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# Korevaar–Schoen–Sobolev spaces and critical exponents in metric measure spaces

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**Abstract.** We survey, unify and present new developments in the theory of Korevaar–Schoen–Sobolev spaces on metric measure spaces. While this theory coincides with those of Cheeger and Shanmugalingam if the space is doubling and supports a Poincaré inequality, it offers new perspectives in the context of fractals for which the approach by weak upper gradients is inadequate.

**Metristen mitta-avaruuksien Korevaarin–Schoenin Sobolevin avaruudet ja kriittiset eksponentit**

**Tiivistelmä.** Tässä työssä tarkastellaan ja yhtenäistetään metristen mitta-avaruuksien Korevaarin–Schoenin Sobolevin avaruuksien teoriaa ja esitellään sen uutta kehitystä. Tämä teoria yhtyy Cheegerin ja Shanmugalingamin vastaaviin, jos avaruus toteuttaa tuplausehdon ja Poincarén epäyhtälön, mutta tarjoaa uusia näkökulmia tarkasteltaessa fraktaaleita, joihin ylägradientteihin perustuva lähestymistapa ei sovellu.

## 1. Introduction

If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ Lipschitz function then for every $x, y \in \mathbb{R}^n$

$$|f(x) - f(y)| \le \|\nabla f\|_\infty \|x - y\|.$$

One can rephrase this inequality as

$$\sup_{r>0} \sup_{x,y,\|x-y\|\le r} \frac{|f(x)-f(y)|}{r} \le \liminf_{r\to 0} \sup_{x,y,\|x-y\|\le r} \frac{|f(x)-f(y)|}{r}.$$

More generally (as follows for instance from [17, Proposition 1.11]) such an inequality still holds if $f$ is a Lipschitz function defined on a length metric space, i.e. a space for which any pair of points $x, y$ can be connected with a rectifiable curve and the infimum of the length of such curves is the distance $d(x, y)$.

In this work, we shall be interested in $L^p$ analogues in the context of a doubling metric measure space $(X, d, \mu)$: For $p \ge 1, r > 0, \alpha > 0$ and $f \in L^p(X, \mu)$, we define

$$E_{p,\alpha}(f, r) = \int_X \frac{1}{\mu(B(x,r))} \left( \int_{B(x,r)} \frac{|f(y) - f(x)|^p}{r^{p\alpha}} \, d\mu(y) \right) d\mu(x)$$

and ask if for some $\alpha > 0$ there is a constant $C \ge 1$ (depending only on the geometry of the space $(X, d, \mu)$) such that

$$\sup_{r>0} E_{p,\alpha}(f, r) \le C \liminf_{r\to 0} E_{p,\alpha}(f, r) \tag{1}$$

holds for some non-constant functions $f$. An underlying insight is that if (1) holds for a large class of functions, then the underlying metric measure space has some level of $L^p$ infinitesimal regularity and global controlled $L^p$ geometry. In particular,

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the value $p = 1$ is related to the existence of a rich theory of BV functions and sets of finite perimeter satisfying isoperimetric estimates, see [4, 5] and [53]. The value $p = 2$ is related to the existence of a nice Laplacian on the space, more precisely the existence of a local Dirichlet form which can be constructed as a Γ-limit of the functionals $E_{2,\alpha}$ as $r \to 0$, see [48] and [63].

In this paper we will see that (1) holds in a large class of spaces of different nature. This class includes doubling spaces satisfying a Poincaré inequality but also some fractals like the Vicsek set and the Sierpiński gasket. For some other spaces like the Sierpiński carpet the validity of (1) is still an open question[1].

We will also show that if (1) holds, a rich theory of Sobolev spaces develops using the scale of the Korevaar–Schoen spaces first introduced in [45] in a Riemannian setting. Assuming doubling and a $p$-Poincaré inequality, that theory is equivalent to the theory of Sobolev spaces built on the notion of weak upper gradients by Cheeger [17] and Shanmugalingam [59] and also to the theory of Hajłasz [30]. However, on spaces like fractals where the set of rectifiable curves is not rich enough in the measure theoretic sense (see Remark 6.3) the theory built on weak upper gradients yields non-useful Sobolev spaces, often the whole $L^p$ space. By contrast the theory we can develop using the Korevaar–Schoen spaces still produces a fruitful set of results which can be used to study further the geometry of the space; In particular a whole scale of Gagliardo–Nirenberg type Sobolev embeddings is available.

Furthermore, an appealing aspect of the theory of Korevaar–Schoen–Sobolev spaces is its close connection to the very rich theory of heat kernels and Dirichlet forms as was developed in [3, 4, 5] after [47], [28], [52], and [58]. Due to this connection, one can hope to export to a general metric measure space setting some of the powerful heat kernel techniques, like the Bakry–Émery–Ledoux machinery, see [4, 5].

The paper is organized as follows. Sections 2 and 3 are preliminary sections, we collect some useful and mostly known results about the class of Besov–Lipschitz functions on a doubling metric measure space. In Section 4, we first introduce and study the Besov critical exponents of a metric measure space and discuss (1) in connection with the Korevaar–Schoen–Sobolev spaces which we define as the Besov–Lipschitz spaces at the critical exponent. Finally, we show how (1) yields Sobolev embeddings and Gagliardo–Nirenberg inequalities. In Section 5, we show after [40] and [46] that (1) holds with $\alpha = 1$ if the space satisfies a $p$-Poincaré inequality and point out that the theory of Korevaar–Schoen–Sobolev spaces is then equivalent to Cheeger and Shanmugalingam theories. We also prove, and this is one of our main contributions, that if the space satisfies a generalized $p$-Poincaré inequality and a controlled cutoff condition similar to that of [10], then (1) holds with a parameter $\alpha$ possibly greater than one. The case $p = 2$ in Dirichlet spaces with sub-Gaussian heat kernel estimates is then discussed as a corollary of this general approach.

In Section 6, we discuss in detail the Korevaar–Schoen–Sobolev spaces in two popular examples of fractals: The Vicsek set and the Sierpiński gasket. We show that those two examples satisfy for every $p \geq 1$ the inequality (1) for some value $\alpha = \alpha_p > 1$ and as a consequence obtain new Nash inequalities. Finally, in Section 7, we review some of the results in [4, 5, 3] about the connection between the Dirichlet

[1]March 2024 update: The validity of (1) in the Sierpiński carpet was recently proved for some $\alpha$ and some range of $p$'s by Yang in [67] and then shortly after, for every $p > 1$ by Murugan and Shimizu in [56].

forms theory and the Korevaar–Schoen–Sobolev spaces. Throughout the text several open problems and possible research directions are discussed.

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## 2. Setup

Our setting is that of [35]. Throughout the paper, let $(X, d, \mu)$ be a metric measure space[2] where $\mu$ is a Borel regular measure. Open metric balls will be denoted by

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Sometimes, when convenient, if $B$ is a ball and $\lambda > 0$ we will denote by $\lambda B$ the ball with same center and radius multiplied by $\lambda$.

We will always assume that the measure $\mu$ is doubling and positive in the sense that there exists a constant $C > 0$, called the doubling constant, such that for every $x \in X, r > 0$,

$$0 < \mu(B(x, 2r)) \le C\mu(B(x, r)) < +\infty.$$

It follows from the doubling property of $\mu$ (see [35, Lemma 8.1.13]) that there is a constant $0 < Q < \infty$ and $C > 0$ such that whenever $0 < r \le R$ and $x \in X$, we have

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \le C \left(\frac{R}{r}\right)^Q . \tag{2}$$

Another well-known consequence of the doubling property is the availability of maximally separated $\varepsilon$-coverings with the bounded overlap property and subordinated Lipschitz partitions of unity (see [35, Pages 102–104] or [29, Appendix B.7]):

**Proposition 2.1.** (Controlled Lipschitz partition of unity) *Let $\varepsilon > 0$. There exists a countable subset $A(\varepsilon)$ of $X$ such that:*

- *$d(a_1, a_2) \ge \varepsilon$ for all $a_1, a_2 \in A(\varepsilon)$ with $a_1 \ne a_2$;*
- *$X = \bigcup_{a \in A(\varepsilon)} B(a, \varepsilon)$.*

*Moreover, for any $k > 0$ there exists a constant $\beta(k) > 0$ depending only on $k$ and on the doubling constant such that:*

- *$\sum_{a \in A(\varepsilon)} 1_{B(a, k\varepsilon)}(x) \le \beta(k)$ for every $x \in X$.*

*In addition, we can find a family $(\phi_a^\varepsilon)_{a \in A(\varepsilon)}$ of real-valued Lipschitz functions on $X$ such that:*

- *The functions $\phi_a^\varepsilon$ have Lipschitz constant not greater than $\lambda/\varepsilon$ where $\lambda > 0$ is a constant depending only on the doubling constant;*
- *$0 \le \phi_a^\varepsilon \le 1_{B(a, 2\varepsilon)}$;*
- *$\sum_{a \in A(\varepsilon)} \phi_a^\varepsilon = 1$.*

We note that under our assumptions $(X, d)$ is in particular separable.

[2] We assume that $X$ has more than one point so that there exist non-constant functions and $\mathrm{diam}(X) > 0$.

Notation:

1. Throughout the notes, we use the letters $c, C, c_1, c_2, C_1, C_2$ to denote positive constants which may vary from line to line.
2. For two non-negative functionals $\Lambda_1, \Lambda_2$ defined on a functional space $\mathcal{F}$, the notation $\Lambda_1(f) \simeq \Lambda_2(f)$ means that there exist two constants $C_1, C_2 > 0$ such that for every $f \in \mathcal{F}$, $C_1\Lambda_1(f) \leq \Lambda_2(f) \leq C_2\Lambda_1(f)$.
3. For any Borel set $A$ and any measurable function $f$, we sometimes write the average of $f$ on the set $A$ as
$$\fint_A f(x)\, d\mu(x) := \frac{1}{\mu(A)} \int_A f(x)\, d\mu(x).$$
4. If $A, B$ are subsets of $X$, then
$$d(A,B) := \inf \{d(x,y),\ x \in A,\ y \in B\}.$$
5. If $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$.

## 3. Besov–Lipschitz spaces

We start with a short review of some properties of the Besov–Lipschitz spaces that will be useful in the sequel. The theory of Besov classes on doubling metric measure spaces is rich and the literature on this topic is nowadays quite large so we will not try to be exhaustive and do not claim originality; For references related to the discussion below, see for instance [1, 4, 5, 3, 28, 33, 66].

**3.1. Some basic properties.** For $p \geq 1, r > 0, \alpha \geq 0$ and $f \in L^p(X,\mu)$, we define
$$E_{p,\alpha}(f,r) = \int_X \frac{1}{\mu(B(x,r))} \left( \int_{B(x,r)} \frac{|f(y) - f(x)|^p}{r^{p\alpha}}\, d\mu(y) \right) d\mu(x)$$
and consider the Besov–Lipschitz space
$$\mathcal{B}^{\alpha,p}(X) = \left\{ f \in L^p(X,\mu) \colon \sup_{r>0} E_{p,\alpha}(f,r) < +\infty \right\}.$$
We equip $\mathcal{B}^{\alpha,p}(X)$ with the norm given by
$$\|f\|^p_{\mathcal{B}^{\alpha,p}(X)} = \|f\|^p_{L^p(X,\mu)} + \sup_{r>0} E_{p,\alpha}(f,r).$$

**Lemma 3.1.** *For every $p \geq 1$, $f \in L^p(X,\mu)$, $r > 0$ and $\alpha \geq 0$*
$$E_{p,\alpha}(f,r) \leq \frac{C}{r^{p\alpha}} \|f\|^p_{L^p(X,\mu)}.$$
*In particular, for $\alpha = 0$, we have $\mathcal{B}^{\alpha,p}(X) = L^p(X,\mu)$.*

*Proof.* We note that
$$\begin{aligned} E_{p,\alpha}(f,r) &= \frac{1}{r^{p\alpha}} \int_X \frac{1}{\mu(B(x,r))} \left( \int_{B(x,r)} |f(y) - f(x)|^p\, d\mu(y) \right) d\mu(x) \\ &\leq \frac{2^{p-1}}{r^{p\alpha}} \int_X \frac{1}{\mu(B(x,r))} \left( \int_{B(x,r)} (|f(y)|^p + |f(x)|^p)\, d\mu(y) \right) d\mu(x) \\ &= \frac{2^{p-1}}{r^{p\alpha}} \left( \|f\|^p_{L^p(X,\mu)} + \int_X \left( \int_{B(y,r)} \frac{d\mu(x)}{\mu(B(x,r))} \right) |f(y)|^p\, d\mu(y) \right). \end{aligned}$$

Using the volume doubling property, one has then

$$\int_{B(y,r)} \frac{d\mu(x)}{\mu(B(x,r))} \leq C \int_{B(y,r)} \frac{d\mu(x)}{\mu(B(x,2r))} \leq C \int_{B(y,r)} \frac{d\mu(x)}{\mu(B(y,r))} = C,$$

and the conclusion follows. □

Using an argument as in the proof of the previous lemma, one also has the following result:

**Lemma 3.2.** *Let $p \geq 1$ and $\alpha \geq 0$. For $f \in L^p(X,\mu)$, and for every $r > 0$, we have*

$$\sup_{\rho>0} E_{p,\alpha}(f,\rho) \leq \frac{C}{r^{p\alpha}} \|f\|^p_{L^p(X,\mu)} + \sup_{\rho\in(0,r]} E_{p,\alpha}(f,\rho).$$

*Therefore,*

$$\mathcal{B}^{\alpha,p}(X) = \left\{ f \in L^p(X,\mu) \colon \limsup_{r\to 0} E_{p,\alpha}(f,r) < +\infty \right\}.$$

It follows that for a fixed $p$, the family of spaces $\mathcal{B}^{\alpha,p}(X)$, $\alpha \geq 0$ is non-increasing:

**Corollary 3.3.** *Let $p \geq 1$ and $\alpha \geq 0$. Then, for $\beta > \alpha$, $\mathcal{B}^{\beta,p}(X) \subset \mathcal{B}^{\alpha,p}(X)$.*

*Proof.* If $f \in \mathcal{B}^{\beta,p}(X)$, one has $\sup_{r>0} E_{p,\beta}(f,r) < +\infty$. This gives

$$\limsup_{r\to 0} E_{p,\alpha}(f,r) = \limsup_{r\to 0} r^{p(\beta-\alpha)} E_{p,\beta}(f,r) = 0 < +\infty.$$ □

Next, we show the Banach space property.

**Theorem 3.1.** *$(\mathcal{B}^{\alpha,p}(X), \|\cdot\|_{\mathcal{B}^{\alpha,p}(X)})$ is a Banach space for every $p \geq 1$ and $\alpha \geq 0$.*

*Proof.* Let $f_n$ be a Cauchy sequence in $\mathcal{B}^{\alpha,p}(X)$. Let $f$ be the $L^p$ limit of $f_n$. From Minkowski's inequality and Lemma 3.1 one has

$$|E_{p,\alpha}(f,r)^{1/p} - E_{p,\alpha}(f_n,r)^{1/p}| \leq E_{p,\alpha}(f-f_n,r)^{1/p} \leq \frac{C}{r^\alpha} \|f-f_n\|_{L^p(X,\mu)}.$$

Thus $E_{p,\alpha}(f_n,r) \to E_{p,\alpha}(f,r)$ from which we deduce

$$E_{p,\alpha}(f,r) = \lim_{n\to+\infty} E_{p,\alpha}(f_n,r) \leq C.$$

This implies that $f \in \mathcal{B}^{\alpha,p}(X)$ with $\|f\|_{\mathcal{B}^{\alpha,p}(X)} \leq \lim_{n\to+\infty} \|f_n\|_{\mathcal{B}^{\alpha,p}(X)}$. Similarly, for every fixed $m$,

$$\|f-f_m\|_{\mathcal{B}^{\alpha,p}(X)} \leq \lim_{n\to+\infty} \|f_n - f_m\|_{\mathcal{B}^{\alpha,p}(X)}$$

and passing to the limit as $m \to +\infty$ together with the fact that $(f_n)$ is Cauchy with respect to $\|\cdot\|_{\mathcal{B}^{\alpha,p}(X)}$ completes the proof that $f_n \to f$ in $\mathcal{B}^{\alpha,p}(X)$ and therefore that $(\mathcal{B}^{\alpha,p}(X), \|\cdot\|_{\mathcal{B}^{\alpha,p}(X)})$ is a Banach space. □

**3.2. Embeddings of Besov–Lipschitz spaces into Hölder spaces.** For a fixed $p \geq 1$, one can think of the parameter $\alpha$ as a regularity parameter: The larger $\alpha$ is, the smoother functions in $\mathcal{B}^{\alpha,p}(X)$ are. The theorem below reflects this fact. Recall that we denote by $Q$ the constant in (2).

**Theorem 3.2.** *Let $\alpha > 0$ and $p \geq 1$ be such that $p > \frac{Q}{\alpha}$. Let $x_0 \in X$ and $R > 0$. There exists $C > 0$ such that for every $f \in \mathcal{B}^{\alpha,p}(X)$,*

$$\mu\otimes\mu - \operatorname*{ess\,sup}_{x,y\in B(x_0,R),\,0<d(x,y)<R/3} \frac{|f(x)-f(y)|}{d(x,y)^\lambda} \leq C \sup_{r\in(0,R]} E_{p,\alpha}(f,r)^{1/p} \tag{3}$$

*where $\lambda = \alpha - \frac{Q}{p}$.*

*Proof.* Let first $0 < r < R/3$ and consider $x, y \in B(x_0, R)$ with $d(x,y) \le r$. Define

$$f_r(x) := \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} f(z)\, d\mu(z)$$

and notice that

$$f_r(x) = \frac{1}{\mu\big(B(x,r)\big)\mu\big(B(y,r)\big)} \int_{B(x,r)} \int_{B(y,r)} f(z)\, d\mu(z')\, d\mu(z).$$

Analogously one defines $f_r(y)$. Hölder's inequality yields

$$\begin{aligned} |f_r(x) - f_r(y)| &= \frac{1}{\mu\big(B(x,r)\big)\mu\big(B(y,r)\big)} \Big| \int_{B(x,r)} \int_{B(y,r)} (f(z) - f(z'))\, d\mu(z')\, d\mu(z) \Big| \\ &\le \left( \frac{1}{\mu\big(B(x,r)\big)\mu\big(B(y,r)\big)} \int_{B(x,r)} \int_{B(y,r)} |f(z) - f(z')|^p\, d\mu(z')\, d\mu(z) \right)^{1/p}. \end{aligned}$$

We now note that if $z \in B(x,r)$, $z' \in B(y,r)$ then one has $d(z,z') < 3r$, $B(z,3r) \subset B(x,4r)$ and moreover from the doubling condition (2)

$$\mu(B(y,r)) \ge C r^Q \frac{\mu(B(y,2R))}{R^Q} \ge C r^Q \frac{\mu(B(x_0,R))}{R^Q}.$$

Hence, we get

$$\begin{aligned} |f_r(x) - f_r(y)|^p &\le \frac{C}{r^Q} \int_X \frac{1}{\mu(B(z,3r))} \int_{B(z,3r)} |f(z) - f(z')|^p\, d\mu(z')\, d\mu(z) \\ &\le C r^{p\alpha - Q} \sup_{\rho \in (0,R/3)} \frac{1}{\rho^{p\alpha}} \int_X \frac{1}{\mu(B(z,3\rho))} \int_{B(z,3\rho)} |f(z) - f(z')|^p\, d\mu(z')\, d\mu(z) \\ &\le C r^{p\alpha - Q} \sup_{\rho \in (0,R]} E_{p,\alpha}(f,\rho), \end{aligned}$$

where the constant $C$ depends on $R$ and $\mu(B(x_0,R))$. Thus,

$$|f_r(x) - f_r(y)| \le C r^{\alpha - \frac{Q}{p}} \sup_{\rho \in (0,R]} E_{p,\alpha}(f,\rho)^{1/p}.$$

Analogously one obtains

$$|f_{2r}(x) - f_r(x)| \le C r^{\alpha - \frac{Q}{p}} \sup_{\rho \in (0,R]} E_{p,\alpha}(f,\rho)^{1/p}. \tag{4}$$

Let now $x \in B(x_0, R)$ be a Lebesgue point of $f$. Setting $r_k = 2^{-k} r$, $k = 0, 1, 2 \ldots$, the latter inequality yields

$$|f(x) - f_r(x)| \le \sum_{k=0}^{\infty} |f_{r_k}(x) - f_{r_{k+1}}(x)| \le C r^{\alpha - \frac{Q}{p}} \sup_{\rho \in (0,R]} E_{p,\alpha}(f,\rho)^{1/p}. \tag{5}$$

Let $y \in B(x_0, R)$ be another Lebesgue point of $f$ such that $d(x,y) < R/3$. Applying the triangle inequality as well as (4) and (5) with $r = d(x,y)$ we obtain

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_r(x)| + |f_r(x) - f_r(y)| + |f_r(y) - f(y)| \\ &\le C d(x,y)^{\alpha - \frac{Q}{p}} \sup_{\rho \in (0,R]} E_{p,\alpha}(f,\rho)^{1/p}. \end{aligned}$$

Then, by virtue of [35, Theorem 3.4.3], the volume doubling property of the space implies the validity of the Lebesgue differentiation theorem, i.e. that $\mu$ a.e. $x \in X$ is a Lebesgue point of $f$. The conclusion follows. □

**Remark 3.4.** If $X$ has maximal volume growth, i.e. $\mu(B(x,R)) \geq cR^Q$ for every $R \geq 0$, and $x \in X$, for some $c > 0$, then after tracking the constants in the previous proof, we can let $R \to +\infty$ in (3) and obtain

$$\mu \otimes \mu - \operatorname*{ess\,sup}_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\lambda} \leq C \sup_{r>0} E_{p,\alpha}(f,r)^{1/p}.$$

The same conclusion holds if $X$ has finite diameter as can be seen by choosing $R$ large enough in Theorem 3.2.

**Remark 3.5.** Theorem 3.2 implies that if $f \in \mathcal{B}^{\alpha,p}(X)$ with $p > \frac{Q}{\alpha}$, then one can find a locally $\left(\alpha - \frac{Q}{p}\right)$ Hölder continuous function $g\colon X \to \mathbb{R}$ such that $f = g$ $\mu$-a.e.

## 4. Korevaar–Schoen–Sobolev spaces

For $\alpha = 1$, Korevaar–Schoen–Sobolev spaces have been introduced in a Riemannian setting in [45] and a presentation in a metric measure space setting is done in [35, Section 10.4]. However, for some spaces like fractals, it turns out that (1) might be satisfied with $\alpha > 1$. The parameter $\alpha$ for which it holds has to be a critical parameter in the scale of the Besov–Lipschitz spaces. In this section we study the critical exponents in the scale of the Besov–Lipschitz spaces, introduce the Korevaar–Schoen–Sobolev spaces as Besov–Lipschitz spaces at the critical parameter and prove that they satisfy Sobolev embeddings and the whole scale of Gagliardo–Nirenberg inequalities if (1) is satisfied.

### 4.1. Critical exponents.

**Definition 4.1.** Let $p \geq 1$. We define the $L^p$ *critical Besov exponent* of $(X,d,\mu)$ by

$$\alpha_p = \sup\left\{\alpha \geq 0\colon \mathcal{B}^{\alpha,p}(X) \text{ contains non-constant functions}\right\}.$$

Here and hereafter, by constant function we mean constant $\mu$-a.e.

**Remark 4.2.** It might be that $\alpha_p = +\infty$ for every $p \geq 1$, as is the case if $(X,d)$ contains one isolated point or is strongly disconnected in the sense that there exist two disjoint non-empty open sets $X_1, X_2$ such that $d(X_1,X_2) > 0$, $\mu(X_1)$ is finite and $X = X_1 \cup X_2$. Indeed, in that case the function $f = 1_{X_1}$ is non-constant and in $\mathcal{B}^{\alpha,p}(X)$ for every $\alpha \geq 0$ and $p \geq 1$. For a sufficient condition ensuring the finiteness of $\alpha_p$, see Theorem 4.2 below.

**Theorem 4.1.**

1. *For every $p \geq 1$, $\alpha_p \geq 1$.*
2. *The map $p \to p\alpha_p$ is non-decreasing.*
3. *The map $p \to \alpha_p$ is non-increasing.*

*In particular, if $\alpha_p$ is finite for some $p \geq 1$, then it is finite for every $p \geq 1$.*

For the first item, consider $x_0 \in X$ and $y_0 \neq x_0$. Denote $r = d(x_0,y_0)$. The function

$$\Psi(x) = d(x, X \setminus B(x_0, r/3))$$

is non-constant, Lipschitz, in $L^p(X,\mu)$ for $p \geq 1$, and seen to be in $\mathcal{B}^{1,p}(X)$. Thus $\alpha_p \geq 1$.

For the second item, let $p \geq 1$ and $\alpha < \alpha_p$. Let $f \in \mathcal{B}^{\alpha,p}(X)$ be non-constant and $q \geq p$. For $n \geq 1$, denote $f_n(x) = \max\{-n, \min\{f(x), n\}\}$. Then, $f_n \in \mathcal{B}^{\alpha,p}(X) \cap$

$L^q(X,\mu)$ and moreover

$$\begin{aligned}
&\int_X \frac{1}{\mu(B(x,r))}\left(\int_{B(x,r)} |f_n(y)-f_n(x)|^q \, d\mu(y)\right) d\mu(x)\\
&\le \int_X \frac{1}{\mu(B(x,r))}\left(\int_{B(x,r)} (|f_n(y)|+|f_n(x)|)^{q-p}|f_n(y)-f_n(x)|^p \, d\mu(y)\right) d\mu(x)\\
&\le 2^{q-p}n^{q-p}r^{\alpha p}\int_X \frac{1}{\mu(B(x,r))}\left(\int_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}} \, d\mu(y)\right) d\mu(x).
\end{aligned}$$

Therefore $f_n \in \mathcal{B}^{\alpha\frac{p}{q},q}(X)$. For $n$ large enough $f_n$ is not constant. Thus, $\alpha_q \ge \alpha\frac{p}{q}$. This is true for all $\alpha < \alpha_p$, so $q\alpha_q \ge p\alpha_p$. The third item follows from the following lemma:

**Lemma 4.3.** *If $1 \le q \le p < \infty$, there exists a constant $C > 0$ such that if $f \in \mathcal{B}^{\alpha,p}(X)$, then $|f|^{p/q} \in \mathcal{B}^{\alpha,q}(X)$ and*

$$\sup_{r>0} E_{q,\alpha}(|f|^{p/q},r) \le C\|f\|_{L^p(X,\mu)}^{p-q} \sup_{r>0} E_{p,\alpha}(f,r)^{q/p}. \tag{6}$$

*Proof.* We use for any $a,b \ge 0$ such that $a \ne b$, the elementary inequality

$$\frac{|a^{p/q}-b^{p/q}|}{|a-b|} \le \frac{p}{q}\max\{a,b\}^{\frac{p}{q}-1}.$$

Equivalently,

$$|a^{p/q}-b^{p/q}|^q \le \left(\frac{p}{q}\right)^q \max\{a,b\}^{p-q}|a-b|^q.$$

Using this elementary inequality, one has

$$\begin{aligned}
&\int_X \frac{1}{\mu(B(x,r))}\int_{B(x,r)} \big||f(x)|^{p/q}-|f(y)|^{p/q}\big|^q \, d\mu(y)\, d\mu(x)\\
&\le \left(\frac{p}{q}\right)^q \int_X \frac{1}{\mu(B(x,r))}\int_{B(x,r)} \big(|f(x)|^{p-q}+|f(y)|^{p-q}\big)\big||f(x)|-|f(y)|\big|^q \, d\mu(y)\, d\mu(x)\\
&\le \left(\frac{p}{q}\right)^q \int_X \frac{1}{\mu(B(x,r))}\int_{B(x,r)} \big(|f(x)|^{p-q}+|f(y)|^{p-q}\big)\big|f(x)-f(y)\big|^q \, d\mu(y)\, d\mu(x). \qquad (7)
\end{aligned}$$

We now observe that by Fubini's theorem and the volume doubling property

$$\begin{aligned}
&\int_X \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)|^{p-q}\big|f(x)-f(y)\big|^q \, d\mu(y)\, d\mu(x)\\
&= \int_X \int_{B(x,r)} \frac{1}{\mu(B(y,r))}|f(x)|^{p-q}\big|f(x)-f(y)\big|^q \, d\mu(y)\, d\mu(x)\\
&\le C\int_X \int_{B(x,r)} \frac{1}{\mu(B(y,2r))}|f(x)|^{p-q}\big|f(x)-f(y)\big|^q \, d\mu(y)\, d\mu(x)\\
&\le C\int_X \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(x)|^{p-q}\big|f(x)-f(y)\big|^q \, d\mu(y)\, d\mu(x).
\end{aligned}$$

Thus, applying Hölder's inequality we have

$$\begin{aligned}
&\int_X \frac{1}{\mu(B(x,r))}\int_{B(x,r)} \big||f(x)|^{p/q}-|f(y)|^{p/q}\big|^q \, d\mu(y)\, d\mu(x)\\
&\le C\int_X \frac{|f(x)|^{p-q}}{\mu(B(x,r))}\Big(\int_{B(x,r)}\big|f(x)-f(y)\big|^q \, d\mu(y)\Big)\, d\mu(x)\\
&\le C\int_X |f(x)|^{p-q}\Big(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}\big|f(x)-f(y)\big|^p \, d\mu(y)\Big)^{q/p} d\mu(x)\\
&\le C\|f\|^{p-q}_{L^p(X,\mu)}\Big(\int_X \frac{1}{\mu(B(x,r))}\int_{B(x,r)}\big|f(x)-f(y)\big|^p \, d\mu(y)\, d\mu(x)\Big)^{q/p},
\end{aligned}$$

which implies (6). □

**Definition 4.4.** The metric space $(X,d)$ is said to satisfy the chaining condition if there exists a constant $C_h \ge 1$ such that for every $x,y \in X$, and $n \ge 1$ there is a family of points $x_0 = x, \cdots, x_n = y$ of $X$ such that for $j = 0, \cdots, n-1$, $d(x_j, x_{j+1}) \le \frac{C_h}{n} d(x,y)$.

For instance, geodesic spaces satisfy the chaining condition.

**Theorem 4.2.** *Assume that $(X,d,\mu)$ satisfies the chaining condition. Then, for every $p \ge 1$ we have $\alpha_p \le 1 + \frac{Q}{p}$.*

*Proof.* Let $f \in \mathcal{B}^{\alpha,p}(X)$ with $\alpha > 1 + \frac{Q}{p}$. From Theorem 3.2 and Remark 3.5, we can assume that the function $f$ is locally Hölder continuous with exponent $> 1$ which implies that $f$ is constant in view of the chaining condition. Indeed, let $z \in X$ and $R > 0$. Let $x,y \in X$ with $d(z,x) < \frac{R}{3}$, $d(z,y) < \frac{R}{3}$ and $d(x,y) < \frac{R}{3C_h}$. From the chaining condition, for an integer $n \ge 2$ one can consider a family of points $x_0 = x, \cdots, x_n = y$ of $X$ such that for $j = 0, \cdots, n-1$, $d(x_j, x_{j+1}) \le \frac{C_h}{n} d(x,y)$. One has $x_j \in B(z,R)$ and $d(x_j,x_{j+1}) < \frac{R}{3}$. Therefore, from Theorem 3.2

$$\begin{aligned}
|f(x)-f(y)| &\le |f(x)-f(x_1)| + \cdots + |f(x_{n-1})-f(y)|\\
&\le C\sum_{j=0}^{n-1} d(x_j,x_{j+1})^{\alpha-Q/p} \le Cn\frac{1}{n^{\alpha-Q/p}} d(x,y)^{\alpha-Q/p}.
\end{aligned}$$

Letting $n \to +\infty$ yields $f(x) = f(y)$. By arbitrariness of $z$ and $R$ one concludes that $f$ is constant. □

**4.2. Korevaar–Schoen spaces and $\mathcal{P}(p,\alpha)$.** For $p \ge 1$, $\alpha \ge 0$, the Korevaar–Schoen space $KS^{\alpha,p}(X)$ is defined as

$$KS^{\alpha,p}(X) = \Big\{ f \in L^p(X,\mu) \colon \limsup_{r\to 0} E_{p,\alpha}(f,r) < +\infty \Big\}$$

equipped with the norm given by

$$\|f\|^p_{KS^{\alpha,p}(X)} = \|f\|^p_{L^p(X,\mu)} + \limsup_{r\to 0} E_{p,\alpha}(f,r).$$

From Lemma 3.2, as a set $KS^{\alpha,p}(X) = \mathcal{B}^{\alpha,p}(X)$ and obviously $\|\cdot\|_{KS^{\alpha,p}(X)} \le \|\cdot\|_{\mathcal{B}^{\alpha,p}(X)}$.

**Definition 4.5.** Let $p \ge 1$, $\alpha \ge 0$. We will say that $\mathcal{P}(p,\alpha)$ holds if $KS^{\alpha,p}(X)$ contains non-constant functions and there exists a constant $C \ge 1$ such that for every

$f \in KS^{\alpha,p}(X)$,

$$\sup_{r>0} E_{p,\alpha}(f,r) \le C \liminf_{r\to 0} E_{p,\alpha}(f,r).$$

**Remark 4.6.** We point out that the weaker property

$$\sup_{r>0} E_{p,\alpha}(f,r) \le C \limsup_{r\to 0} E_{p,\alpha}(f,r)$$

does not suffice to develop a rich theory, since the super-additivity of the lim inf is used in crucial parts of the arguments, for instance to obtain the Sobolev embeddings of Section 4.4.

**Lemma 4.7.** *Let $p \ge 1$, $\alpha \ge 0$. If $\mathcal{P}(p,\alpha)$ holds, then $\alpha = \alpha_p$.*

*Proof.* If $\beta > \alpha$, and $f \in \mathcal{B}^{\beta,p}(X)$, we have $\liminf_{r\to 0} E_{p,\alpha}(f,r) = 0$ and thus $\sup_{r>0} E_{p,\alpha}(f,r) = 0$ which yields that $f$ is constant. □

**Definition 4.8.** When $\alpha = \alpha_p$ the space $KS^{\alpha,p}(X)$ is referred to as the Korevaar–Schoen–Sobolev space and for $f \in KS^{\alpha,p}(X)$ we will denote

$$\mathrm{Var}_p(f) = \liminf_{r\to 0} E_{p,\alpha_p}(f,r)^{1/p}. \tag{8}$$

An important property of $\mathrm{Var}_p$ is that it is a Sobolev quasi-seminorm in the sense of Bakry–Coulhon–Ledoux–Saloff Coste, see [7, Section 2]:

**Theorem 4.3.** *If $\mathcal{P}(p,\alpha)$ holds, then $\mathrm{Var}_p$ is a Sobolev quasi-seminorm, i.e. it satisfies the following properties:*

- *There exists a constant $C \ge 1$ such that for every $f, g \in KS^{\alpha,p}(X)$,*
$$\mathrm{Var}_p(f+g) \le C(\mathrm{Var}_p(f) + \mathrm{Var}_p(g));$$
- *If $f \in KS^{\alpha,p}(X)$ is such that $\mathrm{Var}_p(f) = 0$, then $f$ is constant;*
- *For every $s, t \ge 0$, $\mathrm{Var}_p((f-t)^+ \wedge s) \le \mathrm{Var}_p(f)$;*
- *There exists a constant $C > 0$ such that for any nonnegative $f \in KS^{\alpha,p}(X)$ and any $\rho > 1$,*
$$\left( \sum_{k\in\mathbb{Z}} \mathrm{Var}_p(f_{\rho,k})^p \right)^{1/p} \le C \mathrm{Var}_p(f),$$
*where $f_{\rho,k} := (f-\rho^k)^+ \wedge (\rho^k(\rho-1))$, $k \in \mathbb{Z}$.*

*Proof.* The first two items follow from the fact that $\mathcal{P}(p,\alpha)$ implies

$$\mathrm{Var}_p(f) \le \sup_{r>0} E_{p,\alpha}(f,r)^{1/p} \le C \mathrm{Var}_p(f).$$

The third item is immediate, because $E_{p,\alpha}((f-t)^+ \wedge s, r) \le E_{p,\alpha}(f,r)$. The fourth item can be proved as in the proof of Lemma 7.1 in [7] and by using then the super-additivity property of lim inf. □

**Remark 4.9.** Under $\mathcal{P}(p,\alpha)$ the functionals $\limsup_{r\to 0} E_{p,\alpha}(f,r)^{1/p}$ and $\sup_{r>0} E_{p,\alpha}(f,r)^{1/p}$ are also Sobolev quasi-seminorms.

**Remark 4.10.** If $\mathcal{P}(p,\alpha)$ holds, then one can rewrite the conclusion of Theorem 3.2 as

$$\mu\otimes\mu - \operatorname*{ess\,sup}_{x,y\in B(x_0,R), 0<d(x,y)<R/3} \frac{|f(x)-f(y)|}{d(x,y)^{\alpha-\frac{Q}{p}}} \le C \mathrm{Var}_p(f).$$

This can be interpreted as a Morrey inequality for the Sobolev space $KS^{\alpha_p,p}(X)$.

**4.3. Reflexivity and separability of the Korevaar–Schoen–Sobolev spaces.** In this section we prove that if the property $\mathcal{P}(p,\alpha_p)$ holds with $p>1$, then the space $KS^{\alpha_p,p}(X)$ is reflexive and separable. We use the following lemma.

**Lemma 4.11.** *Let $(Z,\|\cdot\|)$ be a Banach space. If for every $\varepsilon>0$ there exists $\delta>0$ with the property that $\|x+y\|\le 2(1-\delta)$ whenever $x,y\in Z$ satisfy $\|x\|<1$, $\|y\|<1$ and $\|x-y\|>\varepsilon$, then $(Z,\|\cdot\|)$ is reflexive.*

*Proof.* From Milman–Pettis theorem, it is enough to prove that the stated property implies that $(Z,\|\cdot\|)$ is uniformly convex. Let $\varepsilon>0$ and $\delta>0$ be as in the stated property. Suppose that $x,y\in Z$ are such that $\|x\|=1$, $\|y\|=1$ and $\|x-y\|>\varepsilon$. Then for $0<\eta<1$ sufficiently close to 1 we have that $\|\eta x\|<1$, $\|\eta y\|<1$ and $\|\eta x-\eta y\|>\varepsilon$. Therefore we obtain

$$\|\eta x+\eta y\|\le 2(1-\delta).$$

Passing to the limit $\eta\to 1^-$ yields $\|x+y\|\le 2(1-\delta)$. □

**Theorem 4.4.** *Let $p>1$. If $\mathcal{P}(p,\alpha_p)$ holds, then $(KS^{\alpha_p,p}(X),\|\cdot\|_{KS^{\alpha_p,p}(X)})$ is a reflexive and separable Banach space.*

*Proof.* Let $\varepsilon>0$ and suppose that $f,g\in KS^{\alpha_p,p}(X)$ satisfy $\|f\|_{KS^{\alpha_p,p}(X)}<1$, $\|g\|_{KS^{\alpha_p,p}(X)}<1$ and $\|f-g\|_{KS^{\alpha_p,p}(X)}>\varepsilon$. Since $\|f\|_{KS^{\alpha_p,p}(X)}<1$ and $\|g\|_{KS^{\alpha_p,p}(X)}<1$, we first deduce that there exists $r_0>0$ such that for $0<r<r_0$,

$$\|f\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}(f,r)<1$$

and

$$\|g\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}(g,r)<1.$$

Then, we have from the property $\mathcal{P}(p,\alpha_p)$

$$\begin{aligned}\varepsilon^p<\|f-g\|^p_{KS^{\alpha_p,p}(X)}&=\|f-g\|^p_{L^p(X,\mu)}+\limsup_{r\to 0}E_{p,\alpha_p}(f-g,r)\\&\le C\left(\|f-g\|^p_{L^p(X,\mu)}+\liminf_{r\to 0}E_{p,\alpha_p}(f-g,r)\right).\end{aligned}$$

Therefore, there exists $r_1>0$ such that for $0<r<r_1$,

$$\|f-g\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}(f-g,r)>\frac{\varepsilon^p}{C}.$$

We now first assume $p\ge 2$. The Clarkson inequalities for $L^p$ functions yield the following:

$$\begin{aligned}&\|f+g\|^p_{L^p(X,\mu)}+\|f-g\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}(f+g,r)+E_{p,\alpha_p}(f-g,r)\\&\le 2^{p-1}\left(\|f\|^p_{L^p(X,\mu)}+\|g\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}(f,r)+E_{p,\alpha_p}(g,r)\right).\end{aligned}$$

We therefore have for $0<r<r_0\wedge r_1$,

$$\|f+g\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}(f+g,r)\le 2^p-\frac{\varepsilon^p}{C}.$$

This implies that

$$\|f+g\|^p_{KS^{\alpha_p,p}(X)}\le 2^p-\frac{\varepsilon^p}{C}.$$

We then conclude from Lemma 4.11 that $(KS^{\alpha_p,p}(X),\|\cdot\|_{KS^{\alpha_p,p}(X)})$ is reflexive. We now turn to the case $1<p<2$. Let $q$ be the conjugate exponent of $p$, i.e. $q=\frac{p}{p-1}$.

We have from the reverse Minkowski inequality and Clarkson's inequalities for $L^p$ functions

$$\left[\left(\left\|\frac{f+g}{2}\right\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}\left(\frac{f+g}{2},r\right)\right)^{q/p}+\left(\left\|\frac{f-g}{2}\right\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}\left(\frac{f-g}{2},r\right)\right)^{q/p}\right]^{p/q}$$
$$\le\left(\left\|\frac{f+g}{2}\right\|^q_{L^p(X,\mu)}+\left\|\frac{f-g}{2}\right\|^q_{L^p(X,\mu)}\right)^{p/q}+\left(E_{p,\alpha_p}\left(\frac{f+g}{2},r\right)^{q/p}+E_{p,\alpha_p}\left(\frac{f-g}{2},r\right)^{q/p}\right)^{p/q}$$
$$\le\frac{1}{2}\|f\|^p_{L^p(X,\mu)}+\frac{1}{2}\|g\|^p_{L^p(X,\mu)}+\frac{1}{2}E_{p,\alpha_p}(f,r)+\frac{1}{2}E_{p,\alpha_p}(g,r).$$

We therefore have for $0<r<r_0\wedge r_1$,

$$\|f+g\|^p_{L^p(X,\mu)}+E_{p,\alpha_p}(f+g,r)\le 2^p\left(1-\frac{\varepsilon^q}{2^qC^{q/p}}\right)^{p/q}.$$

This implies that

$$\|f+g\|^p_{KS^{\alpha_p,p}(X)}\le 2^p\left(1-\frac{\varepsilon^q}{2^qC^{q/p}}\right)^{p/q},$$

and we conclude as above. It remains to prove separability. The identity map $\iota\colon (KS^{\alpha_p,p}(X),\|\cdot\|_{KS^{\alpha_p,p}(X)})\to(L^p(X,\mu),\|\cdot\|_{L^p(X,\mu)})$ is a linear and bounded injective map. Since the space $(KS^{\alpha_p,p}(X),\|\cdot\|_{KS^{\alpha_p,p}(X)})$ is reflexive and $L^p(X,\mu)$ is separable because $X$ is, it now follows from Proposition 4.1 in [6] that $(KS^{\alpha_p,p}(X),\|\cdot\|_{KS^{\alpha_p,p}(X)})$ is separable. □

**4.4. Sobolev type embeddings / Gagliardo–Nirenberg inequalities.** Let $p\ge 1$, $\alpha>0$. Throughout the section we assume:

- A volume non-collapsing condition: There exists $R>0$ such that

$$\inf_{x\in X}\mu(B(x,R))>0.$$

- The property $\mathcal{P}(p,\alpha)$ holds (see Definition 4.5).

Note that the non-collapsing condition always holds if $X$ has a finite diameter since we can take $R=\mathrm{diam}X$. Also note that the collapsing condition implies from (2) that for every $x\in X$, $0<r\le R$,

$$\mu(B(x,r))\ge cr^Q,$$

with $c>0$ (depending on $R$). For $f\in L^q(X,\mu)$, $q\ge 1$ and $r>0$ we consider the averaging operator

$$\mathcal{M}_rf(x)=\frac{1}{\mu(B(x,r))}\int_{B(x,r)}f(y)\,d\mu(y).$$

**Lemma 4.12.** *There exists a constant $C>0$ such that for every $r\in(0,R]$, $q\ge 1$ and $f\in L^q(X,\mu)$,*

$$\|\mathcal{M}_rf\|_{L^\infty(X,\mu)}\le\frac{C}{r^{Q/q}}\|f\|_{L^q(X,\mu)}.$$

*Proof.* The estimate follows from Hölder's inequality and the non-collapsing condition. □

**Lemma 4.13.** *There exists a constant $C > 0$ such that for every $r > 0$ and $f \in KS^{\alpha,p}(X)$,*

$$\|f - \mathcal{M}_r f\|_{L^p(X,\mu)} \le C r^\alpha \operatorname{Var}_p(f).$$

*Proof.* This follows from the property $\mathcal{P}(p,\alpha)$. Indeed, from Hölder's inequality

$$\|f - \mathcal{M}_r f\|_{L^p(X,\mu)} \le r^\alpha \sup_{\rho>0} E_{p,\alpha_p}(f,\rho)^{1/p} \le C r^\alpha \operatorname{Var}_p(f). \qquad \square$$

Remarkably, together with Theorem 4.3, the two simple previous lemmas are enough to obtain the full scale of Gagliardo–Nirenberg inequalities. The results follow from applying the results of [7, Theorem 9.1], see also [1].

**Theorem 4.5.** *Let $q = \frac{pQ}{Q-\alpha p}$ with the convention that $q = \infty$ if $Q = \alpha p$. Let $r, s \in (0, +\infty]$ and $\theta \in (0,1]$ satisfying*

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s}.$$

*If $Q = \alpha p$ with $p > 1$, we assume $r < +\infty$. Then, there exists a constant $C > 0$ such that for every $f \in KS^{\alpha,p}(X)$,*

$$\|f\|_{L^r(X,\mu)} \le C \left(\|f\|_{L^p(X,\mu)} + \operatorname{Var}_p(f)\right)^\theta \|f\|^{1-\theta}_{L^s(X,\mu)}. \tag{9}$$

We explicitly point out some particular cases of interest.

1. Assume that $p\alpha < Q$. If $r = s$, then $r = \frac{pQ}{Q-p\alpha}$ and (9) yields the Sobolev inequality
$$\|f\|_{L^r(X,\mu)} \le C \left(\|f\|_{L^p(X,\mu)} + \operatorname{Var}_p(f)\right).$$
2. Assume that $p\alpha < Q$. If $s = +\infty$ and $r \ge \frac{pQ}{Q-p\alpha}$, then (9) yields
$$\|f\|_{L^r(X,\mu)} \le C \left(\|f\|_{L^p(X,\mu)} + \operatorname{Var}_p(f)\right)^\theta \|f\|^{1-\theta}_{L^\infty(X,\mu)}$$
with $\theta = \frac{pQ}{r(Q-p\alpha)}$.
3. If $r = p > 1$ and $s = 1$, then (9) yields the Nash inequality
$$\|f\|_{L^p(X,\mu)} \le C \left(\|f\|_{L^p(X,\mu)} + \operatorname{Var}_p(f)\right)^\theta \|f\|^{1-\theta}_{L^1(X,\mu)}$$
with $\theta = \frac{(p-1)Q}{p(\alpha+Q)-Q}$.
4. Assume either $p\alpha > Q$ or $p\alpha = Q$ with $p = 1$. Then, for $s \ge 1$,
$$\|f\|_{L^\infty(X,\mu)} \le C \left(\|f\|_{L^p(X,\mu)} + \operatorname{Var}_p(f)\right)^\theta \|f\|^{1-\theta}_{L^s(X,\mu)},$$
where $\theta = \frac{pQ}{pQ+s(p\alpha-Q)}$. In particular, if $s = 1$, and if $f$ is supported in a set $\Omega$ of finite measure we have $\|f\|_{L^s(X,\mu)} \le \|f\|_{L^\infty(X,\mu)}\mu(\Omega)$ and we get:
$$\|f\|_{L^\infty(X,\mu)} \le C \left(\|f\|_{L^p(X,\mu)} + \operatorname{Var}_p(f)\right) \mu(\Omega)^{\frac{\alpha}{Q}-\frac{1}{p}}.$$

From [7, Corollaries 6.3 & 6.4], in the case $p\alpha = Q$ with $p > 1$ one also obtains Trudinger–Moser type inequalities.

**Corollary 4.14.** *Assume that $p\alpha = Q$ and that $p > 1$. Let $k \ge p-1$ be an integer. Then, there exist constants $c, C > 0$ such that for every $f \in KS^{\alpha,p}(X)$ with $\|f\|_{L^p(X,\mu)} + \operatorname{Var}_p(f) = 1$,*

$$\int_X \exp_k\left(c|f|^{\frac{p}{p-1}}\right) d\mu \le C \|f\|^p_{L^p(X,\mu)},$$

*where* $\exp_k(x) = \sum_{\ell=k}^{+\infty} \frac{x^\ell}{\ell!}$. *Moreover, if* $f \in KS^{\alpha,p}(X)$ *with* $\|f\|_{L^p(X,\mu)} + \mathrm{Var}_p(f) = 1$ *is supported in a set* $\Omega$ *of finite measure, then*

$$\int_\Omega e^{c|f|^{\frac{p}{p-1}}} \, d\mu \le C\mu(\Omega).$$

**Remark 4.15.** In all of those inequalities, it is possible to track the dependence of the constants on $R, Q, \inf_{x\in X} \mu(B(x,R))$ and the constant in the property $\mathcal{P}(p,\alpha)$, see the arguments in [1].

**Remark 4.16.** If $X$ has maximal volume growth, i.e. $\mu(B(x,R)) \ge cR^Q$ for every $R > 0$ and $x \in X$ for some $c > 0$ then we can let $R \to +\infty$ in the arguments yielding the Gagliardo–Nirenberg inequalities and get everywhere $\mathrm{Var}_p(f)$ instead of $\|f\|_{L^p(X,\mu)} + \mathrm{Var}_p(f)$, see the arguments in [1] which follow again from [7, Theorem 9.1].

## 5. Korevaar–Schoen–Sobolev spaces and Poincaré inequalities

**5.1. Poincaré inequalities and $\mathcal{P}(p,1)$.** As before, $(X,d,\mu)$ is a metric measure space where $\mu$ is a positive and doubling Borel regular measure. In this section, under the assumption of a $p$-Poincaré inequality, we prove the property $\mathcal{P}(p,\alpha)$ with $\alpha = 1$. Let $p \ge 1$. Consider the following $p$-Poincaré inequality for locally Lipschitz functions

$$\int_{B(x,r)} |f(y) - f_{B(x,r)}|^p \, d\mu(y) \le Cr^p \int_{B(x,\lambda r)} (\mathrm{Lip} f)(y)^p \, d\mu(y), \tag{10}$$

where we denote

$$(\mathrm{Lip} f)(y) = \limsup_{r\to 0} \sup_{x\in X, d(x,y)\le r} \frac{|f(x)-f(y)|}{r}$$

and

$$f_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y).$$

In the inequality, the constants $C > 0$ and $\lambda \ge 1$ are independent from $x$, $r$ and $f$.

**Remark 5.1.** Poincaré inequalities and their applications in the study of metric spaces have extensively been studied in the literature and are nowadays standard assumptions, see for instance [35] and [34] for detailed accounts. For concrete examples, it is known for instance that if a metric measure space satisfies a measure contraction property $\mathrm{MCP}(0,N)$ for some $N \ge 1$, then the $p$-Poincaré inequality holds for every $p \ge 1$, see [63] and [64, 65]. As a consequence, complete Riemannian manifolds with non-negative Ricci curvature and many sub-Riemannian manifolds support a $p$-Poincaré inequality.

**Remark 5.2.** In view of the Hajłasz–Koskela Sobolev embedding [31, Theorem 5.1] (see also [35, Theorem 9.1.2]), for $p > 1$, one can replace the assumption of a $p$-Poincaré inequality (10) by the assumption of a $(1,p)$ Poincaré inequality:

$$\fint_{B(x,r)} |f(y) - f_{B(x,r)}| \, d\mu(y) \le Cr \left( \fint_{B(x,\lambda r)} (\mathrm{Lip} f)(y)^p \, d\mu(y) \right)^{1/p}. \tag{11}$$

**Remark 5.3.** If $(X,d)$ is complete, the $p$-Poincaré inequality (10) is known to be equivalent to the $p$-Poincaré inequality with upper gradients, cf. [35, Theorem 8.4.2].

The main result in that setting is the following theorem. It follows from a combination of results in [40] and [46] (see also [21]). We define $L^p_{\mathrm{loc}}(X,\mu)$ to be the space of locally $p$-integrable functions and for $f_n, f \in L^p_{\mathrm{loc}}(X,\mu)$ we say that $f_n \to f$ in $L^p_{\mathrm{loc}}(X,\mu)$ if for every ball $B \subset X$ one has $\int_B |f_n - f|^p \, d\mu \to 0$ when $n \to +\infty$.

**Theorem 5.1.** *The p-Poincaré inequality* (10) *implies* $\mathcal{P}(p,1)$. *Moreover, on* $KS^{1,p}(X)$

$$\mathrm{Var}_p(f)^p \simeq \inf_{f_n} \liminf_{n\to+\infty} \int_X (\mathrm{Lip} f_n)(y)^p \, d\mu(y)$$

*where the infimum is taken over the sequences of locally Lipschitz functions* $f_n$ *such that* $f_n \to f$ *in* $L^p_{\mathrm{loc}}(X,\mu)$.

*Proof.* The proof is a minor modification of the proof of Theorem 3.1 in [53]; We however write all details since similar arguments will be used in the next section in a more complicated setting. Fix $r > 0$ and, see Proposition 2.1, consider an $r$-covering of $X$ that consists of balls $\{B(x_i,r)\}_{i\ge1}$ with the property that $\{B(x_i,2\lambda r)\}_{i\ge1}$ have the bounded overlap property, i.e. there exists $C > 0$ (independent from $r$) such that

$$\sum_{i\ge1} \mathbf{1}_{B(x_i,2\lambda r)}(x) < C$$

for all $x \in X$. In addition, for any $x \in B(x_i,r)$ and $y \in B(x,r)$ we note that the doubling property implies

$$\mu(B(x_i,r)) \le \mu(B(y,4r)) \le C\mu(B(y,r)),$$
$$\mu(B(x_i,r)) \le \mu(B(x,2r)) \le C\mu(B(x,r)).$$

Now, let $f$ be a locally Lipschitz function on $X$ which is in $L^p_{\mathrm{loc}}(X,\mu)$. We have

$$\begin{aligned}
&\frac{1}{r^p}\int_X \int_{B(x,r)} \frac{|f(x)-f(y)|^p}{\mu(B(x,r))}\, d\mu(y)\, d\mu(x)\\
&\le \frac{1}{r^p}\sum_{i\ge1}\int_{B(x_i,r)}\int_{B(x,r)} \frac{|f(x)-f(y)|^p}{\mu(B(x,r))}\, d\mu(y)\, d\mu(x)\\
&\le \frac{2^{p-1}}{r^p}\sum_{i\ge1}\int_{B(x_i,r)}\int_{B(x,r)} \frac{|f(x)-f_{B(x_i,r)}|^p}{\mu(B(x,r))} + \frac{|f(y)-f_{B(x_i,r)}|^p}{\mu(B(x,r))}\, d\mu(y)\, d\mu(x).
\end{aligned}$$

We control the first term with the $p$-Poincaré inequality as follows.

$$\begin{aligned}
&\sum_{i\ge1}\int_{B(x_i,r)}\int_{B(x,r)} \frac{|f(x)-f_{B(x_i,r)}|^p}{\mu(B(x,r))}\, d\mu(y)\, d\mu(x)\\
&= \sum_{i\ge1}\int_{B(x_i,r)} |f(x)-f_{B(x_i,r)}|^p \, d\mu(x)\\
&\le Cr^p \sum_{i\ge1}\int_{B(x_i,\lambda r)} (\mathrm{Lip} f)(y)^p \, d\mu(y)\\
&\le Cr^p \int_X (\mathrm{Lip} f)(y)^p \, d\mu(y).
\end{aligned}$$

The second term can be controlled in a similar way. First, by using Fubini's theorem and the volume doubling property one obtains

$$\begin{aligned}&\sum_{i\geq 1}\int_{B(x_i,r)}\int_{B(x,r)}\frac{|f(y)-f_{B(x_i,r)}|^p}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\\&\leq \sum_{i\geq 1}\int_{B(x_i,2r)}\int_{B(y,r)}\frac{|f(y)-f_{B(x_i,r)}|^p}{\mu(B(x,r))}\,d\mu(x)\,d\mu(y)\\&\leq C\sum_{i\geq 1}\int_{B(x_i,2r)}|f(y)-f_{B(x_i,r)}|^p\,d\mu(y).\end{aligned}$$

Then, one has

$$\begin{aligned}&\int_{B(x_i,2r)}|f(y)-f_{B(x_i,r)}|^p\,d\mu(y)\\&\leq 2^{p-1}\left(\int_{B(x_i,2r)}|f(y)-f_{B(x_i,2r)}|^p\,d\mu(y)+\mu(B(x_i,2r))|f_{B(x_i,2r)}-f_{B(x_i,r)}|^p\right)\\&\leq C\left(r^p\int_{B(x_i,2\lambda r)}\mathrm{Lip}(f)(y)^p\,d\mu(y)+\mu(B(x_i,2r))|f_{B(x_i,2r)}-f_{B(x_i,r)}|^p\right).\end{aligned}$$

Finally, from Hölder's inequality and the $p$-Poincaré inequality again we get

$$\begin{aligned}\mu(B(x_i,2r))|f_{B(x_i,2r)}-f_{B(x_i,r)}|^p&\leq C\int_{B(x_i,r)}|f(y)-f_{B(x_i,2r)}|^p\,d\mu(y)\\&\leq C\int_{B(x_i,2r)}|f(y)-f_{B(x_i,2r)}|^p\,d\mu(y)\\&\leq Cr^p\int_{B(x_i,2\lambda r)}(\mathrm{Lip}f)(y)^p\,d\mu(y).\end{aligned}$$

Combining everything together we obtain that for every $r>0$

$$\frac{1}{r^p}\int_X\int_{B(x,r)}\frac{|f(x)-f(y)|^p}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\leq C\int_X(\mathrm{Lip}f)(y)^p\,d\mu(y). \tag{12}$$

We therefore proved that any locally Lipschitz function which is in $L^p(X,\mu)$ and such that $\mathrm{Lip}f\in L^p(X,\mu)$ is in the Besov–Lipschitz space $\mathcal{B}^{1,p}(X)$. In particular, $\mathcal{B}^{1,p}(X)$ contains non-constant functions. The estimate (12) also shows that for every $f\in L^p(X,\mu)$ and every ball $B$

$$\frac{1}{r^p}\int_B\int_{B(x,r)}\frac{|f(x)-f(y)|^p}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\leq C\inf_{f_n}\liminf_{n\to+\infty}\int_X(\mathrm{Lip}f_n)(y)^p\,d\mu(y)$$

where the infimum is taken over the sequences of locally Lipschitz functions $f_n$ such that $f_n\to f$ in $L^p_{\mathrm{loc}}(X,\mu)$. Indeed we have

$$\lim_{n\to+\infty}\int_B\int_{B(x,r)}\frac{|f_n(x)-f_n(y)|^p}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)=\int_B\int_{B(x,r)}\frac{|f(x)-f(y)|^p}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x).$$

This proves that for every $f\in L^p(X,\mu)$

$$\frac{1}{r^p}\int_X\int_{B(x,r)}\frac{|f(x)-f(y)|^p}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\leq C\inf_{f_n}\liminf_{n\to+\infty}\int_X(\mathrm{Lip}f_n)(y)^p\,d\mu(y).$$

We now turn to the second part of the proof where we establish that

$$\inf_{f_n} \liminf_{n\to+\infty} \int_X (\mathrm{Lip} f_n)(y)^p \, d\mu(y) \le C \liminf_{r\to 0} \frac{1}{r^p} \int_X \int_{B(x,r)} \frac{|f(x)-f(y)|^p}{\mu(B(x,r))} \, d\mu(y)\, d\mu(x).$$

Fix $\varepsilon > 0$. Let $\{B_i^\varepsilon = B(x_i,\varepsilon)\}_i$ be an $\varepsilon$-covering of $X$, so that the family $\{B_i^{5\varepsilon}\}_i$ has the bounded overlap property uniformly in $\varepsilon$. Let $\varphi_i^\varepsilon$ be a $(C/\varepsilon)$-Lipschitz partition of unity subordinated to this cover, see Proposition 2.1: that is, $0 \le \varphi_i^\varepsilon \le 1$ on $X$, $\sum_i \varphi_i^\varepsilon = 1$ on $X$, and $\varphi_i^\varepsilon = 0$ in $X \setminus B_i^{2\varepsilon}$. For $f \in KS^{1,p}(X)$, we set

$$f_\varepsilon := \sum_i f_{B_i^\varepsilon} \, \varphi_i^\varepsilon,$$

where $f_{B_i^\varepsilon} = \fint_{B_i^\varepsilon} f \, d\mu$. Then $f_\varepsilon$ is locally Lipschitz. Indeed, for $x, y \in B_j^\varepsilon$ we see that

$$\begin{aligned}
|f_\varepsilon(x) - f_\varepsilon(y)| &= \left| \sum_{i: 2B_i^\varepsilon \cap 2B_j^\varepsilon \ne \emptyset} (f_{B_i^\varepsilon} - f_{B_j^\varepsilon})(\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)) \right| \\
&\le \sum_{i: 2B_i^\varepsilon \cap 2B_j^\varepsilon \ne \emptyset} |f_{B_i^\varepsilon} - f_{B_j^\varepsilon}| |\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)| \\
&\le \frac{C\, d(x,y)}{\varepsilon} \sum_{i: 2B_i^\varepsilon \cap 2B_j^\varepsilon \ne \emptyset} \left( \fint_{B_i^\varepsilon} \fint_{B(w,6\varepsilon)} |f(u)-f(w)|^p \, d\mu(u)\, d\mu(w) \right)^{1/p}.
\end{aligned}$$

Therefore, we see that on $B_j^\varepsilon$

$$\begin{aligned}
\mathrm{Lip}(f_\varepsilon) &\le \frac{C}{\varepsilon} \sum_{i: 2B_i^\varepsilon \cap 2B_j^\varepsilon \ne \emptyset} \left( \fint_{B_i^\varepsilon} \fint_{B(x,6\varepsilon)} |f(y)-f(x)|^p \, d\mu(y)\, d\mu(x) \right)^{1/p} \\
&\le C \left( \fint_{5B_j^\varepsilon} \fint_{B(x,6\varepsilon)} \frac{|f(y)-f(x)|^p}{\varepsilon^p} \, d\mu(y)\, d\mu(x) \right)^{1/p},
\end{aligned}$$

and so by the bounded overlap property of the collection $5B_j^\varepsilon$,

$$\begin{aligned}
\int_X \mathrm{Lip}(f_\varepsilon)^p \, d\mu &\le \sum_j \int_{B_j^\varepsilon} \mathrm{Lip}(f_\varepsilon)^p \, d\mu \\
&\le C \sum_j \int_{5B_j^\varepsilon} \fint_{B(x,6\varepsilon)} \frac{|f(y)-f(x)|^p}{\varepsilon^p} \, d\mu(y)\, d\mu(x) \\
&\le C \int_X \fint_{B(x,6\varepsilon)} \frac{|f(y)-f(x)|^p}{\varepsilon^p} \, d\mu(y)\, d\mu(x).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_X \mathrm{Lip}(f_\varepsilon)^p \, d\mu &\le C \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^p} \int_X \int_{B(x,\varepsilon)} \frac{|f(x)-f(y)|^p}{\mu(B(x,\varepsilon))} \, d\mu(y)\, d\mu(x) \\
&< +\infty.
\end{aligned} \tag{13}$$

In a similar manner, we can also show that

$$\int_X |f_\varepsilon(x) - f(x)|^p \, d\mu(x) \le C\varepsilon^p \int_X \int_{B(x,6\varepsilon)} \frac{|f(x)-f(y)|^p}{\varepsilon^p \mu(B(x,\varepsilon))} \, d\mu(y)\, d\mu(x).$$

Therefore $f_\varepsilon \to f$ in $L^p(X,\mu)$. By now the proof is complete. □

**Remark 5.4.** It follows from Theorem 5.1 and [17] (or [35, Theorem 10.1.1]) that if $p > 1$ and the $p$-Poincaré inequality is satisfied, then the Korevaar–Schoen–Sobolev space $KS^{1,p}(X)$ coincides (with equivalent norm) with the Newtonian Sobolev space $N^{1,p}(X)$ introduced by Shanmugalingam in [59]. On the other hand, if $p = 1$ and the 1-Poincaré inequality is satisfied, then the Korevaar–Schoen–Sobolev space $KS^{1,1}(X)$ coincides (with equivalent norm) with the BV space introduced by Miranda in [54]; This fact was first observed in [53]. It follows that if $p \geq 1$ and the $p$-Poincaré inequality is satisfied then $KS^{1,p}(X)$ is dense in $L^p(X, \mu)$.

**Remark 5.5.** In the previous theorem, the property $\mathcal{P}(p, 1)$ implies in particular that

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_X \int_{B(x,\varepsilon)} \frac{|f(x) - f(y)|^p}{\mu(B(x, \varepsilon))} \, d\mu(y) \, d\mu(x) \\ &\leq C \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_X \int_{B(x,\varepsilon)} \frac{|f(x) - f(y)|^p}{\mu(B(x, \varepsilon))} \, d\mu(y) \, d\mu(x). \end{aligned}$$

It is therefore natural to ask whether or not the limit actually exists, i.e. if the inequality holds with $C = 1$. It has been recently proved in [22] (see also [32]) that under the additional condition that the tangent space in the Gromov–Hausdorff sense is Euclidean with fixed dimension, the limit exists if $p > 1$ and $f \in KS^{1,p}(X)$ and is given by

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_X \int_{B(x,\varepsilon)} \frac{|f(x) - f(y)|^p}{\mu(B(x, \varepsilon))} \, d\mu(y) \, d\mu(x) = \mathrm{Ch}_p(f)$$

where $\mathrm{Ch}_p$ is (a multiple of) the Cheeger $p$-energy.

**Remark 5.6.** In the previous proof, the upper bound

$$\inf_{f_n} \liminf_{n \to +\infty} \int_X (\mathrm{Lip} f_n)(y)^p \, d\mu(y) \leq C \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_X \int_{B(x,\varepsilon)} \frac{|f(x) - f(y)|^p}{\mu(B(x, \varepsilon))} \, d\mu(y) \, d\mu(x)$$

does not use the $p$-Poincaré inequality and therefore always holds in volume doubling metric measure spaces.

**Remark 5.7.** Using heat kernel techniques, the following has been proved in [4]:

- If the 2-Poincaré inequality and a weak Bakry–Émery estimate are satisfied then $\mathcal{P}(1, 1)$ holds;
- If the 2-Poincaré inequality and a quasi Bakry–Émery estimate are satisfied then $\mathcal{P}(p, 1)$ holds for every $p \geq 1$.

**Remark 5.8.** If $(X, d)$ is complete and $p > 1$, then the upper bound (12) also follows from arguments on maximal functions. Indeed, from the Keith–Zhong theorem [35, Theorem 12.3.9], the $p$-Poincaré inequality (10) implies a $(1, q)$-Poincaré inequality for some $1 \leq q < p$. From [35, Theorem 8.1.7], this $q$-Poincaré inequality implies the pointwise estimate

$$|f(x) - f(y)| \leq C d(x, y) (\mathcal{M}((\mathrm{Lip} f)^q)(x) + \mathcal{M}((\mathrm{Lip} f)^q)(y))^{1/q} \tag{14}$$

where

$$\mathcal{M}((\mathrm{Lip} f)^q)(x) = \sup_{r>0} \fint_{B(x,r)} (\mathrm{Lip} f)^q(y) \, d\mu(y)$$

is the maximal function associated to $(\mathrm{Lip} f)^q$. Since $q < p$, from $L^{p/q}$-boundedness of the maximal function one has

$$\int_X \mathcal{M}((\mathrm{Lip} f)^q)^{p/q}(y)\, d\mu(y) \le C \int_X (\mathrm{Lip} f)^p(y)\, d\mu(y) \tag{15}$$

and (12) then directly follows from (14) and (15).

**5.2. Generalized Poincaré inequalities and controlled cutoffs.** In this section, we are interested in sufficient conditions for $\mathcal{P}(p, \alpha)$, where the parameter $\alpha$ is possibly greater than one. We make a further assumption on the space $(X, d)$ and assume that it is compact. Concerning the measure $\mu$ we assume that it is a Radon measure and still assume that it is doubling. We denote by $C(X)$ the space of continuous functions on $X$ and by $\mathcal{B}(X)$ the class of Borel sets in $X$.

**Definition 5.9.** A local transition kernel $\{\rho_n, n \in \mathbb{N}\}$ on $X$ is a sequence of Radon measures

$$\rho_n \colon \mathcal{B}(X) \otimes \mathcal{B}(X) \to \mathbb{R}_{\ge 0}$$

such that for any closed sets $A, B \subset X$ with $d(A, B) > 0$

$$\lim_{n \to +\infty} \int_A \int_B d\rho_n(x, y) = 0.$$

**Example 5.10.** 1. Define for $r > 0$, and $\alpha \ge 0$ the Korevaar–Schoen transition kernel

$$d\rho_r(x, y) = \frac{1_{B(x,r)}(y)}{r^\alpha \mu(B(x, r))}\, d\mu(y)\, d\mu(x).$$

Then, the locality property

$$\lim_{r \to 0} \int_A \int_B d\rho_r(x, y) = 0$$

when $d(A, B) > 0$ is easily checked.

2. Consider the graph approximation $V_r$, $r > 0$ of $(X, d, \mu)$ as in Section 3 of [48]. For $\alpha \ge 0$, consider the transition kernel defined by

$$\rho_r(A, B) = \frac{1}{r^\alpha} \mathrm{Card}\, \{(x, y) \in (A \cap V_r) \times (B \cap V_r) \colon x \sim y\}, \quad r > 0,$$

where $x \sim y$ means that $x$ and $y$ are neighbors in the graph $V_r$, and Card denotes the number of elements in the set. Then, the locality property

$$\lim_{r \to 0} \int_A \int_B d\rho_r(x, y) = 0$$

when $d(A, B) > 0$ is also easily checked.

Given a local transition kernel $\rho_n$ and $p \ge 1$, for $f \in C(X)$ we consider the sequence of Radon measures

$$\nu_{n,p}(f, A) = \int_A \int_X |f(x) - f(y)|^p\, d\rho_n(x, y), \quad A \in \mathcal{B}(X).$$

We will denote

$$\mathcal{F}_p = \left\{ f \in C(X) \colon \sup_n \nu_{n,p}(f, X) < +\infty \right\}.$$

Let $\alpha > 0$. To prove the property $\mathcal{P}(p, \alpha)$ in that setting, we consider the following two conditions:

**Definition 5.11.** We will say that the generalized $p$-Poincaré inequality holds if there exists $C>0$ and $\lambda \ge 1$ such that for every $f \in \mathcal{F}_p$, $x\in X$, $r>0$,

$$\int_{B(x,r)} |f(y)-f_{B(x,r)}|^p \, d\mu(y) \le C r^{p\alpha} \liminf_{n\to+\infty} \nu_{n,p}(f, B(x,\lambda r)) \tag{16}$$

**Definition 5.12.** We will say that the controlled cutoff condition holds if for every $\varepsilon>0$ there exists a covering $\{B_i^\varepsilon = B(x_i,\varepsilon)\}_i$ of $X$, so that the family $\{B_i^{5\varepsilon}\}_i$ has the bounded overlap property (uniformly in $\varepsilon$) and an associated family of functions $\varphi_i^\varepsilon$ such that:

- $\varphi_i^\varepsilon \in \mathcal{F}_p$;
- $0 \le \varphi_i^\varepsilon \le 1$ on $X$;
- $\sum_i \varphi_i^\varepsilon = 1$ on $X$;
- $\varphi_i^\varepsilon = 0$ in $X \setminus B_i^{2\varepsilon}$;
- $\limsup_{n\to+\infty} \nu_{n,p}(\varphi_i^\varepsilon, X) \le C\frac{\mu(B_i^\varepsilon)}{\varepsilon^{\alpha p}}$.

**Remark 5.13.** Even though Definition 5.12 might seem difficult to check at first, it is in essence a capacity estimate requirement for balls. For instance, assume that for every ball $B$ with radius $\varepsilon$ one can find a non-negative $\phi \in \mathcal{F}_p$ supported inside of $B$ with $\phi=1$ on $B/2$ such that

$$\limsup_{n\to+\infty} \nu_{n,p}(\phi, X) \le C\frac{\mu(B)}{\varepsilon^{\alpha p}}.$$

Then the controlled cutoff condition of Definition 5.12 is proved to be satisfied using covering by balls satisfying the bounded overlap property as in Section 4.1 in [35]; see also Lemma 2.5 in [55] for a related discussion.

**Remark 5.14.** Definition 5.11 is a generalized Poincaré on balls and Definition 5.12 involves a covering of the space by balls. However, the examples of nested fractals in Section 6 show that in some situations it is more convenient to work with other basis of the topology, like simplices in the case of fractals.

We now show that the combination of the previous conditions implies the property $\mathcal{P}(p,\alpha)$.

**Theorem 5.2.** *The generalized $p$-Poincaré inequality* (16) *and the controlled cutoff condition imply $\mathcal{P}(p,\alpha)$. Moreover, in that case, on $KS^{\alpha,p}(X)$*

$$\mathrm{Var}_p(f)^p \simeq \inf_{f_m} \liminf_{m\to+\infty} \liminf_{n\to+\infty} \nu_{n,p}(f_m, X),$$

*where the infimum is taken over the sequences of functions $f_m \in \mathcal{F}_p$ such that $f_m \to f$ in $L^p(X,\mu)$.*

*Proof.* Repeating the arguments of the first part of the proof of Theorem 5.1 shows that the generalized $p$-Poincaré inequality implies that for every $f\in\mathcal{F}_p$ and every $r>0$

$$\frac{1}{r^{\alpha p}} \int_X \int_{B(x,r)} \frac{|f(x)-f(y)|^p}{\mu(B(x,r))} \, d\mu(y)\, d\mu(x) \le C \liminf_{n\to+\infty} \nu_{n,p}(f,X). \tag{17}$$

Therefore $\mathcal{F}_p \subset KS^{\alpha,p}(X)$ and for every $f\in KS^{\alpha,p}(X)$ and $r>0$

$$\frac{1}{r^{\alpha p}} \int_X \int_{B(x,r)} \frac{|f(x)-f(y)|^p}{\mu(B(x,r))} \, d\mu(y)\, d\mu(x) \le C \inf_{f_m} \liminf_{m\to+\infty} \liminf_{n\to+\infty} \nu_{n,p}(f_m,X), \tag{18}$$

where the infimum is taken over the sequences of functions $f_m\in\mathcal{F}_p$ such that $f_m\to f$ in $L^p(X,\mu)$. Note that at this point of the proof, we do not know that for an arbitrary

$f \in KS^{\alpha,p}(X)$ there actually exists a sequence $f_m \in \mathcal{F}_p$ such that $f_m \to f$ in $L^p(X,\mu)$ so we do not know yet that the right-hand side of (18) is finite; This will be established below by using the controlled cutoff partitions of unity.

Fix $\varepsilon > 0$. Let $\{B_i^\varepsilon = B(x_i,\varepsilon)\}_i$ be an $\varepsilon$-covering of $X$, so that the family $\{B_i^{5\varepsilon}\}_i$ has the bounded overlap property. Let $\varphi_i^\varepsilon$ be a controlled cutoff partition of unity subordinated to this cover. For $f \in KS^{\alpha,p}(X)$, we set

$$f_\varepsilon := \sum_i f_{B_i^\varepsilon}\, \varphi_i^\varepsilon,$$

where $f_{B_i^\varepsilon} = \fint_{B_i^\varepsilon} f\, d\mu$. We first note that $f_\varepsilon \in \mathcal{F}_p$ because $\mathcal{F}_p$ is a linear space and the above sum is finite since $X$ is compact. We now claim that $f_\varepsilon \to f$ in $L^p(X,\mu)$ when $\varepsilon \to 0$. Indeed, for $x \in B_j^\varepsilon$

$$\begin{aligned}
|f(x) - f_\varepsilon(x)| &\le \sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset} |f(x) - f_{B_i^\varepsilon}|\varphi_i^\varepsilon(x) \\
&\le \sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset} \left(\fint_{B_i^\varepsilon} |f(x)-f(y)|^p\, d\mu(y)\right)^{1/p} \\
&\le C\left(\fint_{B_j^{5\varepsilon}} |f(x)-f(y)|^p\, d\mu(y)\right)^{1/p}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\int_X |f(x)-f_\varepsilon(x)|^p\, d\mu(x) &\le C\sum_j \int_{B_j^\varepsilon}\fint_{B_j^{5\varepsilon}} |f(x)-f(y)|^p\, d\mu(y)\, d\mu(x) \\
&\le C\sum_j \int_{B_j^\varepsilon}\fint_{B(x,6\varepsilon)} |f(x)-f(y)|^p\, d\mu(y)\, d\mu(x) \\
&\le C\int_X\fint_{B(x,6\varepsilon)} |f(x)-f(y)|^p\, d\mu(y)\, d\mu(x) \\
&= C\varepsilon^{p\alpha}\int_X\fint_{B(x,6\varepsilon)} \frac{|f(x)-f(y)|^p}{\varepsilon^{p\alpha}}\, d\mu(y)\, d\mu(x).
\end{aligned}$$

Therefore, since $f \in KS^{\alpha,p}(X)$, we deduce that $f_\varepsilon \to f$ in $L^p(X,\mu)$. In particular, $\mathcal{F}_p$ is therefore $L^p$-dense in $KS^{\alpha,p}(X)$. Now, for $x, y \in 2B_j^\varepsilon$ we see that

$$\begin{aligned}
|f_\varepsilon(x) - f_\varepsilon(y)| &= \left|\sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset} (f_{B_i^\varepsilon} - f_{B_j^\varepsilon})(\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y))\right| \\
&\le \sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset} |f_{B_i^\varepsilon} - f_{B_j^\varepsilon}||\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)|.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\int_{B_j^\varepsilon}\int_{2B_j^\varepsilon} |f_\varepsilon(x)-f_\varepsilon(y)|^p\, d\rho_n(x,y) \\
&\le C\sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset} |f_{B_i^\varepsilon} - f_{B_j^\varepsilon}|^p \int_X\int_X |\varphi_i^\varepsilon(x)-\varphi_i^\varepsilon(y)|^p\, d\rho_n(x,y).
\end{aligned}$$

Using the locality property of $\rho_n$ one has

$$\lim_{n\to+\infty}\int_{B_j^\varepsilon}\int_{X\setminus 2B_j^\varepsilon}|f_\varepsilon(x)-f_\varepsilon(y)|^p\,d\rho_n(x,y)=0.$$

We deduce therefore

$$\begin{aligned}&\limsup_{n\to+\infty}\int_{B_j^\varepsilon}\int_X|f_\varepsilon(x)-f_\varepsilon(y)|^p\,d\rho_n(x,y)\\&\le C\sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset}|f_{B_i^\varepsilon}-f_{B_j^\varepsilon}|^p\limsup_{n\to+\infty}\int_X\int_X|\varphi_i^\varepsilon(x)-\varphi_i^\varepsilon(y)|^p\,d\rho_n(x,y).\end{aligned}$$

From the controlled cutoff condition this yields

$$\limsup_{n\to+\infty}\int_{B_j^\varepsilon}\int_X|f_\varepsilon(x)-f_\varepsilon(y)|^p\,d\rho_n(x,y)\le\frac{C}{\varepsilon^{\alpha p}}\sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset}|f_{B_i^\varepsilon}-f_{B_j^\varepsilon}|^p\mu(B_i^\varepsilon).$$

Using the same arguments as in the second part of the proof of Theorem 5.1 to control the term $\sum_{i:2B_i^\varepsilon\cap 2B_j^\varepsilon\neq\emptyset}|f_{B_i^\varepsilon}-f_{B_j^\varepsilon}|^p\mu(B_i^\varepsilon)$, we thus see that

$$\limsup_{n\to+\infty}\int_{B_j^\varepsilon}\int_X|f_\varepsilon(x)-f_\varepsilon(y)|^p\,d\rho_n(x,y)\le C\int_{5B_j^\varepsilon}\fint_{B(x,6\varepsilon)}\frac{|f(y)-f(x)|^p}{\varepsilon^{\alpha p}}\,d\mu(y)\,d\mu(x).$$

Summing up over $j$ and using the bounded overlap property gives

$$\limsup_{n\to+\infty}\int_X\int_X|f_\varepsilon(x)-f_\varepsilon(y)|^p\,d\rho_n(x,y)\le C\int_X\fint_{B(x,6\varepsilon)}\frac{|f(y)-f(x)|^p}{\varepsilon^{\alpha p}}\,d\mu(y)\,d\mu(x).$$

This implies

$$\liminf_{\varepsilon\to0}\liminf_{n\to+\infty}\nu_{n,p}(f_\varepsilon,X)\le C\liminf_{\varepsilon\to0}\int_X\fint_{B(x,\varepsilon)}\frac{|f(y)-f(x)|^p}{\varepsilon^{\alpha p}}\,d\mu(y)\,d\mu(x).$$

Going back to (18) we conclude that

$$\begin{aligned}&\frac{1}{r^{\alpha p}}\int_X\int_{B(x,r)}\frac{|f(x)-f(y)|^p}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\\&\le C\liminf_{\varepsilon\to0}\int_X\fint_{B(x,\varepsilon)}\frac{|f(y)-f(x)|^p}{\varepsilon^{\alpha p}}\,d\mu(y)\,d\mu(x)\end{aligned}$$

and

$$\mathrm{Var}_p(f)^p\simeq\inf_{f_m}\liminf_{m\to+\infty}\liminf_{n\to+\infty}\nu_{n,p}(f_m,X).$$ □

To illustrate the scope of our results, we now discuss a situation where the above results can be used when $p=2$.

**Generalized Poincaré inequalities and controlled cutoffs in Dirichlet spaces.** Let $(X,d,\mu)$ be a compact metric measure space where $\mu$ is a doubling Radon measure. Let $(\mathcal{E},\mathrm{dom}\,\mathcal{E})$ be a regular and strongly local Dirichlet form on $L^2(X,\mu)$. Let $P_t$ be the semigroup associated with $\mathcal{E}$ and $p_t(x,dy)$ be the associated heat kernel measures, i.e. for $f\in L^\infty(X,\mu)$,

$$P_tf(x)=\int_X f(y)p_t(x,dy),\quad t>0.$$

In that setting, one can construct a local transition kernel by considering the family of Radon measures

$$d\rho_t(x,y)=\frac{1}{t}p_t(y,dx)\,d\mu(y),\quad t>0.$$

Note that the locality property of $\rho_t$:

$$\lim_{t\to 0}\frac{1}{t}\int_A\int_B p_t(y,dx)\,d\mu(y)=0$$

if $d(A,B)>0$ follows from the assumed locality of the Dirichlet form (see for instance the proof of Proposition 2.1 in [38]). If $f\in\mathcal{F}_2=\operatorname{dom}\mathcal{E}\cap C(X)$, the measures

$$\nu_t(f,A)=\frac{1}{t}\int_A\int_X (f(x)-f(y))^2 p_t(y,dx)\,d\mu(y),$$

converge weakly as $t\to 0$ to the so-called energy measure $d\Gamma(f,f)$ of $f$, see e.g. [19, (3.2.14)]. In that setting, the validity of the generalized 2-Poincaré inequality (with $\alpha=d_w/2$, $d_w$ being the walk dimension) and of the controlled cutoff condition can be checked under suitable assumptions by using the results in [10]. In particular, one gets the following result:

**Proposition 5.15.** *Assume that the semigroup $P_t$ has a measurable heat kernel $p_t(x,y)$ satisfying, for some $c_1,c_2,c_3,c_4\in(0,\infty)$ and $d_h\geq 1, d_w\in[2,+\infty)$ the sub-Gaussian estimates:*

$$\begin{aligned}c_1 t^{-d_h/d_w}\exp\Biggl(-c_2\Bigl(\frac{d(x,y)^{d_w}}{t}\Bigr)^{\frac{1}{d_w-1}}\Biggr)&\leq p_t(x,y)\\ &\leq c_3 t^{-d_h/d_w}\exp\Biggl(-c_4\Bigl(\frac{d(x,y)^{d_w}}{t}\Bigr)^{\frac{1}{d_w-1}}\Biggr)\end{aligned}\tag{19}$$

*for $\mu$-a.e. $(x,y)\in X\times X$ and each $t\in\bigl(0,\operatorname{diam}(X)^{1/d_w}\bigr)$. Then we have $KS^{d_w/2,2}(X)=\operatorname{dom}\mathcal{E}$. Moreover, there exist constants $c,C>0$ such that for every $f\in KS^{d_w/2,2}(X)$,*

$$c\sup_{r>0}E_{2,d_w/2}(f,r)\leq\mathcal{E}(f,f)\leq C\liminf_{r\to 0}E_{2,d_w/2}(f,r).$$

*In particular, the property $\mathcal{P}(2,d_w/2)$ holds.*

The identification of the domain of the Dirichlet form as a Besov–Lipschitz space seems to have been first uncovered in [39], see also [25, Theorem 4.2] and [57]. The property $\mathcal{P}(2,d_w/2)$ was first pointed out in [1, Proposition 3.5] where it was proved using completely different methods.

## 6. Korevaar–Schoen–Sobolev spaces and $\mathcal{P}(p,\alpha_p)$ on some simple fractals

The approach to Sobolev spaces based on upper gradients does not work on fractals, see the comments [35, Page 409] and Remark 6.3 below. Therefore fractals offer an interesting playground to test the scope of the Korevaar–Schoen approach. We study here two concrete examples, the Vicsek set and the Sierpiński gasket, and prove that those two examples satisfy the property $\mathcal{P}(p,\alpha_p)$ for a critical exponent $\alpha_p>1$. We moreover prove that the corresponding Korevaar–Schoen–Sobolev spaces are dense in $L^p$. The Vicsek set and the Sierpiński gasket are examples of nested fractals (a concept introduced in [51]), and it appears reasonable to infer that our results could be extended to a large class of such nested fractals[3]. More generally, the theory of Korevaar–Schoen–Sobolev spaces on p.c.f. fractals seems to be promising

[3]March 2024 update: This extension was recently carried out in [16] using similar techniques as we introduce here, see also [20].

to explore in view of the recent results of [13]. For different and interesting other approaches to the theory of Sobolev spaces on fractals we refer the interested reader to [13, 43, 44]. Those approaches define the Sobolev spaces as the domains of limits of discrete $p$-energies. This is a natural approach in view of the case $p = 2$ which yields a rich theory of Dirichlet forms, see [9] and [41]. We explain below how those two approaches coincide on the Vicsek set and the Sierpiński gasket. It is worth mentioning that Kigami's general approach has been recently carried out in great details by Shimizu [60] for the Sierpiński carpet. It has been proved there that the domain of the constructed $p$-energy turns out to be a Korevaar–Schoen–Sobolev space when $p$ is larger than the Ahlfors regular conformal dimension of the carpet. The validity or not of the property $\mathcal{P}(p, \alpha_p)$ would be an interesting question to settle then since the carpet is an example of infinitely ramified fractals, making its geometry very different from the two examples treated below[4]. Finally, let us mention that there also exists a large amount of literature concerning the construction of Sobolev-like and more generally Besov-like functional spaces on fractals using the Laplace operator as a central object, see for instance [14, 15, 23, 61] and the references therein. When $p \neq 2$, some inclusions are known (following for instance from [3]), but making an exact identification between those spaces and the Korevaar–Schoen ones is a challenging interesting problem for the future. In a nutshell, to make such connections, it would be interesting to study the possible continuity properties in the Korevaar–Schoen–Sobolev spaces of some suitable fractional power of the Laplace operator; This is a boundedness of Riesz transform type problem, see [5, Section 3.5].

**6.1. Vicsek set.** Let $q_1 = (-\sqrt{2}/2, \sqrt{2}/2)$,$q_2 = (\sqrt{2}/2, \sqrt{2}/2)$ , $q_3 = (\sqrt{2}/2, -\sqrt{2}/2)$, and $q_4 = (-\sqrt{2}/2, -\sqrt{2}/2)$ be the 4 corners of the unit square and let $q_5 = (0, 0)$ be the center of that square. Define $\psi_i(z) = \frac{1}{3}(z - q_i) + q_i$ for $1 \le i \le 5$. Then the Vicsek set $K$ is the unique non-empty compact set such that

$$K = \bigcup_{i=1}^{5} \psi_i(K).$$

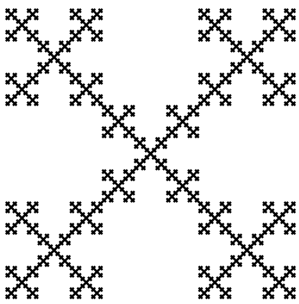

Figure 1. Vicsek set.

Denote $W = \{1, 2, 3, 4, 5\}$ and $W_n = \{1, 2, 3, 4, 5\}^n$ for $n \ge 1$. For any $w = (i_1, \cdots, i_n) \in W_n$, we denote by $\Psi_w$ the contraction mapping $\psi_{i_1} \circ \cdots \circ \psi_{i_n}$ and write $K_w := \Psi_w(K)$. The set $K_w$ is called an $n$-simplex. Let $V_0 = \{q_1, q_2, q_3, q_4, q_5\}$. We

[4]March 2024 update: As already pointed out in the introduction, the validity of $\mathcal{P}(p, \alpha_p)$ in the Sierpiński carpet was recently proved for $p$ greater than the Ahlfors regular conformal dimension in [67] and then shortly after for every $p > 1$ in [56].

define a sequence of sets of vertices $\{V_n\}_{n\geq 0}$ inductively by

$$V_{n+1} = \bigcup_{i=1}^{5} \psi_i(V_n).$$

Let then $\bar{V}_0$ be the cable system included in $K$ with vertices $V_0 = \{q_1, q_2, q_3, q_4, q_5\}$ and consider the sequence of cable systems $\bar{V}_n$ with vertices in $V_n$ inductively defined as follows. The first cable system is $\bar{V}_0$ and then

$$\bar{V}_{n+1} = \bigcup_{i=1}^{5} \psi_i(\bar{V}_n).$$

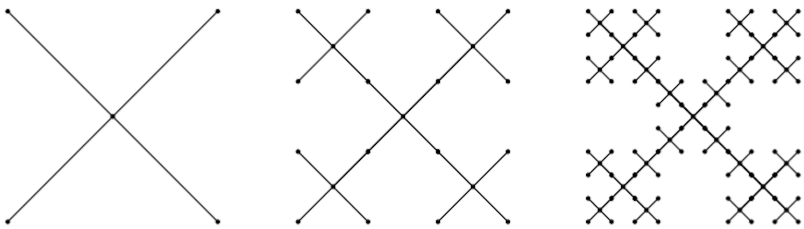

Figure 2. Vicsek approximating cable systems $\bar{V}_0$, $\bar{V}_1$ and $\bar{V}_2$.

Note that $\bar{V}_n \subset K$ and that $K$ is the closure of $\cup_{n\geq 0}\bar{V}_n$. The set

$$\mathcal{S} = \bigcup_{n\geq 0} \bar{V}_n$$

is called the skeleton of $K$ and is dense in $K$. Therefore we have a natural increasing sequence of Vicsek cable systems $\{\bar{V}_n\}_{n\geq 0}$ whose edges have length $3^{-n}$ and whose set of vertices is $V_n$ (see Figure 2). From this viewpoint, the Vicsek set $K$ is seen as a limit of the cable systems $\{\bar{V}_n\}_{n\geq 0}$.

On $K$ we will consider the geodesic distance $d$: For $x, y \in \bar{V}_n$, $d(x, y)$ is defined as the length of the geodesic path between $x$ and $y$ and $d$ is then extended by continuity to $K \times K$. The geodesic distance $d$ is then bi-Lipschitz equivalent to the restriction of the Euclidean distance to $K$. The normalized Hausdorff measure $\mu$ is the unique Borel measure on $K$ such that for every $i_1, \cdots, i_n \in \{1, 2, 3, 4, 5\}$

$$\mu(\psi_{i_1} \circ \cdots \circ \psi_{i_n}(K)) = 5^{-n}.$$

The Hausdorff dimension of $K$ is then $d_h = \frac{\log 5}{\log 3}$ and the metric space $(K, d)$ is $d_h$-Ahlfors regular in the sense that there exist constants $c, C > 0$ such that for every $x \in K$, $r \in (0, \operatorname{diam} K]$,

$$c\, r^{d_h} \leq \mu(B(x, r)) \leq C r^{d_h},$$

where, as before, $B(x, r) = \{y \in K : d(x, y) < r\}$ denotes the ball with center $x$ and radius $r$ and $\operatorname{diam} K = 2$ is the diameter of $K$.

**Theorem 6.1.** *For the Vicsek set, for $p \geq 1$ the $L^p$ critical Besov exponent is given by*

$$\alpha_p = 1 + \frac{d_h - 1}{p} > 1$$

*and the space $KS^{\alpha_p,p}(K)$ is dense in $L^p(K, \mu)$.*

*Proof.* The case $p = 1$ was first treated in [5, Theorem 5.1] where it is actually proved that for any nested fractal $\alpha_1 = d_h$ and that the corresponding Korevaar–Schoen–Sobolev space is always dense in $L^1$. The case $1 < p \leq 2$ was then treated for the first time in [1, Theorem 3.10] where it is proved that in that case $\alpha_p = 1+\frac{d_h-1}{p}$ and that $KS^{\alpha_p,p}(K)$ contains non-constant functions. The case $p>2$ follows from [11, Corollary 4.5]. Finally, the fact that $KS^{\alpha_p,p}(K)$ is dense in $L^p(K,\mu)$ for every $p>1$ follows from Section 2.5 in [11] where it is observed that the set of piecewise affine functions which is dense in $L^p(K,\mu)$, is a subset of $KS^{\alpha_p,p}(K)$. □

Since $\alpha_p > \frac{d_h}{p}$ when $p>1$ it follows from Theorem 3.2 that any function $f \in KS^{\alpha_p,p}(K)$, $p>1$, has a version which is $\left(1-\frac{1}{p}\right)$-Hölder continuous. We will therefore see $KS^{\alpha_p,p}(K)$ as a subset of $C(K)$.

For $1<p<+\infty$, the discrete $p$-energy on $V_m$ of a function $f\in C(K)$ is defined as

$$\mathcal{E}_p^m(f) := \frac{1}{2} 3^{(p-1)m} \sum_{x,y\in V_m, x\sim y} |f(x)-f(y)|^p.$$

As a consequence of the basic inequalities

$$|x+y+z|^p \le 3^{p-1}(|x|^p+|y|^p+|z|^p),$$

and of the tree structure of $V_m$ we always have for $p>1$

$$\mathcal{E}_p^m(f) \le \mathcal{E}_p^n(f), \quad \forall\, m,n\in\mathbb{N},\ m\le n. \tag{20}$$

Moreover, from this fact we deduce that

$$\lim_{n\to\infty} \mathcal{E}_p^n(f) = \sup_{n\ge 0} \mathcal{E}_p^n(f) = \limsup_{n\to\infty} \mathcal{E}_p^n(f) = \liminf_{n\to\infty} \mathcal{E}_p^n(f), \tag{21}$$

where the above quantities are in $\mathbb{R}_{\ge 0}\cup\{+\infty\}$.

**Definition 6.1.** Let $p>1$. For $f\in C(K)$, we define the (possibly infinite) $p$-energy of $f$ by

$$\mathcal{E}_p(f) := \lim_{m\to\infty} \mathcal{E}_p^m(f)$$

and let

$$\mathcal{F}_p = \left\{ f\in C(K)\colon \sup_{m\ge 0} \mathcal{E}_p^m(f) < +\infty \right\}.$$

We consider on $\mathcal{F}_p$ the seminorm

$$\|f\|_{\mathcal{F}_p} = \sup_{m\ge 0} \mathcal{E}_p^m(f)^{1/p}, \quad p>1.$$

We have then the following characterization of the Korevaar–Schoen–Sobolev spaces in terms of the discrete $p$-energies:

**Theorem 6.2.** [11, Theorem 2.9] *Let $p>1$. For $f\in C(K)$ the following are equivalent:*

1. $f\in KS^{\alpha_p,p}(K)$;
2. $f\in \mathcal{F}_p$;

*Moreover, one has*

$$\sup_{r>0} E_{p,\alpha_p}(f,r)^{1/p} \simeq \|f\|_{\mathcal{F}_p}.$$

**Remark 6.2.** In [11] a further characterization of $KS^{\alpha_p,p}(K)$ is given in terms of weak gradients.

**Remark 6.3.** As in Remark 5.4, one might wonder how the theory of Newtonian Sobolev spaces introduced by Shanmugalingam in [59] applies in that setting and if it compares to the Korevaar–Schoen–Sobolev spaces. However, one can see that for every $p\ge 1$, the Newtonian space $N^{1,p}(K)$ is equal to $L^p(K,\mu)$. Indeed, it is well known that there exists a non-negative Borel function $[0,1] \to \mathbb{R}$ whose integral is infinite on any interval of positive length. As a consequence, there exists a non-negative function $\rho$ on the skeleton $\mathcal{S}$ which is measurable with respect to the $\sigma$-algebra generated by the set of geodesic paths included in $\mathcal{S}$ and such that for every geodesic path $\gamma$ connecting two points $x \neq y \in \mathcal{S}$, $\int_\gamma \rho = +\infty$. The function $\rho$ can be extended to be zero on $K \backslash \mathcal{S}$. This function $\rho$ is then non-negative and in $L^p(K,\mu)$ for every $p \geq 1$. According to Lemma 2.1 in [59] one has therefore $N^{1,p}(K) = L^p(K,\mu)$. Intuitively, the issue is that even though the space is geodesic, the Hausdorff measure *does not see* the rectifiable paths because they all are supported on a set of measure zero.

**Theorem 6.3.** *On the Vicsek set the property $\mathcal{P}(p,\alpha_p)$ holds for every $p \geq 1$. Therefore, if $p > 1$, $\mathrm{Var}_p(f)^p \simeq \mathcal{E}_p(f)$.*

*Proof.* If $p = 1$, the validity of $\mathcal{P}(1,\alpha_1)$ is established in [5, Theorem 4.9], so we assume that $p > 1$. We adapt slightly the proof of Theorem 5.4 in [11] and use the notion of piecewise affine function on the Vicsek set. A continuous function $\Phi\colon K \to \mathbb{R}$ is called $n$-piecewise affine, if there exists $n \geq 0$ such that $\Phi$ is piecewise affine on the cable system $\bar{V}_n$ (i.e. linear between the vertices of $\bar{V}_n$) and constant on any connected component of $\bar{V}_m \setminus \bar{V}_n$ for every $m > n$. If $\Phi\colon K \to \mathbb{R}$ is an $n$-piecewise affine function, then, for $p > 1$, $\mathcal{E}_p^0(\Phi) \leq \cdots \leq \mathcal{E}_p^n(\Phi) = \mathcal{E}_p^m(\Phi)$, where $m \geq n$, and therefore $\mathcal{E}_p(\Phi) = \mathcal{E}_p^n(\Phi)$. Let $f \in KS^{\alpha_p,p}(K)$. We define a sequence of piecewise affine functions $\{\Phi_n\}_{n\geq 1}$ associated with $f$ on the cable systems $\{\bar{V}_n\}_{n\geq 1}$ as follows. For any fixed $n \geq 1$, we first define a function $f_n$ on $V_n$ by

$$f_n(v) := \frac{1}{\mu(K^*_{n+1}(v))} \int_{K^*_{n+1}(v)} f\, d\mu, \quad v \in V_n,$$

where $K^*_{n+1}(v)$ is the union of the $(n+1)$-simplices containing $v$. Then let $\Phi_n$ be the unique piecewise affine function that coincides with $f_n$ on $V_n$. It is easy to see that

$$\Phi_n(x) = \sum_{v\in V_n} \left( \frac{1}{\mu(K^*_{n+1}(v))} \int_{K^*_{n+1}(v)} f\, d\mu \right) u_v(x) = \sum_{v\in V_n} f_n(v)\, u_v(x),$$

where $u_v$ is the unique $n$-piecewise affine function that takes the value 1 at $v$ and zero on $V_n \setminus \{v\}$. Note that we have $0 \leq u_v \leq 1$, $\operatorname{supp} u_v \subset K^*_n(v)$, where $K^*_n(v)$ is the union of the $n$-simplices containing $v$, and

$$\sum_{v\in V_n} u_v(x) = 1, \quad \forall\, x \in K.$$

We observe that the covering $\{K^*_n(v)\}_{v\in V_n}$ has the bounded overlap property. Also, for any $x \in K^*_n(v)$, $K^*_{n+1}(v) \subset B(x, 3^{-n+1})$. We note that $\Phi_n$ is an analogue for the Vicsek set of the sequence $f_\varepsilon$ that was considered in the proof of Theorem 5.1. By Hölder's inequality one has:

$$\begin{aligned}
\|f - \Phi_n\|^p_{L^p(K,\mu)} &\leq C \sum_{v\in V_n} \int_{K^*_n(v)} |f(x) - f_n(v)|^p (u_v(x))^p\, d\mu(x) \\
&\leq C \sum_{v\in V_n} \int_{K^*_n(v)} \fint_{K^*_{n+1}(v)} |f(x) - f(y)|^p\, d\mu(y)\, d\mu(x) \\
&\leq C \int_K \fint_{B(x,3^{-n+1})} |f(x) - f(y)|^p\, d\mu(y)\, d\mu(x).
\end{aligned} \tag{22}$$

On the other hand from Theorem 6.2, we have for every $r > 0$

$$\frac{1}{r^{p\alpha_p}} \int_K \fint_{B(z,r)} |\Phi_n(z) - \Phi_n(w)|^p \, d\mu(w) \, d\mu(z) \le C\mathcal{E}_p(\Phi_n) = C\mathcal{E}_p^n(\Phi_n)$$

and $\mathcal{E}_p^n(\Phi_n)$ can be controlled as follows. Observe that for any $x \in V_n$, one has $\Phi_n(x) = f_n(x)$ by definition. Hence

$$\mathcal{E}_p^n(\Phi_n) = \frac{1}{2} 3^{(p-1)n} \sum_{x,y \in V_n, x \sim y} |f_n(x) - f_n(y)|^p.$$

For any neighboring vertices $x, y \in V_n$, Hölder's inequality yields

$$\begin{aligned} |f_n(x) - f_n(y)| &\le \frac{1}{\mu(K_{n+1}^*(x))\mu(K_{n+1}^*(y))} \int_{K_{n+1}^*(x)} \int_{K_{n+1}^*(y)} |f(z) - f(w)| \, d\mu(w) \, d\mu(z) \\ &\le C \left( 5^{2n} \int_{K_{n+1}^*(x)} \int_{K_{n+1}^*(y)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z) \right)^{\frac{1}{p}}. \end{aligned}$$

Thanks to the fact that $x, y \in V_n$ are adjacent $K_{n+1}^*(y) \subset B(z, 3^{-n+1})$ for any $z \in K_{n+1}^*(x)$. Therefore we get

$$|f_n(x) - f_n(y)|^p \le C 5^{2n} \int_{K_{n+1}^*(x)} \int_{B(z,3^{-n+1})} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z).$$

By the bounded overlap property of $\{K_{n+1}^*(v)\}_{v \in V_n}$, we then have

$$\begin{aligned} \mathcal{E}_p^n(\Phi_n) &\le C 3^{(p-1)n} 5^{2n} \sum_{x,y \in V_n, x \sim y} \int_{K_{n+1}^*(x)} \int_{K_{n+1}^*(y)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z) \\ &\le C 3^{(p-1)n} 5^{2n} \int_K \int_{B(z,3^{-n+1})} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z). \end{aligned}$$

Set $r_n = 3^{-n+1}$. We can rewrite the above inequality as

$$\mathcal{E}_p^n(\Phi_n) \le \frac{C}{r_n^{p\alpha_p}} \int_K \fint_{B(z,r_n)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z).$$

Consequently, we have for every $r > 0$

$$\begin{aligned} &\frac{1}{r^{p\alpha_p}} \int_K \fint_{B(z,r)} |\Phi_n(z) - \Phi_n(w)|^p \, d\mu(w) \, d\mu(z) \\ &\le \frac{C}{r_n^{p\alpha_p}} \int_K \fint_{B(z,r_n)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z). \end{aligned}$$

Let now $0 < \varepsilon < 1$ and let $n_\varepsilon$ be the unique integer $\ge 1$ such that $\frac{1}{3^{n_\varepsilon - 1}} < \varepsilon \le \frac{1}{3^{n_\varepsilon - 2}}$. We have for every $r > 0$

$$\begin{aligned} &\frac{1}{r^{p\alpha_p}} \int_K \fint_{B(z,r)} |\Phi_{n_\varepsilon}(z) - \Phi_{n_\varepsilon}(w)|^p \, d\mu(w) \, d\mu(z) \\ &\le \frac{C}{\varepsilon^{p\alpha_p}} \int_K \fint_{B(z,\varepsilon)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z). \end{aligned} \tag{23}$$

However, (22) also gives that

$$\|f - \Phi_{n_\varepsilon}\|_{L^p(K,\mu)}^p \le C\varepsilon^{p\alpha_p} \frac{1}{\varepsilon^{p\alpha_p}} \int_K \fint_{B(z,\varepsilon)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z)$$

which implies that as $\varepsilon \to 0$, $\Phi_{n_\varepsilon} \to f$ in $L^p(K,\mu)$. By taking $\liminf_{\varepsilon\to 0}$ in (23) we obtain then that for every $r>0$

$$\frac{1}{r^{p\alpha_p}}\int_K \fint_{B(z,r)} |f(z)-f(w)|^p \, d\mu(w)\, d\mu(z) \le C\mathrm{Var}_p(f)^p. \qquad \square$$

**Remark 6.4.** In view of the property $\mathcal{P}(p,\alpha_p)$ on the Vicsek set, one might wonder (as in Remark 5.5) whether the limit

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{p\alpha_p}}\int_K \fint_{B(z,\varepsilon)} |f(z)-f(w)|^p \, d\mu(w)\, d\mu(z)$$

actually exists or not. This problem was studied in [2] in the case $p=1$ and it was proved that the limit does not exist in general due to the small-scale oscillations appearing in the geometry of the Vicsek set.

As a consequence of the property $\mathcal{P}(p,\alpha_p)$ and Theorem 4.5, we therefore get the following Nash inequalities on the Vicsek set.

**Corollary 6.5.** *For $p>1$ the following Nash inequality holds for every $f\in KS^{\alpha_p,p}(K)$,*

$$\|f\|_{L^p(K,\mu)} \le C\left(\|f\|_{L^p(K,\mu)} + \mathrm{Var}_p(f)\right)^\theta \|f\|_{L^1(K,\mu)}^{1-\theta}$$

*with $\theta = \frac{(p-1)d_h}{p-1+pd_h}$, while for $p=1$, there exists a constant $C>0$ such that for every $f\in KS^{d_h,1}(K)$*

$$\|f\|_{L^\infty(K,\mu)} \le C\left(\|f\|_{L^1(K,\mu)} + \mathrm{Var}_1(f)\right).$$

**Remark 6.6.** For $p=2$, Nash inequalities have been studied in connection with heat kernel estimates in the more general context of p.c.f. fractals. Corollary 6.5 recovers the special case of [9, Theorem 8.3] for the Vicsek set since $\mathrm{Var}_2(f)^2 \simeq \mathcal{E}_2(f)$ which is the Dirichlet form on the Vicsek set.

**6.2. Sierpiński gasket.** Let $V_0=\{q_1,q_2,q_3\}$ be the set of vertices of an equilateral triangle of side 1 in $\mathbb{C}$. Define

$$\psi_i(z) = \frac{z-q_i}{2} + q_i$$

for $i=1,2,3$. Then the Sierpiński gasket $K$ is the unique non-empty compact subset in $\mathbb{C}$ such that

$$K = \bigcup_{i=1}^3 \psi_i(K).$$

The Hausdorff dimension of $K$ with respect to the geodesic metric $d$ is given by $d_h = \frac{\log 3}{\log 2}$. The (normalized) Hausdorff measure on $K$ is given by the unique Borel measure $\mu$ on $K$ such that for every $i_1,\cdots,i_n \in \{1,2,3\}$,

$$\mu\left(\psi_{i_1}\circ\cdots\circ\psi_{i_n}(K)\right) = 3^{-n}.$$

This measure $\mu$ is $d_h$-Ahlfors regular, i.e. there exist constants $c,C>0$ such that for every $x\in K$ and $r\in(0,\mathrm{diam}(K)]$,

$$cr^{d_h} \le \mu(B(x,r)) \le Cr^{d_h}. \tag{24}$$

We define a sequence of sets $\{V_m\}_{m\ge 0}$ inductively by

$$V_{m+1} = \bigcup_{i=1}^3 \psi_i(V_m).$$

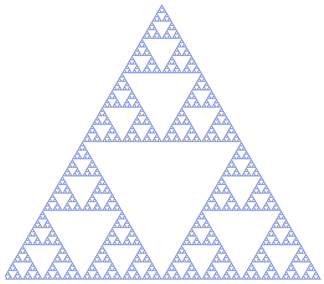

Figure 3. Sierpiński gasket.

Let $p > 1$. The study of $p$-energies on the Sierpiński gasket was undertaken in [36] where the authors introduced a non-linear renormalization problem from which it is possible to compute the $L^p$ critical Besov exponents. After [36] consider the function

$$F_p(a) = |a_1 - a_2|^p + |a_2 - a_3|^p + |a_3 - a_1|^p, \quad a = (a_1, a_2, a_3) \in \mathbb{R}^3.$$

The renormalization problem for the function $F_p$ is to find a non-negative continuous convex function $A_p$ on $\mathbb{R}^3$ and a number $0 < r_p < 1$ such that:

1. $A_p(a) \simeq F_p(a)$;
2. $\min_{b\in\mathbb{R}^3} (A_p(a_1, b_2, b_3) + A_p(b_1, a_2, b_3) + A_p(b_1, b_2, a_3)) = r_p A_p(a)$.

It was shown in [36] that a solution of the renormalization problem exists. While the uniqueness of the function $A_p$ is not known, remarkably the number $r_p$ is. For $p = 2$ the value of $r_p$ is $\frac{3}{5}$. For other values of $p$, the value of $r_p$ is unknown but was proved in [36] to satisfy

$$2^{1-p} \le r_p \le 2^{p-1} \left(1 + \sqrt{1 + 2^{3-1/(p-1)}}\right)^{1-p} < 3 \cdot 2^{-p}.$$

**Theorem 6.4.** *For the Sierpiński gasket, if $p > 1$ the $L^p$ critical Besov exponent is given by*

$$\alpha_p = \frac{1}{p}\left(\frac{\log 3}{\log 2} - \frac{\log r_p}{\log 2}\right) \in \left(\frac{d_h}{p}, 1 + \frac{d_h - 1}{p}\right]$$

*and for $p = 1$,*

$$\alpha_1 = d_h = \frac{\log 3}{\log 2}.$$

*Moreover, for every $p \ge 1$, the Korevaar–Schoen–Sobolev space $KS^{\alpha_p,p}(K)$ is dense in $L^p(K, \mu)$.*

*Proof.* As before, the case $p = 1$ follows from [5, Theorem 5.1]. For $p > 1$ the result follows from the proof of Theorem 2.1 in [37]. Indeed, a thorough inspection shows that it is actually proved there that for $\alpha = \frac{1}{p}\left(\frac{\log 3}{\log 2} - \frac{\log r_p}{\log 2}\right)$ one has

$$\sup_{r>0} E_{p,\alpha}(f, r) \le C \limsup_{r\to 0} E_{p,\alpha}(f, r).$$

See also Corollary 2.1 in the same paper [37]. Finally, $KS^{\alpha_p,p}(K)$ is dense in $L^p(K, \mu)$ from Lemma 6.9 below. □

Since $\alpha_p > \frac{d_h}{p}$ when $p > 1$ it follows from Theorem 3.2 that any function $f \in KS^{\alpha_p,p}(K)$, $p > 1$, has a version which is $\alpha_p - \frac{d_h}{p} = -\frac{\log r_p}{p \log 2}$-Hölder continuous, see Remark 3.5. We will therefore see $KS^{\alpha_p,p}(K)$ as a subset of $C(K)$ when $p > 1$.

As for the Vicsek set, it is possible to characterize the Korevaar–Schoen–Sobolev space as the domain of a limit of discrete $p$-energies. Denote $W = \{1, 2, 3\}$ and

$W_n = \{1,2,3\}^n$ for $n \geq 1$. For any $w = (i_1, \cdots, i_n) \in W_n$, we denote by $\Psi_w$ the contraction mapping $\psi_{i_1} \circ \cdots \circ \psi_{i_n}$ and write $K_w := \Psi_w(K)$. As before, the set $K_w$ is called an $n$-simplex.

For $1 < p < +\infty$, the discrete $p$-energy on $V_m$ of a function $f \in C(K)$ is defined as

$$\begin{aligned}\mathcal{E}_p^m(f) &:= \frac{1}{2} r_p^{-m} \sum_{x,y \in V_m, x \sim y} |f(x) - f(y)|^p \\ &= r_p^{-m} \sum_{w \in W_m} F_p(f(\Psi_w(q_1)), f(\Psi_w(q_2)), f(\Psi_w(q_3))).\end{aligned}$$

Unlike for the Vicsek set, the sequence $\mathcal{E}_p^m(f)$ needs not be non-decreasing, however one may instead consider the modified $p$-energy

$$\mathcal{A}_p^m(f) := r_p^{-m} \sum_{w \in W_m} A_p(f(\Psi_w(q_1)), f(\Psi_w(q_2)), f(\Psi_w(q_3)))$$

where $A_p$ solves the renormalization problem. It is then clear that for every $m$

$$c\mathcal{A}_p^m(f) \leq \mathcal{E}_p^m(f) \leq C\mathcal{A}_p^m(f)$$

and moreover that $\mathcal{A}_p^m(f)$ is non-decreasing. In particular $\sup_{m \geq 0} \mathcal{E}_p^m(f)$ is finite if and only if the limit $\lim_{m \to +\infty} \mathcal{A}_p^m(f)$ is finite.

**Definition 6.7.** Let $p > 1$. For $f \in C(K)$, we define the (possibly infinite) $p$-energy of $f$ by

$$\mathcal{E}_p(f) := \sup_{m \geq 0} \mathcal{E}_p^m(f).$$

We define then

$$\mathcal{F}_p = \left\{ f \in C(K) \colon \sup_{m \geq 0} \mathcal{E}_p^m(f) < +\infty \right\}$$

and consider on $\mathcal{F}_p$ the seminorm

$$\|f\|_{\mathcal{F}_p} = \sup_{m \geq 0} \mathcal{E}_p^m(f)^{1/p}.$$

We have then the following characterization of the Korevaar–Schoen–Sobolev spaces in terms of the discrete $p$-energies:

**Theorem 6.5.** [37, Theorem 2.1] *Let $p > 1$. For $f \in C(K)$ the following are equivalent:*

1. *$f \in KS^{\alpha_p,p}(K)$;*
2. *$f \in \mathcal{F}_p$;*

*Moreover, one has*

$$\sup_{r>0} E_{p,\alpha_p}(f,r)^{1/p} \simeq \|f\|_{\mathcal{F}_p}.$$

One can construct plenty of functions in $\mathcal{F}_p$ and therefore in $KS^{\alpha_p,p}(K)$ by using the notion of $p$-harmonic extension. This extension procedure is explained in detail in Corollary 2.4 in [36] (see also [13]) and can be described as follows. Let $n \geq 0$ and $f_n \colon V_n \to \mathbb{R}$. One can extend $f_n$ into a function $f_{n+1}$ defined on $V_{n+1}$ such that:

- For all $v \in V_n$, $f_{n+1}(v) = f_n(v)$;
- $\mathcal{A}_p^{n+1}(f_{n+1}) = \min\{\mathcal{A}_p^{n+1}(g) \colon g_{|V_n} = f_n\}$;

Indeed, if we fix $w \in W_n$ and minimize over $b_1, b_2, b_3$ the quantity

$$A_p(f_n(\Psi_w(q_1)), b_2, b_3) + A_p(b_1, f_n(\Psi_w(q_2)), b_3) + A_p(b_1, b_2, f_n(\Psi_w(q_3)))$$

a minimizer will assign values of $f_{n+1}$ on $\Psi_w(V_1)$. We note then that

$$\mathcal{A}_p^{n+1}(f_{n+1}) = \mathcal{A}_p^n(f_n).$$

The process can be repeated and we thus get a sequence of functions $f_m \colon V_m \to \mathbb{R}$, $m \geq n$ such that for every $v \in V_m$, $f_{m+1}(v) = f_m(v)$ and

$$\mathcal{A}_p^m(f_m) = \mathcal{A}_p^n(f_n).$$

We denote then by $H_p(f_n)$ the function on $\bigcup_m V_m$ whose restriction to each $V_m$ coincides with $f_m$. The function $H_p(f_n)$ is called a $p$-harmonic extension of $f_n$. We note that for suitable choices of $A_p$ the $p$-harmonic extension of $f_n$ is unique, see the discussion before Lemma A.2. in [13].

**Lemma 6.8.** *There exists a constant $C > 0$ such that for every $u, v \in \bigcup_m V_m$,*

$$|H_p(f_n)(u) - H_p(f_n)(v)|^p \leq C d(u,v)^{-\frac{\log r_p}{\log 2}} \mathcal{A}_p^n(f_n).$$

*In particular $H_p(f_n)$ has a unique Hölder continuous extension to $K$, which we still denote by $H_p(f_n)$.*

*Proof.* It is clear that for $m \geq n$ and $u, v \in V_m, u \sim v$

$$|f_m(u) - f_m(v)|^p \leq r_p^m \mathcal{E}_p^m(f) \leq C r_p^m \mathcal{A}_p^m(f) = C r_p^m \mathcal{A}_p^n(f).$$

Therefore,

$$|H_p(f_n)(u) - H_p(f_n)(v)|^p \leq C r_p^m \mathcal{A}_p^n(f).$$

Since $d(u,v) = \frac{1}{2^m}$ this can be rewritten as

$$|H_p(f_n)(u) - H_p(f_n)(v)|^p \leq C d(u,v)^{-\frac{\log r_p}{\log 2}} \mathcal{A}_p^n(f).$$

Now, for general $u, v \in V_m$ one can use a chaining argument similar to the one described in the third paragraph of Section 1.4 in [62], see also [13, Proof of Proposition 5.3 (d)]. □

The following result follows from Corollary 2.4 in [36].

**Lemma 6.9.** *Let $p > 1$. For every $f \in C(K)$, let $f_n \colon V_n \to \mathbb{R}$ be the unique function on $V_n$ that coincides with $f$. Then*

$$\lim_{n \to +\infty} \sup_{x \in K} |H_p(f_n)(x) - f(x)| = 0.$$

We are now ready to prove the following:

**Theorem 6.6.** *Let $p \geq 1$. On the Sierpiński gasket the property $\mathcal{P}(p, \alpha_p)$ holds. Therefore, if $p > 1$, $\mathrm{Var}_p(f)^p \simeq \mathcal{E}_p(f)$.*

*Proof.* If $p = 1$ the validity of $\mathcal{P}(1, \alpha_1)$ is established in [5, Theorem 4.9], so we assume that $p > 1$. The proof is relatively similar to the proof of Theorem 6.3; the idea is to replace piecewise affine functions by $p$-harmonic extensions. Let $f \in KS^{\alpha_p, p}(K)$. For any fixed $n \geq 1$, we first define a function $\hat{f}_n$ on $V_n$ by

$$\hat{f}_n(v) := \frac{1}{\mu(K_{n+1}^*(v))} \int_{K_{n+1}^*(v)} f \, d\mu, \quad v \in V_n,$$

where $K_{n+1}^*(v)$ is the union of the $(n+1)$-simplices containing $v$. Then, we let $\Phi_n = H_p(\hat{f}_n)$. We observe that the covering $\{K_n^*(v)\}_{v \in V_n}$ has the bounded overlap property. Also, for any $x \in K_n^*(v)$, $K_{n+1}^*(v) \subset B(x, 2^{-n+1})$. From Theorem 6.5, we have for every $r > 0$

$$\frac{1}{r^{p\alpha_p}} \int_K \fint_{B(z,r)} |\Phi_n(z) - \Phi_n(w)|^p \, d\mu(w) \, d\mu(z) \le C\mathcal{E}_p(\Phi_n) \le C\mathcal{A}_p(\Phi_n) \le C\mathcal{E}_p^n(\Phi_n)$$

and $\mathcal{E}_p^n(\Phi_n)$ can be controlled as follows. Observe that for any $x \in V_n$, one has $\Phi_n(x) = \hat{f}_n(x)$ by definition. Hence

$$\mathcal{E}_p^n(\Phi_n) = \frac{1}{2} r_p^{-n} \sum_{x,y \in V_n, x \sim y} |\hat{f}_n(x) - \hat{f}_n(y)|^p.$$

For any neighboring vertices $x, y \in V_n$, Hölder's inequality yields

$$\begin{aligned} |\hat{f}_n(x) - \hat{f}_n(y)| &\le \frac{1}{\mu(K_{n+1}^*(x))\mu(K_{n+1}^*(y))} \int_{K_{n+1}^*(x)} \int_{K_{n+1}^*(y)} |f(z) - f(w)| \, d\mu(w) \, d\mu(z) \\ &\le C \left( 3^{2n} \int_{K_{n+1}^*(x)} \int_{K_{n+1}^*(y)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z) \right)^{\frac{1}{p}}. \end{aligned}$$

Notice that if $x, y \in V_n$ are adjacent then $K_{n+1}^*(y) \subset B(z, 2^{-n+2})$ for any $z \in K_{n+1}^*(x)$. Therefore we get

$$|\hat{f}_n(x) - \hat{f}_n(y)|^p \le C 3^{2n} \int_{K_{n+1}^*(x)} \int_{B(z, 2^{-n+2})} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z).$$

By the bounded overlap property of $\{K_{n+1}^*(v)\}_{v \in V_n}$, we then have

$$\begin{aligned} \mathcal{E}_p^n(\Phi_n) &\le C r_p^{-n} 3^{2n} \sum_{x,y \in V_n, x \sim y} \int_{K_{n+1}^*(x)} \int_{K_{n+1}^*(y)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z) \\ &\le C r_p^{-n} 3^{2n} \int_K \int_{B(z, 2^{-n+2})} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z). \end{aligned}$$

Set $\varepsilon_n = 2^{-n+2}$. We can rewrite the above inequality as

$$\mathcal{E}_p^n(\Phi_n) \le \frac{C}{\varepsilon_n^{p\alpha_p}} \int_K \fint_{B(z,\varepsilon_n)} |f(z) - f(w)|^p d\mu(w) d\mu(z). \tag{25}$$

Consequently, we have for every $r > 0$

$$\begin{aligned} &\frac{1}{r^{p\alpha_p}} \int_K \fint_{B(z,r)} |\Phi_n(z) - \Phi_n(w)|^p \, d\mu(w) \, d\mu(z) \\ &\le \frac{C}{\varepsilon_n^{p\alpha_p}} \int_K \fint_{B(z,\varepsilon_n)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z). \end{aligned}$$

To conclude, it remains therefore to prove that $\Phi_n$ converges to $f$ in $L^p(K, \mu)$. Since $f \in KS^{\alpha_p, p}(K)$, from Theorem 3.2, for every $x, y \in K$

$$|f(x) - f(y)|^p \le C d(x,y)^{-\frac{\log r_p}{\log 2}}.$$

Let now $n \ge 1$ and $w \in W_n$. One has for $x \in K_w$

$$|f(x) - \Phi_n(x)| \le |f(x) - \hat{f}_n(v)| + |\Phi_n(v) - \Phi_n(x)|$$

where $v$ is a vertex of $K_w$. We have first

$$|f(x) - \hat{f}_n(v)| = \left| f(x) - \frac{1}{\mu(K^*_{n+1}(v))} \int_{K^*_{n+1}(v)} f \, d\mu \right| \leq C3^n \int_{K^*_{n+1}(v)} |f(x) - f(y)| \, d\mu(y) \leq C r_p^n.$$

Then, from Lemma 6.8

$$|\Phi_n(v) - \Phi_n(x)| \leq C r_p^n \mathcal{A}_p^n(\Phi_n) \leq C r_p^n \mathcal{E}_p^n(\Phi_n).$$

From (25) we know that

$$\mathcal{E}_p^n(\Phi_n) \leq C \sup_{\varepsilon > 0} \frac{1}{\varepsilon^{p\alpha_p}} \int_K \fint_{B(z,\varepsilon)} |f(z) - f(w)|^p \, d\mu(w) \, d\mu(z) < +\infty.$$

Therefore, for all $x \in K$

$$|f(x) - \Phi_n(x)| \leq C r_p^n,$$

which implies that $\Phi_n \to f$ in $L^p(K, \mu)$. The proof is by now complete. □

As a consequence of the property $\mathcal{P}(p, \alpha_p)$ and Theorem 4.5, we get the following Nash inequalities on the Sierpiński gasket.

**Corollary 6.10.** *For $p > 1$ the following Nash inequality holds for every $f \in KS^{\alpha_p, p}(K)$,*

$$\|f\|_{L^p(K,\mu)} \leq C \left( \|f\|_{L^p(K,\mu)} + \mathrm{Var}_p(f) \right)^\theta \|f\|_{L^1(K,\mu)}^{1-\theta}$$

*with $\theta = \frac{(p-1)d_h}{p(\alpha_p + d_h) - d_h}$, while for $p = 1$ there exists a constant $C > 0$ such that for every $f \in KS^{d_h, 1}(K)$*

$$\|f\|_{L^\infty(K,\mu)} \leq C \left( \|f\|_{L^1(K,\mu)} + \mathrm{Var}_1(f) \right).$$

## 7. Korevaar–Schoen–Sobolev spaces and heat kernels

In this section, for completeness, we now briefly survey some of the results in [1] and [4, 5, 3] where a connection was deepened between the theory of Dirichlet forms and the theory of Korevaar–Schoen–Sobolev spaces following earlier works like [58]. This connection allows to study properties of the Korevaar–Schoen–Sobolev spaces in some settings where Poincaré inequalities might not be available but a rich theory of heat kernels is. We also mention several open questions related to this approach which are connected to the results of the present paper.

**7.1. Dirichlet forms with Gaussian or sub-Gaussian heat kernel estimates.** As before, $(X, d, \mu)$ is a metric measure space where $\mu$ is a positive and doubling Borel regular measure. We use the basic definitions and properties of Dirichlet forms and associated heat semigroups listed in [3, Section 2]. For a complete exposition of the theory we refer to [19].

Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a Dirichlet form on $L^2(X, \mu)$. We call $(X, d, \mu, \mathcal{E}, \mathcal{F})$ a metric measure Dirichlet space. We assume that the semigroup $\{P_t\}$ associated with $\mathcal{E}$ is stochastically complete (i.e. $P_t 1 = 1$) and has a measurable heat kernel $p_t(x, y)$

satisfying, for some $c_1, c_2, c_3, c_4 \in (0, \infty)$ and $d_h \geq 1, d_w \in [2, +\infty)$,

$$(26)\quad \begin{aligned} c_1 t^{-d_h/d_w} \exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) &\leq p_t(x,y) \\ &\leq c_3 t^{-d_h/d_w} \exp\left(-c_4\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \end{aligned}$$

for $\mu$-a.e. $(x, y) \in X \times X$ for each $t \in \big(0, \operatorname{diam}(X)^{d_w}\big)$. As before, $\operatorname{diam}(X)$ is the diameter of $X$ which could possibly be $+\infty$.

The exact values of $c_1, c_2, c_3, c_4$ are irrelevant. However, the parameters $d_h$ and $d_w$ are important. We will see below that the parameter $d_h$ is the Ahlfors dimension (volume exponent). The parameter $d_w$ is called the walk dimension (for its probabilistic interpretation). When $d_w = 2$, one speaks of Gaussian estimates and when $d_w > 2$, one speaks of sub-Gaussian estimates.

In some concrete situations like manifolds or fractals, the estimates (26) might be obtained using geometric, analytic or probabilistic methods. A large amount of literature is devoted to the study of such estimates, see for instance [24, 26, 27, 42]. Therefore, at least for our purpose here, they are a reasonable assumption to work with. In Barlow [9], geodesic complete metric spaces supporting a heat kernel satisfying the estimates (26) are called fractional spaces.

A basic consequence of (26) is the $d_h$-Ahlfors regularity of the space (see [25, Theorem 3.2]): There exist constants $c, C > 0$ such that for every $x \in X$, $r \in (0, \operatorname{diam}(X))$,

$$c r^{d_h} \leq \mu(B(x, r)) \leq C r^{d_h}.$$

The first important result connecting the theory of Dirichlet forms to the theory of Korevaar–Schoen–Sobolev spaces is the following result, which was already mentioned and reproved in the case where $X$ is compact, see Proposition 5.15.

**Theorem 7.1.** *We have $\mathcal{F} = KS^{d_w/2,2}(X)$. Moreover, there exist constants $c, C > 0$ such that for every $f \in KS^{d_w/2,2}(X)$,*

$$c \sup_{r>0} E_{2,d_w/2}(f, r) \leq \mathcal{E}(f, f) \leq C \liminf_{r\to 0} E_{2,d_w/2}(f, r).$$

*In particular, the property $\mathcal{P}(2, d_w/2)$ holds.*

Note that in [48, Theorem 4.1] the domain of the Dirichlet form associated with some fractals satisfying more general heat kernel estimates is identified as a Besov–Lipschitz space with variable regularity. In that framework it would be interesting to investigate a theory of Korevaar–Schoen–Sobolev spaces with variable regularity.

**7.2. Besov–Lipschitz spaces and heat kernels.** Let $p \geq 1$ and $\beta \geq 0$. Similarly to [3], we define the Besov seminorm associated with the heat semigroup as follows

$$(27)\quad \|f\|_{p,\beta} := \sup_{t\in(0,\operatorname{diam}(X)^{d_w})} t^{-\beta} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x,y)\, d\mu(x)\, d\mu(y)\right)^{1/p}$$

and define the heat semigroup-based Besov class by

$$\mathbf{B}^{p,\beta}(X) := \{f \in L^p(X, \mu) \colon \|f\|_{p,\beta} < +\infty\}.$$

The following result is essentially proved in [58], see also [5, Theorem 2.4] and its proof. It establishes the basic and fundamental connection between the Besov–Lipschitz spaces and the study of heat kernels.

**Theorem 7.2.** *Let $p \geq 1$ and $\alpha > 0$. We have $\mathcal{B}^{\alpha,p}(X) = \mathbf{B}^{p,\frac{\alpha}{d_w}}(X)$ and there exist constants $c, C > 0$ such that for every $f \in \mathcal{B}^{\alpha,p}(X)$ and $r \in (0, \mathrm{diam}(X))$,*

$$c \sup_{s\in(0,\mathrm{diam}(X))} E_{p,\alpha}(f,s)^{1/p} \leq \|f\|_{p,\alpha/d_w} \leq C\left(\sup_{s\in(0,r]} E_{p,\alpha}(f,s)^{1/p} + \frac{1}{r^\alpha}\|f\|_{L^p(X,\mu)}\right).$$

*In particular, if* $\mathrm{diam}(X) = +\infty$, *then* $\|f\|_{p,\alpha/d_w} \simeq \sup_{s>0} E_{p,\alpha}(f,s)^{1/p}$.

**7.3. $L^p$ critical Besov exponents in strongly recurrent metric measure Dirichlet spaces.**

**Definition 7.1.** The metric measure Dirichlet space $(X, d, \mu, \mathcal{E}, \mathcal{F})$ with heat kernel estimates (26) is called strongly recurrent if $(X, d)$ is complete, geodesic and $d_h < d_w$.

Strongly recurrent metric measure Dirichlet spaces and their potential theory were extensively studied by Barlow in [9]. The terminology comes from the fact that the Hunt process associated to the heat kernel is strongly recurrent, i.e. visits a given point with probability one. Nested fractals like the Vicsek set or the Sierpiński gasket are examples of strongly recurrent metric measure Dirichlet spaces. The Sierpiński carpet is also a strongly recurrent metric measure Dirichlet space. A key property of the heat semigroup on strongly recurrent metric measure Dirichlet spaces is the following Borel to Hölder uniform regularization property for the heat semigroup that was proved in [5]: For every $g \in L^\infty(X,\mu)$ a continuous version of $P_t g$ exists and there exists a constant $C > 0$ (independent from $g$) such that for every $x, y \in X$ and $t > 0$,

$$|P_t g(x) - P_t g(y)| \leq C\left(\frac{d(x,y)}{t^{1/d_w}}\right)^{d_w - d_h} \|g\|_{L^\infty(X,\mu)}. \tag{28}$$

Such an estimate is called the weak Bakry–Émery estimate in [5]. The functional inequality (28) plays the same role in the fractional space setting as the estimate

$$\|\nabla P_t g\|_\infty \leq \frac{C}{\sqrt{t}}\|g\|_\infty \tag{29}$$

does in the Riemannian setting. The importance of (29) in the study of BV functions and isoperimetric estimates has been recognized in several works of Ledoux [49], [50]. Note that for Riemannian manifolds with non-negative Ricci curvature the inequality (29) is a simple byproduct of the Bakry–Émery machinery. For fractional spaces the proof of (28) we gave in [5] relies on potential theoretical results proved in [9] by using probabilistic methods. It would be very interesting to have a direct analytic proof of (28) which does not rely on probabilistic methods. Using (28) we can prove several results concerning the Korevaar–Schoen–Sobolev spaces and we mention a few below. First, let us introduce some terminology. Let $E \subset X$ be a Borel set. We say $x$ is a Lebesgue density point of $E$ and write $x \in E^*$ if

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0.$$

The measure-theoretic boundary is $\partial^* E = E^* \cap (E^c)^*$. Now for $r > 0$ define the measure-theoretic $r$-neighborhood $\partial^*_r E$ by

$$\partial^*_r E := \big(E^* \cap (E^c)_r\big) \cup \big((E^c)^* \cap E_r\big), \tag{30}$$

where $E_r = \{x \in X \colon \mu(B(x,r) \cap E) > 0\}$ and similarly for $(E^c)_r$. Notice that $\partial^* E \subset \bigcap_{r>0} \partial^*_r E \subset \partial E$, where this last is the topological boundary.

**Theorem 7.3.** [5, Theorem 5.1] *Assume that $(X, d, \mu, \mathcal{E}, \mathcal{F})$ is a strongly recurrent metric measure Dirichlet space and that there exists a non-empty open set $E \subset X$ such that $\mu(E) < +\infty$, $\mu(E^c) > 0$, and*

$$\limsup_{r\to 0} \frac{1}{r^{d_h}} \mu\left(\partial^*_r E\right) < +\infty$$

*where $\partial^*_r E$ is the measure-theoretic $r$-neighborhood of $E$. Then $1_E \in \mathcal{B}^{d_h,1}(X)$ and the property $\mathcal{P}(1, d_h)$ holds.*

In the setting of the previous theorem the Korevaar–Schoen–Sobolev space $KS^{d_h,1}(X)$ was interpreted in [5] as a space of bounded variation (BV) functions and the property $\mathcal{P}(1, d_h)$ allowed to develop a rich theory which for instance applies to any nested fractal. On the other hand, this result does not apply to infinitely ramified fractals like the Sierpiński carpet, see Figure 4.

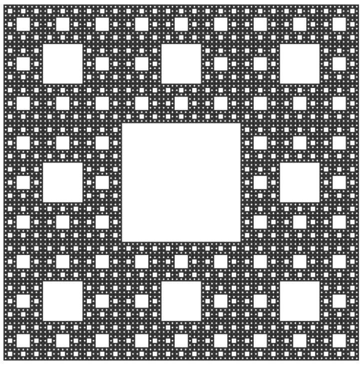

Figure 4. Sierpiński carpet.

It was conjectured in [5] that for the Sierpiński carpet one has instead

$$\alpha_1 = 2\frac{\log 2}{\log 3} = d_h - d_{th} + 1$$

where $d_{th}$ is the topological Hausdorff dimension defined in [8]. This conjecture is still open today as far as we know. The next result provides some estimates on the $L^p$ critical Besov exponents of strongly recurrent metric measure Dirichlet spaces.

**Theorem 7.4.** [5, Theorem 3.11] *Assume that $(X, d, \mu, \mathcal{E}, \mathcal{F})$ is a strongly recurrent metric measure Dirichlet space, then the $L^p$ critical Besov exponents of $(X, d, \mu)$ satisfy:*

- *For $1 \le p \le 2$ we have $\frac{d_w}{2} \le \alpha_p \le \left(1 - \frac{2}{p}\right)(d_w - d_h) + \frac{d_w}{p}$.*
- *For $p \ge 2$ we have $\left(1 - \frac{2}{p}\right)(d_w - d_h) + \frac{d_w}{p} \le \alpha_p \le \frac{d_w}{2}$.*

We note that for the Vicsek set one has for every $p \ge 1$, $\alpha_p = \left(1 - \frac{2}{p}\right)(d_w - d_h) + \frac{d_w}{p}$ because in that case $d_w - d_h = 1$. On the other hand for the real line one has for every $p \ge 1$, $\alpha_p = \frac{d_w}{2} = 1$. Therefore, in a sense, the above estimates are optimal over the range of all possible strongly recurrent metric measure Dirichlet spaces. However, for spaces satisfying $d_w - d_h < 1$, like the Sierpiński gasket, the lower bound $\alpha_p \ge \left(1 - \frac{2}{p}\right)(d_w - d_h) + \frac{d_w}{p}$, $p \ge 2$, is interesting only when $2 \le p \le \frac{2d_h - d_w}{1 - d_w + d_h}$ because we know that we always have $\alpha_p \ge 1$. When $p \ge 2$, the upper bound

$\alpha_p \leq \frac{d_w}{2}$ is not very good for large values of $p$ because we know from Theorem 4.2 that $\alpha_p \leq 1 + \frac{d_h}{p}$. In view of this discussion and of the known results in the Vicsek set and the Sierpiński gasket, we can ask the following question: Is it true that for a nested fractal we have for every $p \geq 1$, $\alpha_p \leq 1 + \frac{d_h - 1}{p}$?

The quantity

$$\delta = \inf\{p \geq 1 \colon p\alpha_p > d_h\}$$

seems to be of significance. In the theory of Korevaar–Schoen–Sobolev spaces, it is the infimum of the exponent $p$ for which the space $KS^{\alpha_p,p}(X)$ can be embedded into the space of continuous functions by using Theorem 3.2; For instance, for the Vicsek set or the Sierpiński gasket, the previous results show that $\delta = 1$. Actually, from Lemma 4.10 in [20], for every nested fractals one has $p\alpha_p > d_h$ for every $p > 1$. Therefore one also has $\delta = 1$ for every nested fractals. A similar exponent appears in Kigami's work [44] where it is conjectured to be equal to the Ahlfors regular conformal dimension of the space. Kigami's conjecture was recently proved in [12]. It is natural to make the same conjecture in our setting.

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