)$$

$$= \sum_{B_{r_i}(x_i) \in \mathfrak{B}} \int \varphi\left(\frac{y-x_i}{r_i}\right) d\mu(y) + \sum_{B_{r_i}(x_i) \in \mathfrak{B}} \int \varphi\left(\frac{y-x_i}{r_i}\right) d\mu_-(y)$$

$$(4.8) \qquad \lesssim \sum_{j=1}^{C_1} \sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} \int \varphi\left(\frac{y-x_i}{r_i}\right) d\mu(y) + C_1 \mu_-(V)$$

$$\lesssim \sum_{j=1}^{C_1} g\left(\sum_{B_{r_i}(x_i) \in \mathfrak{B}^j} r_i^\alpha\right) + C_1 \varepsilon$$

$$\leqslant C_1 g(C_2 a) + C_1 \varepsilon \lesssim g(a) + \varepsilon \underset{\varepsilon \to 0}{\longrightarrow} g(a). \qquad \square$$

Proof of Lemma 2.3. Consider the case $\varphi(x) = o(|x|^{-\alpha})$ first.

Let ϑ be a function such that $\varphi(x) = \vartheta(|x|)$. Let A be a set such that $\mathcal{H}(A) < a$ and let K be a compact set like in the proof of Theorem 2.1. Let $\mathfrak{B}$ be a covering of K provided by Lemma 2.3 such that the radius of any ball does not exceed $\frac{\varepsilon}{N}$. Note that for $r < \frac{\varepsilon}{N}$ we have

$$(4.9) \qquad \int_{\mathbb{R}^d \setminus B_{\frac{1}{N}}(x)} \psi\left(\frac{y-x}{r}\right) d|\mu|(y) \lesssim |\mu|(\mathbb{R}^d) r^\alpha \frac{\vartheta(\frac{1}{Nr})}{r^\alpha} \leqslant r^\alpha \delta(\varepsilon),$$

where $\delta(\varepsilon) = |\mu|(\mathbb{R}^d) \sup_{r \leqslant \frac{\varepsilon}{N}} \frac{\vartheta(\frac{1}{Nr})}{r^\alpha}$. Note that $\delta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0$ because $\varphi(x) = o(|x|^{-\alpha})$. We can write

$$\mu_+(K) \lesssim \sum_{B_{r_i}(x_i) \in \mathfrak{B}} \int_{B_{r_i}(x_i)} \psi\left(\frac{y-x_i}{r_i}\right) d\mu_+(y)$$

$$\leqslant \sum_{B_{r_i}(x_i) \in \mathfrak{B}} \int_{B_{\frac{1}{N}}(x_i)} \psi\left(\frac{y-x_i}{r_i}\right) d\mu_+(y)$$

$$\overset{\text{Lem 4.3}}{\lesssim} \sum_{B_{r_i}(x_i)\in\mathfrak{B}} \int_{B_{\frac{1}{N}}(x_i)} \psi\left(\frac{y-x_i}{r_i}\right) d\mu(y)$$

$$(4.10) \quad \leqslant \sum_{B_{r_i}(x_i)\in\mathfrak{B}} \int_{\mathbb{R}^d} \psi\left(\frac{y-x_i}{r_i}\right) d\mu(y) + \sum_{B_{r_i}(x_i)\in\mathfrak{B}} \left|\int_{\mathbb{R}^d\setminus B_{\frac{1}{N}}(x_i)} \psi\left(\frac{y-x_i}{r_i}\right) d\mu(y)\right|$$

$$\lesssim \sum_{j=1}^{M} \sum_{B_{r_i}(x_i)\in\mathfrak{B}^j} \int_{\mathbb{R}^d} \psi\left(\frac{y-x_i}{r_i}\right) d\mu(y) + \sum_{B_{r_i}(x_i)\in\mathfrak{B}} r_i^\alpha \delta(\varepsilon)$$

$$\lesssim \sum_{j=1}^{M} g\left(\sum_{B_{r_i}(x_i)\in\mathfrak{B}^j} r_i^\alpha\right) + a\delta(\varepsilon) \lesssim g(a) + a\delta(\varepsilon) \underset{\varepsilon\to 0}{\to} g(a).$$

Second case, $\varphi(x) = O(|x|^{-\alpha})$.

Let A be a set such that $\mathcal{H}^\alpha(A) = 0$. We will prove that $\mu_+(A) = 0$. Let K be a compact set like in the previous proof. For any positive a we can write $\mathcal{H}^\alpha(A) < a$. If we make the same estimates we will have the inequality $\mu_+(K) \lesssim g(a) + a\delta(\varepsilon) \underset{a\to 0}{\longrightarrow} 0$. $\qquad\square$

Let φ be a function in $C_0^\infty(\mathbb{R}^{d-1})$ supported in the unit ball. For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ we write $x_{[i]}$ for the $(d-1)$-dimensional vector that is obtained from x by forgetting the i-th coordinate (for example, for $d = 3$, $x_{[2]} = (x_1, x_3)$).

We cite Lemma 2.3 in [15].

Lemma 4.4. *Let $\mathfrak{B}$ be a disjoint family of balls in $\mathbb{R}^{d-1}$, let $\psi \in C_0^\infty(\mathbb{R})$ be a test function. Suppose that f is a compactly supported function. If $\mu = (\partial_1^{m_1} f, \partial_2^{m_2} f, \ldots, \partial_d^{m_d} f)$ is a measure, then, for all $i = 1, 2 \ldots d$ any $\varphi \in C_0^\infty(\mathbb{R}^{d-1})$ supported in unit ball,*

$$(4.11) \qquad \sum_{B_r(x)\in\mathfrak{B}} \left|\int_{\mathbb{R}^d} \psi(x_i)\varphi\left(\frac{y-x}{r}\right) d\mu_i(y)\right| \lesssim \left(\sum_{B_r(x)\in\mathfrak{B}} r^{d-1}\right)^{\frac{1}{q_i'}}$$

for some fixed q_i' (the constants may depend on φ and ψ).

Lemma 4.5. *Let μ be a Borel measure on $\mathbb{R}^{l+k}$. Suppose that $\mu(I \times A) = 0$ for every parallelepiped $I \subset \mathbb{R}^k$ and every Borel $A \subset \mathbb{R}^l$ such that $\mathcal{H}^\alpha(A) = 0$. Then μ is absolutely continuously with respect to $\mathcal{H}^\alpha$.*

Proof of Lemma 4.5 is absolutely similar to the proof of Lemma 2.5 in [15].

Proof of Corollary 2.2. Assume the contrary. Let F be some Borel set such that $\mathcal{H}^{d-1}(F) = 0$, but $\mu(F) \neq 0$. We may assume that $\mu_1(F) \neq 0$ (by symmetry) and F is compact (due to the regularity of the measure). Multiplying f by a test function that equals 1 on F, we make f compactly supported without loosing the condition that its higher order derivatives are signed measures. To get a contradiction, it suffices to prove that for every set $A \subset \mathbb{R}^{d-1}$ such that $\mathcal{H}^{d-1}(A) = 0$ and every function $\psi \in C_0^\infty(\mathbb{R})$, we have:

$$(4.12) \qquad \int_{A\times\mathbb{R}} \psi(x_1) d\mu_1(x) = 0.$$

Then, approximating the characteristic function of an interval I by smooth functions, we get the hypothesis of Lemma 4.5 with $\alpha = d - 1$, which, in its turn, asserts that $\mu_1(F) = 0$.

Consider now a complex measure μ_ψ on $\mathbb{R}^{d-1}$ given by formula

$$\mu_\psi(B) = \int_{B \times \mathbb{R}} \psi(x_1) \, d\mu_1(x)$$

and note that (4.12) holds for any A such that $\mathcal{H}^{d-1}(A) = 0$. By Lemma 4.4, μ_ψ satisfies the hypothesis of Theorem 2.2 with $\alpha = d - 1$. Therefore, μ_ψ is absolutely continuous with respect to $\mathcal{H}^{d-1}$. $\qquad\square$

5. Generalizations and examples

The following lemma allows to construct measures ν violating the Frostman condition, but falling under the scope of Theorems 2.1 and 2.2.

Lemma 5.1. *Suppose that $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a radial non-negative function supported in the unit ball. Assume that φ decreases as the radius grows and $\varphi(x) = 1$ when $|x| \leq \frac{3}{4}$. Let μ be a positive Borel measure such that $\mu(B_r(x)) \leq r^\alpha$ for any open ball $B_r(x)$. Let ν be a signed measure continuous with respect to μ. Then for any disjoint family of balls $\mathfrak{B}$ the inequality*

$$(5.1) \qquad \sum_{B_{r_j}(x_j) \in \mathfrak{B}} \left| \int \varphi\left(\frac{y - x_j}{r_j}\right) d\nu(y) \right| \lesssim g\left(\sum_{B_{r_j}(x_j) \in \mathfrak{B}} r_j^\alpha \right) \quad \text{holds true,}$$

where

$$(5.2) \qquad g(t) = \int_0^t \left| \frac{d\nu}{d\mu} \right|^* (s) ds.$$

Here and in what follows the notation f^* means the monotonic rearrangement of the function f and $\frac{d\nu}{d\mu}$ is the density of ν with respect to μ. The monotonic rearrangement of a function f is defined by the formula:

$$(5.3) \qquad f^*(t) = \sup\{\alpha \mid \mu\{x \mid f(x) > \alpha\} \leq t\}.$$

Proof. Let $\mathfrak{B}$ be a disjoint family of balls and let $\mathcal{B} = \bigcup_{B_{r_j}(x_j) \in \mathfrak{B}} B_{r_j}(x_j)$. Note that $\mu(\mathcal{B}) \leq \sum_{B_{r_j}(x_j) \in \mathfrak{B}} r_j^\alpha$. We can write

$$(5.4) \qquad \sum_{B_r(x) \in \mathfrak{B}} \left| \int \varphi\left(\frac{y - x}{r}\right) d\nu(y) \right| \lesssim |\nu|(\mathcal{B}) \leq \int_0^{\mu(\mathcal{B})} \left| \frac{d\nu}{d\mu} \right|^* (s) \, ds$$

$$= g(\mu(\mathcal{B})) \leq g\left(\sum_{B_r(x) \in \mathfrak{B}} r^\alpha \right). \qquad\square$$

Our lemmas may be generalised for f-Hausdorff measures. We recall the definition of these measures.

Definition 5.1. Let f be a regular function. The f-Hausdorff measure be defined by the formula

$$(5.5) \qquad \Lambda_f(A) = \lim_{\delta \to 0} \inf_{\substack{A \subset \cup B_j \\ \mathrm{diam}(B_j) < \delta}} \sum f(\mathrm{diam}(B_j)).$$

Note that if $f(t) = t^\alpha$ then $\Lambda_f = \mathcal{H}^\alpha$. We formulate more general versions of our lemmas.

Lemma 5.2. *Let φ be a radially symmetric, radially non-increasing function supported in the unit ball. Assume also $\varphi(x) = 1$ when $|x| \leqslant \frac{3}{4}$. Let ψ be equivalent to φ. Let μ be an $\mathbb{R}$-valued signed radon measure, let g be a regular function that satisfies the Dini condition (2.2). Assume that*

$$(5.6) \qquad \sum_{B_{r_j}(x_j) \in \mathfrak{B}} \left| \int_{\mathbb{R}^d} \psi\left(\frac{y - x_j}{r_j}\right) d\mu(y) \right| \lesssim g\left(\sum_{B_{r_j}(x_j) \in \mathfrak{B}} f(r_j) \right)$$

for any disjoint family of balls $\mathfrak{B}$. Then,

$$(5.7) \qquad |\mu|(A) \lesssim h(\Lambda_f(A)),$$

where $h(x) = \int_0^x \frac{g(t)}{t}\, dt$ and A is an arbitrary Borel set $A \subset \mathbb{R}^d$.

Definition 5.2. *A regular function f is d-falling if it satisfies the conditions*

$$(5.8) \qquad \int_0^1 \frac{f(t)}{t^d}\, dt < \infty,$$

$$(5.9) \qquad x^{d-1} \int_0^x \frac{f(t)}{t^d}\, dt \asymp f(x).$$

Lemma 5.3. *Let φ be a bounded function supported in the unit ball and such that $\varphi(x) \geqslant 1$ when $|x| \leqslant \frac{3}{4}$. Let μ be an $\mathbb{R}$-valued signed Radon measure, let g be a regular function. Assume also that f is d-falling. If*

$$(5.10) \qquad \sum_{B_{r_j}(x_j) \in \mathfrak{B}} \left| \int_{\mathbb{R}^d} \varphi\left(\frac{y - x_j}{r_j}\right) d\mu(y) \right| \lesssim g\left(\sum_{B_{r_j}(x_j) \in \mathfrak{B}} f(r_j) \right)$$

for any disjoint family of balls $\mathfrak{B}$, then

$$(5.11) \qquad |\mu|(A) \lesssim g(\Lambda_f(A))$$

for any Borel set $A \subset \mathbb{R}^d$.

Lemma 5.4. *Let φ be a bounded radially symmetric, radially non-increasing function such that $B_3(0) \subset \operatorname{supp} \varphi$. Let ψ be equivalent to φ. Let μ be a signed measure of bounded variation, let g be a regular function. Assume*

$$(5.12) \qquad \sum_{B_{r_j}(x_j) \in \mathfrak{B}} \left| \int_{\mathbb{R}^d} \psi\left(\frac{y - x_j}{r_j}\right) d\mu(y) \right| \lesssim g\left(\sum_{B_{r_j}(x_j) \in \mathfrak{B}} f(r_j) \right)$$

for any disjoint family of balls $\mathfrak{B}$. If

$$(5.13) \qquad |\varphi(x)| = O(f(|x|^{-1})), \quad x \to \infty,$$

then μ is absolutely continuous with respect to Λ_f. If

$$(5.14) \qquad |\varphi(x)| = o(f(|x|^{-1})), \quad x \to \infty,$$

then the inequality

$$(5.15) \qquad |\mu|(A) \lesssim g(\Lambda_f(A))$$

holds true for any Borel set A.

The proofs of Lemmas 5.2, 5.3, 5.4 are exactly the same as the proofs of Theorems 2.1, 2.2 and Lemma 2.3.

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