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Strong barriers for weighted quasilinear equations

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Abstract. In potential theory, use of barriers is one of the most important techniques. We construct strong barriers for weighted quasilinear elliptic operators. There are two applications: (i) solvability of Poisson-type equations with boundary singular data, and (ii) a geometric version of Hardy inequality. Our construction method can be applied to a general class of divergence form elliptic operators on domains with rough boundary.

Painollisten kvasilineaaristen yhtälöiden vahvat esteratkaisut

Tiivistelmä. Esteiden käyttö on potentiaaliteorian tärkeimpiä menetelmiä. Tässä työssä rakennetaan vahvoja esteratkaisuita painollisille kvasilineaarisille elliptisille operaattoreille. Tällä on kaksi sovellusta: (i) Poissonin-tyyppisten yhtälöiden ratkeavuus singulaarisilla reuna-arvoilla ja (ii) Hardyn epäyhtälön geometrinen muotoilu. Työssä esitetty menetelmä soveltuu yleiseen luokkaan karkeareunaisten alueiden lähdemuotoisia elliptisiä operaattoreita.

1. Introduction

Let $\Omega \subsetneq \mathbb{R}^n$ $(n \geq 1)$ be an open set with nonempty boundary, and let $1 < p < \infty$. We consider elliptic differential equations of the type

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u(x)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\operatorname{div} \mathcal{A}(x, \nabla \cdot)$ is a weighted (p, w)-Laplacian type elliptic operator, w is a doubling weight on $\mathbb{R}^n$ which admit a p-Poincaré inequality (see (2.1) and (2.2) for detail), and f is a locally integrable function on Ω such that f/w is locally bounded. The most simple example of w is $w \equiv 1$, and the reason for considering weighted equations will be explained later. The precise assumptions on $\mathcal{A}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are as follows: For each $z \in \mathbb{R}^n$, $\mathcal{A}(\cdot, z)$ is measurable, for each $x \in \Omega$, $\mathcal{A}(x, \cdot)$ is continuous, and there exists $1 \leq L < \infty$ such that

$$\mathcal{A}(x, z) \cdot z \geq w(x)|z|^p, \tag{1.2}$$

$$|\mathcal{A}(x, z)| \leq L w(x)|z|^{p-1}, \tag{1.3}$$

$$(\mathcal{A}(x, z_1) - \mathcal{A}(x, z_2)) \cdot (z_1 - z_2) > 0, \tag{1.4}$$

$$\mathcal{A}(x, tz) = t|t|^{p-2}\mathcal{A}(x, z) \tag{1.5}$$

for all $x \in \Omega$, $z, z_1, z_2 \in \mathbb{R}^n$, $z_1 \neq z_2$ and $t \in \mathbb{R}$. When $\mathcal{A}(x, z) = w(x)|z|^{p-2}z$, the operator $\operatorname{div} \mathcal{A}(x, \nabla u)$ is called as the (p, w)-Laplacian. In particular, if $\mathcal{A}(x, z) = z$, then the operator coincides with the classical Laplacian.

The aim of this paper is to provide an existence result of weak solutions to (1.1) for boundary singular data. The study of equation (1.1) has long history, and the

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standard approach to solve this problem is the variational method and its generalization (see, [Lio69]). However, this method yields only solutions with finite energy. On the other hand, we can confirm existence of infinite energy solutions for boundary singular data by considering Poisson's equation (in the classical sense) or ordinary differential equations. To obtain such solutions, we divide the problem into three steps: (i) find finite energy solutions to approximate problems, (ii) prove compactness of solutions, and (iii) derive a uniform bound for solutions. The part (i) is trivial in our setting, and we can find sufficient results for (ii) in prior work (e.g., [BM92, HKM06, TW02]). This sequel from [Har21, Har22] proposes a new perspective for (iii). More precisely, we construct supersolutions called strong barriers.

If there are known supersolutions to (1.1), they are an effective tool to control boundary behavior of solutions. For concrete problems, direct calculations for the distance from the boundary often yield sharp estimates (see, e.g., [Tol83, MS00, BK06, AKSZ07, FMT07]). Unfortunately, this method can not be applied to general elliptic equations on domains with rough boundary. When conditions of the form (1.2)–(1.5) have to be considered, such as in applications to homogenization problems (e.g., [BLP78]), it is needed to consider a construction method of supersolutions itself.

Ancona [Anc86] defined strong barriers for linear elliptic operators and constructed them under capacity density type conditions. In addition, a Hardy inequality was proved as one application of them. Since other proofs by Lewis [Lew88] and Wannebo [Wan90], many authors have proved more general Hardy-type inequalities under capacity density conditions (see, §2). However, another application of strong barriers, to the Dirichlet problem of the type (1.1), seems not to be discussed sufficiently.

We construct strong barriers for quasilinear operators (Theorem 5.1) and apply the result to (1.1) (Theorem 6.3). Specifically, we make auxiliary functions by a boundary Hölder estimate in the De Giorgi–Nash–Moser theory and construct a global function by gluing them. These results can be regarded as extensions of [Anc86, Theorem 1 and Remark 6.2] or [Har22, Corollary 4.4] as well as analogs of known facts for the p-Laplacian on C^2 domains (see, [GT01, Problem 6.6], [MS00, Theorem 1]). In addition, we prove a Hardy-type inequality (Corollary 5.3) by combining the results and the Picone inequality (see [AH98]). For connection with prior work on Hardy-type inequalities, we consider weighted operators borrowing a framework in [HKM06] (see also [BB11, KLV21] for recent progress). Known results for Hardy-type inequalities that seem to be particularly relevant to this study will be discussed in §2. Throughout the paper, we assume only (1.2)–(1.5), (2.1)–(2.2) and (p, w)-capacity density conditions. The quantitative statements in results are new even for unweighted linear equations (compare with [Anc86, Remark 5.2]). Our method seems to work for Cheeger differential equations on metric measure spaces (see, [Che99, BMS01], [BB11, Chapter B.2], [Har18]). On the other hand, there are large gaps in its direct application to minimizers of variational problems.

Organization of the paper. In §2, we confirm our problem and pick up related results on weighted Sobolev spaces. In §3, we define weak solutions to (1.1) and prove a Kato-type inequality. In §4, we state regularity estimates that will be used in §5. Proofs of two lemmas will be provided in §A. In §5, we construct strong barriers for (1.1). In §6, we apply the result in §5 to (1.1) and achieve our goals.

Notation. Let $\Omega \subsetneq \mathbb{R}^n$ be an open set.

- $\mathbf{1}_E(x) :=$ the indicator function of a set E.

- $C_c^\infty(\Omega)$:= the set of all infinitely-differentiable functions with compact support in Ω.
- $L^p(\Omega;\mu)$:= the L^p space with respect to a measure μ on Ω.
- $f_+ := \max\{f,0\}$ and $f_- := -\min\{f,0\}$.

For a closed set $\Gamma \subset \mathbb{R}^n$, we denote by δ_Γ the distance from Γ. For a ball $B = B(x,R) = \{y\colon \operatorname{dist}(x,y) < R\}$ and $\lambda > 0$, we denote $B(x,\lambda R)$ by λB. The letters c and C denote various constants with and without indices.

2. Setting and related work

2.1. Admissible weights. Let $1 < p < \infty$ be a fixed number. A function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n; dx)$ is called the *weight* if $w(x) > 0$ a.e. in $\mathbb{R}^n$. We write $w(E) = \int_E w\,dx$ for a Lebesgue measurable set $E \subset \mathbb{R}^n$. Throughout the below, we assume that w satisfying the doubling condition

$$w(2B) \le C_D w(B) \tag{2.1}$$

and the p-Poincaré inequality

$$⨍_B |u - u_B|\,dw \le C_P \operatorname{diam}(B) \left(⨍_{\lambda B} |\nabla u|^p\,dw \right)^{1/p}, \quad \forall u \in C_c^\infty(\mathbb{R}^n), \tag{2.2}$$

where B is an arbitrary ball in $\mathbb{R}^n$, $⨍_B := w(B)^{-1}\int_B$, $u_B := ⨍_B u\,dw$ and C_D, C_P and $\lambda \ge 1$ are constants. A weight w satisfying (2.1) and (2.2) is said to be *p-admissible*.

It is well-known that (2.1) and (2.2) yield the following Sobolev type inequality:

$$\left(⨍_B |u|^{\chi p}\,dw \right)^{1/\chi p} \le C \operatorname{diam}(B) \left(⨍_B |\nabla u|^p\,dw \right)^{1/p}, \quad \forall u \in C_c^\infty(B). \tag{2.3}$$

where C and $\chi > 1$ are constants depending only on p, C_D, C_P and λ. For detail, we refer to [HKM06, Chapter 20] and the references cited therein.

Muckenhoupt A_p-weights are one typical example of p-admissible weights ([HKM06, Chapter 15]). The power function $|x|^\mu$ is an A_p-weight on $\mathbb{R}^n$ if and only if $-n < \mu < n(p-1)$. It seems to be known conventionally that if Ω is a bounded Lipschitz domain, then $w(x) = \delta(x)^\mu$ is an A_p-weight on $\mathbb{R}^n$ for $-1 < \mu < p-1$. Finer results can be found in [Hor89, Hor91, DLG10, DIL+19] and [KLV21, Chapter 10]. Roughly speaking, if Γ is an s-dimensional set with $0 < s < n$, then $w(x) = \delta_\Gamma(x)^\mu$ is an A_p-weight on $\mathbb{R}^n$ for $-(n-s) < \mu < (n-s)(p-1)$.

2.2. Sobolev spaces and capacities. The weighted Sobolev space $H^{1,p}(\Omega; w)$ is the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H^{1,p}(\Omega;w)} := \left(\int_\Omega |u|^p + |\nabla u|^p\,dw \right)^{1/p},$$

where ∇u is the gradient of u in the sense of $\mathbb{R}^n$. The corresponding local space $H^{1,p}_{\mathrm{loc}}(\Omega; w)$ is defined in the usual manner. We denote by $H^{1,p}_0(\Omega; w)$ the closure of $C_c^\infty(\Omega)$ in $H^{1,p}(\Omega; w)$. It is well-known that if $u, v \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$, then $\min\{u,v\} \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$.

Let $O \subset \mathbb{R}^n$ be open, and let $K \subset O$ be compact. The *(p,w)-capacity* $\operatorname{cap}_{p,w}(K,O)$ of the *condenser* (K,O) is defined by

$$\operatorname{cap}_{p,w}(K,O) := \inf\left\{ \|\nabla u\|^p_{L^p(O;w)} \colon u \ge 1 \text{ on } K,\ u \in C_c^\infty(O) \right\}. \tag{2.4}$$

For a boundary point $\xi \in \partial\Omega$ (more generally, for $\xi \in \mathbb{R}^n \setminus \Omega$), we consider the following (p, w)-*capacity density condition*: There exists $\gamma > 0$ such that

$$\frac{\operatorname{cap}_{p,w}(\overline{B(\xi, R)} \setminus \Omega, B(\xi, 2R))}{\operatorname{cap}_{p,w}(\overline{B(\xi, R)}, B(\xi, 2R))} \geq \gamma. \tag{2.5}$$

If (2.5) holds for all small $R > 0$, then ξ is a regular point of the corresponding Dirichlet problem (see Lemma 4.3 below, [HKM06, Theorem 6.31] and the references cited therein). Sufficient conditions for (2.5) via exterior corkscrew-type conditions can be found in [HKM06, Theorem 6.31] and [BB11, Corollary 11.25]. In particular, every boundary point of a ball satisfies (2.5) for all $R > 0$.

When (2.5) holds for all $\xi \in \mathbb{R}^n \setminus \Omega$ and $R > 0$, the set $\mathbb{R}^n \setminus \Omega$ is called as *uniformly* (p, w)-*fat*, and this condition is closely related to Hardy-type inequalities. The study of Hardy-type inequalities on uniformly fat sets started with Ancona's work [Anc81]. Three proofs have been proposed that differ in their use of the uniform assumption. (i) Ancona [Anc86] constructed strong barriers using decay of harmonic measures. (ii) Lewis [Lew88] developed theory of self-improvement property of uniformly $(p, 1)$-fat sets using Havin–Maz'ya potentials. (iii) Wannebo [Wan90] proved higher order Hardy inequalities using an estimate for the overlap of Whitney-type cubes. Later, Mikkonen [Mik96] proved self-improvement property of uniformly (p, w)-fat sets using a boundary Hölder estimates of (p, w)-harmonic functions. See also [BMS01] for the extension to the metric measure space setting. We also refer to [Haj99, KM97, Leh08, KLT11] for further properties of uniformly fat sets. Wannebo's method was revisited in [EHS05, BK04], and Korte, Lehrbäck and Tuominen [KLT11] gave a new proof of Mikkonen's result. Further related work can be found in [KLV21, Chapter 6] and [BEL15, Chapter 3]. However, the proof in [Anc86] is not very similar to any of the others.

The study of cases where (2.5) holds on a part of the boundary is more limited. The most similar known result to Corollary 5.3 below is [LTV17, Proposition 5.4] (see also [KLV21, Theorem 7.31]). They have used Wannebo's idea. In [EHDR15], another form of Hardy inequality and its applications are discussed under assumptions on Sobolev extensibility of domains.

3. Quasilinear elliptic operator

Assume that $\mathcal{A}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.2)–(1.5). For simplicity, we denote the following extended function $\overline{\mathcal{A}}$ by $\mathcal{A}$ again:

$$\overline{\mathcal{A}}(x, z) = \begin{cases} \mathcal{A}(x, z) & x \in \Omega, \\ w(x)|z|^{p-2}z & \text{otherwise}, \end{cases} \qquad z \in \mathbb{R}^n.$$

Let $f \in L^1_{\mathrm{loc}}(\Omega)$. A function $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$ is called weak (super-, sub-)solution to $-\operatorname{div} \mathcal{A}(x, \nabla u) = f$ in Ω if

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx = (\geq, \leq) \int_\Omega \varphi f \, dx \tag{3.1}$$

for any nonnegative $\varphi \in C_c^\infty(\Omega)$.

If u is a weak supersolution to $-\operatorname{div} \mathcal{A}(x, \nabla u) = 0$ in Ω, then, by the Riesz representation theorem, there is a unique Radon measure $\nu[u]$ in Ω such that

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx = \int_\Omega \varphi \, d\nu[u]$$

for any $\varphi \in C_c^\infty(\Omega)$. The measure $\nu[u]$ is called the *Riesz measure* of u. We refer to [HKM06, Chapter 21] for further detail.

By (1.2), if u is a solution to $-\operatorname{div}\mathcal{A}(x,\nabla u) = f$ in Ω, then its truncation $\min\{u,k\}$ is a supersolution to $-\operatorname{div}\mathcal{A}(x,\nabla u) = \min\{f,0\}$ in Ω. The monotonicity condition (1.4) yields the following more general result.

Lemma 3.1. *Assume that u and v are weak supersolutions to $-\operatorname{div}\mathcal{A}(x,\nabla u) = 0$ in Ω. Then,*

$$\int_\Omega \varphi\, d\nu[\min\{u,v\}] \geq \int_\Omega \varphi \mathbf{1}_{\{u\leq v\}}\, d\nu[u] + \int_\Omega \varphi \mathbf{1}_{\{u>v\}}\, d\nu[v] \tag{3.2}$$

for any nonnegative $\varphi \in C_c^\infty(\Omega)$. Moreover, if $u_1, \cdots, u_k$ are weak supersolutions to $-\operatorname{div}\mathcal{A}(x,\nabla u) = 0$ in Ω, and if there is a Radon measure σ such that $\nu[u_k] \ll \sigma$ for all k, then

$$\nu[\min\{u_1,\cdots,u_k\}] \geq \min\{f_1,\cdots,f_k\}\sigma \quad \text{in } \Omega \tag{3.3}$$

in the sense of distributions, where f_k is the Radon–Nikodým derivative of $\nu[u_k]$ with respect to σ.

Proof. Note that $u, v \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$. For $\epsilon > 0$, consider the functions

$$\Phi_\epsilon(u-v) := \frac{\epsilon}{(u-v)_+ + \epsilon} \quad \text{and} \quad \Psi_\epsilon(u-v) := 1 - \Phi_\epsilon(u-v).$$

These functions are globally Lipschitz continuous with respect to $u - v$; therefore, $\Phi_\epsilon(u-v), \Psi_\epsilon(u-v) \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$. We also note that $\Phi_\epsilon(u-v)(x) \to \mathbf{1}_{\{u\leq v\}}(x)$ for all x. Fix a nonnegative function $\varphi \in C_c^\infty(\Omega)$. Then,

$$\int_\Omega \varphi\, d\nu[\min\{u,v\}] = \int_\Omega \varphi\Phi_\epsilon(u-v)\, d\nu[\min\{u,v\}] + \int_\Omega \varphi\Psi_\epsilon(u-v)\, d\nu[\min\{u,v\}].$$

By the definition of $\nu[\min\{u,v\}]$, we have

$$\begin{aligned}\int_\Omega \varphi\Phi_\epsilon(u-v)\, d\nu[\min\{u,v\}] &= \int_{\{u\leq v\}} \mathcal{A}(x,\nabla u)\cdot\nabla(\varphi\Phi_\epsilon(u-v))\, dx \\ &\quad + \int_{\{u>v\}} \mathcal{A}(x,\nabla v)\cdot\nabla(\varphi\Phi_\epsilon(u-v))\, dx.\end{aligned}$$

Note that $(\mathcal{A}(x,\nabla v) - \mathcal{A}(x,\nabla u))\cdot\nabla\Phi_\epsilon(u-v) \geq 0$ by (1.4). Therefore,

$$\int_{\{u>v\}} \mathcal{A}(x,\nabla v)\cdot\nabla(\varphi\Phi_\epsilon(u-v))\, dx + \mathbf{I}_\epsilon \geq \int_{\{u>v\}} \mathcal{A}(x,\nabla u)\cdot\nabla(\varphi\Phi_\epsilon(u-v))\, dx,$$

where

$$\mathbf{I}_\epsilon := \int_{\{u>v\}} (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla v))\cdot\nabla\varphi\, \Phi_\epsilon(u-v)\, dx.$$

Adding the two inequalities, we get

$$\int_\Omega \varphi\Phi_\epsilon(u-v)\, d\nu[\min\{u,v\}] + \mathbf{I}_\epsilon \geq \int_\Omega \varphi\Phi_\epsilon(u-v)\, d\nu[u]. \tag{3.4}$$

Similarly, since

$$\begin{aligned}\int_\Omega \varphi\Psi_\epsilon(u-v)\, d\nu[\min\{u,v\}] &= \int_{\{u\leq v\}} \mathcal{A}(x,\nabla u)\cdot\nabla(\varphi\Psi_\epsilon(u-v))\, dx \\ &\quad + \int_{\{u>v\}} \mathcal{A}(x,\nabla v)\cdot\nabla(\varphi\Psi_\epsilon(u-v))\, dx\end{aligned}$$

and $(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla v))\cdot\nabla\Psi_\epsilon(u-v)\geq 0$, we have

$$\int_\Omega \varphi\Psi_\epsilon(u-v)\,d\nu[\min\{u,v\}]+\mathbf{II}_\epsilon\geq\int_\Omega\varphi\Psi_\epsilon(u-v)\,d\nu[v], \tag{3.5}$$

where

$$\mathbf{II}_\epsilon:=\int_{\{u\leq v\}}(\mathcal{A}(x,\nabla v)-\mathcal{A}(x,\nabla u))\cdot\nabla\varphi\,\Psi_\epsilon(u-v)\,dx.$$

Combining (3.4) and (3.5), we obtain

$$\int_\Omega\varphi\,d\nu[\min\{u,v\}]+\mathbf{I}_\epsilon+\mathbf{II}_\epsilon\geq\int_\Omega\varphi\Phi_\epsilon(u-v)\,d\nu[u]+\int_\Omega\varphi\Psi_\epsilon(u-v)\,d\nu[v].$$

Take the limit $\epsilon\to 0$. By the dominated convergence theorem, the right-hand side of this inequality goes to the right-hand of (3.2). By the same reason, $\mathbf{I}_\epsilon,\mathbf{II}_\epsilon\to 0$. Therefore, (3.2) holds. The latter statement is a consequence of induction. $\square$

4. Regularity estimates

By standard techniques in the De Giorgi–Nash–Moser theory, the Sobolev inequality (2.3) yields the following global L^∞ estimate and weak Harnack inequality. Abbreviated proofs of them will be provided in §A.

Lemma 4.1. *Let $u\in H^{1,p}(\Omega;w)$ be a weak subsolution to $-\operatorname{div}\mathcal{A}(x,\nabla u)=f$ in Ω. Let $F_+=\left(\operatorname{diam}(\Omega)^p\|f_+/w\|_{L^\infty(\Omega)}\right)^{1/(p-1)}$. Then,*

$$\operatorname*{ess\,sup}_\Omega u\leq\sup_{\partial\Omega}u+CF_+,$$

where $\sup_{\partial\Omega}u:=\inf\{k\in\mathbb{R}^n\colon(u-k)_+\in H_0^{1,p}(\Omega;w)\}$ and C is a constant depending only on p, C_D and $\{C_P,\lambda\}$. In particular, there is a constant $c_1=c_1(p,C_D,\{C_P,\lambda\})$ such that if $\|f_+/w\|_{L^\infty(\Omega)}\leq c_1\operatorname{diam}(\Omega)^{-p}$, then $\operatorname{ess\,sup}_\Omega u\leq\sup_{\partial\Omega}u+1/4$.

Lemma 4.2. *Let $u\in H^{1,p}(2B;w)$ be a nonnegative weak supersolution to $-\operatorname{div}\mathcal{A}(x,\nabla u)=f$ in $2B$, and let $F_-=\left(R^p\|f_-/w\|_{L^\infty(2B)}\right)^{1/(p-1)}$. Then,* (i) *for each $0<s<\chi(p-1)$, there exists a constant C depending only on p, C_D, $\{C_P,\lambda\}$, L and s such that*

$$\left(\fint_B u^s\,dw\right)^{1/s}\leq C\left(\operatorname*{ess\,inf}_B u+F_-\right). \tag{4.1}$$

(ii) *there exists a constant C depending only on p, C_D, $\{C_P,\lambda\}$ and L such that*

$$R^{p-1}\fint_B|\nabla u|^{p-1}\,dw\leq C\left(\operatorname*{ess\,inf}_B u+F_-\right)^{p-1}. \tag{4.2}$$

It is well-known that Lemma 4.2 yields a local Hölder estimate (see, e.g., [GT01, Theorem 8.24]). Below, we always assume that f/w is locally bounded. Under this assumption, any weak solution u to $-\operatorname{div}\mathcal{A}(x,\nabla u)=f$ can be regard as a locally Hölder continuous function.

We use the exterior condition (2.5) via the following boundary Hölder estimate. The proof below is the same as [GT01, pp. 206–209] (see also [GZ77] and [Str84]).

Lemma 4.3. *Let B be a ball centered at $\xi\in\partial\Omega$ with radius R. Assume that (2.5) holds. Let $u\in H^{1,p}(\Omega;w)\cap L^\infty(\Omega)$ satisfy $-\operatorname{div}\mathcal{A}(x,\nabla u)=f$ in $\Omega\cap 4B$. Let*

$F_\pm = \big(R^p \|f_\pm/w\|_{L^\infty(2B)}\big)^{1/(p-1)}$. *Then,*

$$\operatorname*{osc}_{\Omega\cap B} u \le \left(1 - \frac{\gamma^{1/(p-1)}}{C}\right) \operatorname*{osc}_{\Omega\cap 4B} u + \frac{\gamma^{1/(p-1)}}{C} \operatorname*{osc}_{\partial\Omega\cap B} u + F_+ + F_-,$$

where C is a constant depending only on p, C_D, $\{C_P, \lambda\}$ and L and $\operatorname{osc} u := \sup u - \inf u$.

Proof. We first assume that u is a nonnegative supersolution to $-\operatorname{div}\mathcal{A}(x,\nabla u) = f$ in $\Omega\cap 4B$. Let $m = \inf_{\partial\Omega\cap B} u$, and let

$$u_m^-(x) = \begin{cases} \min\{u(x), m\} & x\in\Omega, \\ m & \text{otherwise.}\end{cases}$$

Note that $0 \le u_m^- \le m$ and $u_m^- = m$ on $\overline{B}\setminus\Omega$. Take $\eta \in C_c^\infty(2B)$ such that $0\le\eta\le 1$ in $2B$, $\eta = 1$ on B and $|\nabla\eta| \le C/R$. Then, (2.5) gives

$$m^p\gamma \le \frac{m^p \mathrm{cap}_{p,w}(\overline{B}\setminus\Omega, 2B)}{\mathrm{cap}_{p,w}(\overline{B}, 2B)} \le CR^p \fint_{2B} |\nabla(u_m^-\eta)|^p\, dw.$$

By the product rule, $\nabla(u_m^-\eta) = \nabla u_m^-\eta + u_m^-\nabla\eta$ a.e. Therefore,

$$\int_{2B} |\nabla(u_m^-\eta)|^p\, dw \le C\left(R^{-p}\int_{2B} (u_m^-)^p d\mu + \int_{2B} |\nabla u_m^-|^p\eta^p\, dw\right). \tag{4.3}$$

By Lemma 4.2 (i), the former term on the right-hand side is estimated by

$$\fint_{2B} (u_m^-)^p\, dw \le m\fint_{2B} (u_m^-)^{p-1}\, dw \le Cm\left(\inf_B u_m^- + F_-\right)^{p-1}. \tag{4.4}$$

Consider the test function $(m-u_m^-)\eta^p$. Since u_m^- is a supersolution to $-\operatorname{div}\mathcal{A}(x,\nabla u) = \min\{f,0\}$ in Ω, we have

$$\begin{aligned}&\int_\Omega \mathcal{A}(x,\nabla u_m^-)\cdot\nabla(m-u_m^-)\eta^p\, dx + p\int_\Omega \mathcal{A}(x,\nabla u_m^-)\cdot\nabla\eta\eta^{p-1}(m-u_m^-)\, dx\\ &\ge \int_\Omega (m-u_m^-)\eta^p\frac{\min\{f,0\}}{w}\, dw.\end{aligned}$$

By (1.2) and (1.3), this inequality yields

$$\begin{aligned}\int_{2B} |\nabla u_m^-|^p\eta^p\, dw &= \int_{2B} |\nabla(m-u_m^-)|^p\eta^p\, dw\\ &\le Cm\left(R^{-1}\int_{2B}|\nabla u_m^-|^{p-1}\, dw + F_-^{p-1}R^{-p}w(2B)\right).\end{aligned}$$

By Lemma 4.2 (ii), the former term on the right-hand side is estimated by

$$R^{-1}\fint_{2B} |\nabla u_m^-|^{p-1}\, dw \le CR^{-p}\left(\inf_B u_m^- + F_-\right)^{p-1}. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we obtain

$$m \le \frac{C}{\gamma^{1/(p-1)}}\left(\inf_B u_m^- + F_-\right). \tag{4.6}$$

Let $M(R) = \sup_{\Omega\cap B(\xi,R)} u$ and $m(R) = \inf_{\Omega\cap B(\xi,R)} u$. Applying (4.6) to $M(4R) - u$ and $u - m(4R)$, we obtain

$$M(4R) - \sup_{\partial\Omega\cap B} u \le \frac{C}{\gamma^{1/(p-1)}}\left(M(4R) - M(R) + F_+\right), \tag{4.7}$$

$$\inf_{\partial\Omega\cap B} u - m(4R) \le \frac{C}{\gamma^{1/(p-1)}} \left(m(R) - m(4R) + F_-\right).$$

Adding the two inequalities, we arrive at the desired estimate. □

Using (4.7) in Lemma 4.3 iteratively, we get the following lemma.

Lemma 4.4. *Let B be a ball centered at $\xi \in \partial\Omega$ with radius R_0. Assume that (2.5) holds for all $0 < R \le R_0$. Then, there are positive constants c_2 and $\theta \in (0,1)$ depending only on p, C_D, $\{C_P, \lambda\}$, L and γ such that if $u \in H_0^{1,p}(\Omega; w)$ is a nonnegative bounded weak subsolution to $-\operatorname{div}\mathcal{A}(x, \nabla u) = f$ in $\Omega \cap B$, and if $\|f_+/w\|_{L^\infty(\Omega)} \le c_2 R_0^{-p} \left(\sup_{\Omega\cap B} u\right)^{p-1}$, then $\sup_{\Omega\cap\theta B} u \le (1/4)\sup_{\Omega\cap B} u$.*

5. Construction of strong barriers

We prove Theorem 5.1 using Lemma 5.2 infinitely many times. Lemma 5.2 itself is a consequence of regularity estimates in the previous section. Also, we prove Corollary 5.3 as a corollary of Theorem 5.1.

Theorem 5.1. *Let $\Gamma \subset \partial\Omega$ be a closet set, and assume that (2.5) holds for all $\xi \in \Gamma$ and $R > 0$. Then, there exists a nonnegative function $s_\Gamma \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\Omega)$ satisfying*

$$-\operatorname{div}\mathcal{A}(x, \nabla s_\Gamma) \ge c_H \frac{s_\Gamma(x)^{p-1}}{\delta_\Gamma(x)^p} w(x) \quad \text{in } \Omega \tag{5.1}$$

and

$$\delta_\Gamma(x)^\alpha \le s_\Gamma(x) \le 30\, \delta_\Gamma(x)^\alpha \tag{5.2}$$

for all $x \in \Omega$, where c_H and $\alpha > 0$ are positive constants depending only on p, C_D, $\{C_P, \lambda\}$, L and γ.

Lemma 5.2. *Let B be a ball centered at $\xi \in \partial\Omega$ with radius R_0. Assume that (2.5) holds for all $0 < R \le R_0$. Let $c_3 := \min\{c_1, c_2\}$, where c_1 and c_2 are constants in Lemmas 4.1 and 4.4. Then, there exists a function $u_B \in H^{1,p}(\Omega; w) \cap C(\Omega)$ satisfying*

$$-\operatorname{div}\mathcal{A}(x, \nabla u_B) = c_3 R_0^{-p} w \quad \text{in } \Omega \cap B, \tag{5.3}$$

$$\frac{1}{4} \le u_B \le \frac{5}{4} \quad \text{in } \Omega, \tag{5.4}$$

$$u_B = 1 \quad \text{on } \Omega \setminus B, \quad u_B \le \frac{1}{2} \quad \text{on } \Omega \cap \theta B, \tag{5.5}$$

where θ is a constant in Lemma 4.4.

Proof. We follow the method in [Lio69, p. 177]. Take $\eta_B \in C^\infty(\overline{\Omega})$ such that

$$\eta_B = \frac{1}{4} \text{ on } \overline{\Omega \cap B/2}, \quad \eta_B = 1 \text{ on } \overline{\Omega} \setminus B, \quad \frac{1}{4} \le \eta_B \le 1 \text{ in } \overline{\Omega}.$$

Consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla(v_B + \eta_B)) = c_3 R_0^{-p} w & \text{in } \Omega \cap B, \\ v_B \in H_0^{1,p}(\Omega \cap B; w). \end{cases}$$

By (2.3), the right-hand side belongs to the dual of $H_0^{1,p}(\Omega \cap B; w)$. Therefore, by the Minty–Browder theorem, there exists a unique solution $u_B \in \eta_B + H_0^{1,p}(\Omega \cap B; w)$ to (5.3). By Lemma 4.2, u_B is continuous in $\Omega \cap B$. Meanwhile, if $x \in \Omega \cap \partial B$, then u_B is continuous at x by Lemma 4.3. Consequently, u_B is continuous in Ω. By the comparison principle, $u_B \ge 1/4$ in Ω. Also, by Lemma 4.1, $u_B \le \sup_{\partial\Omega} u + 1/4 \le 5/4$ in Ω. The latter bound in (5.5) follows from Lemma 4.4. $\square$

Proof of Theorem 5.1. First, we construct a function s by the following two steps. (i) For $k \in \mathbb{Z}$, set $E_k = \{x \in \Omega\colon \delta_\Gamma(x) \le (\theta/2)^k\}$, where θ is a constant in Lemma 5.2. Choosing $\{\xi_j\}_{j\in J_k} \subset \Gamma$, we construct a locally finite covering

$$\{B(\xi_j, 2(\theta/2)^{k+1})\}_{j\in J_k}$$

of E_{k+1}. Note that

$$E_{k+1} \subset D_k := \Omega \cap \bigcup_{j\in J_k} B(\xi_j, (\theta/2)^k) \subset E_k.$$

Using Lemma 5.2, we define a function $v_k \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\Omega)$ by

$$v_k(x) = \min_{\substack{\xi_j \in J_k \\ B(\xi_j,(\theta/2)^k)\ni x}} u_{B(\xi_j,(\theta/2)^k)}(x) \quad \text{for } x \in D_k$$

and $v_k = 1$ on $\Omega \setminus D_k$. By (5.4) and (5.5), we have

$$\frac{1}{4} \le v_k \le \frac{5}{4} \text{ in } \Omega \tag{5.6}$$

and

$$v_k \le \frac{1}{2} \text{ on } E_{k+1}. \tag{5.7}$$

Moreover, by Lemma 3.1, we have

$$-\operatorname{div} \mathcal{A}(x, \nabla v_k) \ge c_3 \left(\frac{2}{\theta}\right)^{kp} w \quad \text{in } D_k. \tag{5.8}$$

(ii) Define a function s on Ω by

$$s(x) = \inf_{E_k \ni x} \left(\frac{3}{4}\right)^k v_k(x).$$

By (5.6) and the inequality $(3/4)^6 \le 1/5 < (3/4)^5$,

$$\left(\frac{3}{4}\right)^{k-6} v_{k-6}(x) \ge \left(\frac{3}{4}\right)^{k-6} \frac{1}{4} \ge \left(\frac{3}{4}\right)^k \frac{5}{4} \ge \left(\frac{3}{4}\right)^k v_k(x)$$

for any $x \in E_k$. Therefore,

$$s(x) = \min\left\{\left(\frac{3}{4}\right)^{k-5} v_{k-5}(x), \cdots, \left(\frac{3}{4}\right)^k v_k(x)\right\} \tag{5.9}$$

for all $x \in E_k \setminus E_{k+1}$. In particular, $s \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\Omega)$.

Next, we claim that

$$-\operatorname{div} \mathcal{A}(x, \nabla s) \ge c_3 \left(\frac{3}{4}\right)^{k(p-1)} \left(\frac{2}{\theta}\right)^{(k-5)p} w \tag{5.10}$$

in an open neighborhood of $E_k \setminus E_{k+1}$. By Lemma 3.1 and (5.8), this inequality holds in $D_k \setminus E_{k+1}$. Meanwhile, by (5.7), we have

$$v_{k-1} \le \frac{1}{2} < \frac{3}{4} = \frac{3}{4} v_k \quad \text{in } E_k \setminus D_k.$$

By continuity of v_k and v_{k-1}, we can take an open set O_k such that $v_{k-1} < (3/4)v_k$ in O_k. Note that

$$s(x) = \min \left\{ \left(\frac{3}{4}\right)^{k-5} v_{k-5}(x), \cdots, \left(\frac{3}{4}\right)^{k-1} v_{k-1}(x) \right\}$$

for all $x \in O_k$. Since $v_{k-5}, \dots, v_{k-1}$ satisfy (5.8) in D_{k-1}, (5.10) holds in $O_k \cap D_{k-1}$ by the same reason as above. The open set $(D_k \setminus E_{k+1}) \cup (O_k \cap D_{k-1})$ has the desired property.

Finally, we consider pointwise behavior of s. Fix $x \in E_k \setminus E_{k+1}$. By (5.9), we have

$$\frac{1}{4}\left(\frac{3}{4}\right)^k \leq s(x) \leq \frac{5}{4}\left(\frac{4}{3}\right)^5 \left(\frac{3}{4}\right)^k. \tag{5.11}$$

By the latter inequality and the definition of E_k, the right-hand side of (5.10) is estimated from below by

$$c_3 \left(\frac{3}{4}\right)^{k(p-1)} \left(\frac{2}{\theta}\right)^{(k-5)p} w(x) \geq s^{p-1} \frac{c_H}{\delta_\Gamma(x)^p} w(x),$$

where $c_H = c_3 \left\{(5/4)\,(4/3)^5\right\}^{1-p} (\theta/2)^{6p}$. Therefore, s satisfies (5.1). Take $\alpha > 0$ such that $3/4 = (\theta/2)^\alpha$. By (5.11), we have

$$\frac{1}{4}\delta_\Gamma(x)^\alpha \leq s(x) \leq \frac{5}{4}\left(\frac{4}{3}\right)^6 \delta_\Gamma(x)^\alpha \leq \frac{30}{4}\delta_\Gamma(x)^\alpha$$

for all $x \in \Omega$. Thus, the function $s_\Gamma := 4s$ has the desired properties. □

Corollary 5.3. *Assume that w is a p-admissible weight on $\mathbb{R}^n$. Let $\Gamma \subset \partial\Omega$ be a nonempty closed set, and assume that (2.5) holds for all $\xi \in \Gamma$ and $R > 0$. Let $c_H = c_H(p, C_D, \{C_P, \lambda\}, L = 1, \gamma) > 0$ be the constant in Theorem 5.1. Then,*

$$c_H \int_\Omega \frac{|\varphi|^p}{\delta_\Gamma^p}\, dw \leq \int_\Omega |\nabla \varphi|^p \, dw$$

for all $\varphi \in C_c^\infty(\Omega)$.

Proof. Applying Theorem 5.1 to $\mathcal{A}(x,z) = w(x)|z|^{p-2}z$, we get a nonnegative function $s_\Gamma \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\Omega)$ satisfying

$$-\Delta_{p,w} s_\Gamma \geq c_H \frac{s_\Gamma(x)^{p-1}}{\delta_\Gamma(x)^p} w(x) \quad \text{in } \Omega.$$

Take $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq 0$. Then,

$$c_H \int_\Omega \frac{\varphi^p}{\delta_\Gamma^p}\, dw = c_H \int_\Omega \frac{\varphi^p}{s_\Gamma^{p-1}} \frac{s_\Gamma^{p-1}}{\delta_\Gamma^p}\, dw \leq \int_\Omega \varphi^p \frac{d\nu[s_\Gamma]}{s_\Gamma^{p-1}},$$

where $\nu[s_\Gamma]$ is the Riesz measure of s_Γ with respect the (p,w)-Laplacian. Applying the Picone inequality ([AH98, Theorem 1.1]) to the right-hand side (for detail, see also [HS20, Lemma 3.2]), we get

$$\int_\Omega \varphi^p \frac{d\nu[s_\Gamma]}{s_\Gamma^{p-1}} \leq \int_\Omega |\nabla\varphi|^p \, dw.$$

Combining the two inequalities, we obtain the desired inequality. □

6. Applications to Dirichlet problems

We discuss the substitutions into concave functions of the barriers constructed in the previous section. Taking them as super-solutions, we derive Theorem 6.3.

Lemma 6.1. *Assume that there exists a function s_Γ on Ω satisfying* (5.1) *and* (5.2). *Let $h\colon (0,\infty) \to (0,\infty)$ be a continuously differentiable nondecreasing concave function such that*

$$g(s) := \int_0^s h(t)\frac{dt}{t} < \infty$$

for some $s > 0$. Then, $v(x) := g(s_\Gamma(x)) \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\Omega)$ is a nonnegative weak supersolution to

$$-\operatorname{div} \mathcal{A}(x, \nabla v) \geq c_H \frac{h(\delta_\Gamma(x)^\alpha)^{p-1}}{\delta_\Gamma(x)^p} w(x) \quad \text{in } \Omega.$$

Moreover, $g(\delta_\Gamma(x)^\alpha) \leq v(x) \leq g(30\,\delta_\Gamma(x)^\alpha)$ for all $x \in \Omega$.

Proof. For the sake of simplicity, we denote s_Γ by s and the derivative of $f(t)$ by $f'(t)$. By assumption, $h'(t)t \leq h(t)$ for all $t \in (0,\infty)$; therefore, g is increasing and concave on $(0,\infty)$. Fix a nonnegative function $\varphi \in C_c^\infty(\Omega)$. Since $g'' \leq 0$, we have

$$\int_\Omega \mathcal{A}(x, \nabla s) \cdot \nabla \left(g'(s)^{p-1}\right) \varphi \, dx \leq 0.$$

By the chain rule, (1.5) and (5.1), we get

$$\begin{aligned}
\int_\Omega \mathcal{A}(x, \nabla g(s)) \cdot \nabla\varphi \, dx &= \int_\Omega \mathcal{A}(x, g'(s)\nabla s) \cdot \nabla\varphi \, dx \\
&\geq \int_\Omega \mathcal{A}(x, \nabla s) \cdot \nabla \left(g'(s)^{p-1}\varphi\right) \, dx \\
&\geq c_H \int_\Omega \frac{s^{p-1}}{\delta_\Gamma^p} \left(g'(s)^{p-1}\varphi\right) \, dw.
\end{aligned}$$

Since h is nondecreasing, (5.2) gives

$$s^{p-1} g'(s)^{p-1} = h\,(s)^{p-1} \geq h(\delta_\Gamma^\alpha)^{p-1}.$$

The latter statement is a consequence of the monotonicity of g. □

Proposition 6.2. *Let Ω be a bounded open set, and let $\Gamma \subset \partial\Omega$ be a nonempty closed set satisfying* (2.5) *for all $\xi \in \Gamma$ and $R > 0$. Let $f \in L^1_{\mathrm{loc}}(\Omega)$, and assume that there exists a function h satisfying the assumption in Lemma 6.1 and*

$$|f(x)| \leq c_H \frac{h(\delta_\Gamma(x)^\alpha)^{p-1}}{\delta_\Gamma(x)^p} w(x)$$

for almost every $x \in \Omega$, where c_H and α are constants in Theorem 5.1. Then, there exists a nonnegative unique weak solution $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\Omega)$ to (1.1) *in the sense that* (i) *the zero extension of u belongs to $H^{1,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Gamma; w)$, and* (ii) *$\lim_{x\to\xi} u(x) = 0$ for all $\xi \in \Gamma$.*

Proof. Using Theorem 5.1 and Lemma 6.1, we get a nonnegative supersolution $v(x) = g(s(x))$ to $-\operatorname{div} \mathcal{A}(x, \nabla u) = |f|$ in Ω. Set $D_k = \{x \in \Omega\colon\ \operatorname{dist}(x, \Gamma) > 1/k\}$, and consider the sequence of weak solutions $\{v_k\}_{k=1}^{\infty} \subset H_0^{1,p}(\Omega; w) \cap C(\Omega)$ to

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v_k) = |f| \mathbf{1}_{D_k} & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Since Ω is bounded, the right-hand side belongs to the dual of $H_0^{1,p}(\Omega; w)$. By the comparison principle in [Har22, Theorem 3.5],

$$0 \le v_k(x) \le v(x) \tag{6.1}$$

for all $x \in \Omega$. Also, consider the sequence of weak solutions $\{u_k\}_{k=1}^{\infty} \subset H_0^{1,p}(\Omega; w) \cap C(\Omega)$ to

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_k) = f \mathbf{1}_{D_k} & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

By the comparison principle for weak solutions, we have

$$-v_k(x) \le u_k(x) \le v_k(x)$$

for all $x \in \Omega$; therefore,

$$|u_k(x)| \le v(x) \tag{6.2}$$

for all $x \in \Omega$. Fix $j \ge 1$, and take a nonnegative function $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ on $\overline{D_j}$ and $\eta \equiv 0$ on $\Omega \setminus D_{j+1}$. Testing the equation of u_k with $u_k \eta^p$, we get

$$\int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla u_k \eta^p \, dx + p \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla \eta \eta^{p-1} u_k \, dx = \int_\Omega f u_k \eta^p \, dx.$$

By (1.2) and (1.3), this inequality yields

$$\int_{D_j} |\nabla u_k|^p \, dw \le C \int_{D_{j+1}} |u_k|^p \, dw + C \left\| \frac{f}{w} \right\|_{L^\infty(D_{j+1})}^{p/(p-1)} w(D_{j+1}). \tag{6.3}$$

By (6.2) and assumption on f, the right-hand side is bounded with respect to k. Meanwhile, Lemma 4.2 yields a local Hölder estimate of u_k. Therefore, for each $j \ge 1$, there are constants C_j and $\alpha \in (0, 1)$ independent of k such that

$$\|u_k\|_{C^\alpha(\overline{D_j})} + \|\nabla u_k\|_{L^p(D_j, w)} \le C_j.$$

Take a subsequence of $\{u_k\}_{k=1}^{\infty}$ and a function u on Ω such that $u_k \to u$ locally uniformly in Ω and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(D_j; w)$ for all j. Let η be the function defined before. By the product rule, we have

$$\int_\Omega |\nabla(u_k \eta)|^p \, dw \le C \left(\int_{D_{j+1}} |\nabla u_k|^p \eta^p \, dw + \int_{D_{j+1}} |u_k|^p |\nabla \eta|^p \, dw \right).$$

The left-hand side belong to $H_0^{1,p}(\Omega; w)$, and thus, $u\eta \in H_0^{1,p}(\Omega; w)$. This implies that the zero extension of u belongs to $H_{\mathrm{loc}}^{1,p}(\mathbb{R}^n \setminus \Gamma; w)$. Using the test function $u\eta - u_k \eta \in H_0^{1,p}(\Omega; w)$, we obtain

$$\begin{aligned} \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla (u - u_k) \eta \, dx &= - \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla \eta (u - u_k) \, dx \\ &\quad + \int_\Omega f (u - u_k) \eta \, dx. \end{aligned} \tag{6.4}$$

Take the limit $k \to \infty$. The latter term on the right-hand side goes to zero. Moreover, by (1.3), the former term is estimated by

$$\begin{aligned}
&\left| \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla \eta (u - u_k)\, dx \right| \\
&\le C \left(\int_{D_{j+1}} |\nabla u_k|^p \, dw \right)^{(p-1)/p} \left(\int_{D_{j+1}} |u - u_k|^p \, dw \right)^{1/p}.
\end{aligned}$$

The right-hand side goes to zero because $u_k \to u$ uniformly in D_{j+1}. Thus, the left-hand side of (6.4) goes to zero. Meanwhile, by the weak convergence of ∇u_k in $L^p(D_{j+1}; w)$, we have

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla (u - u_k) \eta \, dx \to 0. \tag{6.5}$$

Combining (6.4), (6.5) and (1.4), we find that

$$\int_{D_j} \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_k) \cdot \nabla (u - u_k) \eta \, dx \to 0.$$

It follows from [HKM06, Lemma 3.73] that u satisfies $-\operatorname{div} \mathcal{A}(x, \nabla u) = f$ in Ω. Interior regularity of u follows from Lemma 4.2. If $\xi \in \Gamma$, then u is continuous at ξ by the upper bound (6.2).

Let $u, v \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\Omega)$ be weak solutions to (1.1) satisfying the Dirichlet boundary condition in the statement. Assume that $u(x) > v(x)$ for some $x \in \Omega$. Then, $D = \{x \in \Omega \colon u(x) > v(x) + \epsilon\}$ is a nonempty bounded open set for some $\epsilon > 0$. If $\operatorname{dist}(\overline{D}, \Gamma) = 0$, then there is a boundary point $\xi \in \overline{D} \cap \Gamma$. This contradicts to assumption because $(u - v)(\xi) = 0$ and $\inf_D (u - v) \ge \epsilon > 0$. Therefore, $\operatorname{dist}(\overline{D}, \Gamma) > 0$ and $u, v \in H^{1,p}(D; w)$. By density,

$$\int_D (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v)) \cdot \nabla \varphi \, dx = 0$$

for all $\varphi \in H^{1,p}_0(D; w)$. Testing this equation with $\varphi = u - v - \epsilon$ and using (1.4), we find that $u = v$ on D. This contradicts to the assumption. $\square$

Theorem 6.3. *Assume that Ω is a bounded open set and that $\mathbb{R}^n \setminus \Omega$ is uniformly (p, w)-fat. Let $f \in L^1_{\mathrm{loc}}(\Omega)$, and assume that there are constants $K > 0$ and $0 < \beta \le \alpha$ such that*

$$|f(x)| \le K \delta_{\partial\Omega}(x)^{\beta(p-1)-p}\, w(x)$$

for almost every $x \in \Omega$, where α is a positive number in Theorem 5.1. Then, there exists a unique weak solution $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w) \cap C(\overline{\Omega})$ to (1.1). Moreover,

$$|u(x)| \le C K^{1/(p-1)} \delta_{\partial\Omega}(x)^\beta \tag{6.6}$$

for all $x \in \Omega$, where $C = c_H^{1-p} 30^{\beta/\alpha} (\alpha/\beta)$.

Proof. Applying to Proposition 6.2 to $\Gamma = \partial\Omega$ and $h(t) = (K/c_H)^{1/(p-1)} t^{\beta/\alpha}$, we obtain the desired unique weak solution u. The upper bound (6.6) follows from (6.2). $\square$

Appendix A. Proofs of Lemmas 4.1 and 4.2

The proofs below are combinations of [KS80, p. 63, Lemma B.2], [GT01, Theorem 8.18] and [HKM06, Theorems 3.59 and 7.46]. See also [Str84, Chapter 3.1.0].

Proof of Lemma 4.1. Fix any $k > \sup_{\partial\Omega} u$, and consider the test function $(u-k)_+ \in H_0^{1,p}(\Omega; w)$. By (1.2), we have

$$\int_\Omega |\nabla (u-k)_+|^p \, dw \le \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla (u-k)_+ \, dx = \int_\Omega f(u-k)_+ \, dx.$$

By Hölder's inequality, the right-hand side is estimated by

$$\int_\Omega f(u-k)_+ \, dx \le \left(\int_\Omega (u-k)_+^{\chi p} \, dw \right)^{1/\chi p} \left\| \frac{f_+}{w} \right\|_{L^\infty(\Omega)} w([u>k])^{1-1/\chi p}.$$

Meanwhile, by taking a ball B with radius $\operatorname{diam}(\Omega)$ such that $\Omega \subset B$, (2.3) yields

$$\left(\frac{1}{w(B)} \int_\Omega (u-k)_+^{\chi p} \, dw \right)^{1/\chi} \le C \operatorname{diam}(\Omega)^p \frac{1}{w(B)} \int_\Omega |\nabla (u-k)_+|^p \, dw.$$

Combining the three inequalities, we obtain

$$\left(\frac{1}{w(B)} \int_\Omega (u-k)_+^{\chi p} \, dw \right)^{1/\chi p} \le A \left(\frac{w([u>k])}{w(B)} \right)^{(1-1/\chi p)/(p-1)},$$

where $A = C \operatorname{diam}(\Omega)^{p/(p-1)} \|f_+/w\|_{L^\infty(\Omega)}^{1/(p-1)}$. By Chebyshev's inequality,

$$\frac{w([u>h])}{w(B)} \le \frac{A^{\chi p}}{(h-k)^{\chi p}} \left(\frac{w([u>k])}{w(B)} \right)^{(\chi p-1)/(p-1)}$$

for any $h > k$. From [KS80, p. 63, Lemma B.1], the desired estimate follows. □

Proof of Lemma 4.2. Set $\bar{u} = u + F_-$. Consider the test function $\bar{u}^\beta \eta^p$, where $\beta < 0$ and $\eta \in C_c^\infty(2B)$. Since $\nabla \bar{u} = \nabla u$, we have

$$\beta \int_{2B} \mathcal{A}(x, \nabla \bar{u}) \cdot \nabla \bar{u} \bar{u}^{\beta-1} \eta^p \, dx + p \int_{2B} \mathcal{A}(x, \nabla \bar{u}) \cdot \nabla \eta \eta^{p-1} \bar{u}^\beta \, dx \ge \int_{2B} f \bar{u}^\beta \eta^p \, dx. \tag{A.1}$$

By the definition of F_-, the right-hand side is estimated from below by

$$\int_{2B} f_- \bar{u}^\beta \eta^p \, dx \le \frac{C}{R^p} \int_{2B} \bar{u}^{p-1+\beta} \eta^p \, dw.$$

Hence, by (1.2) and (1.3), we obtain

$$|\beta| \int_{2B} |\nabla \bar{u}|^p \bar{u}^{\beta-1} \eta^p \, dw \le C \left(|\beta|^{1-p} \|\nabla \eta\|_{L^\infty(2B)}^p + R^{-p} \right) \int_{\operatorname{supp} \eta} \bar{u}^{p-1+\beta} \, dw. \tag{A.2}$$

Fix a ball $B(x,r)$ such that $B(x, 4\lambda r) \subset 2B$, where λ is a constant in (2.2). Assume that $\beta \neq 1-p$. Let $r \le r_1 < r_2 \le 2r$. By the chain rule, (A.2) and (2.3), we have

$$\left(\fint_{B(x,r_1)} \bar{u}^{\chi(p-1+\beta)} \, dw \right)^{1/\chi} \le \frac{C \cdot c(\beta)}{(r_2 - r_1)^p} \fint_{B(x,r_2)} \bar{u}^{p-1+\beta} \, dw, \tag{A.3}$$

where $c(\beta) = |(p-1+\beta)/p|^p (|\beta|^{-p} + |\beta|^{-1})$. First, we consider the case $\beta > 1-p$. Using (A.3) finitely many times, we get

$$\left(\fint_{B(x,r)} \bar{u}^s \, dw \right)^{1/s} \le C \left(\fint_{B(x,2r)} \bar{u}^{s_0} \, dw \right)^{1/s_0}, \tag{A.4}$$

where $s > s_0 > 0$ and C is a constant depending also on s_0. Next, for $k = 0, 1, 2, \cdots$, we consider $\beta_k = (1-p) - \chi^k s_0$ and the sequence of balls $B^k = B(x, (1+2^{-k})r)$. Then, (A.3) gives

$$\left(\fint_{B^{k+1}} \bar{u}^{-\chi^{k+1}s_0} \, dw \right)^{-1/\chi^{k+1}s_0} \geq \frac{1}{(C_1 C_2^k)^{1/\chi^k}} \left(\fint_{B^k} \bar{u}^{-\chi^k s_0} \, dw \right)^{-1/\chi^k s_0},$$

where C_1 and C_2 are constants depending also on s_0, but independent of k. Since $\sum_k 1/\chi^k$ and $\sum_k k/\chi^k$ are finite, we obtain

$$\operatorname*{ess\,inf}_{B(x,r)} \bar{u} \geq \frac{1}{C} \left(\fint_{B(x,2r)} \bar{u}^{-s_0} \, dw \right)^{-1/s_0}. \tag{A.5}$$

Finally, we consider $\beta = 1 - p$. Then, (A.2) yields

$$\int_{B(x,2\lambda r)} |\nabla \log \bar{u}|^p \, dw \leq C r^{-p} w(B(x, 4\lambda r)).$$

Applying (2.2) to the left-hand side, we obtain

$$\fint_{B(x,2r)} |\log \bar{u} - c|^p \, dw \leq C r^p \fint_{B(x,2\lambda r)} |\nabla \log \bar{u}|^p \, dw \leq C, \tag{A.6}$$

where $c = \fint_{B(x,2r)} \log \bar{u} \, dw$. Combining (A.4), (A.5) and (A.6) and using the John-Nirenberg lemma in [HKM06, Chapter 19] (see also [SC02, Lemma 2.2.6]), we arrive at

$$\left(\fint_{B(x,r)} \bar{u}^s \, dw \right)^{1/s} \leq C \operatorname*{ess\,inf}_{B(x,r)} \bar{u}.$$

The desired inequality (4.1) follows from this inequality and a covering argument. The latter inequality (4.2) is a consequence of (4.1) and (A.2). □

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References

[AKSZ07] Aikawa, H., T. Kilpeläinen, N. Shanmugalingam, and X. Zhong: Boundary Harnack principle for p-harmonic functions in smooth Euclidean domains. - Potential Anal. 26:3, 2007, 281–301.

[AH98] Allegretto, W., and Y. X. Huang: A Picone's identity for the p-Laplacian and applications. - Nonlinear Anal. 32:7, 1998, 819–830.

[Anc81] Ancona, A.: Une propriété des espaces de Sobolev. - C. R. Acad. Sci. Paris Sér. I Math. 292:9, 1981, 477–480.

[Anc86] Ancona, A.: On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$. - J. London Math. Soc. (2) 34:2, 1986, 274–290.

[BEL15] Balinsky, A. A., W. D. Evans, and R. T. Lewis: The analysis and geometry of Hardy's inequality. - Universitext, Springer, Cham, 2015.

[BLP78] Bensoussan, A., J.-L. Lions, and G. Papanicolaou: Asymptotic analysis for periodic structures. - Studies in Mathematics and its Applications 5, North-Holland Publishing Co., Amsterdam-New York, 1978.

[BB11] Björn, A., and J. Björn: Nonlinear potential theory on metric spaces. - EMS Tracts Math. 17, Eur. Math. Soc. (EMS), Zürich, 2011.

[BMS01] Björn, J., P. MacManus, and N. Shanmugalingam: Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces. - J. Anal. Math. 85, 2001, 339–369.

[BM92] Boccardo, L., and F. Murat: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. - Nonlinear Anal. 19:6, 1992, 581–597.

[BK06] Borsuk, M., and V. Kondratiev: Elliptic boundary value problems of second order in piecewise smooth domains. - North-Holland Mathematical Library 69, Elsevier Science B.V., Amsterdam, 2006.

[BK04] Buckley, S. M., and P. Koskela: Orlicz–Hardy inequalities. - Illinois J. Math. 48:3, 2004, 787–802.

[Che99] Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. - Geom. Funct. Anal. 9:3, 1999, 428–517.

[DLG10] Durán, R. G., and F. López García: Solutions of the divergence and analysis of the Stokes equations in planar Hölder-α domains. - Math. Models Methods Appl. Sci. 20:1, 2010, 95–120.

[DIL+19] Dyda, B., L. Ihnatsyeva, J. Lehrbäck, H. Tuominen, and A. V. Vähäkangas: Muckenhoupt A_p-properties of distance functions and applications to Hardy–Sobolev-type inequalities. - Potential Anal. 50:1, 2019, 83–105.

[EHS05] Edmunds, D. E., and R. Hurri-Syrjänen: Weighted Hardy inequalities. - J. Math. Anal. Appl. 310:2, 2005, 424–435.

[EHDR15] Egert, M., R. Haller-Dintelmann, and J. Rehberg: Hardy's inequality for functions vanishing on a part of the boundary. - Potential Anal. 43:1, 2015, 49–78.

[FMT07] Filippas, S., V. Maz'ya, and A. Tertikas: Critical Hardy–Sobolev inequalities. - J. Math. Pures Appl. (9) 87:1, 2007, 37–56.

[GZ77] Gariepy, R., and W. P. Ziemer: A regularity condition at the boundary for solutions of quasilinear elliptic equations. - Arch. Rational Mech. Anal. 67:1, 1977, 25–39.

[GT01] Gilbarg, D., and N. S. Trudinger: Elliptic partial differential equations of second order. - Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[Haj99] Hajłasz, P.: Pointwise Hardy inequalities. - Proc. Amer. Math. Soc. 127:2, 1999, 417–423.

[Har18] Hara, T,: Wolff potential estimates for Cheeger p-harmonic functions. - Collect. Math. 69:3, 2018, 407–426.

[Har21] Hara, T.: Quasilinear elliptic equations with sub-natural growth terms in bounded domains. - NoDEA Nonlinear Differential Equations Appl. 28:6, 2021, Paper No. 62.

[Har22] Hara, T.: Trace inequalities of the Sobolev type and nonlinear Dirichlet problems. - Calc. Var. Partial Differential Equations 61:6, 2022, Paper No. 216.

[HS20] Hara, T., and A. Seesanea: Existence of minimal solutions to quasilinear elliptic equations with several sub-natural growth terms. - Nonlinear Anal. 197, 2020, 111847, 21.

[HKM06] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear potential theory of degenerate elliptic equations. - Dover Publications, Inc., Mineola, NY, 2006.

[Hor89] Horiuchi, T.: The imbedding theorems for weighted Sobolev spaces. - J. Math. Kyoto Univ. 29:3, 1989, 365–403.

[Hor91] Horiuchi, T.: The imbedding theorems for weighted Sobolev spaces. II. - Bull. Fac. Sci. Ibaraki Univ. Ser. A 23, 1991, 11–37.

[KS80] Kinderlehrer, D., and G. Stampacchia: An introduction to variational inequalities and their applications. - Pure and Applied Mathematics 88, Academic Press, Inc., New York-London, 1980.

[KLV21] Kinnunen, J., J. Lehrbäck, and A. Vähäkangas: Maximal function methods for Sobolev spaces. - Math. Surveys Monogr. 257, Amer. Math. Soc., Providence, RI, 2021.

[KM97] Kinnunen, J., and O. Martio: Hardy's inequalities for Sobolev functions. - Math. Res. Lett. 4:4, 1997, 489–500.

[KLT11] Korte, R., J. Lehrbäck, and H. Tuominen: The equivalence between pointwise Hardy inequalities and uniform fatness. - Math. Ann. 351:3, 2011, 711–731.

[Leh08] Lehrbäck, J.: Pointwise Hardy inequalities and uniformly fat sets. - Proc. Amer. Math. Soc. 136:6, 2008, 2193–2200.

[LTV17] Lehrbäck, J., H. Tuominen, and A. V. Vähäkangas: Self-improvement of uniform fatness revisited. - Math. Ann. 368:3-4, 2017, 1439–1464.

[Lew88] Lewis, J. L.: Uniformly fat sets. - Trans. Amer. Math. Soc. 308:1, 1988, 177–196.

[Lio69] Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. - Dunod, Gauthier-Villars, Paris, 1969.

[MS00] Marcus, M., and I. Shafrir: An eigenvalue problem related to Hardy's L^p inequality. - Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29:3, 2000, 581–604.

[Mik96] Mikkonen, P.: On the Wolff potential and quasilinear elliptic equations involving measures. - Ann. Acad. Sci. Fenn. Math. Diss. 104, 1996, 71 pp.

[SC02] Saloff-Coste, L.: Aspects of Sobolev-type inequalities. - London Math. Soc. Lecture Note Ser. 289, Cambridge Univ. Press, Cambridge, 2002.

[Str84] Stredulinsky, E. W.: Weighted inequalities and degenerate elliptic partial differential equations. - Lecture Notes in Math. 1074, Springer-Verlag, Berlin, 1984.

[Tol83] Tolksdorf, P.: On the Dirichlet problem for quasilinear equations in domains with conical boundary points. - Comm. Partial Differential Equations 8:7, 1983, 773–817.

[TW02] Trudinger, N. S., and X.-J. Wang: On the weak continuity of elliptic operators and applications to potential theory. - Amer. J. Math. 124:2, 2002, 369–410.

[Wan90] Wannebo, A.: Hardy inequalities. - Proc. Amer. Math. Soc. 109:1, 1990, 85–95.

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