Annales Fennici Mathematici
Volumen 49, 2024, 337–348

A quantitative version of the Hopf–Oleinik lemma for a quasilinear non-uniformly elliptic operator

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Abstract. This paper establishes a quantitative version of the Hopf–Oleinik lemma (HOL) for a quasilinear non-uniformly elliptic operator of the form $\mathcal{L}_\infty u := 2\Delta_\infty u + \Delta u$. One key point in the proof is the passage from non-uniformly elliptic operators to locally uniformly ones via a new, uniform, and, rescaled version of the gradient estimate obtained by Evans and Smart for solutions to a family of non-uniformly quasilinear elliptic operators.

Hopfin–Oleinikin lemman suuruusarviollinen muotoilu kvasilineaariselle epätasaisesti elliptiselle operaattorille

Tiivistelmä. Tässä työssä todistetaan Hopfin–Oleinikin lemman suuruusarviollinen muotoilu kvasilineaariselle epätasaisesti elliptiselle operaattorille $\mathcal{L}_\infty u := 2\Delta_\infty u + \Delta u$. Yksi todistuksen avainkohta on siirtyminen epätasaisesti elliptisistä operaattoreista paikallisesti tasaisesti elliptisiin käyttäen uutta, tasaista ja uudelleen skaalattua muotoilua epätasaisesti kvasilineaaristen elliptisten operaattoreiden perheiden ratkaisuita koskevasta Evansin ja Smartin gradienttiarviosta.

1. Introduction

An essential result in the theory of elliptic partial differential equations is the Hopf–Oleinik lemma (HOL), which was proven independently by Hopf in [6] and Oleinik in [12]. Among the many applications of this lemma in the field, the most known one is the proof of the strong maximum principle for second-order uniformly elliptic operators, and its use to study boundary regularity issues for solutions to elliptic equations and free boundary problems. The HOL is essentially a qualitative result which establishes that a nonnegative supersolution that is not identically zero reaches the boundary, wherever it vanishes, with a nontrivial slope. We refer the reader to Theorem 2.2 in [2] where some ideas related to HOL were an inspiration to this quantitative version here. More precisely, if A is (λ,Λ)-uniformly elliptic matrix[1] in the unit ball $B_1 \subset \mathbb{R}^N$ and $Lu = Tr(A(x)D^2u) = 0$ in B_1, where $0 \leq u \in C^2(B_1) \cap C(\overline{B_1})$ is not identically zero, and $\partial u/\partial\nu(x_0)$ exists, where $\nu = -x_0$ is the inner unit normal to ∂B_1 at $x_0 \in \partial B_1$ and $u(x_0) = 0$, then

$$\frac{\partial u}{\partial \nu}(x_0) \geq Cu(0),$$

where $C = C(\lambda,\Lambda,N) > 0$ is a universal constant. Now, let $g \in C([0,\infty)) \cap C^1((0,\infty))$ be a positive nondecreasing function satisfying $g(0) = 0$, $g(t) > 0$ for

https://doi.org/10.54330/afm.146035
2020 Mathematics Subject Classification: Primary 35J94; Secondary 35J60, 35J67.
Key words: Hopf–Oleinik lemma, quasilinear non-uniformly elliptic operators.
[1]This means that $A(x)$ is symmetric and $\lambda \cdot I \leq A(x) \leq \Lambda \cdot I, \forall x \in B_1$ where $0 < \lambda \leq \Lambda$.

$t > 0$, and

$$0 < \delta_0 \leq \frac{g'(t)t}{g(t)} \leq g_0, \quad \forall\, t > 0 \tag{1}$$

for fixed positive constants δ_0 and g_0. These assumptions on the function g establish the uniform ellipticity condition to operators in divergence form of the type

$$\mathcal{L}_g u := \operatorname{div}\left(g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right). \tag{2}$$

A thorough study of this theory was developed by Lieberman in the early 90's in [7]. A particular case of the results proved by Braga and Moreira [3] is the following quantitative Hopf–Oleinik lemma for quasilinear uniformly elliptic operators in divergence form satisfying (1).

Theorem 1.1. *Let $0 \leq u \in C^0(\overline{B_R}) \cap W^{1,G}_{\mathrm{loc}}(B_R)$ be a weak solution to*

$$\mathcal{L}_g u = 0 \quad \text{in } B_R,$$

where g satisfies (1) and $G(t) = \int_0^t g(s)\, ds$ for $t \geq 0$. Then,

$$u(x) \geq \frac{C}{R} \sup_{B_{R/2}} u \operatorname{dist}(x, \partial B_R), \quad \forall\, x \in \overline{B_R}. \tag{3}$$

Moreover, if $x_0 \in \partial B_R$ is a point such that $u(x_0) = 0$ and there exists the inner normal derivative $\frac{\partial u}{\partial \nu}(x_0)$, where ν is the corresponding inner unit normal at x_0, then

$$\frac{\partial u}{\partial \nu}(x_0) \geq \frac{C}{R} \sup_{B_{R/2}} u, \tag{4}$$

in a case where $C = C(N, \delta_0, g_0) > 0$.

Remark 1.2. We point out that interesting results have been developed recently related to Hopf–Oleinik lemma's quantitative version. Namely, the results of Sirakov et al in [14, 15] on the divergence case for linear uniformly elliptic operators with general coefficients involving even low-order terms. These imply boundary Harnack-type inequality, which by its turn encompasses HOL in quantitative form in this uniformly elliptic and linear case.

The purpose of this paper is to prove Theorem 1.1 where the operator in focus is given by $\mathcal{L}_\infty u := 2\Delta_\infty u + \Delta u$. As a matter of fact, this is somehow equivalent to treat (2) for the case where $g(t) = G'(t)$ with $G(t) = e^{t^2} - 1$. In this case $\delta_0 = 1$ and $g_0 = \infty$ (see (5)–(8) and Lemma 1.6). The proof is not immediate as one can readily suspect looking at the dependence $C = C(N, \delta_0, g_0) > 0$ in the estimates (3) and (4). The research on this type of equation has been active for several years now with recent developments (see [1, 8, 10, 11]). Our strategy here is to mix the ideas from [17], [3], and [5] in order to prove the result. More precisely, we obtain a precise and rescaled version of the gradient estimate found in [5]. This allows one to pass from non-uniform ellipticity to local uniform ellipticity in a precisely quantified way. Moreover, here we also construct barriers that are more suitable for our operator adapting ideas due to Vázquez in [17] and Braga and Moreira in [3, 4] regarding the geometric behavior of the barriers. Under the possession of those ingredients, we follow the geometric strategy developed in [3] to implement the proof of the quantitative version of HOL via Harnack-type arguments combined with comparison principles.

We point out that the quantitative version of HOL obtained here was recently used in [16] to prove Lipschitz regularity for local minimizers of variational two-phase Bernoulli-type free boundary problems where the energy functionals are associated with non-uniformly elliptic operators. The optimal regularity was obtained under some restrictions on the size/geometry of the negative phase. This suggests a potential role that might be played by this quantitative version of the HOL to deal with the study of free boundary problems or boundary regularity issues in the context where non-uniformly elliptic operators are present.

From now on, we denote $\Phi(t) = e^{t^2} - 1$ and $\phi(t) = \Phi'(t) = 2te^{t^2}$. Note that

$$0 < 1 \le \frac{\phi'(t)t}{\phi(t)} = 1 + 2t^2, \quad \forall\, t > 0. \tag{5}$$

Thus

$$g_0 = \sup_{t>0} \frac{\phi'(t)t}{\phi(t)} = \infty \tag{6}$$

and the strongly degenerate operator defined by

$$\mathcal{L}_\phi u := \operatorname{div}\left(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = \operatorname{div}\left(2e^{|\nabla u|^2}\nabla u\right) \tag{7}$$

is a quasilinear non-uniformly elliptic operator that does not satisfy the Lieberman's condition (1) since $g_0 = \infty$. A straightforward computation on the divergence above shows that this operator can be written in nondivergence form as

$$\mathcal{L}_\phi u = 2e^{|\nabla u|^2}\left\{2\Delta_\infty u + \Delta u\right\} = 2e^{|\nabla u|^2}\mathcal{L}_\infty u, \tag{8}$$

where

$$\Delta_\infty u := \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j},$$

and

$$\mathcal{L}_\infty u := 2\Delta_\infty u + \Delta u.$$

In what follows, we state a quantitative version of the Hopf–Oleinik lemma for a quasilinear non-uniformly elliptic operator in nondivergence form.

Theorem 1.3. *Let B_R be an open ball of radius $R > 0$ in $\mathbb{R}^N$ $(N \ge 2)$, and let $0 \le w \in C^2(B_R) \cap C^0(\overline{B}_R)$ be such that*

$$\mathcal{L}_\infty w \le 0 \quad \text{in } B_R. \tag{9}$$

Suppose that there are $\sigma \in (0,1)$ and $M > 0$ such that

$$\|\nabla w\|_{L^\infty(B_{\sigma R})} \le M. \tag{10}$$

Then for any $\tau \in (0,\sigma)$ and $p \in (0, N/(N-2))$ if $N > 2$, $p \in (0,\infty)$ if $N = 2$, we have

(i) *There exists a constant $C > 0$ depending on N, p, σ, τ, and M such that*

$$w(x) \ge \frac{C}{R}\left(\fint_{B_{\tau R}} w^p\, dx\right)^{\frac{1}{p}} d(x, \partial B_R), \ \forall x \in \overline{B_R},$$

where $d(x, \partial B_R)$ is the distance between x and the boundary ∂B_R of B_R.

(ii) *If $x_0 \in \partial B_R$ is a point such that $w(x_0) = 0$ and the inner normal derivative $\frac{\partial w}{\partial \nu}(x_0)$ exists, then*

$$\frac{\partial w}{\partial \nu}(x_0) \geq \frac{C}{R}\left(\fint_{B_{\tau R}} w^p \, dx \right)^{\frac{1}{p}},$$

where $C > 0$ is the constant given by item (i). In particular, if $\frac{\partial w}{\partial \nu}(x_0) = 0$, then $w \equiv 0$ in $B_{\sigma R}$.

Corollary 1.4. *Let B_R be an open ball of radius $R > 0$ in $\mathbb{R}^N$ $(N \geq 2)$, and let $0 \leq w \in C^2(B_R) \cap C^0(\overline{B}_R)$ be such that*

$$\mathcal{L}_\infty w = 0 \quad \text{in } B_R. \tag{11}$$

Then for any $\tau \in (0, 1)$ we have

(i) *There exists a constant $C > 0$ depending on N, τ and $\sup_{B_R} w/R$ such that*

$$w(x) \geq \frac{C}{R}\left(\sup_{B_{\tau R}} w \right) d(x, \partial B_R), \ \forall x \in \overline{B_R},$$

where $d(x, \partial B_R)$ is the distance between x and the boundary ∂B_R of B_R.

(ii) *If $x_0 \in \partial B_R$ is a point such that $w(x_0) = 0$ and the inner normal derivative $\frac{\partial w}{\partial \nu}(x_0)$ exists, then*

$$\frac{\partial w}{\partial \nu}(x_0) \geq \frac{C}{R} \sup_{B_{\tau R}} w,$$

where $C > 0$ is the constant given by item (i). In particular, if $\frac{\partial w}{\partial \nu}(x_0) = 0$, then $w \equiv 0$ in B_R.

To prove these results, we need first to develop some ingredients. We start by stating the following gradient estimate.

Theorem 1.5. (The gradient estimate) *Let $w \in C^2(B_R) \cap C^0(\overline{B}_R)$ be a solution of*

$$2\Delta_\infty u + \Delta u = 0 \quad \text{in } B_R. \tag{12}$$

Then for any $\sigma \in (0, 1)$ there is a positive constant C_0 depending only on N and σ such that

$$\|\nabla w\|_{L^\infty(B_{\sigma R})} \leq C_0 \left(1 + \frac{\|w\|_{L^\infty(B_R)}}{R} \right).$$

The following lemma establishes the equivalence of (classical) solutions to $\mathcal{L}_\infty u = 0$ in B_R and (classical) weak solutions to $\text{div}(2e^{|\nabla u|^2} \nabla u) = 0$ in B_R.

Lemma 1.6. *Let $u \in C^2(B_R)$. Then, the following identity holds pointwise everywhere*

$$\text{div}\left(2e^{|\nabla u|^2} \nabla u\right) = 2e^{|\nabla u|^2} \mathcal{L}_\infty u, \quad \text{in } B_R. \tag{13}$$

Moreover,

$$\mathcal{L}_\infty u = 0 \ \ (\leq 0, \ \geq 0) \quad \text{in } B_R \tag{14}$$

if and only if the following equation is satisfied in the weak sense

$$\mathcal{L}_\phi u = \text{div}\left(2e^{|\nabla u|^2} \nabla u\right) = 0 \ \ (\leq 0, \ \geq 0),$$

i.e.

$$\int_{B_R} 2e^{|\nabla u|^2}\nabla u \nabla\psi\, dx = 0 \quad (\geq 0,\ \leq 0)$$

for all $\psi \in H^1_c(B_R)$ $(0 \leq \psi \in H^1_c(B_R))$.

Proof. The identity (13) is a straightforward computation. From (13) and divergence theorem,

$$-\int_{B_R} 2e^{|\nabla u|^2}\nabla u \nabla\psi\, dx = \int_{B_R} \operatorname{div}(2e^{|\nabla u|^2}\nabla u)\psi\, dx = \int_{B_R} \psi 2e^{|\nabla u|^2}\mathcal{L}_\infty u\, dx, \tag{15}$$

for all $\psi \in C^\infty_c(B_R)$. Now, once $e^{|\nabla u|^2}|\nabla u| \in L^\infty_{\mathrm{loc}}(B_R) \subset L^2_{\mathrm{loc}}(B_R)$, by Lemma 14.2 in [4], the identity (15) holds for every $\psi \in H^1_c(B_R)$. This finishes the proof. □

In what follows, we will construct barriers using some ideas of Vázquez [17]. They are subsolutions for the operators $\mathcal{L}_\phi$ and $\mathcal{L}_\infty$. Moreover, we recover the geometry of the barriers as presented in [3, 4]. We postpone the proof until Section 4.

Theorem 1.7. (Existence and geometry of barriers) *Assume* $\rho \in (0,1)$, $R > 0$ *and* $\mathcal{A}_{\rho,R} := B_R \setminus B_{\rho R}$. *Given* $\mathcal{M} \geq 0$, *there exists* $\Gamma = \Gamma^{\mathcal{M}}_R \in C^\infty(\overline{\mathcal{A}_{\rho,R}})$ *such that:*

i) $\Gamma|_{\partial B_R} = 0$ *and* $\Gamma|_{\partial B_{\rho R}} = \mathcal{M}$;

ii) *There exists a constant* $C > 0$ *depending only on* N *and* ρ *such that*

$$\mathcal{L}_\infty \Gamma \geq C\frac{\mathcal{M}}{R^2} \geq 0 \quad \text{in } \overline{\mathcal{A}_{\rho,R}}.$$

In particular, we also have

$$\mathcal{L}_\phi \Gamma \geq C\frac{\mathcal{M}}{R^2} \geq 0 \quad \text{in } \overline{\mathcal{A}_{\rho,R}};$$

iii) *There exist constants* $C_1, C_2 > 0$ *depending only on* N *and* ρ *such that*

$$0 \leq C_1\frac{\mathcal{M}}{R}d(x,\partial B_R) \leq \Gamma(x) \leq C_2\frac{\mathcal{M}}{R}d(x,\partial B_R),\ \forall x \in \overline{\mathcal{A}_{\rho,R}};$$
$$0 \leq C_1\frac{\mathcal{M}}{R} \leq |\nabla\Gamma(x)| \leq C_2\frac{\mathcal{M}}{R},\ \forall x \in \overline{\mathcal{A}_{\rho,R}}.$$

Remark 1.8. We highlight the dependence of the constants describing the geometry of the barriers given in the Theorem 1.7 on the annulus radii ratio ρ. More precisely,

$$C_1 := \frac{2(N-1)}{\rho[e^{2(N-1)(1-\rho)/\rho} - 1]} > 0,$$
$$C_2 := \frac{2(N-1)e^{2(N-1)(1-\rho)/\rho}}{\rho[e^{2(N-1)(1-\rho)/\rho} - 1]} > 0.$$

Regarding the asymptotics of those constants, $C_1(\rho), C_2(\rho) \to +\infty$ as $\rho \to 1^-$. Geometrically, this seems natural, as the annulus gets narrower and narrower, the solution gets steeper and steeper. Moreover, when $\rho \to 0^+$, we have $C_1 \to 0$ and $C_2 \to +\infty$. This degenerate geometry somehow resembles the famous nonexistence example given by Zaremba in [18] on the solution to the Dirichlet problem involving harmonic functions on a punctured ball.

2. Proofs of Theorem 1.3 and Corollary 1.4

We first prove the theorem for the case where $R = 1$. From (9) and Lemma 1.6, w is a supersolution of $\mathcal{L}_\phi u = 0$ in B_σ, i.e.,

$$0 \geq \mathcal{L}_\phi w = \operatorname{div}\left(2e^{|\nabla w|^2}\nabla w\right) = \sum_{i,j=1}^{N} (a_{ij}(x)w_{x_i})_{x_j} \quad \text{in } B_\sigma$$

where $a_{ij} := 2\delta_{ij}e^{|\nabla w|^2}$. From (10) there exists $M > 0$ such that

$$2|\eta|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\eta_i\eta_j \leq 2e^{M^2}|\eta|^2 \quad \text{in } B_\sigma, \tag{16}$$

for all $\eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$. By the weak Harnack inequality (Theorem 3.13 in [9]), for any $\tau \in (0, \sigma)$ and $p \in (1, N/(N-2))$ if $N > 2$, $p \in (0, \infty)$ if $N = 2$, there is a positive constant $C = C(N, p, \tau, \sigma, M)$ verifying

$$T := \left(\fint_{B_\tau} w^p \, dx\right)^{\frac{1}{p}} \leq C \inf_{B_{\tau/2}} w, \tag{17}$$

which implies

$$w(x) \geq \inf_{B_{\tau/2}} w \geq C^{-1}T, \quad \forall\, x \in \overline{B}_{\tau/2}. \tag{18}$$

Now, applying Theorem 1.7 for $\mathcal{M} = C^{-1}T \geq 0$ and $\rho = \tau/2$, there exists a barrier $\Gamma \in C^\infty(\overline{\mathcal{A}_{\tau/2,1}})$ satisfying the properties (i)–(iii). By (i)–(ii), we have

$$\begin{cases} -\mathcal{L}_\phi\Gamma \leq 0 \leq -\mathcal{L}_\phi w & \text{in } \mathcal{A}_{\tau/2,1}, \\ \Gamma \leq w & \text{on } \partial\mathcal{A}_{\tau/2,1}. \end{cases}$$

Since $\Gamma, w \in C^0(\overline{\mathcal{A}_{\tau/2,1}})$, we can use the comparison principle (see [13, Theorem 2.4.1 and Proposition 2.4.2]), to obtain

$$w(x) \geq \Gamma(x), \quad \forall\, x \in \overline{\mathcal{A}_{\tau/2,1}}.$$

From Theorem 1.7(iii), we obtain

$$w(x) \geq C_1C^{-1}Td(x, \partial B_1), \quad \forall\, x \in \overline{\mathcal{A}_{\tau/2,1}}.$$

On the other hand, by (18), we have

$$w(x) \geq C^{-1}T \geq C^{-1}Td(x, \partial B_1), \quad \forall\, x \in \overline{B_{\tau/2}}.$$

Taking $C_0 = \min\{C^{-1}, C_1C^{-1}\}$, we have

$$w(x) \geq C_0Td(x, \partial B_1), \quad \forall\, x \in \overline{B_1}. \tag{19}$$

Note that C_0 is a positive constant depending only on N, p, τ, σ, and M, and (i) is proved in the case $R = 1$. Let $x_0 \in \partial B_1$ be a point such that $w(x_0) = 0$ and the unit inner normal derivative $\frac{\partial w}{\partial \nu}(x_0)$ exists. By (19), since $\nu = -x_0$, we have

$$\begin{aligned} \frac{\partial w}{\partial \nu}(x_0) &= \lim_{t\to 0^+} \frac{w(x_0 + t\nu) - w(x_0)}{t} \geq \lim_{t\to 0^+} \frac{C_0Td(x_0 + t\nu, \partial B_1)}{t} \\ &= \lim_{t\to 0^+} \frac{C_0T(1 - |x_0 + t\nu|)}{t} = \lim_{t\to 0^+} \frac{C_0Tt}{t} \\ &= \lim_{t\to 0^+} C_0T = C_0T. \end{aligned} \tag{20}$$

In particular, if $\frac{\partial w}{\partial \nu}(x_0) = 0$, then $w \equiv 0$ in B_τ. As $\tau \in (0, \sigma)$ is arbitrary, $w \equiv 0$ in B_σ, and (ii) is proved in the case $R = 1$. In order to prove the general case $R > 0$, we consider the following rescaled function

$$u(x) = \frac{1}{R} w(Rx), \quad x \in B_1.$$

Thus, we have

a) $\sup_{B_1} u = \sup_{B_R} w/R$;
b) $\nabla u(x) = \nabla w(Rx)$, $x \in B_1$;
c) $\frac{\partial u}{\partial \nu}\left(\frac{x_0}{R}\right) = \frac{\partial w}{\partial \nu}(x_0)$, $x_0 \in \partial B_R$ and $\nu = -\frac{x_0}{|x_0|}$;
d) $d(x, \partial B_R) = Rd\left(\frac{x}{R}, \partial B_1\right)$, $x \in \overline{B_R}$;
e) $\mathcal{L}_\infty u(x) = R\mathcal{L}_\infty w(Rx)$, $x \in B_1$.

From (19), it follows that

$$\frac{1}{R} w(Rx) \geq C_0 \left(\fint_{B_\tau} \left(\frac{w(Rx)}{R} \right)^p dx \right)^{\frac{1}{p}} d(x, \partial B_1), \quad \forall\, x \in \overline{B_1}$$

where C_0 is a positive constant depending only on N, p, τ, and M. Consequently,

$$w(x) \geq \frac{C_0}{R} \left(\fint_{B_{\tau R}} w^p(x)\, dx \right)^{\frac{1}{p}} d(x, \partial B_R), \quad \forall\, x \in \overline{B_R},$$

which gives (i) of Theorem 1.3. If in addition, $w(x_0) = 0$ and $\frac{\partial w}{\partial \nu}(x_0)$ exists, by applying (20) to u at $x_0/R \in \partial B_1$, we have

$$\frac{\partial u}{\partial \nu}\left(\frac{x_0}{R}\right) \geq C_0 \left(\fint_{B_\tau} u^p(x)\, dx \right)^{\frac{1}{p}}.$$

By translating this back in terms of w, we obtain the estimate (ii) for the general case $R > 0$. The case where there is a vanishing of normal derivative of w at x_0 in the boundary of B_R follows as before. □

Proof of Corollary 1.4. The proof of Corollary 1.4 goes similarly to the proof of Theorem 1.3. Once more, by scaling, it is enough to treat the case where $R = 1$. A careful inspection of that proof reveals that once the estimates (16) and (17) are recovered, the proof can be repeated *ipsis litteris*. Since w is a classical solution to $\mathcal{L}_\infty w = 0$ in B_1, the interior gradient estimate (Theorem 1.5) assures that we can take $M := C_0(1 + \|w\|_{L^\infty(B_1)})$, for some positive constant C_0 depending on N and σ, where σ is an arbitrary number in $(0, 1)$. This renders (16). By Lemma 1.6, $\mathcal{L}_\phi w = 0$ in B_1 in a weak sense. In fact, this is a uniform elliptic equation with ellipticity constants given by $\lambda = 2$ and $\Lambda = 2e^{M^2}$, as guaranteed by (16). Now, we can use Harnack inequality (Theorem 3.14 in [9]) to obtain, for $\tau \in (0, \sigma)$,

$$T = \sup_{B_\tau} w \leq C \inf_{B_\tau} w,$$

for some positive constant C depending only on N, τ, and σ. This recovers (17). From this point on, the proof goes exactly as the remaining part of the proof of Theorem 1.3. □

3. Proof of Theorem 1.5

We begin by stating an essential result due to Evans and Smart [5].

Proposition 3.1. *Given $\epsilon \in (0,1]$, assume that $u_\epsilon \in C^2(B_1)\cap C^0(\overline{B}_1)$, with $\|u_\epsilon\|_{L^\infty(B_1)} \le 1$, is a solution of the quasilinear equation*

$$\mathcal{L}_\epsilon u := \Delta_\infty u + \epsilon \Delta u = 0 \quad \text{in } B_1. \tag{21}$$

Then for any $\sigma \in (0,1)$ there is a positive constant C depending only N and σ such that

$$\|\nabla u_\epsilon\|_{L^\infty(B_\sigma)} \le C, \quad \forall \epsilon \in (0,1].$$

We first prove Theorem 1.5 in the case $R = 1$ and $\|w\|_{L^\infty(B_1)} \le 1$. By applying Proposition 3.1 with $\epsilon = 1/2$ to w, we obtain

$$\|\nabla w\|_{L^\infty(B_\sigma)} \le C.$$

In the case of $\|w\|_{L^\infty(B_1)} > 1$, we consider $\sqrt{\epsilon} := 1/\|w\|_{L^\infty(B_1)} \in (0,1)$ and $u_\epsilon := \sqrt{\epsilon} w$. Thus, $\|u_\epsilon\|_{L^\infty(B_1)} = 1$ and

$$\mathcal{L}_\epsilon u_\epsilon = 2\Delta_\infty(\sqrt{\epsilon} w) + \epsilon\Delta(\sqrt{\epsilon} w) = 2\epsilon^{3/2}\Delta_\infty w + \epsilon^{3/2}\Delta w = \epsilon^{3/2}\mathcal{L}_\infty w = 0 \quad \text{in } B_1.$$

Proposition 3.1 again shows that

$$\left\| \nabla\left(\frac{w}{\|w\|_{L^\infty(B_1)}}\right)\right\|_{L^\infty(B_\sigma)} = \|\nabla u_\epsilon\|_{L^\infty(B_\sigma)} \le C,$$

which gives

$$\|\nabla w\|_{L^\infty(B_\sigma)} \le C\|w\|_{L^\infty(B_1)}.$$

Therefore, in any case,

$$\|\nabla w\|_{L^\infty(B_\sigma)} \le C\left(1 + \|w\|_{L^\infty(B_1)}\right), \tag{22}$$

and the theorem is proved if $R = 1$. In order to prove the general case $R > 0$, we set $v(x) = w(Rx)/R$ for $x \in B_1$ and observe that

$$\mathcal{L}_\infty v = \mathcal{L}_\infty\left(\frac{w(Rx)}{R}\right) = 2R\Delta_\infty w(Rx) + R\Delta w(Rx) = R\mathcal{L}_\infty w(Rx) = 0 \text{ in } B_1.$$

From (22), we have

$$\|\nabla w\|_{L^\infty(B_{\sigma R})} = \|\nabla v\|_{L^\infty(B_\sigma)} \le C\left(1 + \|v\|_{L^\infty(B_1).}\right) = C\left(1 + \frac{\|w\|_{L^\infty(B_R)}}{R}\right),$$

and the proof is complete. □

4. Proof of Theorem 1.7

Let $\rho \in (0,1)$, $R > 0$, and $A_{\rho,R}$ be the set given by

$$A_{\rho,R} = \left\{x \in \mathbb{R}^N \colon \rho R <| x |< R\right\} \subset B_R.$$

Define the constants

$$\alpha := \frac{\mathcal{M}}{e^{\beta(1-\rho)R} - 1} > 0, \quad \beta := \frac{2(N-1)}{\rho R}. \tag{23}$$

Following the arguments of Vázquez in [17], we define the barrier Γ by setting

$$\Gamma(x) := v(R- \mid x \mid) := \alpha(e^{\beta(R-|x|)} - 1), \quad x \in \overline{A_{\rho,R}},$$

where $v(t) := \alpha(e^{\beta t} - 1)$. Clearly $\Gamma \in C^\infty(\overline{A_{\rho,R}})$. For $x \in \partial A_{\rho,R}$, we have

$$\Gamma(x) = v(R- \mid x \mid) = \begin{cases} v(0) = 0, & \mid x \mid = R, \\ v((1-\rho)R) = \mathcal{M}, & \mid x \mid = \rho R \end{cases}$$

this proves (i). We now observe that Γ satisfies for each $i, j \in \{1, \dots, N\}$

$$\begin{aligned} \Gamma_{x_i} &= -v'(R- \mid x \mid)\frac{x_i}{\mid x \mid}, \\ \Gamma_{x_i x_i} &= v''(R- \mid x \mid)\frac{x_i^2}{\mid x \mid^2} - v'(R- \mid x \mid)\frac{\mid x \mid^2 - x_i^2}{\mid x \mid^3}, \\ \Gamma_{x_i x_j} &= v''(R- \mid x \mid)\frac{x_i x_j}{\mid x \mid^2} - v'(R- \mid x \mid)\frac{\delta_{ij}|x|^2 - x_i x_j}{\mid x \mid^3}. \end{aligned}$$

Thus,

$$\mid \nabla\Gamma \mid^2 = \sum_{i=1}^N (\Gamma_{x_i})^2 = \sum_{i=1}^N v'(R- \mid x \mid)^2 \frac{x_i^2}{\mid x \mid^2} = v'(R- \mid x \mid)^2, \quad x \in \overline{A_{\rho,R}},$$

and

$$\begin{aligned} \Delta\Gamma &= \sum_{i=1}^N \Gamma_{x_i x_i} = \sum_{i=1}^N \left(v''(R- \mid x \mid)\frac{x_i^2}{\mid x \mid^2} - v'(R- \mid x \mid)\frac{\mid x \mid^2 - x_i^2}{\mid x \mid^3} \right) \\ &= \left(v''(R- \mid x \mid) - \frac{(N-1)}{\mid x \mid} v'(R- \mid x \mid) \right), \quad x \in \overline{A_{\rho,R}}. \end{aligned}$$

In particular,

$$\mid \nabla\Gamma \mid = v'(R- \mid x \mid), \quad x \in \overline{A_{\rho,R}}. \tag{24}$$

Furthermore,

$$\begin{aligned} \Delta_\infty\Gamma_{x_i} &= \sum_{i,j=1}^N \Gamma_{x_i}\Gamma_{x_j}\Gamma_{x_i x_j} \\ &= \sum_{i,j=1}^N v'(R-|x|)^2 \frac{x_i x_j}{|x|^2} \left(v''(R- \mid x \mid)\frac{x_i x_j}{\mid x \mid^2} - v'(R- \mid x \mid)\frac{\delta_{ij}|x|^2 - x_i x_j}{\mid x \mid^3} \right) \\ &= \sum_{i,j=1}^N \left(v'(R-|x|)^2 v''(R- \mid x \mid)\frac{x_i^2 x_j^2}{\mid x \mid^4} - v'(R- \mid x \mid)^3 \frac{x_i x_j \delta_{ij}|x|^2 - x_i^2 x_j^2}{\mid x \mid^5} \right) \\ &= v'(R-|x|)^2 v''(R-|x|) - v'(R-|x|)^3 \frac{|x|^4 - |x|^4}{|x|^5} \\ &= v'(R-|x|)^2 v''(R-|x|). \end{aligned}$$

As a consequence, given $x \in \overline{A_{\rho,R}}$, we have

$$\mathcal{L}_\infty\Gamma = 2v'(R-|x|)^2 v''(R-|x|) + \left(v''(R- \mid x \mid) - \frac{(N-1)}{\mid x \mid} v'(R- \mid x \mid) \right).$$

Since $v''(t) = \beta v'(t)$ for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned} \mathcal{L}_\infty \Gamma &= 2\beta v'(R - |x|)^3 + \left(\beta - \frac{(N-1)}{\mid x \mid}\right) v'(R- \mid x \mid) \\ &= \left(2\beta v'(R-|x|)^2 + \left(\beta - \frac{(N-1)}{\mid x \mid}\right)\right) v'(R- \mid x \mid) \\ &\geq \left(\beta - \frac{(N-1)}{\mid x \mid}\right) v'(R- \mid x \mid) \\ &\geq \left(\beta - \frac{(N-1)}{\rho R}\right) v'(R- \mid x \mid) \end{aligned} \tag{25}$$

for every $x \in \overline{A_{\rho,R}}$, once for every for every $x \in \overline{A_{\rho,R}}$, by (23),

$$\left(\beta - \frac{(N-1)}{|x|}\right) \geq \beta - \frac{(N-1)}{\rho R} = \frac{\beta}{2}. \tag{26}$$

Since $v'(t) = \alpha\beta e^{\beta t} \geq \alpha\beta$ for every $t \in \mathbb{R}$, from (25) and (26), we obtain

$$\mathcal{L}_\infty \Gamma \geq \frac{\beta^2 \alpha}{2} = C\frac{\mathcal{M}}{R^2}, \quad \forall\, x \in \overline{A_{\rho,R}}, \tag{27}$$

where

$$C = \frac{2(N-1)^2}{\rho^2(e^{2(N-1)(1-\rho)/\rho} - 1)}.$$

From (8) and (27), we conclude the proof of (ii). Now, defining

$$\varphi(t) := v(R-t), \quad t \in [\rho R, R],$$

we have, $\varphi(R) = v(0) = 0$ and

$$\varphi'(t) = -v'(R-t) < 0, \quad t \in [\rho R, R].$$

Hence $\varphi'(R) = -v'(0) = -\alpha\beta < 0$. Since $v''(t) = \alpha\beta^2 e^{\beta t} > 0$ for all $t \in \mathbb{R}$,

$$\varphi''(t) = v''(R-t) > 0, \quad t \in [\rho R, R].$$

Therefore, φ is a convex function in $[\rho R, R]$, which implies

$$\varphi(t) \geq \varphi'(R)(t-R) = \alpha\beta(R-t) = \frac{2(N-1)\mathcal{M}}{\rho R(e^{\beta(1-\rho)R} - 1)}(R-t),$$

for every $t \in [\rho R, R]$. Consequently,

$$\Gamma(x) \geq \frac{2(N-1)\mathcal{M}}{\rho R(e^{\beta(1-\rho)R} - 1)} d(x, \partial B_R), \quad \forall\, x \in \overline{A_{\rho,R}}, \tag{28}$$

and since $\beta = \dfrac{2(N-1)}{\rho R}$, we obtain

$$\Gamma(x) \geq \frac{2(N-1)}{\rho(e^{2(N-1)(1-\rho)/\rho} - 1)} \frac{\mathcal{M}}{R} d(x, \partial B_R), \quad \forall\, x \in \overline{A_{\rho,R}},$$

which shows that (iii) is true for $C_1 = 2(N-1)/(\rho(e^{2(N-1)(1-\rho)/\rho} - 1)) > 0$. On the other hand, given $t \in (\rho R, R)$ by mean value theorem there exists $\xi_t \in (t, R)$ such that

$$\begin{aligned}\varphi(t) = \varphi(t) - \varphi(R) = \varphi'(\xi_t)(t-R) &= \alpha\beta e^{\beta(R-\xi_t)}(R-t)\\ &= \frac{2(N-1)\mathcal{M}}{\rho R(e^{\beta(1-\rho)R}-1)} e^{\beta(R-\xi_t)}(R-t)\\ &= \frac{2(N-1)}{\rho(e^{\frac{2(N-1)(1-\rho)}{\rho}}-1)}\frac{\mathcal{M}}{R} e^{\frac{2(N-1)}{\rho R}(R-\xi_t)}(R-t)\\ &\le \frac{2(N-1)e^{2(N-1)(1-\rho)/\rho}}{\rho(e^{\frac{2(N-1)(1-\rho)}{\rho}}-1)}\frac{\mathcal{M}}{R}(R-t),\end{aligned}$$

where in the last inequality we have used that $|\xi_t - R| \le |R-\rho R| = (1-\rho)R$. Therefore,

$$\Gamma(x) \le \frac{2(N-1)e^{2(N-1)(1-\rho)/\rho}}{\rho(e^{\frac{2(N-1)(1-\rho)}{\rho}}-1)}\frac{\mathcal{M}}{R} d(x,\partial B_R), \quad \forall\, x \in \overline{A_{\rho,R}},$$

and so (iii) is true for $C_2 = 2(N-1)e^{2(N-1)(1-\rho)/\rho}/\rho(e^{\frac{2(N-1)(1-\rho)}{\rho}}-1) > 0$. In order to show the gradient estimates in (iii), note that

$$\begin{aligned}|\nabla\Gamma(x)| = |v'(R-|x|)| &= \alpha\beta e^{\beta(R-|x|)}\\ &= \frac{2(N-1)}{\rho(e^{2(N-1)(1-\rho)/\rho}-1)}\frac{\mathcal{M}}{R} e^{\frac{2(N-1)(R-|x|)}{\rho R}}, \quad \forall\, x \in \overline{A_{\rho,R}},\end{aligned}$$

whence, once again, since $R-|x| \le R-\rho R = (1-\rho)R$,

$$\frac{2(N-1)}{\rho[e^{2(N-1)(1-\rho)/\rho}-1]}\frac{\mathcal{M}}{R} \le |\nabla\Gamma(x)| \le \frac{2(N-1)e^{2(N-1)(1-\rho)/\rho}}{\rho[e^{2(N-1)(1-\rho)/\rho}-1]}\frac{\mathcal{M}}{R},$$

for every $x \in \overline{A_{\rho,R}}$. Finally,

$$\mathcal{L}_\phi u = 2e^{|\nabla u|^2}\mathcal{L}_\infty u \ge \mathcal{L}_\infty u \ge C\frac{\mathcal{M}}{R^2}.$$

This finishes the proof of Theorem 1.7. □

Acknowledgements. The authors thank the editor and anonymous referee for the comments, suggestions, and corrections. Jefferson Abrantes Santos was partially supported by CNPq grant 304774/2022-7 and Paraíba State Research Foundation (FAPESQ) grant 3031/2021. Diego Moreira was partially supported by grants UNI-00210.00496.01.00/23, FUNCAP-Ceará, and 310292/2023-9 and 406692/2023-8, CNPq. This work was supported by grants 2022/16407-1 and 2023/14636-6, São Paulo Research Foundation (FAPESP).

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Received 9 December 2023 • Accepted 2 May 2024 • Published online 31 May 2024

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