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Erratum to "The multiple-slit version of Loewner's differential equation and pointwise Hölder continuity of driving functions"

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Abstract. The proof of Theorem 1.2 of [S. Schleissinger, Ann. Acad. Sci. Fenn. Math. 37:1, 2012, 191–201] contains a gap and one implication in Theorem 1.3 of the same paper is wrong.

Oikaisu artikkeliin *The multiple-slit version of Loewner's differential equation and pointwise Hölder continuity of driving functions* ("Loewnerin differentiaaliyhtälö usean viillon tapauksessa ja ohjausfunktioiden pisteittäinen Hölderin jatkuvuus")

Tiivistelmä. Artikkelin [S. Schleissinger, Ann. Acad. Sci. Fenn. Math. 37:1, 2012, 191–201] lauseen 1.2 todistuksessa on aukko, ja yksi lauseen 1.3 väitteistä on väärin.

1. Theorem 1.2

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$. Theorem 1.2 considers the conformal mappings $g_t \colon \mathbb{H} \setminus K_t \to \mathbb{H}$, $t \in [0, E]$, generated by the multiple-slit equation (1.3), and claims that K_E consists of n simple curves if the driving functions do not intersect and satisfy a certain pointwise regularity condition.

The proof of Theorem 1.2 contains a gap. It ends (see p. 201) with showing that for each $j = 1, \dots, n$ and $T \in (0, E]$, there exist two real numbers x_0 and y_0 such that the solutions $x(t)$ and $y(t)$ of the backward equation (3.6) with initial value x_0 and y_0 respectively satisfy $x(T) = y(T) = U_j(0)$. In this sense, each connected component of the hull K_E is "welded" together from two intervals.

However, this property alone does not allow the conclusion that the components are simple curves. It still remains to show that g_E^{-1} can be extended continuously to $\mathbb{R}$, which is not clear (see, e.g., the criterion in [RS05, Theorem 4.1]). In other words, it is not shown that the hulls are "generated by curves", i.e. that the domains $\mathbb{H} \setminus K_t$ are the unbounded connected components of $\mathbb{H}$ minus n disjoint, continuous non-crossing (but possibly self-touching) curves in $\mathbb{H}$. (This includes the case $n = 1$. The statement at (1.2) is not justified.)

For literature concerning this problem, we refer to [Lin05, MR05] for the case of 1/2-Hölder continuous driving functions, and to [STW19] for certain locally regular driving functions. See also [LMR10, LR12, Zha18, MM24].

Remark. The author does not know an explicit example for the setting of Theorem 1.2, which does not generate simple curves. We note that the proof of Proposition 3.1 also contains a deficient argument. The mappings f_t do not generate simple curves and they are obtained as the limit of $f_t(z, d)$ as $d \to \infty$. It is not clear, however, why the mappings $f_t(z, d)$ do not generate simple curves neither.

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2. Theorem 1.3

Theorem 1.3 claims that if the hulls K_t approach $\mathbb{R}$ at $U_j(0)$ in φ–direction, then $\lim_{t\downarrow 0}(U_j(t) - U_j(0))/\sqrt{t}$ exists. The author would like to thank Huy Tran for pointing out that this implication is wrong. (In the proof, the application of [LMR10, Theorem 4.3] is not valid in general.)

We consider the case $n = 1$, $U := U_1$ with $U(0) = 0$, and the angle $\varphi = \pi/2$. Theorem 1.3 claims that $\lim_{t\downarrow 0} U(t)/\sqrt{t} = 0$. This is not true in the following example.

We denote by $\operatorname{diam}(A) := \sup_{z,w\in A} |z - w|$ the diameter of a subset $A \subset \mathbb{C}$, and by $B(z, r)$ the open disk with center $z \in \mathbb{C}$ and radius $r > 0$.

Consider the region R in $\mathbb{H}$ between the two curves

$$\{x + iy \in \mathbb{H} \mid x, y \in \mathbb{R},\ x = y^2\} \quad \text{and} \quad \{x + iy \in \mathbb{H} \mid x, y \in \mathbb{R},\ x = -y^2\}.$$

Any hull inside this region clearly approaches 0 in $\pi/2$-direction. Now, for $n \in \mathbb{N}$, let a_n be the intersection point of $L_n := \{x + iy \in \mathbb{H} \mid x, y \in \mathbb{R},\ x = -y^{n+2}\}$ and $C_n := \{x + \frac{1}{2^n} i \in \mathbb{H} \mid x \in \mathbb{R}\}$ and let b_n be the intersection point of $R_n := \{x + iy \in \mathbb{H} \mid x, y \in \mathbb{R},\ x = y^{n+2}\}$ and C_n. First, we construct a curve $\hat{\gamma}\colon [0, 1] \to \mathbb{C}$ in the following way:

Connect 0 to a_1 via L_1 (i.e. by the subcurve of L_1 connecting 0 to a_1), then a_1 to b_1 via C_1 and b_1 to 0 via R_1. Assume that $\hat{\gamma}\colon [0, 1/2] \to \mathbb{C}$ parametrizes this "triangle" T_1. Next we increase n and construct another, smaller triangle T_2 (parametrized in opposite direction): connect 0 to b_2 via R_2, b_2 to a_2 via C_2 and then a_2 to 0 via L_2. Now we may assume that $\hat{\gamma}\colon [1/2, 3/4] \to \mathbb{C}$ parametrizes this curve. Now we continue inductively and obtain a sequence $(T_n)_{n\in\mathbb{N}}$ of nested triangles, each parametrized by $\hat{\gamma}\colon [1 - 1/2^{n-1}, 1 - 1/2^n] \to \mathbb{C}$. As

$$\operatorname{diam}(T_n) \to 0 \text{ for } n \to \infty \quad \text{and} \quad 0 \in T_n \text{ for all } n \in \mathbb{N},$$

we can extend $\hat{\gamma}$ continuously to the interval $[0, 1]$ by setting $\hat{\gamma}(1) = 0$.

Now consider the curve $\hat{\gamma}(1-t)$ and let $\gamma\colon [0, T] \to \mathbb{C}$ be a parametrization of this curve such that $\operatorname{hcap}(K_t) = 2t$, where we denote by K_t the smallest hull containing $\gamma[0, t]$ (and K_0 is the empty set).

The family $(K_t)_{t\in[0,T]}$ satisfies the local growth property and thus, see [LSW01, Theorem 2.6] or [Law05, p. 96], it can be generated by the one-slit equation with a continuous driving function $U\colon [0, T] \to \mathbb{R}$. Finally, let $t_1 > t_2 > t_3 > \dots$ be the decreasing sequence of zeros of $\gamma(t)$, $t > 0$. We have $U(t_n) = \lim_{x\uparrow 0} g_{K_{t_n}}(x)$ or $U(t_n) = \lim_{x\downarrow 0} g_{K_{t_n}}(x)$. However, as K_{t_n} is symmetric with respect to the imaginary axis, we certainly have

$$2|U(t_n)| = \lim_{x\downarrow 0} g_{K_{t_n}}(x) - \lim_{x\uparrow 0} g_{K_{t_n}}(x) =: \pi \cdot \operatorname{cap}_{\mathbb{H}}(K_{t_n}).$$

The quantity $\operatorname{cap}_{\mathbb{H}}$ is introduced in [Law05], p. 73, see also the first equation on p. 74. There, it is shown that there exists a constant $c_1 > 0$ such that

$$\operatorname{cap}_{\mathbb{H}}(K_{t_n}) \geq c_1 \cdot \operatorname{diam}(K_{t_n}),$$

see (3.14) on p. 74 in [Law05].

Finally, consider $\operatorname{hsiz}(K_{t_n}) = \operatorname{area}\left(\bigcup_{x+iy\in K_{t_n}} B(x+iy, y)\right)$. Then we find another constant $c_2 > 0$ such that $\operatorname{diam}(K_{t_n}) \geq c_2 \cdot \sqrt{\operatorname{hsiz}(K_{t_n})}$. By [LLN09, Theorem 1], the quantity hsiz is comparable to the half-plane capacity, i.e. there exists $c_3 > 0$ such that $\operatorname{hsiz}(K_{t_n}) \geq c_3 \cdot t_n$ and we arrive at

$$|U(t_n)| = \pi/2 \cdot \operatorname{cap}_{\mathbb{H}}(K_{t_n}) \geq c_1 \cdot \pi/2 \cdot \operatorname{diam}(K_{t_n}) \geq c_1 c_2 \sqrt{c_3} \cdot \pi/2 \cdot \sqrt{t_n}.$$

We conclude that $(K_t)_{t\in[0,T]}$ approaches $\mathbb{R}$ at 0 in $\pi/2$-direction, but

$$\limsup_{t\downarrow 0} \frac{|U(t)|}{\sqrt{t}} \geq c_1 c_2 \sqrt{c_3} \cdot \pi/2 > 0.$$

References

[LLN09] Lalley, S., G. F. Lawler, and H. Narayanan: Geometric interpretation of half-plane capacity. - Electron. Commun. Probab. 14, 2009, 566–571.

[Law05] Lawler. G. F.: Conformally invariant processes in the plane. - Math. Surveys Monogr. 114, Amer. Math. Soc., Providence, RI, 2005.

[LSW01] Lawler, G. F., O. Schramm, and W. Werner; Values of Brownian intersection exponents. I. Half-plane exponents. - Acta Math. 187:2, 2001, 237–273.

[Lin05] Lind, J.: A sharp condition for the Loewner equation to generate slits. - Ann. Acad. Sci. Fenn. Math. 30, 2005, 143–158.

[LMR10] Lind, J., D. E. Marshall, and S. Rohde: Collisions and spirals of Loewner traces. - Duke Math. J. 154, 2010, 527–573.

[LR12] Lind, J., and S. Rohde: Space-filling curves and phases of the Loewner equation. - Indiana Univ. Math. J. 61, 2012, 2231–2249.

[MM24] Margarint, V., and T. Mesikepp: Drivers, hitting times, and weldings in Loewner's equation. - J. Lond. Math. Soc. 109:1, 2024.

[MR05] Marshall, D. E., and S. Rohde: The Loewner differential equation and slit mappings. - J. Amer. Math. Soc. 18, 2005, 763–778.

[RS05] Rohde, S., and O. Schramm: Basic properties of SLE. - Ann. of Math. (2) 161, 2005, 883–924.

[Sch12] Schleissinger, S.: The multiple-slit version of Loewner's differential equation and pointwise Hölder continuity of driving functions. - Ann. Acad. Sci. Fenn. Math. 37:1, 2012, 191–201.

[STW19] Shekhar, A., H. Tran, and Y. Wang: Remarks on Loewner chains driven by finite variation functions. - Ann. Acad. Sci. Fenn. Math. 44:1, 2019, 311–327.

[WD14] Wu, H., and X. Dong: Driving functions and traces of the Loewner equation. - Sci. China Math. 57, 2014, 1615–1624.

[Zha18] Zhang, H.: Local analysis of Loewner equation. - PhD thesis, Université d'Orléans, 2018.

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