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On Ramanujan's modular equations and Hecke groups

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Abstract. Inspired by the work of Ramanujan, many people have studied generalized modular equations and the numerous identities found by Ramanujan. These identities known as modular equations can be transformed into polynomial equations. There is no developed theory about how to find the degrees of these polynomial modular equations explicitly. In this paper, we determine the degrees of the polynomial modular equations explicitly and study the relation between Hecke groups and modular equations in Ramanujan's theories of signatures 2, 3, and 4.

Ramanujanin moduliyhtälöt ja Hecken ryhmät

Tiivistelmä. Ramanujanin työn innoittamina monet ovat tutkineet yleistettyjä moduliyhtälöitä ja lukuisia Ramanujanin löytämiä yhtäsuuruuksia. Nämä moduliyhtälöinä tunnetut yhtäsuuruudet voidaan muuntaa polynomiyhtälöiksi, joiden asteen selvittämiseksi ei kuitenkaan ole kehittynyttä teoriaa. Tässä työssä määritetään polynomimuotoisten moduliyhtälöiden asteet sekä tutkitaan Hecken ryhmien ja moduliyhtälöiden välistä yhteyttä Ramanujanin teorioissa, joiden kertaluku on 2, 3 tai 4.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$. For complex numbers a, b, c with $c \neq 0, -1, -2, \ldots,$ and nonnegative integer n, the Gaussian hypergeometric function, ${}_2F_1(a, b; c; z)$, is defined as

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad z \in \mathbb{D}, $$

where $(a)_n$ is the Pochhammer symbol or shifted factorial function given by

$$ (a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)\cdots(a+n-1), & \text{if } n \geq 1. \end{cases} $$

By analytic continuation, ${}_2F_1(a, b; c; z)$ may be extended to a cut plane $\mathbb{C} \setminus [1, \infty)$. For more details, see [6, Chapter II] and [22, Chapter XIV].

For $t \in (0, \frac{1}{2}]$, α, $\beta \in (0, 1)$ and a given integer $p > 1$, we say that β has *order* or *degree* p over α in the theory of signature $\frac{1}{t}$, if

$$ \frac{{}_2F_1(t, 1-t; 1; 1-\beta)}{{}_2F_1(t, 1-t; 1; \beta)} = p \, \frac{{}_2F_1(t, 1-t; 1; 1-\alpha)}{{}_2F_1(t, 1-t; 1; \alpha)}. \tag{1.1} $$

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Equation (1.1) is known as the generalized modular equation. In this article, we will use the terminology *order* to avoid the confusion between the degree of the polynomial $P(\alpha,\beta)$ (see Theorem A) and the degree of the modulus β over the modulus α.

A modular equation of order p in the theory of signature $\frac{1}{t}$ is an explicit relation between α and β induced by (1.1) (see [9]). The great Indian mathematician Ramanujan extensively studied the generalized modular equation (1.1) and gave many identities involving α and β for some rational values of t. Without original proofs, these identities were listed in Ramanujan's unpublished notebooks (see, e.g., [7]). There were no developed theories related to Ramanujan's modular equations before the 1980s. Some mathematicians, for example, Berndt, Bhargava, J. M. Borwein, P. B. Borwein, Garvan developed and organized the theories and tried to give the proofs of many identities recorded by Ramanujan (see [7, 9, 10]). Also, Anderson, Vamanamurthy, Vuorinen and others have investigated the theory of Ramanujan's modular equations from different perspectives (see, e.g., [3, 5]).

In this paper, we will consider the modular equations in the theories of signatures 2, 3, and 4. There are different forms of modular equations for the same order of β over α in the theory of signature $\frac{1}{t}$. For example,

$$(\alpha\beta)^{1/3} + \left\{(1-\alpha)(1-\beta)\right\}^{1/3} = 1, \tag{1.2}$$

$$\left\{\frac{(1-\beta)^2}{1-\alpha}\right\}^{\frac{1}{3}} - \left(\frac{\beta^2}{\alpha}\right)^{\frac{1}{3}} = m \tag{1.3}$$

and

$$\left(\frac{\alpha^2}{\beta}\right)^{\frac{1}{3}} + \left\{\frac{(1-\alpha)^2}{1-\beta}\right\}^{\frac{1}{3}} = \frac{4}{m^4}, \tag{1.4}$$

where $m = \frac{{}_2F_1(1/3,\,2/3;1;\alpha)}{{}_2F_1(1/3,\,2/3;1;\beta)}$, are the modular equations when the modulus β has order 2 over the modulus α in the theory of signature 3 (see [9, Theorem 7.1]). Note that (1.2) can be transformed to the following polynomial equation (see [2])

$$(2\alpha-1)^3\beta^3 - 3\alpha(4\alpha^2-13\alpha+10)\beta^2 + 3\alpha(2\alpha^2-10\alpha+9)\beta - \alpha^3 = 0.$$

There is an intimate relation between the modular equations in Ramanujan's theories of signatures $\frac{1}{t} = 2, 3, 4$ and the Hecke groups. The motivation of our present study comes from this relationship. The author and Sugawa [2] offered a geometric approach to the proof of Ramanujan's identities for the solutions (α,β) to the generalized modular equation (1.1). They proved that the solution (α,β) satisfies a polynomial equation $P(\alpha,\beta)=0$. In this paper, we compute the degrees $\deg_\alpha P(\alpha,\beta)$ and $\deg_\beta P(\alpha,\beta)$ explicitly based on the relation between the Hecke groups and modular equations. We prove by geometric approach that if (α,β) is a solution to the generalized modular equation (1.1), then $(1-\beta, 1-\alpha)$ is also a solution to (1.1) and $P(1-\beta,1-\alpha)=0$. Note that by the degree μ of the polynomial $P(\alpha,\beta)$, we will mean that $\mu = \deg_\alpha P(\alpha,\beta) = \deg_\beta P(\alpha,\beta)$.

For $t \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$, let

$$\lambda_t = 2\cos\frac{(1-2t)\pi}{2}. \tag{1.5}$$

Let $H(\lambda_t)$ be a Hecke group in $\mathrm{PSL}(2,\mathbb{R})$ generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & \lambda_t \\ 0 & 1 \end{pmatrix}.$$

Then

$$H_e(\lambda_t) = \left\{ \begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc\lambda_t^2 = 1 \right\} \tag{1.6}$$

is a subgroup of $H(\lambda_t)$ of index 2 (see [12]). Note that $H_e(\lambda_t)$ is called the even subgroup of $H(\lambda_t)$ for $\lambda_t = \sqrt{2}$ and $\sqrt{3}$. We will consider $H_e(\lambda_t)$ for $\lambda_t = \sqrt{2}, \sqrt{3}$ and 2. Let $\mathbb{H}$ denote the upper half-plane $\{\tau \in \mathbb{C}\colon \operatorname{Im} \tau > 0\}$. Then the quotient Riemann surface $H_e(\lambda_t)\backslash\mathbb{H}$ is $\widehat{\mathbb{C}} \setminus \{0, 1\}$ for $t \in \{\frac{1}{3}, \frac{1}{4}\}$ and $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ for $t = \frac{1}{2}$. The following theorem asserts that the solution (α, β) to the generalized modular equation (1.1) satisfies a polynomial equation in α and β.

Theorem A. [2, Theorem 1.8] *For integers $p, n > 1$ and $t \in (0, 1/2]$, let*

$$H'_e(\lambda_t) = M_p^{-1} H_e(\lambda_t) M_p \quad \textit{and} \quad H_{M_p}(\lambda_t) = H_e(\lambda_t) \cap H'_e(\lambda_t),$$

where $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. Then, the solution (α, β) to the generalized modular equation (1.1) in $H_e(\lambda_t)\backslash\mathbb{H}$ satisfies the polynomial equation $P(\alpha, \beta) = 0$ for an irreducible polynomial $P(x, y)$ of degree $\mu = \deg_x P(x, y) = \deg_y P(x, y)$ if and only if $H_{M_p}(\lambda_t)$ is a subgroup of $H_e(\lambda_t)$ of index μ.

Chan and Liaw [13] studied modular equations in the theory of signature 3 based on the modular equations studied by Russell [18].

Theorem B. [13, Theorems 2.1, 3.1] *If $p > 2$ is a prime, $u = (\alpha\beta)^{l/8}$ and $v = \{(1-\alpha)(1-\beta)\}^{l/8}$, where $(p+1)/8 = m/l$ in lowest terms, then (u, v) satisfies a polynomial equation $Q(u, v) = 0$, where $Q(x, y)$ is of degree $m = \deg_x Q(x, y) = \deg_y Q(x, y)$ in the theory of signature 2. If $p > 3$ is a prime, $u = (\alpha\beta)^{l/6}$ and $v = \{(1-\alpha)(1-\beta)\}^{l/6}$, where $(p+1)/3 = m/l$ in lowest terms, then (u, v) satisfies a polynomial equation $Q(u, v) = 0$, where $Q(x, y)$ is of degree $m = \deg_x Q(x, y) = \deg_y Q(x, y)$ in the theory of signature 3.*

Remark 1. In the theory of signature 3, the degree μ of the polynomial $P(\alpha, \beta)$ in Theorem A and the degree m of the polynomial $Q(u, v)$ in Theorem B are related as follows:

(i) $\mu = 3m$ when $p \equiv 2 \pmod 3$,
(ii) $\mu = m$ when $p \equiv 1 \pmod 3$.

The remainder of this article is organized as follows. In Section 2, we state our main results. Some basic facts related to modular groups and Hecke groups are discussed in Section 3. Finally, the proofs of the main results are given in Section 4.

2. Main results

Let $\Psi(N)$ denote the Dedekind psi function given by

$$\Psi(N) = N \prod_{\substack{q \mid N \\ q \text{ prime}}} \left(1 + \frac{1}{q}\right), \quad N \in \mathbb{N} \tag{2.1}$$

(see [14, p. 123]). Our first result is for determining the degree $\mu = \deg_\alpha P(\alpha, \beta) = \deg_\beta P(\alpha, \beta)$ of the polynomial $P(\alpha, \beta)$ in Theorem A explicitly in Ramanujan's theories of signatures $\frac{1}{t} = 2$, 3 and 4.

Theorem 1. *For an integer $p > 1$, suppose β has order p over α in the theories of signatures $\frac{1}{t} = 2$, 3 and 4. Let $\mu\left(p, \frac{1}{t}\right) = \deg_\alpha P(\alpha, \beta) = \deg_\beta P(\alpha, \beta)$, then*

$$\mu(p, 2) = \mu(p, 4) = \frac{1}{3}\Psi(2p)$$

and

$$\mu(p, 3) = \frac{1}{4}\Psi(3p).$$

Remark 2. If p is an odd prime, then $\mu(p, 2) = \mu(p, 4) = p + 1$. If $p \neq 3$ is a prime, then $\mu(p, 3) = p + 1$.

We compute the degree $\mu\left(p, \frac{1}{t}\right)$ for some small values of p and $t \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$ in Table 1. Even if one does not know the corresponding Hecke subgroups, he/she can compute the degree of modular equations in the theories of signatures 2, 3, and 4 using the formulas in Theorem 1.

The following result establishes some statements related to the Hecke subgroups and the modular equations in the theories of signatures 2, 3, and 4.

Theorem 2. *For a given integer $p > 1$, suppose that β has order p over α in the theories of signatures $\dfrac{1}{t} = 2$, 3 and 4. Let $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, then*

(i) *there exists a Hecke subgroup $H_e(\lambda_t)$ of finite index in $H(\lambda_t)$,*

(ii) *$H_e(\lambda_t) \cap \left(M_p^{-1} H_e(\lambda_t) M_p\right)$ has finite index in $H(\lambda_t)$,*

(iii) *the degree of the branched covering*

$$H_{M_p}(\lambda_t)\backslash\mathbb{H} \to H_e(\lambda_t)\backslash\mathbb{H}$$

is finite, where $H_{M_p}(\lambda_t) = H_e(\lambda_t) \cap \left(M_p^{-1} H_e(\lambda_t) M_p\right)$,

(iv) *there is a polynomial equation $P(\alpha, \beta) = 0$ such that $\mu = \deg_\alpha P(\alpha, \beta) = \deg_\beta P(\alpha, \beta) = \left|H_e(\lambda_t) \colon H_{M_p}(\lambda_t)\right|$.*

Remark 3. In fact, the statements in Theorem 2 are mutually equivalent.

We can express the generalized modular equation (1.1) as $f_t(\beta) = p f_t(\alpha)$, where f_t is defined by

$$f_t(z) = i\,\frac{{}_2F_1(t, 1-t; 1; 1-z)}{{}_2F_1(t, 1-t; 1; z)}, \tag{2.2}$$

$t \in (0, 1/2]$ and p is an integer > 1. Consider the canonical projection $\pi_t \colon \mathbb{H} \to H_e(\lambda_t)\backslash\mathbb{H}$, where π_t is the inverse of f_t (a detailed discussion will be given in Section 3). The moduli $\alpha, \beta \in \widehat{\mathbb{C}} \setminus \{0, 1\}$ satisfy (1.1) if and only if $\alpha = \pi_t(\tau)$ and $\beta(\tau) = \pi_t(p\tau)$ for $\tau \in \mathbb{H}$ (see [2]) and we have the following theorem.

Theorem 3. *For the canonical projection $\pi_t \colon \mathbb{H} \to H_e(\lambda_t)\backslash\mathbb{H}$, let $\alpha = \pi_t(\tau)$ and $\beta = \pi_t(p\tau)$, where $\tau \in \mathbb{H}$ and p is an integer > 1. If the solution (α, β) to the generalized modular equation (1.1) satisfies the equation $P(x, y) = 0$, then $(1 - \beta, 1 - \alpha)$ is also a solution to (1.1) and satisfies the equation $P(x, y) = 0$, where $P(x, y)$ is the polynomial in Theorem A.*

Remark 4. The equation $P(1 - \beta, 1 - \alpha) = 0$ is the reciprocal of the equation $P(\alpha, \beta) = 0$ and this process is known as the method of reciprocation (see [8, Theorem 6.3.2]). We use Lemma 3.4 of [2] in the proof of Theorem 3 and our proof is geometric.

p	$\mu(p,2)$ and $\mu(p,4)$	$\mu(p,3)$
2	2	3
3	4	3
4	4	6
5	6	6
6	8	9
7	8	8
8	8	12
9	12	9
10	12	18
11	12	12
12	16	18
13	14	14
14	16	24
15	24	18
16	16	24
17	18	18
18	24	27
19	20	20
20	24	36

Table 1. Values of $\mu\left(p, \frac{1}{t}\right)$ for some small values of p and $t \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$.

3. Preliminaries

The group $\mathrm{SL}(2,\mathbb{R})$ is defined by

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R}, ad-bc=1 \right\}$$

and is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let I_2 denote the 2×2 identity matrix, then the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\}$. For $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$, the group $\mathrm{PSL}(2,\mathbb{R})$ acts on the upper half-plane $\mathbb{H}$ as follows

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}$$

and $\mathrm{PSL}(2,\mathbb{R})$ is the group of automorphisms of the upper half-plane $\mathbb{H}$. All transformations of $\mathrm{PSL}(2,\mathbb{R})$ are conformal. Assume that Γ is a Fuchsian group of the first kind which leaves the upper half-plane $\mathbb{H}$ or the unit disc $\mathbb{D}$ invariant. Then, Γ is a discrete subgroup of the group of orientation-preserving isometries of $\mathbb{H}$, i.e., Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ (see [16]).

Let m be a positive integer, then the congruence subgroup $\Gamma_0(m)$ is defined as

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})\colon c \equiv 0 \pmod{m} \right\}.$$

We now construct a connection between the Schwarz triangle function and the Gaussian hypergeometric function, ${}_2F_1(a, b; c; z)$. Since

$$w_1 = {}_2F_1(a, b; c; z)$$

and

$$w_2 = {}_2F_1(a, b; a + b + 1 - c; 1 - z)$$

are two linearly independent solutions of the following hypergeometric differential equation

$$z(1 - z)\frac{d^2w}{dz^2} + \{c - (a + b + 1)z\}\frac{dw}{dz} - abw = 0, \tag{3.1}$$

it is a well-known fact that the Schwarz triangle function defined by

$$S(z) = i\,\frac{{}_2F_1(a, b; a + b + 1 - c; 1 - z)}{{}_2F_1(a, b; c; z)}$$

maps the upper half-plane $\mathbb{H}$ conformally onto a curvilinear triangle Δ_t, which has interior angles $(1-c)\pi$, $(c-a-b)\pi$ and $(b-a)\pi$ at the vertices $S(0)$, $S(1)$ and $S(\infty)$, respectively. For details, we recommend the readers to go through [17, Chapter V, Section 7]. For $t \in (0, \frac{1}{2}]$, let $a = t$, $b = 1 - a = 1 - t$ and $c = 1$, then $S(z)$ can be expressed as

$$S(z) = f_t(z) = i\,\frac{{}_2F_1(t, 1 - t; 1; 1 - z)}{{}_2F_1(t, 1 - t; 1; z)}. \tag{3.2}$$

Let $\theta_1 = \frac{\pi}{m_1}$, $\theta_2 = \frac{\pi}{m_2}$ and $\theta_3 = \frac{\pi}{m_3}$ be the interior angles of the triangle Δ_t, then the inverse map $f_t^{-1}\colon \Delta_t \to \mathbb{H}$ can be analytically continued as a single-valued function if and only if m_j is an integer greater than 1 or $m_j = \infty$ for $j = 1,\ 2,\ 3$ (see [19, p. 416]). Therefore,

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1. \tag{3.3}$$

The following lemma is related to the above facts.

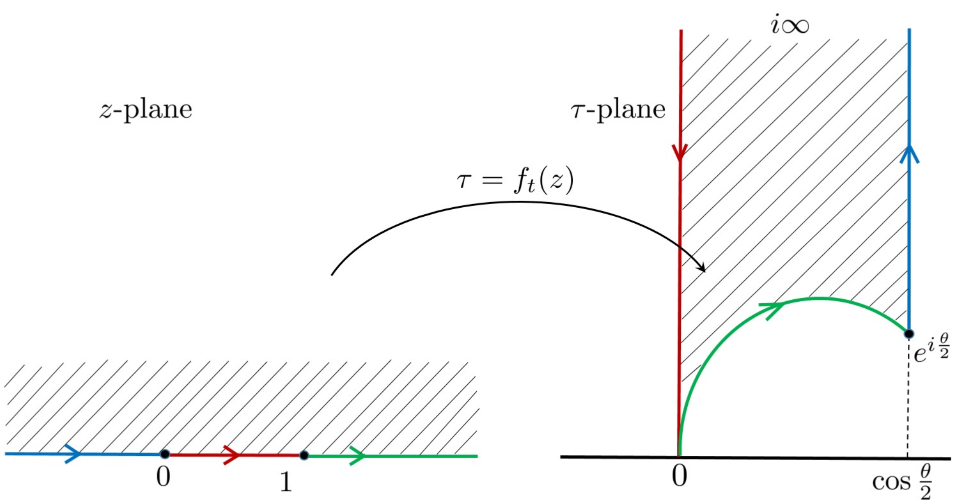

Figure 1. Mapping of the upper half-plane $\mathbb{H}$ onto Δ_t by f_t.

Lemma 1. [4, Lemma 4.1] *Let the map f_t be defined by (3.2) for $t \in (0, \frac{1}{2}]$. Then, the upper half-plane $\mathbb{H}$ is mapped by f_t onto the hyperbolic triangle Δ_t given by*

$$\Delta_t = \left\{\tau \in \mathbb{H}\colon 0 < \operatorname{Re}\tau < \cos\frac{\theta}{2},\ \left|2\tau\cos\frac{\theta}{2} - 1\right| > 1\right\},$$

where $\theta = (1-2t)\pi$. The interior angles of Δ_t are $0, 0$, and $\theta = (1-2t)\pi$ at the vertices $f_t(0) = i\infty$, $f_t(1) = 0$ and $f_t(\infty) = e^{i\frac{\theta}{2}}$, respectively.

By Lemma 1, the condition (3.3) becomes $\frac{1}{m_3} < 1$, i.e., it depends only on the third fixed point $f_t(\infty) = e^{i\frac{\theta}{2}}$. Remember that $m_j(t) = \frac{1}{1-2t}$ and $t \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$. Thus $m_j(1/2) = \infty$, $m_j(1/3) = 3$ and $m_j(1/4) = 2$. Let $\pi_t\colon \Delta_t \to \mathbb{H}$ be the inverse map of f_t, then we can extend $\pi_t(\tau)$ analytically to a single-valued function on $\mathbb{H}$ with the real axis as its natural boundary by applying the Schwarz reflection principle repeatedly. The covering group of π_t is the Hecke subgroup $H_e(\lambda_t)$. For more details, see Section 2 of [2], where $H_e(\lambda_t)$ is denoted by G_q.

The subgroup $H_e(\lambda_t)$ has two cusps and one elliptic point for $t \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$ and has three cusps for $t = \frac{1}{2}$. Thus, the quotient Riemann surface $H_e(\lambda_t)\backslash\mathbb{H}$ is the twice punctured Riemann sphere $\widehat{\mathbb{C}} \setminus \{0, 1\}$ for $t \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$ and the thrice punctured Riemann sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ for $t = \frac{1}{2}$. The set of cusps of the Hecke group $H(\lambda_t)$ is $\mathbb{Q}[\lambda_t] \cup \{\infty\}$. In order to compactify the quotient Riemann surface $H_e(\lambda_t)\backslash\mathbb{H}$, let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q}[\lambda_t] \cup \{\infty\}$. Then, $H_e(\lambda_t)\backslash\mathbb{H}^*$ is a compact Riemann surface. For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(\lambda_t)$ and $\tau \in \mathbb{H}$, the meromorphic function $g\colon \mathbb{H} \to H(\lambda_t)\backslash\mathbb{H}^*$ is called an automorphic function if $g\left(\dfrac{a\tau + b}{c\tau + d}\right) = g(\tau)$ (see [11]).

4. Proofs of main results

Let $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. For an integer $p > 1$, let $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, then the transformation group of order p (see [20, Chapter VI]), Γ_{M_p}, is given by

$$\Gamma_{M_p} := \Gamma \cap \left(M_p^{-1}\Gamma M_p\right),$$

which can be written as the group of Möbius transformations

$$\Gamma_{M_p} := \left\{\gamma \in \Gamma:\ M_p\gamma M_p^{-1} \in \Gamma\right\}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then $M_p\gamma M_p^{-1} = \begin{pmatrix} a & pb \\ \frac{c}{p} & d \end{pmatrix}$. Hence, $M_p\gamma M_p^{-1} \in \Gamma$ only when $c \equiv 0 \pmod{p}$, and we have $\Gamma_{M_p} = \Gamma_0(p)$.

The following lemma is a well-known result, e.g., see Proposition 1.43 in [21] or [20, p. 79].

Lemma 2. *For any positive integer N, $\left|\Gamma\colon \Gamma_0(N)\right| = \Psi(N)$.*

Proof of Theorem 1. For $t \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$ and $\lambda_t = 2\cos\frac{(1-2t)\pi}{2}$, let

$$H_{M_p}(\lambda_t) = \left\{\gamma \in H_e(\lambda_t)\colon M_p\gamma M_p^{-1} \in H_e(\lambda_t)\right\},$$

where p is an integer > 1 and $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. If $\gamma = \begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} \in H_e(\lambda_t)$, then

$$M_p \gamma M_p^{-1} = \begin{pmatrix} a & p b\lambda_t \\ \frac{c}{p}\lambda_t & d \end{pmatrix}.$$

Therefore, $M_p\gamma M_p^{-1} \in H_e(\lambda_t)$ only when $c \equiv 0 \pmod{p}$ and we have

$$H_{M_p}(\lambda_t) = \left\{ \begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} \in H_e(\lambda_t)\colon c \equiv 0 \pmod{p} \right\}. \tag{4.1}$$

Consequently,

$$H_e(\lambda_t) \cap \left(M_p^{-1} H_e(\lambda_t) M_p \right) = H_{M_p}(\lambda_t) \tag{4.2}$$

and

$$H_{M_p}(\lambda_t) < H_e(\lambda_t) < H(\lambda_t).$$

Let π_t and π_t' denote the canonical projections $\mathbb{H} \to H_e(\lambda_t)\backslash\mathbb{H}$ and $\mathbb{H} \to H_{M_p}(\lambda_t)\backslash\mathbb{H}$, respectively. From the subgroup relation $H_{M_p}(\lambda_t) < H_e(\lambda_t)$, we have the branched covering map $\varphi\colon H_{M_p}(\lambda_t)\backslash\mathbb{H} \to H_e(\lambda_t)\backslash\mathbb{H}$ and the following commutative diagram:

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\pi_t'"'] \arrow[dr, "\pi_t"] & \\ H_{M_p}(\lambda_t)\backslash\mathbb{H} \arrow[r, "\varphi"'] & H_e(\lambda_t)\backslash\mathbb{H}. \end{tikzcd}$$

The degree of the branched covering $H_{M_p}(\lambda_t)\backslash\mathbb{H} \to H_e(\lambda_t)\backslash\mathbb{H}$ is $\left|H_e(\lambda_t) : H_{M_p}(\lambda_t)\right|$, which is the degree $\mu\left(p, \frac{1}{t}\right) = \deg_\alpha P(\alpha,\beta) = \deg_\beta P(\alpha,\beta)$ of the polynomial $P(\alpha,\beta)$ by Theorem A.

Further, we have

$$\Gamma \cap \left(M_p^{-1} \Gamma M_p \right) = \Gamma_0(p),$$

and

$$\Gamma_0(\lambda_t^2 p) < \Gamma_0(\lambda_t^2) < \Gamma.$$

Let us consider the mapping

$$\Theta\colon H_e(\lambda_t) \to \Gamma_0(\lambda_t^2)$$

defined by

$$\Theta(A) = M_{\lambda_t}^{-1} A M_{\lambda_t},$$

where $A = \begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} \in H_e(\lambda_t)$ and $M_{\lambda_t} = \begin{pmatrix} \lambda_t & 0 \\ 0 & 1 \end{pmatrix}$. Then, we have

$$\Theta(H_e(\lambda_t)) = \Gamma_0(\lambda_t^2) \quad \text{and} \quad \Theta(H_{M_p}(\lambda_t)) = \Gamma_0(\lambda_t^2 p).$$

Therefore, $H_e(\lambda_t) \cong \Gamma_0(\lambda_t^2)$, $H_{M_p}(\lambda_t) \cong \Gamma_0(\lambda_t^2 p)$, and we have

$$\left|H_e(\lambda_t) : H_{M_p}(\lambda_t)\right| = \left|\Gamma_0(\lambda_t^2) : \Gamma_0(\lambda_t^2 p)\right| = \frac{|\Gamma : \Gamma_0(\lambda_t^2 p)|}{|\Gamma : \Gamma_0(\lambda_t^2)|}.$$

By Lemma 2, $|\Gamma : \Gamma_0(\lambda_t^2 p)| = \Psi(\lambda_t^2 p)$ and $|\Gamma : \Gamma_0(\lambda_t^2)| = \Psi(\lambda_t^2)$. Hence

$$\mu\left(p, \frac{1}{t}\right) = \left|H_e(\lambda_t) : H_{M_p}(\lambda_t)\right| = \frac{\Psi(\lambda_t^2 p)}{\Psi(\lambda_t^2)},$$

which implies $\mu(p,2) = \frac{1}{6}\Psi(4p)$, $\mu(p,3) = \frac{1}{4}\Psi(3p)$, and $\mu(p,4) = \frac{1}{3}\Psi(2p)$. By (2.1), it is easy to show that $\Psi(4p) = 2\Psi(2p)$. Thus, $\mu(p,2) = \mu(p,4)$ as required. $\square$

Let $X_1 = H_e(\lambda_t)\backslash\mathbb{H}$ and $X_2 = H_{M_p}(\lambda_t)\backslash\mathbb{H}$. For the canonical projections $\pi_t\colon \mathbb{H} \to X_1$ and $\pi_t'\colon \mathbb{H} \to X_2$, consider the mappings $\varphi\colon X_2 \to X_1$ and $\psi\colon X_2 \to X_1$ such that $\pi_t = \varphi \circ \pi_t'$ and $\pi_t \circ M_p = \psi \circ \pi_t'$, i.e., the following diagrams commute:

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\pi_t'"'] \arrow[dr, "\pi_t"] & \\ X_2 \arrow[r, "\varphi"'] & X_1 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{H} \arrow[r, "M_p"] \arrow[d, "\pi_t'"'] & \mathbb{H} \arrow[d, "\pi_t"] \\ X_2 \arrow[r, "\psi"'] & X_1. \end{tikzcd}$$

Thus, for $z \in X_2$, the solution (α,β) to the generalized modular equation (1.1) is parametrized by $\alpha = \varphi(z)$ and $\beta = \psi(z)$. Before giving the proofs of Theorems 2 and 3, we recall the following two lemmas from [2].

Lemma 3. [2, Lemma 2.3] *For an integer $p > 1$, let $t \in (0, \frac{1}{2}]$ and $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. Write $H_e'(\lambda_t) = M_p^{-1} H_e(\lambda_t) M_p$ and $H_{M_p}(\lambda_t) = H_e(\lambda_t) \cap H_e'(\lambda_t)$, then*

$$\left|H_e(\lambda_t) : H_{M_p}(\lambda_t)\right| = \left|H_e'(\lambda_t) : H_{M_p}(\lambda_t)\right|.$$

Lemma 4. [2, Lemma 3.4] *Let $W_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ and $\tau \in \mathbb{H}$, then $W_p\tau = -\dfrac{1}{p\tau}$ induces an automorphism ω on X_2 such that $\pi_t' \circ W_p = \omega \circ \pi_t'$ and $\psi = 1 - \varphi \circ \omega$.*

Now, we are ready to prove Theorems 2 and 3.

Proof of Theorem 2. First, recall that the covering group of the map π_t is the Hecke subgroup $H_e(\lambda_t)$ and it is well-known that the index of $H_e(\lambda_t)$ in the Hecke group $H(\lambda_t)$ is 2 (see [12, p. 61]). Thus, (i) follows easily from this fact.

From (4.2), we have $H_{M_p}(\lambda_t) = H_e(\lambda_t) \cap \left(M_p^{-1} H_e(\lambda_t) M_p\right)$. By virtue of the proof of Theorem 1, we have $H_{M_p}(\lambda_t) \cong \Gamma_0(\lambda_t^2 p)$, and hence, $H_{M_p}(\lambda_t)$ is isomorphic to $\Gamma_0(4p)$, $\Gamma_0(3p)$ and $\Gamma_0(2p)$ for $t = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$, respectively. Each of $\Gamma_0(4p)$, $\Gamma_0(3p)$ and $\Gamma_0(2p)$ has finite index in $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. Therefore, $H_{M_p}(\lambda_t)$ has finite index in $H(\lambda_t)$, which implies (ii).

Let $X_1 = H_e(\lambda_t)\backslash\mathbb{H}$ and $X_2 = H_{M_p}(\lambda_t)\backslash\mathbb{H}$, then the degree of the branched covering $\varphi\colon X_2 \to X_1$ is equal to the index of $H_{M_p}(\lambda_t)$ in $H_e(\lambda_t)$. Since each of $H_e(\lambda_t)$ and $H_{M_p}(\lambda_t)$ has finite index in $H(\lambda_t)$, the index of $H_{M_p}(\lambda_t)$ in $H_e(\lambda_t)$ is finite. Therefore, (iii) holds.

The functions $\alpha(\tau)$ and $\beta(\tau) = \alpha(p\tau)$ are automorphic on $H_e(\lambda_t)$ and $H_e'(\lambda_t) := M_p^{-1} H_e(\lambda_t) M_p$, respectively (see Lemma 3.2 of [1]). Recall that the quotient Riemann surface $X_1 = H_e(\lambda_t)\backslash\mathbb{H}$ is $\widehat{\mathbb{C}} \setminus \{0,1\}$ for $t \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$ and $\widehat{\mathbb{C}} \setminus \{0,1,\infty\}$ for $t = \frac{1}{2}$. If $\widehat{X}_1$ is the compactification of X_1, then $\widehat{X}_1$ is the Riemann sphere $\widehat{\mathbb{C}}$. Thus, the field of automorphic functions for $H_e(\lambda_t)$ is $\mathbb{C}(\alpha(\tau))$. Let $X_1' = H_e'(\lambda_t)\backslash\mathbb{H}$. If $\widehat{X}_1'$ is the compactification of X_1', then $\widehat{X}_1' = \widehat{\mathbb{C}}$. The field of automorphic functions for $H_e'(\lambda_t)$ is $\mathbb{C}(\beta(\tau))$. Since $H_{M_p}(\lambda_t) < H_e(\lambda_t)$ and $H_{M_p}(\lambda_t) < H_e'(\lambda_t)$, both $\mathbb{C}(\alpha(\tau))$ and $\mathbb{C}(\beta(\tau))$ are subfields of the field of automorphic functions for $H_{M_p}(\lambda_t) = H_e(\lambda_t) \cap H_e'(\lambda_t)$, i.e., $\mathbb{C}(\alpha(\tau),\beta(\tau))$. Since $\mu = \left|H_e(\lambda_t) : H_{M_p}(\lambda_t)\right|$, so $\varphi\colon X_2 \to X_1$ is a μ-sheeted branched covering map. For any function $g \in \mathbb{C}(\alpha(\tau))$, we have a function $f \in \mathbb{C}(\alpha(\tau),\beta(\tau))$ by virtue of the pullback $\varphi^*(g) = g \circ \varphi = f$, where $\varphi^*\colon \mathbb{C}(\alpha(\tau)) \to \mathbb{C}(\alpha(\tau),\beta(\tau))$ is an algebraic field extension of degree μ (see [15, Theorem 8.3]). Similarly, if ψ

is the branched covering map $X_2 \to X_1'$, then ψ is also a μ-sheeted covering map, since $|H_e'(\lambda_t) : H_{M_p}(\lambda_t)| = \mu$ by Lemma 3. Hence, $\psi^*\colon \mathbb{C}(\beta(\tau)) \to \mathbb{C}(\alpha(\tau), \beta(\tau))$ is an algebraic field extension of degree μ. Consequently, there is a polynomial $P(\alpha(\tau), \beta(\tau))$ which has degree $\mu = \deg_\alpha P(\alpha, \beta) = \deg_\beta P(\alpha, \beta)$. The polynomial $P(\alpha(\tau), \beta(\tau))$ is determined up to a scalar factor so that $P(\alpha(\tau), \beta(\tau)) = 0$, which implies (iv) and completes the proof. □

Recall that the Hecke subgroup $H_{M_p}(\lambda_t)$ is given by

$$H_{M_p}(\lambda_t) = \left\{ \begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} \in H_e(\lambda_t) \colon c \equiv 0 \pmod p \right\}.$$

Let $W_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$, then

$$W_p^{-1} \begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} W_p = \begin{pmatrix} d & -\frac{c}{p}\lambda_t \\ -pb\lambda_t & a \end{pmatrix}, \tag{4.3}$$

where $\begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} \in H_{M_p}(\lambda_t)$.

Proof of Theorem 3. Since $c \equiv 0 \pmod p$, it follows from (4.3) that

$$W_p^{-1} \begin{pmatrix} a & b\lambda_t \\ c\lambda_t & d \end{pmatrix} W_p \in H_{M_p}(\lambda_t).$$

Thus, H_{M_p} is normalized by W_p in $\mathrm{PSL}(2, \mathbb{R})$. The Möbius transformation $W_p\tau = -\dfrac{1}{p\tau}$ induces an automorphism ω on $X_2 = H_{M_p} \backslash \mathbb{H}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{W_p} & \mathbb{H} \\ {\scriptstyle \pi_t'}\downarrow & & \downarrow{\scriptstyle \pi_t'} \\ X_2 & \xrightarrow[\omega]{} & X_2. \end{array}$$

Moreover, by Lemma 4, $\omega\colon X_2 \to X_2$ satisfies the following functional equations:

$$\varphi \circ \omega = 1 - \psi,$$
$$\psi \circ \omega = 1 - \varphi.$$

Hence, for $z \in X_2$, we have $\varphi(\omega(z)) = 1 - \psi(z) = 1 - \beta$ and $\psi(\omega(z)) = 1 - \varphi(z) = 1 - \alpha$, i.e., ω interchanges α and $1 - \beta$, and β and $1 - \alpha$. Thus, we deduce that $(1 - \beta, 1 - \alpha)$ is also a solution to (1.1) and $P(1 - \beta, 1 - \alpha) = 0$. □

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