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Exceptional set estimates for radial projections in $\mathbb{R}^n$

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Abstract. We prove two conjectures in this paper. The first conjecture is by Lund, Pham and Thu: Given a Borel set $A \subset \mathbb{R}^n$ such that $\dim A \in (k, k+1]$ for some $k \in \{1, \ldots, n-1\}$. For $0 < s < k$, we have

$$\dim(\{y \in \mathbb{R}^n \setminus A \mid \dim(\pi_y(A)) < s\}) \le \max\{k + s - \dim A, 0\}.$$

The second conjecture is by Liu: Given a Borel set $A \subset \mathbb{R}^n$, then

$$\dim(\{x \in \mathbb{R}^n \setminus A \mid \dim(\pi_x(A)) < \dim A\}) \le \lceil \dim A \rceil.$$

Avaruuden $\mathbb{R}^n$ säteittäisten projektioiden poikkeusjoukkoarvioita

Tiivistelmä. Tässä työssä vahvistetaan kaksi Borelin joukkoja $A \subset \mathbb{R}^n$ koskevaa konjektuuria. Ensimmäisen esittivät Lund, Pham ja Thu: Jos $\dim A \in (k, k+1]$ jollakin $k \in \{1, \ldots, n-1\}$ ja $0 < s < k$, niin

$$\dim(\{y \in \mathbb{R}^n \setminus A \mid \dim(\pi_y(A)) < s\}) \le \max\{k + s - \dim A, 0\}.$$

Toisen konjektuurin esitti Liu:

$$\dim(\{x \in \mathbb{R}^n \setminus A \mid \dim(\pi_x(A)) < \dim A\}) \le \lceil \dim A \rceil.$$

1. Introduction

In this paper, we study the radial projections in $\mathbb{R}^n$. Let $G(m, n)$ be the set of m-dimensional subspaces in $\mathbb{R}^n$, which is also known as the Grassmannian. For $V \in G(m, n)$, define $\pi_V : \mathbb{R}^n \to V$ to be the orthogonal projection onto V. Given $x \in \mathbb{R}^n$, define $\pi_x : \mathbb{R}^n \setminus \{x\} \to \mathbb{S}^{n-1}$ to be the radial projection centered at x:

$$\pi_x(y) = \frac{y - x}{|y - x|}.$$

We first discuss some background of the projection theory. We use $\dim X$ to denote the Hausdorff dimension of the set X. There is a classical result proved by Marstrand [9], who showed that if A is a Borel set in $\mathbb{R}^2$, then the projection of A onto almost every line through the origin has Hausdorff dimension $\min\{1, \dim A\}$. This was generalized to higher dimensions by Mattila [10], who showed that if A is a Borel set in $\mathbb{R}^n$, then the projection of A onto almost every k-plane through the origin has Hausdorff dimension $\min\{k, \dim A\}$. It turns out that one can obtain some finer results which are known as the exceptional set estimates. The exceptional set estimates give a bound on the set of directions where the projection is small. There are two types of exceptional set estimates known as the Falconer-type estimate and Kaufman-type estimate.

Suppose $A \subset \mathbb{R}^n$ is a Borel set of Hausdorff dimension α. For $0 \le s < \min\{m, \alpha\}$, define the exceptional set

$$E_s(A) = \{V \in G(m, n) \mid \dim(\pi_V(A)) < s\}.$$

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Then we have

(i) (Falconer-type) $\dim(E_s(A)) \leq \max\{m(n-m)+s-\alpha, 0\}$,
(ii) (Kaufman-type) $\dim(E_s(A)) \leq m(n-m-1)+s$.

The original papers of Falconer and Kaufman are [2, 6]. The more general version was proved by Peres and Schlag [14]. We also recommend Theorem 5.10 in [11] for the proofs of these two types of the exceptional set estimates.

In this paper, we study the exceptional set estimates for the radial projections. We first state our theorems.

Theorem 1. *Let $A \subset \mathbb{R}^n$ be a Borel set such that $\alpha = \dim A \in (k, k+1]$ for some $k \in \{1, \dots, n-1\}$. Fix $0 < s < k$ and let*

$$E_s(A) := \{y \in \mathbb{R}^n \setminus A \mid \dim(\pi_y(A)) < s\}.$$

Then,

$$\dim(E_s(A)) \leq \max\{k+s-\alpha, 0\}.$$

Theorem 2. *Let $A \subset \mathbb{R}^n$ be a Borel set such that $\alpha = \dim A \in (k-1, k]$ for some $k \in \{1, \dots, n-1\}$. Define the exceptional set*

$$E(A) := \{x \in \mathbb{R}^n \setminus A \mid \dim(\pi_x(A)) < \alpha\}.$$

Then we have

$$\dim(E(A)) \leq k.$$

Theorem 2 is sharp: if we let A be an α-dimensional subset of $\mathbb{R}^k$, we see that $E(A) = \mathbb{R}^k \setminus A$ which has dimension at most k.

We remark that Theorem 1 answers a conjecture by Lund, Pham and Thu (see [8, Conjecture 1.2]); Theorem 2 answers a conjecture by Liu (see [7, Conjecture 1.2]).

Recently, Orponen and Shmerkin [12] proved the $n = 2$ case for both Theorem 1 and Theorem 2. Their proof of Theorem 1 (when $n = 2$) is based on a Furstenberg-type estimate due to Fu and Ren [4]. Then by a swapping trick, they are able to prove Theorem 2 (when $n = 2$). In this paper, we prove the Theorems for all dimensions. We remark that the upper bound in Theorem 1 is a Falconer-type bound.

We talk about the structure of the paper. In Section 2, we prove Theorem 1 as a result of Proposition 17 and Proposition 19. In Section 3, we prove Theorem 2 based on Proposition 23 and a trick of Orponen and Shmerkin [12].

We note that Orponen, Shmerkin and Wang also proved Theorem 1 and Theorem 2. See [13].

1.1. Some notations. We use "$A \lesssim B$" to denote $A \leq CB$ for some universal constant C.

Definition 3. For a number $\delta > 0$ and any set X (in a metric space), we use $|X|_\delta$ to denote the maximal number of δ-separated points in X.

Definition 4. Let $\delta, s > 0$. We say $A \subset \mathbb{R}^n$ is a (δ, s, C)-*set* if it is δ-separated and satisfies the following estimate:

$$\#(A \cap B_r(x)) \leq C(r/\delta)^s \tag{1}$$

for any $x \in \mathbb{R}^n$ and $1 \geq r \geq \delta$. When the constant C is universal or clear from the context, the condition (1) will be shortened to

$$\#(A \cap B_r(x)) \lesssim (r/\delta)^s.$$

Also, (δ, s, C)-set will be shortened to (δ, s)-set.

Remark 5. Since the condition (1) is for scales $\geq \delta$, we can abuse the notation to define: for $A' = \sqcup B_\delta$ being a union of disjoint δ-balls, we say A is a (δ, s)-set if

$$\#\{B_\delta : B_\delta \subset A \cap B_r(x)\} \lesssim (r/\delta)^s.$$

This definition is consistent with the previous definition: If A is a (δ, s)-set, then the δ-neighborhood of A is also a (δ, s)-set in the new sense; conversely, if A' is a disjoint union of δ-balls and is a (δ, s)-set in the new sense, then the set of centers of the δ-balls in A' is a (δ, s)-set in the old sense. Therefore, it makes sense to say a set A is a (δ, s)-set if A is δ-separated or A is a disjoint union of δ-balls.

Remark 6. Throughout the rest of this paper, We will use $\#E$ to denote the cardinality of a set E and $|\cdot|$ to denote the measure of a region.

1.2. δ-tube and δ-slab. One of the main geometric objects we will study is the so-called δ-tube. In $\mathbb{R}^n$, we call T a δ-tube, if T is a cylinder of radius δ and length 1. For $C > 0$, we use CT to denote the C-dilation of T with respect to the center of T. For two δ-tubes T and T', we say they are comparable, if $10^{-1}T \subset T' \subset 10T$. We say they are essentially distinct, if they are not comparable. If T' is a convex set that is comparable to a δ-tube T, then we also call T' a δ-tube. Therefore, if T is a rectangle of dimensions $\sim \delta \times \cdots \times \delta \times 1$, then T is also a δ-tube.

In this paper, we will frequently encounter the following situation. There are two finite sets $E, F \subset B^n(0, 1)$ (here $B^n(0, 1)$ is the unit ball in $\mathbb{R}^n$ centered at the origin). Each of E and F is contained in a ball of radius $1/8$, and $\mathrm{dist}(E, F) > 1/2$. We use letter y to denote the points in E, x to denote the points in F, and assume F is a (δ, α)-set, and E is a (δ, t)-set. We can view F (or E) as a δ-discretized version of A (or $E_s(A)$) in Theorem 1. If $y \in E$ is in the exceptional set, then the maximal δ-separated subset of $\pi_y(F)$ is roughly a (δ, s)-set in $\mathbb{S}^{n-1}$. We would like to use another geometric object to characterize $\pi_y(F)$. For every $\omega \in \mathbb{S}^{n-1}$, we can define a tube T_ω which is the δ-neighborhood of the line segment $\{y + t\omega : t \in [0, 1]\}$. Roughly speaking, T_ω is a tube of dimensions $\sim \delta \times \cdots \times \delta \times 1$ pointing to the direction ω and passing through y. In this correspondence, a maximal δ-separated subset of $\pi_y(F)$ gives rise a set of δ-tubes $\mathbb{T}^y$ that pass through y, and $\bigcup_{T \in \mathbb{T}^y} T \supset F$. We call $\mathbb{T}^y$ a *bush* centered at y. Additionally, the (δ, s) condition of $\pi_y(F)$ transfers to $\mathbb{T}^y$ which says that: If T_r is a $r \times \cdots \times r \times 1$-tube passing through y, then there are $\lesssim (r/\delta)^s$ many tubes in $\mathbb{T}^y$ that are contained in T_r ($\delta \leq r \leq 1$). When we call a bush $\mathbb{T}^y$ centered at y a (δ, s)-set, we mean that $\pi_y\left(\bigcup_{T \in \mathbb{T}^y} T \setminus B(y, 1/2)\right) \subset \mathbb{S}^{n-1}$ is a (δ, s)-set. Here, $B(y, 1/2)$ is a ball of radius $1/2$ centered at y.

We have discussed the notion of a bush centered at y and the definition for a bush to be a (δ, s)-set. We also need to consider another type of bush called the *truncated bush.* If $\mathbb{T}^y$ is a bush centered at y, then for each $T \in \mathbb{T}^y$ we define the truncated tube

$$\widetilde{T} = T \setminus B_{1/2}(y).$$

By truncation, $\mathbb{T}^y$ gives rise a truncated bush $\widetilde{\mathbb{T}}^y$ centered at y. The reason we do this truncation is that the tubes in $\widetilde{\mathbb{T}}^y$ are now essentially disjoint. This will be helpful in estimating the upper bound of integrals like $\int_{\mathbb{R}^n} \left(\sum_y \sum_{T \in \widetilde{\mathbb{T}}^y} 1_T\right)^2$.

We will also study the geometric object called the k-dimensional δ-slab. They are of dimensions $\underbrace{\delta \times \cdots \times \delta}_{n-k \text{ times}} \times \underbrace{1 \times \cdots \times 1}_{k \text{ times}}$. In particular, a δ-tube is a 1-dimensional δ-slab.

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2. Falconer-type estimates for radial projections

In this section of the paper, we prove Theorem 1. We introduce some notations. Fix $0 \le \sigma$, $\delta > 0$. For a bounded set $E \subset \mathbb{R}^n$, define

$$\mathcal{H}^s_{\delta,\infty}(E) := \inf\left\{\sum_j r(D_j)^s \colon E \subset \cup_j D_j\right\}, \tag{2}$$

where the infimum runs over the coverings of E by dyadic cubes $\{D_j\}$ with side length $\ge \delta$, and $r(D)$ denotes the side length of the cube. One may compare with the definition of

$$\mathcal{H}^s_{\infty}(E) := \inf\left\{\sum_j r(D_j)^s \colon E \subset \cup_j D_j\right\}, \tag{3}$$

where the infimum runs over the coverings of E by dyadic cubes $\{D_j\}$ (without assuming side length $\ge \delta$).

We state some useful lemmas.

Lemma 7. *Let $\delta, s > 0$ and let $B \subset \mathbb{R}^n$ be any set with $\mathcal{H}^s_\infty(B) =: \kappa > 0$ (see the definition in (3)). Then, there exists a (δ, s)-set $P \subset B$ with $\#P \gtrsim \kappa\delta^{-s}$.*

Proof. See [3, Lemma 3.13]. □

Lemma 8. *Fix $a > 0$. Let ν be a probability measure satisfying $\nu(B_r) \lesssim r^a$ for any B_r being a ball of radius r. If A is a set satisfying $\nu(A) \ge \kappa$ ($\kappa > 0$), then for any $\delta > 0$ there exists a subset $F \subset A$ such that F is a (δ, a)-set and $\#F \gtrsim \kappa\delta^{-a}$.*

Proof. By the previous lemma, we just need to show $\mathcal{H}^a_\infty(A) \gtrsim \kappa$. We just check it by definition. For any covering $\{B\}$ of A, we have

$$\kappa \le \sum_B \nu(B) \lesssim \sum_B r(B)^a.$$

Ranging over all the covering of A and taking infimum, we get

$$\kappa \lesssim \mathcal{H}^a_\infty(A).$$ □

For any dyadic number $\delta \le 1$, let $\mathcal{D}_\delta$ be the set of δ-cubes of form $[i_1\delta, (i_1 + 1)\delta] \times \cdots \times [i_m\delta, (i_m + 1)\delta]$ where $0 \le i_1, \ldots, i_m \le \delta^{-1} - 1$. The cubes in $\mathcal{D}_\delta$ form a covering of $[0, 1]^m$ with the overlap only on their boundaries.

Lemma 9. *Suppose $X \subset [0, 1]^m$ with $\dim X < s$. Then for any $\varepsilon > 0$, there exist dyadic cubes $\mathcal{C}_{2^{-k}} \subset \mathcal{D}_{2^{-k}}$ ($k > 0$) so that*

(1) $X \subset \bigcup_{k>0} \bigcup_{D\in\mathcal{C}_{2^{-k}}} D$,
(2) $\sum_{k>0} \sum_{D\in\mathcal{C}_{2^{-k}}} r(D)^s \le \varepsilon$,
(3) *$\mathcal{C}_{2^{-k}}$ satisfies the s-dimensional condition: For $l < k$ and any $D \in \mathcal{D}_{2^{-l}}$, we have $\#\{D' \in \mathcal{C}_{2^{-k}} \colon D' \subset D\} \le 2^{(k-l)s}$.*

Proof. See [5, Lemma 2]. $\square$

Remark 10. Besides $[0,1]^m$, this Lemma also works for other compact metric spaces, for example $\mathbb{S}^n$ and $G(m,n)$. Each of them is locally diffeomorphic to $[0,1]^l$, so we can pull back the structure of dyadic cubes on $[0,l]^l$ to $\mathbb{S}^n$ and $G(m,n)$.

Lemma 11. *Suppose $X\subset[0,1]^m$. Then there exist dyadic cubes*

$$\mathcal{C}=\bigsqcup_{k=0}^{\log_2\delta^{-1}}\mathcal{C}_{2^{-k}}$$

(with $\mathcal{C}_{2^{-k}}\subset\mathcal{D}_{2^{-k}}$) that cover X and

(1) $\sum_{D\in\mathcal{C}}r(D)^s=\mathcal{H}^s_{\delta,\infty}(X)$,
(2) *$\mathcal{C}_{2^{-k}}$ satisfies the s-dimensional condition: For $l<k$ and any $D\in\mathcal{D}_{2^{-l}}$, we have $\#\{D'\in\mathcal{C}_{2^{-k}}: D'\subset D\}\le 2^{(k-l)s}$. In particular, $\mathcal{H}^s_{2^{-k},\infty}\big(\bigcup_{D\in\mathcal{C}_{2^{-k}}}D\big)=\#\mathcal{C}_{2^{-k}}2^{-ks}$.*

Proof. This lemma looks like Lemma 9, but it is much easier since we only care about the scales $\ge\delta$. We just choose $\mathcal{C}$ to be the covering that attain the "inf" in the definition of $\mathcal{H}^s_{\delta,\infty}(X)$. It is not hard to check the two properties are satisfied. $\square$

The next lemma is [3, Proposition A.1]. Though it is stated for $\mathcal{H}^s_\infty$ there, the proof also works for $\mathcal{H}^s_{\delta,\infty}$.

Lemma 12. *Suppose $X\subset[0,1]^m$, with $\mathcal{H}^s_{\delta,\infty}(X)=\kappa>0$. Then there exists a (δ,s)-subset of X with cardinality $\gtrsim\kappa\delta^{-s}$.*

We also have the following lemma saying that the lemma above can be reversed.

Lemma 13. *Suppose $X\subset[0,1]^m$ is a (δ,s)-set with $\#X\ge\kappa\delta^{-s}$. Then, $\mathcal{H}^s_{\delta,\infty}(X)\gtrsim\kappa$. In particular, by Lemma 12, this implies that for any $\delta\le\Delta\le1$, X contains a subset X' which is a (Δ,s)-set and satisfies $\#X'\gtrsim\kappa\Delta^{-s}$; and also implies that for any $u\le s$, X contains a subset X' which is a (Δ,u)-set and satisfies $\#X'\gtrsim\kappa\Delta^{-u}$.*

Proof. Assuming our (δ,s)-set X satisfies $\#X\ge\kappa\delta^{-s}$, we are going to show $\mathcal{H}^s_{\delta,\infty}(X)\gtrsim\kappa$. Let $\mathcal{C}$ be the covering of X that attains "inf" in the definition of $\mathcal{H}^s_{\delta,\infty}(X)$. Also let $\mathcal{C}_\Delta\subset\mathcal{C}$ be the set of Δ-cubes. We write $X=\bigsqcup_\Delta X_\Delta$, where X_Δ is the points in X covered $\mathcal{C}_\Delta$. By the definition of (δ,s)-set, each Δ-cube contains $\lesssim(\frac{\Delta}{\delta})^s$ many points from X_Δ. We have $\#\mathcal{C}_\Delta\gtrsim(\frac{\delta}{\Delta})^s\#X_\Delta$. We see that

$$\mathcal{H}^s_{\delta,\infty}(X)=\sum_{\Delta\ge\delta}\Delta^s\#\mathcal{C}_\Delta\gtrsim\delta^s\#X=\kappa. \qquad\square$$

Remark 14. We see that when X is a (δ,s)-set, then $\#X\gtrsim\delta^{-s+\varepsilon}$ and $\mathcal{H}^s_{\delta,\infty}(X)\gtrsim\delta^\varepsilon$ are equivalent.

We recall Theorem 1 here.

Theorem 15. *Let $A\subset\mathbb{R}^n$ be a Borel set such that $\alpha=\dim A\in(k,k+1]$ for some $k\in\{1,\dots,n-1\}$. Fix $0<s<k$ and let*

$$E_s(A):=\{y\in\mathbb{R}^n\setminus A\mid \dim(\pi_y(A))<s\}.$$

Then,

$$\dim(E_s(A))\le\max\{k+s-\alpha,0\}.$$

We will actually prove the following δ-discretized version which is a generalization of [12, Proposition 4.2].

Theorem 16. *Let $0<\sigma<k$, $a\in(k,k+1]$ for some $k\in\{1,\dots,n-1\}$ and $t>\max\{k+\sigma-a,0\}$. Let $\eta\in(0,1/10)$. Then for ε and δ small enough depending on η,σ,a, and t, we have the following result.*

Let $E,F\subset B^n(0,1)$ be a (δ,t)-set and a (δ,a)-set respectively, with $\#E\gtrsim\delta^{-t+\varepsilon}$, $\#F\gtrsim\delta^{-a+\varepsilon}$. We also assume that each of E and F lies in a ball of radius $1/1000$ and $\operatorname{dist}(E,F)\geq 3/4$. Then, there exists $y\in E$ such that for all $F'\subset F$ with $\#F'\geq\delta^{\varepsilon}\#F$, we have

$$\mathcal{H}^{\sigma}_{\delta,\infty}(\pi_y(F'))>\delta^{\eta}.$$

We first show that Theorem 16 implies Theorem 15.

Proof that Theorem 16 implies Theorem 15. Let $A\subset\mathbb{R}^n$ be a Borel set such that $\alpha=\dim A\in(k,k+1]$ for some $k\in\{1,\dots,n-1\}$. We first do a reduction to localize A. For $\alpha_1<\alpha$, we say $x\in A$ is an α_1-dense point of A if $\dim(A\cap B_r(x))\geq\alpha_1$ for any $r>0$. We notice a fact: for $\alpha_1<\alpha$, A has infinitly many α_1-dense points; otherwise, A can be covered by a finite set and countable union of sets with dimension less than α_1, which contradicts $\dim A=\alpha$.

Fix $\alpha_1<\alpha$ that is sufficiently close to α (we will later let $\alpha_1\to\alpha$). We can find α_1-dense points x_1,x_2 of A. Since our problem is scaling-invariant, we can assume $|x_1-x_2|=99/100$. We let $A_1=A\cap B_{1/1000}(x_1)$, $A_2=A\cap B_{1/1000}(x_2)$, and then $\dim(A_1),\dim(A_2)\geq\alpha_1$. We only need to show for any ball $B_{1/1000}$ of radius $1/1000$, $E_s(A)\cap B_{1/1000}$ has dimension $\leq\max\{k+s-\alpha,0\}$. Since $\operatorname{dist}(A_1,A_2)>98/100$, we have either $\operatorname{dist}(B_{1/1000},A_1)>3/4$ or $\operatorname{dist}(B_{1/1000},A_2)>3/4$. We may assume $\operatorname{dist}(B_{1/1000},A_1)>3/4$. It suffices to show that the set

$$E':=E_s(A_1)\cap B_{1/1000}=\{y\in B_{1/1000}\colon \dim(\pi_y(A_1))<s\}$$

has dimension $\leq\max\{k+s-\dim(A_1),0\}$. From the reduction, these sets satisfy certain separation properties:

$$\text{each of } A_1 \text{ and } E' \text{ lies in some ball of radius } 1/1000, \tag{4}$$

$$A_1,E'\subset B^n(0,1),\quad \operatorname{dist}(A_1,E')\geq 3/4. \tag{5}$$

(We remark that the numerology about the radii of balls or the distance between sets are not important. For example, we only need A_1,E' to be contained in a ball of bounded radius and the distance between A_1 and E' are bigger than some nonzero constant.)

We choose $t<\dim(E')$, $a<\dim(A_1)$. Then $\mathcal{H}^t_\infty(E')>0$, and by Frostman's lemma there exists a probability measure ν_{A_1} supported on A_1 satisfying $\nu_{A_1}(B_r)\lesssim r^a$ for any B_r being a ball of radius r. We only need to prove $t\leq\max\{k+s-a,0\}$, since then we can send $a\to\dim(A_1)$, $t\to\dim(E')$. For the sake of contradiction, assume that $t>\max\{k+s-a,0\}$. Thus, we can find $\sigma>s$ so that $t>\max\{k+\sigma-a,0\}$. Set $\eta=\sigma-s>0$.

We remark that here we did a two-step reduction: we first localize A to be A_1 so the index α becomes α_1; next we use Frostman's lemma to find a a-dimensional measure so that the index α_1 becomes a. Though a is less than α, we can make a arbitrarily close to α.

Now we fix a,t, so we may assume $\mathcal{H}^t_\infty(E')\sim 1$ is a constant. In the following estimates, the "$\lesssim$" notation is allowed to depend on $\mathcal{H}^t_\infty(E')$.

Fix a $y \in E'$. applying Lemma 9 to $\pi_y(A_1)$, we obtain a set of dyadic caps $\mathcal{C}_y = \bigsqcup_j \mathcal{C}_{y,j}$ in $\mathbb{S}^{n-1}$ that cover $\pi_y(A_1)$. Here each $\mathcal{C}_{y,j}$ is a set of 2^{-j}-caps that satisfy the s-dimensional condition (see Lemma 9 (3)) as $\dim(\pi_y(A_1)) < s$. Also, the radius of these caps is less than $\varepsilon_\circ$ which is any given small number.

By the s-dimensional condition of $\mathcal{C}_{y,j}$, we have

$$\mathcal{H}^s_{2^{-j},\infty}\left(\bigcup_{C\in\mathcal{C}_{y,j}} C\right) = \#\mathcal{C}_{y,j}2^{-js} \leq 1.$$

Therefore, we have

$$\mathcal{H}^\sigma_{2^{-j},\infty}\left(\bigcup_{C\in\mathcal{C}_{y,j}} C\right) \leq \#\mathcal{C}_{y,j}2^{-j\sigma} \leq 2^{-j\eta}. \tag{6}$$

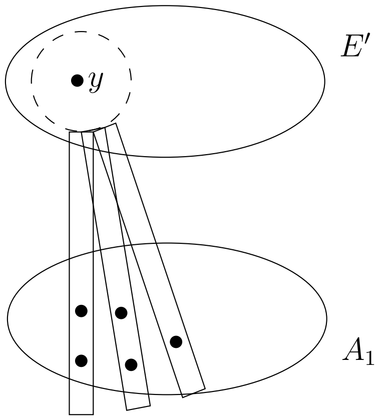

Figure 1. $\mathbb{T}_{y,j}$ in the radial projection.

For each cap $C \in \mathcal{C}_y$, consider $\pi_y^{-1}(C) \cap \{x \in \mathbb{R}^n \colon 1 - \frac{1}{100} \leq |x-y| \leq 1\}$ which is a tube. We obtain a collection of finitely overlapping tubes

$$\mathbb{T}_y = \bigsqcup_j \mathbb{T}_{y,j}$$

that cover A_1 (see Figure 1). This is a truncated bush centered at y. Here, each tube has its coreline passing through y and at distance ~ 1 from y. The tubes in $\mathbb{T}_{y,j}$ have dimensions $\sim 2^{-j} \times \cdots \times 2^{-j} \times 1$.

For this fixed $y \in E'$, there exists a $j(y) \geq |\log_2 \varepsilon_\circ|$ such that

$$\nu_{A_1}\left(A_1 \cap \bigcup_{T\in\mathbb{T}_{y,j(y)}} T\right) \geq \frac{1}{10j(y)^2}\nu_{A_1}(A_1) = \frac{1}{10j(y)^2}. \tag{7}$$

We have a partition $E' = \bigsqcup_j E'_j$ where $E'_j = \{y \in E' \colon j(y) = j\}$. We choose j such that $\mathcal{H}^t_\infty(E'_j) \gtrsim \frac{1}{j^2}$. We let $\delta = 2^{-j}$. Note that $\delta \leq \varepsilon_\circ$ by assumption. By Lemma 7, there exists a subset $E'' \subset E'_j$ which is a (δ, t)-set and $\#E'' \gtrsim |\log\delta|^{-2}\delta^{-t}$. We use μ to denote the counting measure on E''.

Next, we consider the set $S = \{(y,x) \in E'' \times A_1 \colon x \in \bigcup_{T\in\mathbb{T}_{y,j}} T\}$. We also denote the y-section and x-section of S by S^y and S_x. (In Figure 1, E' is drawn above A_1, so we let y be the superscript in S^y). By (7), we have $\nu_{A_1}(S^y) \geq \frac{1}{10j(y)^2}$, so we have

$$(\mu \times \nu_{A_1})(S) \geq \frac{1}{10j^2}\mu(E''). \tag{8}$$

This implies

$$(\mu \times \nu_{A_1})\left(\left\{(y,x) \in S\colon \mu(S_x) \geq \frac{1}{20j^2}\mu(E'')\right\}\right) \geq \frac{1}{20j^2}\mu(E''). \tag{9}$$

Therefore, we have

$$\nu_{A_1}\left(\left\{x \in A_1\colon \mu(S_x) \geq \frac{1}{20j^2}\mu(E'')\right\}\right) \geq \frac{1}{20j^2} \sim |\log\delta|^{-2}. \tag{10}$$

By Lemma 8, we can find a subset F of $\left\{x \in A_1\colon \mu(S_x) \geq \frac{1}{20j^2}\mu(E'')\right\}$, so that F is a (δ, a)-set and $\#F \gtrsim |\log\delta|^{-2}\delta^{-a}$.

Hence,

$$\begin{aligned} |\log\delta|^{-2}\#F\#E'' &\lesssim \sum_{x\in F}\#\left\{y \in E''\colon x \in \bigcup_{T\in\mathbb{T}_{y,j}} T\right\} \\ &= \sum_{y\in E''}\#\left\{x \in F\colon x \in \bigcup_{T\in\mathbb{T}_{y,j}} T\right\}. \end{aligned} \tag{11}$$

By pigeonholing, there exists a subset $E \subset E''$ with $\#E \gtrsim |\log\delta|^{-2}\#E'' \gtrsim \delta^{\varepsilon/2}\delta^{-t}$, so that for any $y \in E$:

$$\#\{x \in F\colon x \in \bigcup_{T\in\mathbb{T}_{y,j}} T\} \gtrsim \delta^{\varepsilon/2}\#F \geq \delta^{\varepsilon}\#F,$$

when δ is small enough.

We set $F_y := \left\{x \in F\colon x \in \bigcup_{T\in\mathbb{T}_{y,j}} T\right\}$. Now we use Theorem 16 to derive a contradiction. Since E is a (δ, t)-set with $\#E \gtrsim \delta^\varepsilon\delta^{-t}$ and F is a (δ, a)-set with $\#F \gtrsim \delta^{-a+\varepsilon}$, Theorem 16 yields the existence of an $y \in E$ such that $\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F_y)) > \delta^\eta$. This contradicts (6). $\square$

Before proving Theorem 16, we prove two propositions. Then we show Theorem 16 is a result of them. The first proposition is a quantitative version of Marstrand's projection theorem. The second proposition is a special case of Theorem 16 when $k = n - 1$.

Proposition 17. *Set $d_{m,n} = m(n-m) = \dim(G(m,n))$. Let $0 < a < m$. Let $\eta \in (0, 1/10)$. Then for ε and δ small enough depending on η, a, we have the following result.*

Let $F \subset B^n(0,1)$ be a (δ, a)-set with $\#F \gtrsim \delta^\varepsilon\delta^{-a}$. Let $G \subset G(m,n)$ be a $(\delta, d_{m,n})$-set, with $\#G \gtrsim \delta^\varepsilon\delta^{-d_{m,n}}$. Then, there exists $G_1 \subset G$ with $\#G_1 \geq 1/2\#G$ such that for any $F' \subset F$ with $\#F' \geq \delta^\varepsilon\#F$ and any $V \in G_1$, we have

$$\mathcal{H}^a_{\delta,\infty}(\pi_V(F')) > \delta^\eta.$$

Here, π_V is the orthogonal projection onto V.

Proof. Actually, we just need to show the existence of a single $V \in G$ that satisfies the property above. Then we can construct $G_1 = \{V_1, \ldots, V_N\}$ inductively and replace G by $G \setminus G_1$, and check whether $\#(G \setminus G_1) \gtrsim \delta^\varepsilon\delta^{-d_{m,n}}$, and then repeat again.

Suppose there does not exist such V. By contradiction, for any $V \in G$, there exists $F_V \subset F$ with $\#F_V \geq \delta^{\varepsilon}\#F$ and

$$\mathcal{H}^a_{\delta,\infty}(\pi_V(F_V)) \leq \delta^{\eta}. \tag{12}$$

By the definition of $\mathcal{H}^a_{\delta,\infty}$, we can find a covering of $\pi_V(F_V)$ by dyadic cubes $\{D\}$ so that $\mathcal{H}^a_{\delta,\infty}(\pi_V(F_V)) = \sum_D r(D)^a$. Consider $\{\pi_V^{-1}(D) \cap B^n(0,1)\}$ which are the preimages of these $\{D\}$ under π_V truncated in the unit ball. They actually form a covering of F_V:

$$F_V \subset \bigsqcup_{\delta \leq \Delta \leq 1} \bigcup_{T \in \mathbb{T}_{V,\Delta}} T.$$

Here, each $\mathbb{T}_{V,\Delta}$ consists of planks of dimensions $\underbrace{\Delta \times \Delta \times \cdots \times \Delta}_{m \text{ times}} \times \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ times}}$ that are orthogonal to V. By Lemma 11, $\mathbb{T}_{V,\Delta}$ satisfies the a-dimensional spacing condition (inherited from $\{D\}$): For $\Delta \leq r \leq 1$, if T_r is a plank of dimensions $\underbrace{r \times r \times \cdots \times r}_{m \text{ times}} \times \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ times}}$ that is orthogonal to V, then T_r contains $\lesssim (r/\Delta)^a$ many planks from $\mathbb{T}_{V,\Delta}$. Also by (12),

$$\#\mathbb{T}_{V,\Delta} \lesssim \delta^{\eta}\Delta^{-a}. \tag{13}$$

We see that $\mathbb{T}_{V,\Delta}$ is non-empty only for $\Delta \leq \delta^{\eta/a}$.

Next, we will apply a standard pigeonholing argument to find a scale Δ. Note that

$$F_V \subset \bigsqcup_{\delta \leq \Delta \leq \delta^{\eta/a}} \bigcup_{T \in \mathbb{T}_{V,\Delta}} T.$$

For each $V \in G$, we can find a dyadic $\Delta(V) \in [\delta, \delta^{\eta/a}]$ so that

$$\#\left(F_V \cap \bigcup_{T \in \mathbb{T}_{V,\Delta(V)}} T\right) \gtrsim |\log\delta|^{-1}\#F_V \gtrsim \delta^{\varepsilon}\#F. \tag{14}$$

Define $G_\Delta = \{V \in G \colon \Delta(V) = \Delta\}$. We see that

$$G = \bigsqcup_{\delta \leq \Delta \leq \delta^{\eta/a}} G_\Delta.$$

By pigeonholing again, we can find a scale Δ, such that

$$\#G_\Delta \gtrsim \delta^{\varepsilon}\#G. \tag{15}$$

We fix this Δ. Noting that G is a $(\delta, d_{m,n})$-set with $\#G \gtrsim \delta^{\varepsilon}\delta^{-d_{m,n}}$, we have that G_Δ is also a $(\delta, d_{m,n})$-set with $\#G_\Delta \gtrsim \delta^{2\varepsilon}\delta^{-d_{m,n}}$. By Lemma 13, we can find a subset G' of G_Δ so that G' is a $(\Delta, d_{m,n})$-set with $\#G' \gtrsim \delta^{2\varepsilon}\Delta^{-d_{m,n}}$. From (14), we have for any $V \in G'$

$$\#\left(F \cap \bigcup_{T \in \mathbb{T}_{V,\Delta}} T\right) \gtrsim \delta^{\varepsilon}\#F \gtrsim \delta^{2\varepsilon - a}. \tag{16}$$

Next, we consider the set

$$S := \left\{(x, V) \in F \times G' \colon x \in \bigcup_{T \in \mathbb{T}_{V,\Delta}} T\right\}.$$

Define the sections of S:

$$S_x := \{V \in G' \colon (x,V) \in S\}, \quad S_V := \{x \in F \colon (x,V) \in S\}.$$

By (16), we have $\#S_V \gtrsim \delta^\varepsilon \#F$ for $V \in G'$. Then we have

$$\#S = \sum_{V \in G'} \#S_V \geq C^{-1}\delta^\varepsilon \#G' \#F. \tag{17}$$

Since

$$\#\{(x,V) \in S \colon \#S_x \leq (2C)^{-1}\delta^{2\varepsilon}\#G'\} \leq (2C)^{-1}\delta^{2\varepsilon}\#G'\#F \leq \frac{1}{2}\#S,$$

we have

$$\#\{(x,V) \in S \colon \#S_x \geq (2C)^{-1}\delta^{2\varepsilon}\#G'\} \gtrsim \delta^\varepsilon \#G'\#F.$$

The inequality above implies

$$\#\{x \in F \colon \#S_x \geq (2C)^{-1}\delta^{2\varepsilon}\#G'\} \gtrsim \delta^\varepsilon \#F.$$

We define

$$F_\Delta := \{x \in F \colon \#S_x \geq (2C)^{-1}\delta^{2\varepsilon}\#G'\}. \tag{18}$$

Noting that F_Δ is a (δ, a)-set with $\#F_\Delta \gtrsim \delta^{2\varepsilon - a}$, by Lemma 13, we can find a subset $F' \subset F_\Delta$ such that F' is a (Δ, a)-set with

$$\#F' \gtrsim \delta^{2\varepsilon}\Delta^{-a}. \tag{19}$$

Let us summarize what we obtained. We find a scale $\Delta \in [\delta, \delta^{\eta/a}]$, a $(\Delta, d_{m,n})$-set $G' \subset G$ with $\#G' \gtrsim \delta^{2\varepsilon}\Delta^{-d_{m,n}}$, and a (Δ, a)-set $F' \subset F$ with $\#F' \gtrsim \delta^{2\varepsilon}\Delta^{-a}$, so that

(i) for each $V \in G'$, we have a set of tubes $\mathbb{T}_{V,\Delta}$ that satisfy the a-dimensional spacing condition and $\#\mathbb{T}_{V,\Delta} \lesssim \delta^\eta \Delta^{-a}$ (see paragraph before (13)),

(ii) each $x \in F'$ is contained in $\gtrsim \delta^{2\varepsilon}\#G' \gtrsim \delta^{4\varepsilon}\Delta^{-d_{m,n}}$ planks from $\bigcup_{V \in G'} \mathbb{T}_{V,\Delta}$ (see (18)).

In the rest of the proof, we fix Δ and simply write $\mathbb{T}_{V,\Delta}$ as $\mathbb{T}_V$.

For each $V \in G'$, let D_V be a

$$\underbrace{\Delta^{-1} \times \Delta^{-1} \times \cdots \times \Delta^{-1}}_{m \text{ times}} \times \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ times}}$$

slab centered at the origin such that the $1 \times 1 \times \cdots \times 1$-side is orthogonal to V. Then, D_V is the dual rectangle of the slabs in $\mathbb{T}_V$.

For all $T \in \mathbb{T}_V$, choose a smooth bump function ψ_T adapted to T such that $\psi_T \geq 1$ on T, ψ_T decays rapidly outside of T, and supp $\widehat{\psi}_T \subset D_V$. Define

$$f_V = \sum_{T \in \mathbb{T}_V} \psi_T \quad \text{and} \quad f = \sum_{V \in G'} f_V.$$

Then by the condition (ii) above, for $x \in N_\Delta(F')$, we have

$$f(x) \gtrsim \delta^{4\varepsilon}\Delta^{-d_{m,n}}.$$

So,

$$\int_{N_\Delta(F')} |f|^2 \gtrsim \delta^{O(\varepsilon)}\Delta^n \Delta^{-a-2d_{m,n}}. \tag{20}$$

We are going to find an upper bound of $\int_{N_\Delta(F')} |f|^2$ using the high-low method. Let K be a large number to be determined later (we will actually choose $K \sim \delta^{-O(\varepsilon)}$).

Let $\eta_{\text{low}}(\xi)$ be a smooth bump function on $B^n(0,(K\Delta)^{-1})$ and $\eta_{\text{high}}(\xi)=1-\eta_{\text{low}}(\xi)$. We have the following high-low decomposition for f:

$$f=f_{\text{low}}+f_{\text{high}},$$

where $\widehat{f}_{\text{low}}=\eta_{\text{low}}\widehat{f}$ and $\widehat{f}_{\text{high}}=\eta_{\text{high}}\widehat{f}$. See Figure 2 for a diagram of the high part and low part and the dual slabs.

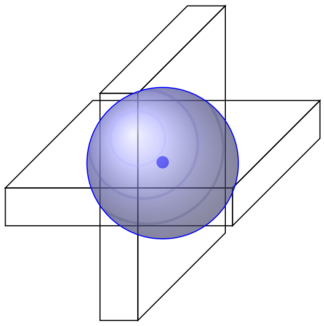

Figure 2. Dual slabs.

For $x\in N_\Delta(F')$, we have

$$\delta^{4\varepsilon}\Delta^{-d_{m,n}}\lesssim f(x)\le|f_{\text{high}}(x)|+|f_{\text{low}}(x)|. \tag{21}$$

We will show that the high part dominates for $x\in N_\Delta(F')$, i.e., $|f_{\text{high}}(x)|\gtrsim\delta^{4\varepsilon}\Delta^{-d_{m,n}}$. It suffices to show

$$|f_{\text{low}}(x)|\le C^{-1}\delta^{4\varepsilon}\Delta^{-d_{m,n}}, \tag{22}$$

for a large constant C.

Recall that $f_{\text{low}}=\sum_{V\in G'}f_V*\eta^\vee_{\text{low}}$. Since η_{low} is a bump function on $B^n(0,(K\Delta)^{-1})$, we see that $\eta^\vee_{\text{low}}$ is an L^1-normalized bump function essentially supported in $B^n(0,K\Delta)$. Let $\chi(x)$ be a positive function $=1$ on $B^n(0,K\Delta)$ and decays rapidly outside $B^n(0,K\Delta)$. We have

$$|\eta^\vee_{\text{low}}|\lesssim\frac{1}{|B^n(0,K\Delta)|}\chi.$$

Therefore,

$$|f_{\text{low}}(x)|\lesssim\sum_{V\in G'}\sum_{T\in\mathbb{T}_V}\psi_T*\frac{1}{|B^n(0,K\Delta)|}\chi(x)\lesssim\sum_{V\in G'}\sum_{T\in\mathbb{T}_V}K^{-m}\chi_{T_K}(x). \tag{23}$$

Here, each T_K is a plank of dimensions

$$\underbrace{K\Delta\times K\Delta\times\cdots\times K\Delta}_{m\text{ times}}\times\underbrace{1\times1\times\cdots\times1}_{n-m\text{ times}}$$

which is the K-thickening of the $\Delta\times\cdots\times\Delta$-side of T, and χ_{T_K} is a bump function $=1$ on T_K and decays rapidly outside T_K. The rapidly decaying tail is negligible, so we can think of each χ_{T_K} as the indicator function of T_K. For a fixed $V\in G'$, we note that $\{T\colon T\in\mathbb{T}_V\}$ are orthogonal to V. Therefore, if we let $P_{K\Delta}$ be a plank of dimensions

$$\underbrace{K\Delta\times K\Delta\times\cdots\times K\Delta}_{m\text{ times}}\times\underbrace{1\times1\times\cdots\times1}_{n-m\text{ times}}$$

that is orthogonal to V and contains x, then by condition (i),

$$\sum_{T\in\mathbb{T}_V}\chi_{T_K}(x)\lesssim\#\{T\in\mathbb{T}_V\colon T\subset P_{K\Delta}\}\lesssim K^a,$$

where the last inequality is by the a-dimensional condition of $\mathbb{T}_V$. Plugging this back into (23), we obtain

$$|f_{\mathrm{low}}(x)| \lesssim K^{a-m} \# G' \lesssim K^{a-m} \Delta^{-d_{m,n}}.$$

(G' is a $(\Delta, d_{m,n})$-set, so $\#G' \lesssim \Delta^{-d_{m,n}}$.)

Noting that $a < m$, we may choose $K \sim_{a,m} \delta^{-O_{a,m}(\varepsilon)}$ so that (22) holds. Plugging back to (20), we have

$$\Delta^{n-a-2d_{m,n}} \lessapprox \int |f_{\mathrm{high}}|^2 = \int \left| \sum_{V \in G'} \widehat{f_V} \eta_{\mathrm{high}} \right|^2 .$$

Here, $A \lessapprox B$ means $A \lesssim \delta^{-O(\varepsilon)} B$. It is good to mention that since $\Delta \le \delta^{\eta/a}$, by choosing ε small enough depending on η, a, we have that K is much smaller than Δ^{-1}.

We use the following lemma to estimate the overlap of $\{\mathrm{supp}(\widehat{f_V}\eta_{\mathrm{high}})\}_{V \in G'}$, or more precisely $\{D_V \setminus B^n(0, (K\Delta)^{-1})\}_{V \in G'}$. After rescaling $x \mapsto \Delta x$, each D_V becomes a $1 \times \cdots \times 1 \times \Delta \times \cdots \times \Delta$-plank with m many 1's and $(n-m)$ many Δ's in the expression. We denote this rescaled plank by P_V. We can see that P_V is morally $N_\Delta(V) \cap B^n(0,1)$. It is harmless to just assume

$$P_V = N_\Delta(V) \cap B^n(0,1).$$

We also see that after rescaling, $D_V \setminus B^n(0,(K\Delta)^{-1})$ becomes $P_V \setminus B^n(0, K^{-1})$. We will bound the overlaps of $\{P_V \setminus B^n(0,K^{-1})\}$ where $\{P_V\}$ are essentially distinct.

Lemma 18. *$\{P_V \setminus B^n(0,K^{-1})\}_{V \in G'}$ is $\lesssim K^{O(1)} \Delta^{-\dim(G(m-1,n-1))}$-overlapping.*

Proof. We will estimate the number of overlaps at the point $\boldsymbol{\xi}_0 = (0, \dots, 0, \lambda)$ with $\lambda \in [K^{-1}, 1]$. We just need to show that the number of planks P_V that pass through 0 and $\boldsymbol{\xi}_0$ is $\lesssim K^{O(1)} \Delta^{-\dim(G(m-1,n-1))}$.

We first talk about some properties for the smooth manifold $G(m,n)$. For $V_1, V_2 \in G(m,n)$, define $d(V_1,V_2) = \|\pi_{V_1} - \pi_{V_2}\|$. Then $d(\cdot,\cdot)$ gives a metric on $G(m,n)$. We need another characterization for this distance. Define $\rho(V_1,V_2)$ to be the smallest number ρ such that $B^n(0,1) \cap V_1 \subset N_\rho(B^n(0,1) \cap V_2)$. We claim that $\rho(V_1,V_2) \sim d(V_1,V_2)$. Suppose $B^n(0,1) \cap V_1 \subset N_\rho(B^n(0,1) \cap V_2)$, then for any $v \in \mathbb{R}^n$ we have

$$|\pi_{V_1}(v) - \pi_{V_2}(v)| \lesssim \rho |v|,$$

which implies $d(V_1,V_2) \lesssim \rho$. On the other hand, if for any $|v| \le 1$ we have

$$|\pi_{V_1}(v) - \pi_{V_2}(v)| \le d|v|,$$

then we obtain that $\pi_{V_1}(v) \subset N_{Cd}(B^n(0,1) \cap V_2)$. Letting v range over the unit ball in V_1, we get $B^n(0,1) \cap V_1 \subset N_{Cd}(B^n(0,1) \cap V_2)$.

Consider the $\widetilde{G} = \{W \in G(m,n) \colon 0, \boldsymbol{\xi}_0 \in W\}$ which is a submanifold of $G(m,n)$. $\widetilde{G}$ is the set of m-subspaces that contain the n-th axis. One can see that $\widetilde{G}$ is isomorphic to $G(m-1,n-1)$. We return back to $P_V = N_\Delta(V) \cap B^n(0,1)$. We make the following geometric observation: if $\boldsymbol{\xi}_0 \subset P_V$, then there exists $W \in \widetilde{G}$ so that $W \cap B^n(0,1) \subset N_{CK\Delta}(V) \cap B^n(0,1)$. Recall the length of $\boldsymbol{\xi}_0$ is $\lambda \in [K^{-1}, 1]$, so the angle between $\boldsymbol{\xi}_0$ and V is $\lesssim \Delta K$. Therefore the unit vector $\lambda^{-1}\boldsymbol{\xi}_0$ is contained in $N_{CK\Delta}(V) \cap B^n(0,1)$. It suffices to find an m-dimensional space W such that $\lambda^{-1}\boldsymbol{\xi}_0 \in W$ and $W \cap B^n(0,1) \subset N_{10CK\Delta}(V) \cap B^n(0,1)$. Let v be the projection of $\lambda^{-1}\boldsymbol{\xi}_0$ onto V, then the angle between v and $\lambda^{-1}\boldsymbol{\xi}_0$ is $\lesssim \Delta K$. Imagine we choose a family of vectors $v(\theta), \theta \in [0, \Delta K]$ so that $v(0) = v$, $v(\Delta K) = \lambda^{-1}\boldsymbol{\xi}_0$ and also

$|v(\theta_1) - v(\theta_2)| \lesssim |\theta_1 - \theta_2|$. Actually, we can choose them so that $v(\theta)$ lies on the line segment connecting $v(0)$ and $v(\Delta K)$. Starting with $\theta = 0$, we choose the m-dimensional space $V(0) = V$ so that $v(0) \in V(0)$. When θ changes we get a family of m-dimensional subspaces $V(\theta)$ by rotating $V(0) = V$ so that $v(\theta) \in V(\theta)$. When θ changes from 0 to ΔK, we see we rotate V to another space $W = V(\Delta K)$ within angle $\lesssim \Delta K$. Therefore we find the desired subspace W.

We proved that there exists $W \in \widetilde{G}$ so that $W \cap B^n(0,1) \subset N_{CK\Delta}(V) \cap B^n(0,1)$. By the comparability of the metric discussed in the previous two paragraphs, we see that $d(V, \widetilde{G}) \lesssim \Delta K$. In other word, those $V \in G(m,n)$ satisfying $\boldsymbol{\xi}_0 \in P_V$ is contained in the $C\Delta K$-neighborhood of $\widetilde{G}$ in $G(m,n)$. We denote this neighborhood by $N_{C\Delta K}(\widetilde{G})$. Noting that $\widetilde{G}$ is submanifold of dimension $\dim(G(m-1,n-1)) = (m-1)(n-m)$ and G' is a Δ-separated subset of $G(m,n)$, we get the number of overlaps of G' at $\boldsymbol{\xi}_0$ is

$$\lesssim \text{measure}\big(N_{C\Delta K}(\widetilde{G})\big)/\Delta^{\dim(G(m,n))} \sim K^{O(1)}\Delta^{-\dim(G(m-1,n-1))}.$$

Note that we use a simple fact: If $\widetilde{M}$ is an m-dimensional smooth submanifold of the n-dimensional manifold M, then

$$\text{measure}\big(N_r(\widetilde{M})\big) \lesssim_{M,\widetilde{M}} r^{n-m},$$

for $0 \le r \le 1$. $\square$

We are now able to find an upper bound to the high part of the integral. We have

$$\Delta^{n-a-2d_{m,n}} \lessapprox \int |f_{\text{high}}|^2 = \int |\widehat{f}_{\text{high}}|^2 \lessapprox \Delta^{-\dim(G(m-1,n-1))} \sum_{V \in G'} \int |\eta_{\text{high}} \widehat{f}_V|^2$$

by Lemma 18. Since $|\eta_{\text{high}}| \lesssim 1$ and the planks in $\mathbb{T}_V$ (for a fixed V) are essentially disjoint, we have

$$\int |\eta_{\text{high}} \widehat{f}_V|^2 \lesssim \sum_{V \in G'} \int |f_V|^2 \lesssim \sum_{V \in G'} \sum_{T \in \mathbb{T}_V} \int |\psi_T|^2 \le (\#G')(\#\mathbb{T}_V)\Delta^m \lesssim \delta^\eta \Delta^{-d_{m,n}-a+m}.$$

Here we have a factor δ^η because of the upper bound (13), and we remark that δ^η is quite important to get a contradiction.

Combining everything and noting that $\dim(G(m-1,n-1)) = (m-1)(n-m)$, we have that

$$1 \lessapprox \delta^\eta.$$

Unwrapping the notation, we get

$$1 \lesssim \delta^{-O(\varepsilon)+\eta}.$$

This is impossible if we choose δ, ε small enough depending on η. We get a contradiction. $\square$

Proposition 19. *Let $0 < \sigma < n-1$, $a \in (n-1, n]$ and $t > \max\{n-1+\sigma-a, 0\}$. Let $0 < \eta < 1$. Then for ε and δ small enough depending on σ, t, and η, we have the following result.*

Let $E, F \subset B^n(0,1)$ so that E is a (δ, t)-set with $\#E \gtrsim \delta^\varepsilon \delta^{-t}$ and F satisfies $\mathcal{H}^a_{\delta,\infty}(F) \gtrsim \delta^\varepsilon$, $\#F \lesssim \delta^{-a}$. (We remark that we did not assume F is δ-separated.) We *also assume that each of E and F lies in a ball of radius $1/1000$ and $\operatorname{dist}(E,F)\ge 1/2$. Then, there exists $y\in E$ such that for all $F'\subset F$ with $\mathcal{H}^a_{\delta,\infty}(F')\ge\delta^\varepsilon$, we have*

$$\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F'))>\delta^\eta.$$

Proof. Since $n-1+\sigma-a<\sigma$, it suffices to prove the proposition for $t<\sigma$. Assume for the sake of contradiction that for all $y\in E$ there exists $F_y\subset F$ with $\mathcal{H}^a_{\delta,\infty}(F_y)\ge\delta^\varepsilon$ such that

$$\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F_y))\le\delta^\eta.$$

We first reduce F to a (δ,a)-set. The algorithm goes as follows. By the condition that $\mathcal{H}^a_{\delta,\infty}(F)\gtrsim\delta^\varepsilon$ and Lemma 12, we can find a (δ,a)-set $F_1\subset F$ with $\#F_1\gtrsim\delta^{-a+2\varepsilon}$. We look at $F\setminus F_1$. If $\mathcal{H}^a_{\delta,\infty}(F\setminus F_1)\le\delta^{2\varepsilon}$, we stop; If $\mathcal{H}^a_{\delta,\infty}(F\setminus F_1)\ge\delta^{2\varepsilon}$, we find a (δ,a)-set $F_2\subset F\setminus F_1$ with $\#F_2\gtrsim\delta^{-a+2\varepsilon}$. Repeating the algorithm until we stop, we obtain a decomposition

$$F=\left(\bigsqcup_{i=1}^N F_i\right)\sqcup F_0,$$

where each F_i $(1\le i\le N)$ is a (δ,s)-set with cardinality $\gtrsim\delta^{-a+2\varepsilon}$, and F_0 satisfies $\mathcal{H}^a_{\delta,\infty}(F_0)\le\delta^{2\varepsilon}$. We also see that $N\lesssim\#F/\delta^{-a+2\varepsilon}\lesssim\delta^{-2\varepsilon}$.

For any $y\in E$, we have $\delta^\varepsilon\le\sum_{i=0}^N\mathcal{H}^a_{\delta,\infty}(F_y\cap F_i)\le\delta^{2\varepsilon}+\sum_{i=1}^N\mathcal{H}^a_{\delta,\infty}(F_y\cap F_i)$. By pigeonholing, there exists $i=i(y)$ such that $\mathcal{H}^a_{\delta,\infty}(F_y\cap F_i)\gtrsim\delta^{3\varepsilon}$. By another pigeonholing, there exists $i\in[1,N]$, such that

$$\#\{y\in E\colon i(y)=i\}\gtrsim\delta^{2\varepsilon}\#E.$$

For simplicity, we will still use the old notation. We replace E by $\{y\in E\colon i(y)=i\}$, F by F_i, F_y by $F_y\cap F_i$, and ε by $\varepsilon/10$. Then, E is still a (δ,t)-set with $\#E\gtrsim\delta^\varepsilon\delta^{-t}$; F is a (δ,a)-set with $\#F\gtrsim\delta^\varepsilon\delta^{-a}$; $F_y\subset F$ and $\#F_y\gtrsim\delta^\varepsilon\#F$ for each $y\in E$ (since $\mathcal{H}^a_{\delta,\infty}(F_y)\gtrsim\delta^{3\varepsilon/10}$ implies $\#F_y\gtrsim\delta^{-a+\varepsilon}\gtrsim\delta^\varepsilon\#F$); moreover,

$$\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F_y))\le\delta^\eta.$$

We will derive a contradiction.

By the definition of $\mathcal{H}^\sigma_{\delta,\infty}$, we can find a covering of $\pi_y(F_y)$ by dyadic caps $\{D\}$ in $\mathbb{S}^{n-1}$, so that $\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F_y))=\sum_D r(D)^\sigma$. For each such D, consider $\pi_y^{-1}(D)\cap\big(B^n(y,2)\setminus B^n(y,1/4)\big)$. It is roughly a tube of length ~ 1 and radius comparable to the radius of D. We put those tubes with radius comparable to Δ together, which is the following set

$$\mathbb{T}_{y,\Delta}:=\left\{\pi_y^{-1}(D)\cap\big(B^n(y,2)\setminus B^n(y,1/4)\big)\colon \Delta/2\le\operatorname{diam}(D)\le\Delta\right\}.$$

(See Figure 3 for the configuration of these tubes.)

By the separation of E,F and noting E,F are contained in $B^n(0,1)$, we see that the tubes obtained in this way form a covering of F_y:

$$F_y\subset\bigsqcup_{\delta\le\Delta\le1}\ \bigcup_{T\in\mathbb{T}_{y,\Delta}}T.$$

$\mathbb{T}_{y,\Delta}$ satisfies the σ-dimensional spacing condition (inherited from $\{D\}$): For $\Delta\le r\le1$, if T_r is a tube of radius r length 1 that passes through y, then T_r contains $\lesssim(r/\Delta)^\sigma$ many tubes from $\mathbb{T}_{y,\Delta}$. Since $\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F_y))\le\delta^\eta$, we have

$$\#\mathbb{T}_{y,\Delta}\lesssim\delta^\eta\Delta^{-\sigma}. \tag{24}$$

We see that $\mathbb{T}_{y,\Delta}$ is non-empty only for $\Delta \le \delta^{\eta/\sigma}$.

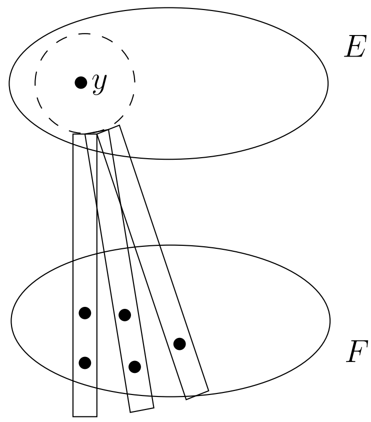

Figure 3. $\mathbb{T}_{y,\Delta}$ in the radial projection.

Next, we will apply a standard pigeonholing argument to find a scale Δ. Note that

$$F_y \subset \bigsqcup_{\delta \le \Delta \le \delta^{\eta/a}} \bigcup_{T \in \mathbb{T}_{y,\Delta}} T.$$

For each $y \in E$, we can find a dyadic $\Delta(y) \in [\delta, \delta^{\eta/a}]$ so that

$$\#\left(F_y \cap \bigcup_{T \in \mathbb{T}_{y,\Delta(y)}} T \right) \gtrsim |\log \delta|^{-1} \# F_y \gtrsim \delta^{\varepsilon} \# F. \tag{25}$$

Define $E_\Delta = \{y \in E \colon \Delta(y) = \Delta\}$. We see that

$$E = \bigcup_{\delta \le \Delta \le \delta^{\eta/a}} E_\Delta.$$

By pigeonholing again, we can find a scale Δ, such that

$$\# E_\Delta \gtrsim \delta^{\varepsilon} \# E. \tag{26}$$

We fix this Δ. Noting that E is a (δ, t)-set with $\#E \gtrsim \delta^{\varepsilon}\delta^{-t}$, we have that E_Δ is also a (δ, t)-set with $\#E_\Delta \gtrsim \delta^{2\varepsilon}\delta^{-t}$. By Lemma 13, we can find a subset E' of E_Δ so that E' is a (Δ, t)-set with $\#E' \gtrsim \delta^{2\varepsilon}\Delta^{-t}$. From (25), we have for any $y \in E'$ that

$$\#\left(F \cap \bigcup_{T \in \mathbb{T}_{y,\Delta}} T \right) \gtrsim \delta^{\varepsilon} \# F \gtrsim \delta^{2\varepsilon - a}. \tag{27}$$

Next, we consider the set

$$S := \left\{ (x, y) \in F \times E' \colon x \in \bigcup_{T \in \mathbb{T}_{y,\Delta}} T \right\}.$$

Define the sections of S:

$$S_x := \{y \in E' \colon (x, y) \in S\}, \quad S^y := \{x \in F \colon (x, y) \in S\}.$$

By (27), we have $\#S^y \gtrsim \delta^{\varepsilon} \# F$ for $y \in E'$. Then we have

$$\# S = \sum_{y \in E'} \# S^y \ge C^{-1} \delta^{\varepsilon} \# E' \# F. \tag{28}$$

Since

$$\#\{(x,y)\in S\colon \#S_x\le (2C)^{-1}\delta^{2\varepsilon}\#E'\}\le (2C)^{-1}\delta^{2\varepsilon}\#E'\#F\le \frac{1}{2}\#S,$$

we have

$$\#\{(x,y)\in S\colon \#S_x\ge (2C)^{-1}\delta^{2\varepsilon}\#E'\}\gtrsim \delta^{\varepsilon}\#E'\#F.$$

The inequality above implies

$$\#\{x\in F\colon \#S_x\ge (2C)^{-1}\delta^{2\varepsilon}\#E'\}\gtrsim \delta^{\varepsilon}\#F.$$

We define

$$F_\Delta := \{x\in F\colon \#S_x\ge (2C)^{-1}\delta^{2\varepsilon}\#E'\}. \tag{29}$$

Noting that F_Δ is a (δ,a)-set with $\#F_\Delta\gtrsim \delta^{2\varepsilon-a}$, by Lemma (13), we can find a subset $F'\subset F_\Delta$ such that F' is a (Δ,a)-set with

$$\#F'\gtrsim \delta^{2\varepsilon}\Delta^{-a}. \tag{30}$$

Let us summarize what we obtained. We find a scale $\Delta\in[\delta,\delta^{\eta/\sigma}]$, a (Δ,t)-set $E'\subset E$ with $\#E'\gtrsim \delta^{2\varepsilon}\Delta^{-t}$, and a (Δ,a)-set $F'\subset F$ with $\#F'\gtrsim\delta^{2\varepsilon}\Delta^{-a}$, so that

- (i) for each $y\in E'$, we have a set of tubes $\mathbb{T}_{y,\Delta}$ that satisfy the σ-dimensional spacing condition with $\#\mathbb{T}_{y,\Delta}\lesssim \delta^{\eta}\Delta^{-\sigma}$ (see paragraph before (24)),
- (ii) each $x\in F'$ is contained in $\gtrsim \delta^{2\varepsilon}\#E'\gtrsim \delta^{4\varepsilon}\Delta^{-t}$ tubes from $\bigcup_{y\in E'}\mathbb{T}_{y,\Delta}$ (see (29)).

We see that we have reduced the problem to the following lemma, from which we will obtain a contradiction. ☐

Lemma 20. *Let $0<t<\sigma<n-1$, $a\in(n-1,n]$. Let $0<\eta<1/100$. Let $0<\delta\le\Delta\le\delta^{\eta/\sigma}$, $\varepsilon>0$, where δ,ε are small enough depending on η,t,σ,a. Let $E,F\subset B^n(0,1)$ be non-empty Δ-separated sets where*

1. *E is a (Δ,t)-set with cardinality $\#E\gtrsim\Delta^{-t}\delta^{\varepsilon}$,*
2. *F is a (Δ,a)-set with cardinality $\#F\gtrsim\Delta^{-a}\delta^{\varepsilon}$,*
3. *each of E and F lies in a ball of radius $1/1000$ and $\mathrm{dist}(E,F)\ge 1/2$.*

For all $y\in E$, we assume there exists a collection of Δ-tubes $\mathbb{T}_y$, such that

1. *each $T\in\mathbb{T}_y$ is of form $\pi_y^{-1}(C)\cap\{x\in\mathbb{R}^n\colon 1-\frac{1}{100}\le|x-y|\le 1\}$ for some dyadic Δ-cap $C\subset\mathbb{S}^{n-1}$,*
2. *$\mathbb{T}_y$ is a (Δ,σ)-set of tubes with cardinality $\#\mathbb{T}_y\lesssim\delta^{\eta}\Delta^{-\sigma}$,*
3. *and for all $x\in F$, $\#\{y\in E\colon \exists T\in\mathbb{T}_y \text{ such that } x\in T\}\gtrsim\Delta^{-t}\delta^{\varepsilon}$.*

Then,

$$\delta^{O(\varepsilon)}\Delta^{-t}\lesssim \delta^{\frac{\sigma-t}{\sigma}\eta}\Delta^{-(n-1)-\sigma+a},$$

which implies that $t\le n-1+\sigma-a$ (if δ is small enough and ε is very small depending on η,t,σ). This contradicts the assumption $t>\max\{n-1+\sigma-a,0\}$ in Proposition 19.

Proof. We will modify $\mathbb{T}_y$ a little bit. Since we will consider the interplay among $\{\mathbb{T}_y\}_{y\in E}$, we want to make the comparable tubes to be exactly the same. Note that F is contained in a ball $B_{1/1000}$ of radius $1/1000$. We choose a set of $\delta/100$-separated directions in $\mathbb{S}^{n-1}$, denoted by $\Theta=\{\theta\}$. For each direction $\theta\in\Theta$, we choose $\mathbb{T}_\theta$ to be a set of 100δ-tubes that point to the direction θ and form a finitely overlapping covering of $B_{1/1000}$. Denote $\mathbb{T}=\cup_\theta\mathbb{T}_\theta$. If $\{\mathbb{T}_\theta\}_{\theta\in\Theta}$ are chosen properly, then for any

δ-tubes T, there exists T' in some $\mathbb{T}_\theta$ such that $T \cap B_{1/1000} \subset T'$. We modify every $\mathbb{T}_y$ in this way. Let us call the original $\mathbb{T}_y$ to be $\mathbb{T}_y^{\mathrm{old}}$. We redefine

$$\mathbb{T}_y := \{T' \in \mathbb{T} \colon T \cap B_{1/1000} \subset T' \text{ for some } T \in \mathbb{T}_y^{\mathrm{old}}\}.$$

Now $\mathbb{T}_y \subset \mathbb{T}$. Also, the new $\mathbb{T}_y$ inherits the properties of the old $\mathbb{T}_y$: $\mathbb{T}_y$ is a (Δ, σ)-set with $\#\mathbb{T}_y \lesssim \delta^\eta \Delta^{-\sigma}$; for any $x \in F$, $\#\{y \in E \colon \exists T \in \mathbb{T}_y$ such that $x \in T\} \gtrsim \Delta^{-t}\delta^\varepsilon$. After the modification, $\mathbb{T}_y$ is still a truncated bush centered at y.

Fix a $y \in E$. For any $T \in \mathbb{T}_y$, choose a bump function ψ_T such that $\psi_T \geq 1$ on T, ψ_T decays rapidly outside of T, and supp $\widehat{\psi_T}$ is contained in the dual rectangle of T which is a $\Delta^{-1} \times \cdots \times \Delta^{-1} \times 1$-slab. Define

$$f_y = \sum_{T \in \mathbb{T}_y} \psi_T \quad \text{and} \quad f = \sum_{y \in E} f_y.$$

Then, for $x \in N_\Delta(F)$, $f(x) \gtrsim \#\{y \in E \colon \exists T \in \mathbb{T}_y \text{ such that } x \in T\} \gtrsim \Delta^{-t}\delta^\varepsilon$ by assumption. Therefore,

$$\delta^{O(\varepsilon)}\Delta^{-2t-a+n} \lesssim \delta^{2\varepsilon}\Delta^{-2t}(\#F)\Delta^n \lesssim \int_{N_\Delta(F)} |f|^2. \tag{31}$$

We will use the high-low method. Let $\eta_{\mathrm{low}}(\xi)$ be a smooth bump function on $B^n(0, (K\Delta)^{-1})$ and $\eta_{\mathrm{high}} = 1 - \eta_{\mathrm{low}}$. We will choose $K \sim \delta^{-O(\varepsilon)}$. Define $f_{\mathrm{low}} = \eta_{\mathrm{low}}^\vee * f$ and $f_{\mathrm{high}} = \eta_{\mathrm{high}}^\vee * f$.

For $x \in N_\Delta(F)$, we have

$$\Delta^{-t}\delta^\varepsilon \lesssim f(x) \leq |f_{\mathrm{low}}(x)| + |f_{\mathrm{high}}(x)|.$$

We claim that

$$|f_{\mathrm{low}}(x)| \lesssim K^{\sigma-(n-1)}\#E \leq C^{-1}\Delta^{-t}\delta^\varepsilon,$$

if $K \sim \delta^{-O(\varepsilon)}$ is properly chosen. To show the claim, we write

$$|f_{\mathrm{low}}(x)| \leq \sum_{y \in E} |\eta_{\mathrm{low}}^\vee| * f_y(x) \leq \sum_{y \in E}\sum_{T \in \mathbb{T}_y} |\eta_{\mathrm{low}}^\vee| * \psi_T(x).$$

Note that $|\eta_{\mathrm{low}}^\vee|(x) \lesssim (K\Delta)^{-n}\chi(x)$, where $\chi(x)$ is a positive function $= 1$ on $B^n(0, K\Delta)$ and decays rapidly outside $B^n(0, K\Delta)$. Therefore,

$$|\eta_{\mathrm{low}}^\vee| * \psi_T(x) \lesssim K^{-(n-1)}\chi_{KT}(x),$$

where $\chi_{KT}(x) = 1$ on KT and decays rapidly outside KT. Since $\mathbb{T}_y$ is a (Δ, σ)-set, we have for $x \in F$,

$$\#\{T \in \mathbb{T}_y \colon x \in 100KT\} \lesssim K^\sigma.$$

Therefore, $\sum_{T \in \mathbb{T}_y} |\eta_{\mathrm{low}}^\vee| * \psi_T(x) \lesssim K^{\sigma-(n-1)}$. Summing over $y \in E$, we prove the claim.

Therefore, we have $|f(x)| \lesssim |f_{\mathrm{high}}(x)|$ on $N_\Delta(F)$. We have

$$\int_{N_\Delta(F)} |f|^2 \lesssim \int |f_{\mathrm{high}}|^2.$$

Here is where things become a little more different than the high-low argument in the proof of Proposition 17. A tube may belong to many different $\mathbb{T}_y$. For each $T \in \mathbb{T}$, define

$$n_T := \#\{y \in E \mid T \in \mathbb{T}_y\}.$$

n_T can be 0, which means $T \notin \mathbb{T}_y$ for any $y \in E$. We have

$$\int |f_{\text{high}}|^2 = \int \left| \sum_{T \in \mathbb{T}} n_T \cdot \psi_{T,\text{high}} \right|^2.$$

Here, $\psi_{T,\text{high}} = \eta_{\text{high}}^{\vee} * \psi_T$. If $T \in \mathbb{T}_\theta$, let S_θ be the slab centered at the origin, of dimensions $\Delta^{-1} \times \cdots \times \Delta^{-1} \times 1$, which is the dual of T. We also see that S_θ is the dual of any $T \in \mathbb{T}_\theta$. Now we have

$$\operatorname{supp}(\widehat{\psi}_{T,\text{high}}) \subset S_\theta \setminus B^n(0, (K\Delta)^{-1}).$$

Applying Lemma 18 at the special case that $m = n-1$ we see that $\{S_\theta \setminus B^n(0,(K\Delta)^{-1})\}_{\theta\in\Theta}$ are $\lesssim K^{O(1)}\Delta^{-\dim(G(n-2,n-1))}$-overlapping. We thus have

$$\begin{aligned}
\int |f_{\text{high}}|^2 = \int \left| \sum_{T\in\mathbb{T}} n_T \cdot \psi_{T,\text{high}} \right|^2 &= \int \left| \sum_{\theta\in\Theta} \sum_{T\in\mathbb{T}_\theta} n_T \cdot \psi_{T,\text{high}} \right|^2 \\
&\lesssim \delta^{-O(\varepsilon)} \Delta^{-(n-2)} \sum_{\theta\in\Theta} \int \left| \sum_{T\in\mathbb{T}_\theta} n_T \psi_{T,\text{high}} \right|^2 \\
&\lesssim \delta^{-O(\varepsilon)} \Delta^{-(n-2)} \sum_{T\in\mathbb{T}} n_T^2 \int |\psi_{T,\text{high}}|^2 \\
&\lesssim \delta^{-O(\varepsilon)} \Delta^{-(n-2)} \sum_{T\in\mathbb{T}} n_T^2 \int |\psi_T|^2 \lesssim \delta^{-O(\varepsilon)} \Delta \sum_{T\in\mathbb{T}} n_T^2.
\end{aligned}$$

In the second to last equation above, we use the fact that tubes in $\mathbb{T}_\theta$ are parallel and finitely overlapping, and hence the essential supports of $\{\psi_{T,\text{high}}\}_{T\in\mathbb{T}_\theta}$, $\{KT\}_{T\in\mathbb{T}_\theta}$, are at most $K^{O(1)}$-overlapping. In the last row above, we use Plancherel: $\int |\eta_{\text{high}}^{\vee} * \psi_T|^2 = \int |\eta_{\text{high}} \widehat{\psi}_T|^2 \le \int |\widehat{\psi}_T|^2 \le \int |\psi_T|^2$.

We are going to find an upper bound to $\sum_{T\in\mathbb{T}} n_T^2$. The intuition is that $n_T = 1$ for $T \in \cup_{y\in E}\mathbb{T}_y$, and $= 0$ for other $T \in \mathbb{T}$. Therefore $\sum_{T\in\mathbb{T}} n_T^2 = \sum_{T\in\mathbb{T}} n_T = \sum_{y\in E} \#\mathbb{T}_y \lesssim \#E\#\mathbb{T}_y \lesssim \Delta^{-t}\delta^{\eta}\Delta^{-\sigma}$. We verify this intuition.

$$\sum_{T\in\mathbb{T}} n_T^2 = \sum_{T\in\mathbb{T}} \#\{y, y' \in E \mid T \in \mathbb{T}_y \cap \mathbb{T}_{y'}\} = \sum_{y\in E} \sum_{y'\in E} \#\{T \in \mathbb{T} \mid T \in \mathbb{T}_y \cap \mathbb{T}_{y'}\}.$$

Given that each $\mathbb{T}_y$ is a (Δ, σ)-set, the above expression is bounded by

$$\lesssim \sum_{y\in E} \sum_{y'\in E\setminus\{y\}} \min\left\{|y-y'|^{-\sigma}, \#\mathbb{T}_y\right\} + \sum_{y\in E} \#\mathbb{T}_y.$$

The second term is bounded by

$$\sum_{y\in E} \#\mathbb{T}_y \lesssim \delta^{\eta}\Delta^{-t-\sigma}.$$

For the first term, we have

$$
\begin{aligned}
&\lesssim \sum_{y\in E}\sum_{k=0}^{\log_2\Delta^{-1}}\sum_{|y-y'|\le 2^{-k}}\min\{|y-y'|^{-\sigma},\delta^\eta\Delta^{-\sigma}\}\\
&\lesssim \sum_{y\in E}\sum_{k=0}^{\log_2\Delta^{-1}}\#\{y'\in E\cap B^n(y,2^{-k})\}\min\{2^{k\sigma},\delta^\eta\Delta^{-\sigma}\}\\
&\lesssim \Delta^{-t}\sum_{k=0}^{\log_2\Delta^{-1}}(\Delta^{-1}2^{-k})^t\min\{2^{k\sigma},\delta^\eta\Delta^{-\sigma}\}\\
&= \Delta^{-t}\sum_{k=0}^{\log_2\Delta^{-1}}\Delta^{-t}\min\{2^{k(\sigma-t)},\delta^\eta\Delta^{-\sigma}2^{-kt}\}.
\end{aligned}
$$

When $2^{k(\sigma-t)}=\delta^\eta\Delta^{-\sigma}2^{-kt}$ or equivalently $2^{k\sigma}=\delta^\eta\Delta^{-\sigma}$, the value of "min" dominates. (Actually, we used the condition $\Delta\le\delta^{\eta/\sigma}$ here in order to find $k\ge 0$ so that $2^{k\sigma}\sim\delta^\eta\Delta^{-\sigma}$.) The expression above is therefore bounded by $\delta^{\frac{\sigma-t}{\sigma}\eta}\Delta^{-t-\sigma}$.

Combining all the estimates, we have

$$\sum_{T\in\mathbb{T}}n_T^2\lesssim(\delta^{\frac{\sigma-t}{\sigma}\eta}+\delta^\eta)\Delta^{-t-\sigma}.$$

Plugging into (31), we have

$$\delta^{O(\varepsilon)}\Delta^{-t}\lesssim\delta^{\frac{\sigma-t}{\sigma}\eta}\Delta^{-(n-1)-\sigma+a}.$$

By choosing ε small enough such that $\delta^{O(\varepsilon)}\ge\delta^{\frac{\sigma-t}{\sigma}\eta}$, then we get

$$\Delta^{-t}\lesssim\Delta^{-(n-1)-\sigma+a}.$$

Since $\Delta\le\delta^{\eta/\sigma}$, if δ is small enough we obtain $t\le n-1+\sigma-a$. □

We now prove Theorem 16.

Proof of Theorem 16. We will show that the result holds for $\varepsilon\le\varepsilon_0(\eta,\sigma,a,t)$, $\delta\le\delta_0(\eta,\sigma,a,t)$, where $\varepsilon_0(\eta,\sigma,a,t),\delta_0(\eta,\sigma,a,t)$ depend on Propositions 17 and 19. The key idea is to project the sets to a lower dimensional subspace. Similar ideas has appeared in [1].

Since $k+\sigma-a<k$, we may assume $t<k+1$ so that we can apply Proposition 17 with $(a,m)=(t,k+1)$. We will apply Proposition 19 with $n=k+1$. For our purpose, we determine the parameters of Proposition 19 in advance. For fixed η, we first choose small number ε' so that Proposition 19 holds for $\varepsilon=\varepsilon'$. Then let the parameter η in Proposition 17 be ε'. We choose ε so that Proposition 17 holds for this ε.

Recall the condition in Theorem 16 that each of E and F lies in a ball of radius $1/1000$ and $\mathrm{dist}(E,F)\ge 3/4$. By the separation of E and F, we can find $\widetilde{G}\subset G(n,k+1)$ which has measure $\ge 10^{-10}$, such that any $V\in\widetilde{G}$ satisfies

$$\mathrm{dist}(\pi_V(E),\pi_V(F))\ge\frac{1}{2}.$$

We choose G to be a maximal δ-separated subset of $\widetilde{G}$. Then G is a $(\delta,d_{k+1,n})$-set with $\#G\gtrsim\delta^{-d_{k+1,n}}$.

By Proposition 17, there exists a subset $G_1 \subset G$ with $\#G_1 \gtrsim \delta^{-d_{k+1,n}}$, so that for any $V \in G_1$ we have

$$\mathcal{H}^a_{\delta,\infty}(\pi_V(F')) > \delta^{\varepsilon'}, \quad \text{for any } F' \subset F \text{ with } \#F' \geq \delta^{\varepsilon}\#F. \tag{32}$$

Similarly, there exists $V \in G_1$, so that

$$\mathcal{H}^t_{\delta,\infty}(\pi_V(E)) > \delta^{\varepsilon'}. \tag{33}$$

We just fix this V for which (32) and (33) hold.

We are about to apply Proposition 19 (with $\varepsilon = \varepsilon'$). By (33), there exists a (δ, t)-set $E_V \subset \pi_V(E)$ with $\#E \gtrsim \delta^{\varepsilon' - t}$. We also check that $\pi_V(F)$ satisfies the requirement in Proposition 19: $\mathcal{H}^a_{\delta,\infty}(\pi_V(F)) \gtrsim \delta^{\varepsilon'}$, $\#\pi_V(F) \leq \#F \lesssim \delta^{-a}$.

We find a point $\widetilde{y} \in E_V$ such that: for all $\widetilde{F} \subset \pi_V(F)$ with $\mathcal{H}^a_{\delta,\infty}(\widetilde{F}) \geq \delta^{\varepsilon'}$, we have

$$\mathcal{H}^\sigma_{\delta,\infty}\big(\pi_{\widetilde{y}}(\widetilde{F})\big) > \delta^{\eta}. \tag{34}$$

We use this property to finish the proof. We choose $y \in E$ so that $\pi_V(y) = \widetilde{y}$. We show that this y satisfies the requirement in Theorem 16. For any $F' \subset F$ with $\#F' \geq \delta^{\varepsilon}\#F$, by (32) we have $\mathcal{H}^a_{\delta,\infty}(\pi_V(F')) \geq \delta^{\varepsilon'}$. Plug in $\widetilde{F} = \pi_V(F')$ into (34):

$$\mathcal{H}^\sigma_{\delta,\infty}\left(\pi_{\widetilde{y}}\left(\pi_V(F')\right)\right) > \delta^{\eta}.$$

Note that

$$\mathcal{H}^\sigma_{\delta,\infty}\left(\pi_y(F')\right) \geq \mathcal{H}^\sigma_{\delta,\infty}\left(\pi_{\widetilde{y}}\left(\pi_V(F')\right)\right),$$

as any covering of $\pi_y(F')$ naturally gives rise to a covering of $\pi_{\widetilde{y}}\big(\pi_V(F')\big)$ by the separation of E, F. Therefore, we have

$$\mathcal{H}^\sigma_{\delta,\infty}\left(\pi_y(F')\right) > \delta^{\eta}. \qquad \square$$

3. Liu's conjecture on radial projections

In this section, we prove Liu's conjecture (Theorem 2). The idea is the same as in [12], but we still provide full details to clarify the numerology since we are in higher dimensions.

We repeat Theorem 2 here.

Theorem 21. *Given a Borel set $E \subset \mathbb{R}^n$, with $\dim E \in (k-1, k]$ for some $k \in \{1, \ldots, n-1\}$, then*

$$\dim\{x \in \mathbb{R}^n \setminus E \mid \dim(\pi_x(E)) < \dim E\} \leq k.$$

It suffices to prove

Proposition 22. *Given a Borel set $E_0 \subset \mathbb{R}^n$, with $\dim E_0 \in (k-1, k]$ for some $k \in \{1, \ldots, n-1\}$, and $\tau_0 > 0$, then we have*

$$\dim\{x \in \mathbb{R}^n \setminus E_0 \mid \dim(\pi_x(E_0)) < \dim E_0 - 10\tau_0\} \leq k.$$

Since the proof of this proposition is technical, we start with a heuristic proof. One of the key tools utilized is Theorem 15.

A heuristic proof of Proposition 22. We just need to prove this for $\dim E_0 < k$. We set $s = \dim E_0$. Let

$$F_0 = \{x \in \mathbb{R}^n \setminus E_0 \mid \dim(\pi_x(E_0)) < \dim E_0 - 10\tau_0\}.$$

By contradiction, we assume $t = \dim F_0 > k$. Also, by passing to a subset of F_0, we may assume $t \in (k, k+1)$. Now we let this F_0 be the set A in Theorem 15. Since $s < k$, we have that the s-exceptional set

$$E_s(F) = \{y \in \mathbb{R}^n \setminus F_0 \colon \dim(\pi_y(F_0)) < s\}$$

has dimension $\le k+s-t < s = \dim E_0$. Subtracting this small exceptional part from E_0, we may pass to a subset of E_0 (still denoted by E_0) with the same dimension s and satisfying

$$\dim(\pi_y(F_0)) \ge s,$$

for any $y \in E_0$.

By δ-discretization, we may assume F_0 is a t-dimensional set of points and E is an s-dimensional set of points. (Here, when we say F_0 is a t-dimensional set, it means that F_0 is a (δ, t)-set and $\#F_0 \gtrsim \delta^{-t}$.) For each $x \in F_0$ and $y \in E_0$, we connect them by a δ-tube. Let $\mathbb{T}$ be the set of δ-tubes produced in this way. We also identify comparable tubes. Roughly speaking, we define

$$\mathbb{T} := \{T \colon T \text{ connects some } x \in F,\ y \in E\}.$$

We also define $\mathbb{T}_x := \{T \in \mathbb{T} \colon x \in T\}$ for $x \in F_0$, and $\mathbb{T}^y := \{T \in \mathbb{T} \colon y \in T\}$ for $y \in E_0$. By definition, we have $\dim(\pi_x(E_0)) \le s - \tau_0$ for $x \in F_0$. This condition morally says that $\mathbb{T}_x$ is an $(s-\tau_0)$-dimensional set. Since the tubes in $\mathbb{T}_x$ are finitely overlapping at the portion away from x, we have

$$\delta^{-s} \le \#E_0 \lesssim \sum_{T \in \mathbb{T}_x} \#(T \cap E_0).$$

Since $\#\mathbb{T}_x \le \delta^{-s+\tau_0}$, we may morally assume $\#(T_x \cap E_0) \gtrsim \delta^{-\tau_0/2}$ for any $T_x \in \mathbb{T}_x$. Morally, we may further assume for any $T \in \mathbb{T}$, we have $\#(T \cap E_0) \gtrsim \delta^{-\tau_0/2}$. The condition $\dim(\pi_y(F_0)) \ge s$ morally says that $\mathbb{T}^y$ is at least an s-dimensional set.

We consider the incidences between E_0 and $\mathbb{T}$. We will derive a contradiction by comparing the upper and lower bounds of $I(E_0, \mathbb{T}) := \{(y, T) \in E_0 \times \mathbb{T} \colon y \in T\}$. First, we have

$$I(E_0, \mathbb{T}) = \sum_{T \in \mathbb{T}} \#(T \cap E_0) \gtrsim \#\mathbb{T}\delta^{-\tau_0/2}.$$

For the upper bound of the incidence, we have

$$\begin{aligned} I(E_0, \mathbb{T}) = \sum_{T \in \mathbb{T}} \#(T \cap E_0) &\le (\#\mathbb{T})^{1/2} \left(\sum_{T \in \mathbb{T}} \#(T \cap E_0)^2 \right)^{1/2} \\ &= (\#\mathbb{T})^{1/2} \left(\sum_{y, y' \in E_0} \#\{T \in \mathbb{T} \colon y, y' \in T\} \right)^{1/2} \\ &= (\#\mathbb{T})^{1/2} \left(\sum_{y \in E_0} \sum_{y' \in E_0} \#\{T \in \mathbb{T}^y \colon y' \in T\} \right)^{1/2}. \end{aligned}$$

By the s-dimensional condition for $\mathbb{T}^y$, for $y \ne y'$ we have

$$\#\{T \in \mathbb{T}^y \colon y' \in T\} \lesssim \left(\frac{\delta}{|y - y'|} \right)^s \#\mathbb{T}^y.$$

Therefore, we have

$$I(E_0,\mathbb{T}) = (\#\mathbb{T})^{1/2}\left(\sum_{y\in E_0}\sum_{y'\in E_0\setminus\{y\}} \#\{T\in\mathbb{T}^y\colon y'\in T\} + \sum_{y\in E_0}\#\mathbb{T}^y\right)^{1/2}$$
$$\lesssim (\#\mathbb{T})^{1/2}\left(\sum_{y\in E_0}\sum_{y'\in E_0\setminus\{y\}} \left(\frac{\delta}{|y-y'|}\right)^s \#\mathbb{T}^y + I(E_0,\mathbb{T})\right)^{1/2}.$$

Using that E_0 is an s-dimensional set, we have

$$\sum_{y'\in E_0\setminus\{y\}}\left(\frac{\delta}{|y-y'|}\right)^s \lessapprox 1,$$

so we have

$$I(E_0,\mathbb{T}) \lessapprox (\#\mathbb{T})^{1/2}\big(\#E_0\cdot\max_y(\#\mathbb{T}^y) + I(E_0,\mathbb{T})\big)^{1/2}.$$

Let us first ignore $\#E_0\cdot\max_y(\#\mathbb{T}^y)$ (we will carefully analyze this term later), and pretend

$$I(E_0,\mathbb{T}) \lessapprox (\#\mathbb{T})^{1/2}\big(I(E_0,\mathbb{T})\big)^{1/2}.$$

This means $I(E_0,\mathbb{T}) \lessapprox \#\mathbb{T}$, which contradicts the lower bound of $I(E_0,\mathbb{T})$. $\square$

We start the rigorous proof. The proof is by contradiction to assume the set

$$F_0 = \{x\in\mathbb{R}^n\setminus E_0 \mid \dim(\pi_x(E_0)) < \dim E_0 - 10\tau_0\} \tag{35}$$

satisfies $t=\dim F_0 > k$. We will derive a contradiction through the following proposition and a standard reduction. It has the same idea in the proof that Theorem 16 implies Theorem 15.

Proposition 23. *Let $k\in\{1,\cdots,n-1\}$. Let $0<s<k$, $t>k$ and $\tau_0>0$. For ε,δ small enough depending on s,t,τ_0, the following holds. Let $E,F\subset B^n(0,1)$ be (δ,s)-set and (δ,t)-set, with $\#E\gtrsim\delta^{-s+\varepsilon}$, $\#F\gtrsim\delta^{-t+\varepsilon}$. We also assume that each one of E,F is contained in a ball of radius* $1/1000$, *and* $\operatorname{dist}(E,F)\ge 1/2$. *Then there exists $x\in F$ such that*

$$|\pi_x(E')|_\delta \ge \delta^{-s+\tau_0}, \quad \text{for all } E'\subset E \text{ with } \#E'\ge\delta^\varepsilon\#E. \tag{36}$$

Proof that Proposition 23 implies Proposition 22. We will do a same reduction as in the proof that Theorem 16 implies Theorem 15. Suppose E_0 is given in Proposition 22, and F_0 is given by (35). Fix $\dim E_0-\tau_0<s_1<\dim E_0$. We can find s_1-dense points y_1,y_2 of E_0. Since our problem is scaling-invariant, we can assume $|y_1-y_2|=99/100$. We let $E_1=E_0\cap B_{1/1000}(y_1)$, $E_2=E_0\cap B_{1/1000}(y_2)$, and then $\dim(E_1),\dim(E_2)\ge s_1$. We only need to show for any ball $B_{1/1000}$ of radius $1/1000$, $F_0\cap B_{1/1000}$ has dimension $\le k$. Since $\operatorname{dist}(E_1,E_2)>98/100$, either $\operatorname{dist}(B_{1/1000},E_1)>3/4$ or $\operatorname{dist}(B_{1/1000},E_2)>3/4$. We may assume $\operatorname{dist}(B_{1/1000},E_1)>3/4$. We will show that the set

$$F' := \{x\in B_{1/1000}\colon \dim(\pi_x(E_1)) < \dim E_1 - 9\tau_0\} (\supset F_0\cap B_{1/1000})$$

has dimension $\le k$. From the reduction, these sets satisfy certain separation properties:

$$\text{each one of } E_1 \text{ and } F' \text{ lies in some ball of radius } 1/1000, \tag{37}$$

$$E_1,F'\subset B^n(0,1), \quad \operatorname{dist}(E_1,F')\ge 1/2. \tag{38}$$

We choose $t < \dim(F')$, $s = \dim(E_1) - \tau_0 < \dim(E_1)$. Then $\mathcal{H}^t_\infty(F') > 0$, and by Frostman's lemma there exists a probability measure ν_{E_1} supported on E_1 satisfying $\nu_{E_1}(B_r) \lesssim r^s$ for any B_r being a ball of radius r. We can rewrite F' as

$$F' = \{x \in B_{1/1000} \colon \dim(\pi_x(E_1)) < s - 8\tau_0\} \tag{39}$$

We only need to prove $t \le k$, since then we can send $t \to \dim(F')$. For the sake of contradiction, assume that $t > k$. Now we fix t, so we may assume $\mathcal{H}^t_\infty(F') \sim 1$ is a constant.

Fix an $x \in F'$. Using Lemma 9 to $\pi_x(E_1)$, we obtain a set of dyadic caps $\mathcal{C}_x = \bigsqcup_j \mathcal{C}_{x,j}$ in $\mathbb{S}^{n-1}$ that cover $\pi_x(E_1)$. Here each $\mathcal{C}_{x,j}$ is a set of 2^{-j}-caps that satisfy the $(s-8\tau_0)$-dimensional condition (see Lemma 9 (3)) because of $\dim(\pi_x(E_1)) < s - 8\tau_0$. Also, the radius of these caps is less than $\varepsilon_\circ$, which is any given small number.

By the $(s - 8\tau_0)$-dimensional condition of $\mathcal{C}_{x,j}$, we have

$$\#\mathcal{C}_{x,j} \le 2^{j(s-8\tau_0)}. \tag{40}$$

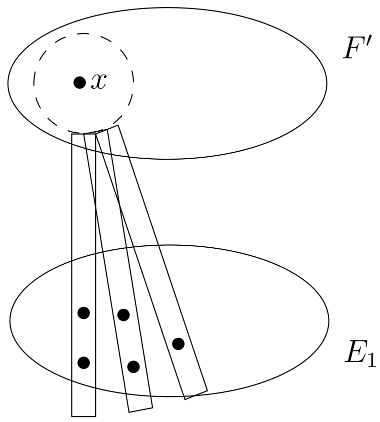

Figure 4. $\mathbb{T}_{x,j}$ in the radial projection.

For each cap $C \in \mathcal{C}_x$, consider $\pi_x^{-1}(C) \cap \{x \in \mathbb{R}^n \colon 1 - \frac{1}{100} \le |x - y| \le 1\}$ which is a tube. We obtain a collection of finitely overlapping tubes

$$\mathbb{T}_x = \bigsqcup_j \mathbb{T}_{x,j}$$

that cover E_1 (see Figure 4). Here, each tube has its coreline passing through x and at distance ~ 1 from x. The tubes in $\mathbb{T}_{x,j}$ have dimensions $2^{-j} \times \cdots \times 2^{-j} \times 1$. Also, $\mathbb{T}_{x,j}$ inherits the property (40) from $\mathcal{C}_{x,j}$:

$$\#\mathbb{T}_{x,j} \le 2^{j(s-8\tau_0)}. \tag{41}$$

For a fixed $x \in F'$, there exists a $j(x) \ge |\log_2 \varepsilon_\circ|$ such that

$$\nu_{E_1}\left(E_1 \cap \bigcup_{T \in \mathbb{T}_{x,j(x)}} T\right) \ge \frac{1}{10 j(x)^2} \nu_{E_1}(E_1) = \frac{1}{10 j(x)^2}. \tag{42}$$

We have a partition $F' = \bigsqcup_j F'_j$ where $F'_j = \{x \in F' \colon j(x) = j\}$. We choose j such that $\mathcal{H}^t_\infty(F'_j) \gtrsim \frac{1}{j^2}$. We let $\delta = 2^{-j}$. Note that $\delta \le \varepsilon_\circ$ by assumption. By Lemma 7, there exists a subset $F'' \subset F'_j$ which is a (δ, t)-set and $\#F'' \gtrsim |\log \delta|^{-2} \delta^{-t}$. We use μ to denote the counting measure on F''.

Next, we consider the set $S = \big\{(y, x) \in E_1 \times F'' \colon y \in \bigcup_{T \in \mathbb{T}_{x,j}} T\big\}$. We also denote the x-section and y-section of S by S^x and S_y. (In Figure 4, F' is drawn above E_1, so we use the convention that x appears as the superscript in S^x.) By (42), we have $\nu_{E_1}(S^x) \geq \frac{1}{10j(x)^2}$, so we have

$$(\nu_{E_1} \times \mu)(S) \geq \frac{1}{10j^2}\mu(F''). \tag{43}$$

This implies

$$(\nu_{E_1} \times \mu)\left(\left\{(y,x) \in S\colon \mu(S_y) \geq \frac{1}{20j^2}\mu(F'')\right\}\right) \geq \frac{1}{20j^2}\mu(F''). \tag{44}$$

Therefore, we have

$$\nu_{E_1}\left(\left\{y \in E_1\colon \mu(S_y) \geq \frac{1}{20j^2}\mu(F'')\right\}\right) \geq \frac{1}{20j^2} \sim |\log\delta|^{-2}. \tag{45}$$

By Lemma 8, we can find a subset E of $\{y \in E_1\colon \mu(S_y) \geq \frac{1}{20j^2}\mu(F')\}$, so that E is a (δ, s)-set and $\#E \gtrsim |\log\delta|^{-2}\delta^{-s}$.

Hence,

$$\begin{aligned} |\log\delta|^{-2}\#E\#F'' &\lesssim \#\left\{(y,x) \in E \times F''\colon y \in \bigcup_{T\in\mathbb{T}_{x,j}} T\right\} \\ &= \sum_{x\in F''} \#\left\{y \in E\colon y \in \bigcup_{T\in\mathbb{T}_{x,j}} T\right\}. \end{aligned} \tag{46}$$

By pigeonholing, there exists a subset $F \subset F''$ with $\#F \gtrsim |\log\delta|^{-2}\#F'' \gtrsim \delta^{\varepsilon/2}\delta^{-t}$, so that for any $x \in F$:

$$\#\left\{y \in E\colon y \in \bigcup_{T\in\mathbb{T}_{x,j}} T\right\} \gtrsim \delta^{\varepsilon}\#E.$$

We set $E_x := \{y \in E\colon y \in \bigcup_{T\in\mathbb{T}_{x,j}} T\}$.

Now we use Proposition 23 to derive a contradiction. We just plug in the E, F and check they satisfy the conditions of Proposition 23. Then it yields the existence of an $x \in F$ such that $|\pi_x(E')|_\delta \geq \delta^{-s+\tau_0}$, for any $E' \subset E$ with $\#E' \geq \delta^\varepsilon\#E$. We just put $E' = E_x$, and see that $\delta^{-s+\tau_0} \leq |\pi_x(E_x)|_\delta \lesssim \#\mathbb{T}_{x,j} \leq \delta^{-s+8\tau_0}$ by (41). This gives a contradiction if δ is small enough depending on τ_0. □

Remark 24. (36) roughly says there exists $x \in F$ such that $\dim(\pi_x(E)) > \dim E - \tau_0$, contradicts the definition of F_0 in (35). Throughout the proof of Proposition 23, we will use x to denote points in F and y to denote points in E.

It remains to prove Proposition 23.

3.1. Proof of Proposition 23. We provide the full details for the proof of Proposition 23. We remark that the proof has the same idea as in [12]. We include here just for completeness.

In [12], Orponen and Shmerkin derive their Corollary 4.5 from Proposition 4.2. By the same argument, we can derive the following corollary from Theorem 16.

Corollary 25. *Let $0 \leq \sigma \leq s < k$, $t \in (k, k+1]$, $0 < \eta < 1/100$, and $s > \max\{k+\sigma-t, 0\}$. Then, for sufficiently small ε, δ depending on s, σ, t, η, the following holds.*

Let $E, F \subset B^n(0,1)$ be (δ,s)-set and (δ,t)-set, with $\#E \gtrsim \delta^{-s+\varepsilon}$ and $\#F \gtrsim \delta^{-t+\varepsilon}$. Each of E and F lies in a ball of radius $1/1000$ and $\mathrm{dist}(E,F) \ge 1/2$. Then, there exists a subset $E' \subset E$ with $\#E' \ge (1-\delta^\varepsilon)\#E$, and for every point $y \in E'$, there exist disjoint families of δ-tubes $\mathbb{T}^y = \mathbb{T}_1^y \sqcup \cdots \sqcup \mathbb{T}_L^y$ (where $L = 3\log(1/\delta)$, and some $\mathbb{T}_j^y$ may be empty), with the following properties:

(i) *The tubes in $\mathbb{T}^y$ form a bush centered at y.*
(ii) *Each $\mathbb{T}_j^y$, if non-empty, can be writen as $\mathbb{T}_j^y = \bigsqcup_i \mathbb{T}_{j,i}^y$, where each $\mathbb{T}_{j,i}^y$ is a (δ,σ)-set with cardinality $\gtrsim \delta^{-\sigma+\eta}$.*
(iii) *$\#(T\cap F) \sim 2^j$, for $T \in \mathbb{T}_j^y$.*
(iv) *$\mathbb{T}_j^y$ is either empty, or $\#\big(F \cap \bigcup_{T\in\mathbb{T}_j^y} T\big) \ge \delta^{2\varepsilon}\#F$ in which case $\#\mathbb{T}_j^y \ge \delta^{2\varepsilon}2^{-j}\#F$; we also trivially have $\#\mathbb{T}_j^y \le 2^{-j}\#F$ by (iii).*
(v) *$\#\big(F\cap\bigcup_{T\in\mathbb{T}^y} T\big) \ge (1-\delta^\varepsilon)\#F$.*

Proof of Corollary 25. We will apply Theorem 16. Since there are many parameters, to make less confusion, we denote the parameters appeared in Theorem 16 by $\sigma(\mathrm{Thm}), t(\mathrm{Thm}), a(\mathrm{Thm}), E(\mathrm{Thm}), F(\mathrm{Thm})$. We write the parameters appearing in Corollary 25 in the usual way as σ, s, t, E, F.

We first talk about the idea. To apply Theorem 16, we let $\sigma(\mathrm{Thm}) = \sigma$, $t(\mathrm{Thm}) = s$, $a(\mathrm{Thm}) = t$, and $E(\mathrm{Thm}) = E, F(\mathrm{Thm}) = F$. We can check that the conditions in Theorem 16 are satisfied. As a result, there exists $y \in E$ such that for all $F' \subset F$ with $\#F' \ge \delta^{2\varepsilon}\#F$ (it is harmless to use 2ε instead of ε), we have

$$\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F')) > \delta^{\eta/2}. \tag{47}$$

We will iteratively use Theorem 16 to obtain a lot of y that satisfies (47). We will let E' to be the set of these y's and our $\mathbb{T}^y$ will be constructed using (47).

We talk about the details. Suppose we have obtained $\{y_1,\dots,y_N\}$ such that (47) is true for each of these y_i. If $N < (1-\delta^\varepsilon)\#E$, then we let $E(\mathrm{Thm}) = E \setminus \{y_1,\dots,y_N\}$. We see that $\#E(\mathrm{Thm}) > \delta^{-s+2\varepsilon}$. If we let $\varepsilon(\mathrm{Thm}) = 2\varepsilon$, then we can apply Theorem 16 and obtain y_{N+1} that satisfies (47). By iteration, we obtain $E' \subset E$ with $\#E' \ge (1-\delta^\varepsilon)\#E$ such that for each $y \in E'$, we have: if $F' \subset F$ with $\#F' \ge \delta^{2\varepsilon}\#F$, then

$$\mathcal{H}^\sigma_{\delta,\infty}(\pi_y(F')) > \delta^{\eta/2}. \tag{48}$$

Our next step is to construct $\mathbb{T}^y = \bigsqcup_{1\le j\le L}\mathbb{T}_j^y$ for each $y \in E'$. We fix a $y \in E'$ in the rest of proof. The idea is to iteratively use (48).

We first choose a set of δ-caps $\mathcal{C} = \{C\} \subset \mathbb{S}^{n-1}$ that forms a partition of $\mathbb{S}^{n-1}$. For each cap C, let T be a δ-tube that passes through y and points to direction C. In this way, $\mathcal{C}$ naturally corresponds to $\mathbb{T}$ which is a full bush centered at y. The reader can check $\#\mathbb{T} \sim \delta^{-(n-1)}$. Our $\mathbb{T}^y$ will be constructed as a subset of $\mathbb{T}$.

We first let $F' = F$, and then of course $\#F' \ge \delta^{2\varepsilon}\#F$, so we have (48). Let $\mathbb{T}' \subset \mathbb{T}$ be the tubes that intersect F', then (48) implies that $\mathbb{T}'$ satisfies that

$$\mathcal{H}^\sigma_{\delta,\infty}\left(\pi_y\left(\textstyle\bigcup_{T\in\mathbb{T}'} T \setminus B_{1/10000}(y)\right)\right) \gtrsim \delta^{\eta/2}.$$

For $1 \le j \le L = 3\log(1/\delta)$, define

$$\mathbb{T}'_j = \{T \in \mathbb{T}' \colon \#(T\cap F') \sim 2^j\}.$$

We obtain a partition $\mathbb{T}' = \bigsqcup_j \mathbb{T}'_j$. By pigeonholing, there exists j such that

$$\mathcal{H}^\sigma_{\delta,\infty}\left(\pi_y\left(\textstyle\bigcup_{T\in\mathbb{T}'_j} T \setminus B_{1/10000}(y)\right)\right) \gtrsim \delta^{\eta}.$$

By Lemma 12, we obtain a subset $\mathbb{T}^y_{j,1} \subset \mathbb{T}'_j$, so that $\mathbb{T}^y_{j,1}$ is a (δ,σ)-set with cardinality $\gtrsim \delta^{-\sigma+\eta}$. Next, we let $F' = F \setminus \bigcup_{T\in\mathbb{T}^y_{j,1}} T$ and then check whether $\#F' \geq \delta^{2\varepsilon}\#F$. If not, we stop. If yes, we repeat the argument above and obtain $\mathbb{T}^y_{j',1}$ or $\mathbb{T}^y_{j,2}$.

Suppose that we have obtained $\mathbb{T}^y_j$ $(j = 1,\dots,L)$, where each $\mathbb{T}^y_j = \bigsqcup_{1\leq i\leq i(j)} \mathbb{T}^y_{j,i}$. Also, each $\mathbb{T}^y_{j,i}$ satisfies (i), and each $T \in \mathbb{T}^y_j$ satisfies (iii). We let

$$F' = F \setminus \bigcup_{T\in\cup_j\mathbb{T}^y_j} T.$$

If $\#F' \geq \delta^{2\varepsilon}\#F$, then we repeat the argument and obtain some $\mathbb{T}^y_{j,i(j)+1}$. We redefine $\mathbb{T}^y_j$ to be the disjoint union $\mathbb{T}^y_{j,i(j)+1} \sqcup \mathbb{T}^y_j$, and redefine $i(j)$ to be $i(j)+1$. If $\#F' < \delta^{2\varepsilon}\#F$, then we stop.

Suppose we stop. For the purpose of (iv), define the significant set of j to be

$$J = \left\{ j \colon \#(F \cap \bigcup_{T\in\mathbb{T}^y_j} T) \geq \delta^{2\varepsilon}\#F \right\}.$$

We throw away those $\mathbb{T}^y_j$ for $j \notin J$, and let $\mathbb{T}^y = \bigsqcup_{j\in J} \mathbb{T}^y_j$. Finally, we check (v). We note that

$$\#\left(F \cap \bigcup_{T\in\mathbb{T}^y} T\right) = \#F - \#\left(F \setminus \bigcup_{T\in\cup_j\mathbb{T}^y_j} T\right) - \sum_{j\notin J} \#\left(F \cap \textstyle\bigcup_{T\in\mathbb{T}^y_j} T\right).$$

This is bounded from below by $\#F - \delta^{2\varepsilon}\#F - 3\log(1/\delta)\delta^{2\varepsilon}\#F \geq (1-\delta^{\varepsilon})\#F$, when δ is small enough. □

Let us return to the proof of Proposition 23. Since

$$s > \max\{k+s-t, 0\},$$

we can apply Corollary 25 with $\sigma := s$. We obtain a set $E' \subset E$ with $\#E' \geq (1-\delta^{4\varepsilon})\#E$, and for all $y \in E'$ the tubes $\mathbb{T}^y = \mathbb{T}^y_1 \sqcup \cdots \sqcup \mathbb{T}^y_L$ $(L = 3\log(1/\delta))$ satisfying the properties in Corollary 25. $\mathbb{T}^y$ is a bush of tubes centered at y. Next, we will estimate the number of pairs $(y,x) \in E' \times F$ that satisfy certain properties. To make the expression easier, for any set of tubes $\mathbb{T}'$, we write $\bigcup\mathbb{T}' := \bigcup_{T\in\mathbb{T}'} T$.

We can also do the same reduction as in the beginning of the proof of Lemma 20. We first choose a maximal set of of $\delta/100$-separated tubes $\mathbb{T} = \cup_\theta \mathbb{T}_\theta$ as in the proof of Lemma 20. All the tubes appear in this proof can be thought of as an element in $\mathbb{T}$, since we can replace a tube by another comparable tube which does not affect the proof.

By (v), we have

$$\#\left\{(y,x) \in E' \times F \colon x \in \bigcup\mathbb{T}^y\right\} = \sum_{y\in E'} \#\left(F \cap \bigcup\mathbb{T}^y\right) \geq (1-\delta^{4\varepsilon})\#E'\#F. \tag{49}$$

Now, we make a counter assumption: (36) fails for all $x \in F$. Thus for every $x \in F$, there exists a subset $E'_x \subset E$ such that $\#E'_x \geq \delta^{\varepsilon}\#E$, and

$$|\pi_x(E'_x)|_\delta < \delta^{-s+\tau_0}. \tag{50}$$

Since $\#E' \geq (1-\delta^{4\varepsilon})\#E$, we have $\#(E'_x \cap E') \gtrsim \delta^{\varepsilon}\#E$. We may assume $E'_x \subset E'$ by replacing E'_x with $E'_x \cap E'$. For each $x \in F$, we choose a bush $\mathcal{T}_x$ centered at x, consisting of δ-tubes, so that $\mathcal{T}_x$ covers E'_x and

$$\#\mathcal{T}_x = |\pi_x(E'_x)|_\delta < \delta^{-s+\tau_0}. \tag{51}$$

We immediately have

$$\#\left\{(y,x) \in E' \times F\colon y \in \bigcup \mathcal{T}_x\right\} = \sum_{x\in F} \#\left(E' \cap \bigcup \mathcal{T}_x\right) \geq \delta^{\varepsilon}\#E'\#F. \tag{52}$$

The inequalities (49) and (52) together imply

$$\#\left\{(y,x) \in E' \times F\colon x \in \bigcup \mathbb{T}^y,\ y \in \bigcup \mathcal{T}_x\right\} \geq (\delta^{\varepsilon} - \delta^{4\varepsilon})\#E'\#F. \tag{53}$$

By pigeonholing, there exists a j such that

$$\#\left\{(y,x) \in E' \times F\colon x \in \bigcup \mathbb{T}^y_j, y \in \bigcup \mathcal{T}_x\right\} \gtrsim \delta^{2\varepsilon}\#E'\#F. \tag{54}$$

Next, we introduce the high-density tubes:

$$\mathbb{T}^{y,h}_j := \{T \in \mathbb{T}^y_j\colon \#\{y' \in E'\colon T \in \mathbb{T}^{y'}_j\} \geq \delta^{-\tau_0/2}\}. \tag{55}$$

Also define the low-density tubes $\mathbb{T}^{y,l}_j := \mathbb{T}^y_j \setminus \mathbb{T}^{y,h}_j$. We want to show that

$$\#\left\{(y,x) \in E' \times F\colon x \in \bigcup \mathbb{T}^{y,h}_j, y \in \bigcup \mathcal{T}_x\right\} \gtrsim \delta^{2\varepsilon}\#E'\#F. \tag{56}$$

To show this, it suffices to show

$$\begin{aligned} &\#\{(y,x) \in E' \times F\colon x \in \bigcup \mathbb{T}^{y,l}_j,\ y \in \bigcup \mathcal{T}_x\} \\ &= \sum_{x\in F} \#\left\{y \in E'\colon x \in \bigcup \mathbb{T}^{y,l}_j,\ y \in \bigcup \mathcal{T}_x\right\} \lesssim \delta^{3\varepsilon}\#E'\#F. \end{aligned} \tag{57}$$

For fixed $x \in F$, we note that if $y \in \left\{y \in E'\colon x \in \bigcup \mathbb{T}^{y,l}_j,\ y \in \bigcup \mathcal{T}_x\right\}$, then there exists $T \in \mathbb{T}^{y,l}_j$ such that $x \in T$, and $T_1 \in \mathcal{T}_x$ such that $y \in T_1$. This means that T is comparable to T_1. Therefore, we can bound (57) by

$$\leq \sum_{x\in F} \sum_{T\in\widetilde{\mathcal{T}}_x} \#\{y \in E'\colon T \in \mathbb{T}^{y,l}_j\}.$$

Here, $\widetilde{\mathcal{T}}_x$ is the set of tubes from $\mathbb{T}$ (recall $\mathbb{T}$ from the paragraph above (49)) that are comparable to some tube in $\mathcal{T}_x$. We have $\#\mathcal{T}_x \sim \#\widetilde{\mathcal{T}}_x$.

By the definition of $\mathbb{T}^{y,l}_j$, we see that if $T \in \mathbb{T}^{y,l}_j$ for some y, then $\#\{y' \in E'\colon T \in \mathbb{T}^{y'}_j\} < \delta^{-\tau_0/2}$. Therefore, we bound the inequality above by

$$\lesssim \sum_{x\in F} \#\mathcal{T}_x\delta^{-\tau_0/2} \lesssim \#F\delta^{-s+\tau_0/2} \lesssim \delta^{3\varepsilon}\#E'\#F,$$

if ε is small enough depending on τ_0. This proves (57) and hence (56).

Next, we show that there exists $E'' \subset E'$ with $\#E'' \gtrsim \delta^{2\varepsilon}\#E'$, such that for $y \in E''$:

$$\#\mathbb{T}^{y,h}_j \geq \delta^{2\varepsilon}\#\mathbb{T}^y_j.$$

Note that

$$\begin{aligned} \delta^{2\varepsilon}\#E'\#F &\lesssim \#\left\{(y,x) \in E' \times F\colon x \in \bigcup \mathbb{T}^{y,h}_j,\ y \in \bigcup \mathcal{T}_x\right\} \\ &= \sum_{y\in E'} \#\left\{x \in F\colon x \in \bigcup \mathbb{T}^{y,h}_j,\ y \in \bigcup \mathcal{T}_x\right\}. \end{aligned}$$

By pigeonholing, we can choose $E'' \subset E'$ with $\#E'' \gtrsim \delta^{2\varepsilon}\#E'$ so that for $y \in E''$,

$$\#\left\{x \in F \colon x \in \bigcup \mathbb{T}_j^{y,h},\ y \in \bigcup \mathcal{T}_x\right\} \gtrsim \delta^{2\varepsilon}\#F.$$

Since $\mathbb{T}_j^{y,h} \subset \mathbb{T}_j^y$ and each $T \in \mathbb{T}_j^y$ satisfies $\#(F \cap T) \sim 2^j$, we have

$$\#\left\{x \in F \colon x \in \bigcup \mathbb{T}_j^{y,h},\ y \in \bigcup \mathcal{T}_x\right\} \le \#\left(F \cap \bigcup \mathbb{T}_j^{y,h}\right) \sim 2^j \#\mathbb{T}_j^{y,h},$$

which implies for $y \in E''$,

$$\#\mathbb{T}_j^{y,h} \gtrsim \delta^{2\varepsilon} 2^{-j} \#F \gtrsim \delta^{2\varepsilon}\#\mathbb{T}_j^y. \tag{58}$$

Define $\mathbb{T}_j^h = \bigcup_{y \in E''} \mathbb{T}_j^{y,h}$. Potentially, there could be a large intersection between $\mathbb{T}_j^{y,h}$ for different y. However, we claim the following estimate:

$$\#\mathbb{T}_j^h \gtrsim \delta^{O(\eta+\varepsilon)} \sum_{y \in E''} \#\mathbb{T}_j^{y,h}. \tag{59}$$

We prove the claim. Recall that $\mathbb{T}_j^y = \bigsqcup_{i=1}^{i(y)} \mathbb{T}_{j,i}^y$ where each $\mathbb{T}_{j,i}^y$ is a (δ, s)-set with cardinality $\gtrsim \delta^{-s+\eta}$ (see (ii)). Here $i(y)$ is the number that indicates the cardinality of $\mathbb{T}_j^y$:

$$i(y)\delta^{-s+\eta} \lesssim \mathbb{T}_j^y \lesssim i(y)\delta^{-s}. \tag{60}$$

Note that E'' is a (δ, s)-set with $\#E'' \gtrsim \delta^{O(\varepsilon)-s}$, by Lemma 13, we can choose a $(\delta^{1-C(\varepsilon+\eta)}, s)$-set $E^* \subset E''$ with $\#E^* \gtrsim \delta^{O(\varepsilon+\eta)-s}$ where C is some large number to be determined later. Actually, we will choose $C = 100s^{-1}$. We also remark that the notation $O(\varepsilon+\eta)$ is actually $O_s(\varepsilon+\eta)$, but we just leave out s since s is a fixed number at the beginning.

By (58), $\#\mathbb{T}_j^{y,h}$ $(y \in E'')$ are comparable up to $\delta^{2\varepsilon}$ factor, so we have

$$\sum_{y \in E''} \#\mathbb{T}_j^{y,h} \lesssim \delta^{-O(\varepsilon+\eta)} \sum_{y \in E^*} \#\mathbb{T}_j^{y,h}. \tag{61}$$

We are ready to estimate the lower bound of $\#\mathbb{T}_j^h$. We have

$$\#\mathbb{T}_j^h \ge \#\left(\bigcup_{y \in E^*} \mathbb{T}_j^{y,h}\right) \ge \#\left(\bigcup_{y \in E^*}\left(\mathbb{T}_j^{y,h} \setminus \bigcup_{y' \in E^* \setminus \{y\}} \mathbb{T}_j^{y',h}\right)\right) \tag{62}$$

$$(\text{by the disjointness}) = \sum_{y \in E^*} \#\left(\mathbb{T}_j^{y,h} \setminus \bigcup_{y' \in E^* \setminus \{y\}} \mathbb{T}_j^{y',h}\right) \tag{63}$$

$$\ge \sum_{y \in E^*} \#\left(\mathbb{T}_j^{y,h} \setminus \bigcup_{y' \in E^* \setminus \{y\}} \mathbb{T}_j^{y'}\right) \tag{64}$$

$$\ge \sum_{y \in E^*}\left(\#\mathbb{T}_j^{y,h} - \sum_{y' \in E^* \setminus \{y\}} \#\left(\mathbb{T}_j^{y'} \cap \mathbb{T}_j^y\right)\right). \tag{65}$$

We show that

$$\#\mathbb{T}_j^{y,h} - \sum_{y' \in E^* \setminus \{y\}} \#\left(\mathbb{T}_j^{y'} \cap \mathbb{T}_j^y\right) \ge \frac{1}{2}\#\mathbb{T}_j^{y,h}. \tag{66}$$

For fixed y, y', we want to find an upper bound for

$$\#\left(\mathbb{T}_j^y \cap \mathbb{T}_j^{y'}\right).$$

This is less than

$$\#\{T \in \mathbb{T}_j^y \colon y' \in T\}.$$

Since $\mathbb{T}_j^y = \bigsqcup_{i=1}^{i(y)} \mathbb{T}_{j,i}^y$ where each $\mathbb{T}_{j,i}^y$ is a (δ, s)-set, we have

$$\#\{T \in \mathbb{T}_j^y \colon y' \in T\} \leq \sum_{i=1}^{i(y)} \#\{T \in \mathbb{T}_{j,i}^y \colon y' \in T\} \lesssim i(y)|y-y'|^{-s}.$$

So, we have

$$\begin{aligned}\sum_{y' \in E^* \setminus \{y\}} \#(\mathbb{T}_j^y \cap \mathbb{T}_j^{y'}) &\lesssim i(y) \sum_{y' \in E^* \setminus \{y\}} |y-y'|^{-s} \\ &= i(y) \sum_{\delta^{1-C(\varepsilon+\eta)} \leq d \leq 1} \ \sum_{y' \in E^*, |y-y'| \sim d} d^{-s}.\end{aligned}$$

Here the summation over d is over dyadic numbers. Since E^* is a $(\delta^{1-C(\varepsilon+\eta)}, s)$-set, we have $\#(E^* \cap B_d(y)) \lesssim (\frac{d}{\delta^{1-C(\varepsilon+\eta)}})^s$, the expression above is bounded by

$$\lesssim i(y) \sum_{\delta^{1-O(\varepsilon+\eta)} \leq d \leq 1} \delta^{sC(\varepsilon+\eta)} (\frac{d}{\delta})^s d^{-s} \lesssim i(y) \delta^{-s} \delta^{sC(\varepsilon+\eta)} |\log \delta|. \tag{67}$$

On the other hand, by (58) and (60), we get $\#\mathbb{T}_j^{y,h} \gtrsim \delta^{2\varepsilon} \#\mathbb{T}_j^y \gtrsim i(y)\delta^{2\varepsilon+\eta-s}$. Therefore, if we choose $C = 100s^{-1}$, then the right hand side of (67) is $\leq \frac{1}{2}\#\mathbb{T}_j^{y,h}$ when δ is small. So, we finish the proof of (66). If we look back to (62), we obtain

$$\#\mathbb{T}_j^h \geq \frac{1}{2} \sum_{y \in E^*} \#\mathbb{T}_j^{y,h}.$$

Combining with (61), we proved the claim (59).

Estimating the right hand side of (59) using (58) and (iv), we obtain

$$\#\mathbb{T}_j^h \gtrsim \delta^{O(\eta+\varepsilon)} \delta^{-s} 2^{-j} \#F. \tag{68}$$

Finally, we estimate $I(E', \mathbb{T}_j^h) := \{(y, T) \in E' \times \mathbb{T}_j^h \colon T \in \mathbb{T}_j^y\}$. Recalling $\mathbb{T}_j^h = \bigcup_{y \in E''} \mathbb{T}_j^{y,h}$ and the definition of $\mathbb{T}_j^{y,h}$ in (55), we have the lower bound

$$I(E', \mathbb{T}_j^h) = \sum_{T \in \mathbb{T}_j^h} \#\{y \in E' \colon T \in \mathbb{T}_j^y\} \geq \#\mathbb{T}_j^h \delta^{-\tau_0/2}. \tag{69}$$

We have the upper bound

$$\begin{aligned} I(E', \mathbb{T}_j^h) &\leq (\#\mathbb{T}_j^h)^{1/2} \left(\sum_{T \in \mathbb{T}_j^h} \#\{y \in E' \colon T \in \mathbb{T}_j^y\}^2 \right)^{1/2} \\ &= (\#\mathbb{T}_j^h)^{1/2} \left(\sum_{y, y' \in E'} \#\{T \in \mathbb{T}_j^h \colon T \in \mathbb{T}_j^y \cap \mathbb{T}_j^{y'}\} \right)^{1/2} \\ &= (\#\mathbb{T}_j^h)^{1/2} \left(\sum_{y \neq y' \in E'} \#\{T \in \mathbb{T}_j^h \colon T \in \mathbb{T}_j^y \cap \mathbb{T}_j^{y'}\} + I(E', \mathbb{T}_j^h) \right)^{1/2} \end{aligned}$$

Note that $\#\{T \in \mathbb{T}^h_j \colon T \in \mathbb{T}^y_j \cap \mathbb{T}^{y'}_j\} \le \#\{T \in \mathbb{T}^y_j \colon y' \in T\}$. We claim that it further has the bound $\lesssim \delta^{-O(\eta)}(\delta/|y-y'|)^s \#\mathbb{T}^y_j$. We use the fact that the tubes in $\{T \in \mathbb{T}^y_j \colon y' \in T\}$ are contained in a $(\delta/|y-y'|)$-tube, and the Frostman's condition (ii) (with $\sigma = s$). Therefore,

$$\begin{aligned}\#\{T \in \mathbb{T}^y_j \colon y' \in T\} &= \sum_i \#\{T \in \mathbb{T}^y_{j,i} \colon y' \in T\} \lesssim \sum_i (1/|y-y'|)^s \\ &\lesssim \sum_i (1/|y-y'|)^s \delta^{-\eta+s} \#\mathbb{T}^y_{j,i} = \delta^{-\eta}(\delta/|y-y'|)^s \#\mathbb{T}^y_j.\end{aligned}$$

We see that

$$\sum_{y \ne y' \in E'} \#\{T \in \mathbb{T}^y_j \colon T \in \mathbb{T}^y_j \cap \mathbb{T}^{y'}_j\} \lesssim \delta^{-O(\eta)} \sum_{y \ne y' \in E'} \left(\frac{\delta}{|y-y'|}\right)^s \#\mathbb{T}^y_j.$$

Noting that from (iv) that $\#\mathbb{T}^y_j \le 2^{-j}\#F$ and

$$\sum_{y \ne y' \in E'} \left(\frac{\delta}{|y-y'|}\right)^s = \sum_{y \in E'} \sum_{y' \in E' \setminus \{y\}} \left(\frac{\delta}{|y-y'|}\right)^s \lesssim \log \delta^{-1} \#E' \lesssim \delta^{-\varepsilon}\delta^{-s},$$

we obtain

$$I(E', \mathbb{T}^h_j) \lesssim (\#\mathbb{T}^h_j)^{1/2} \left(\delta^{-O(\varepsilon+\eta)}\delta^{-s}2^{-j}\#F + I(E', \mathbb{T}^h_j)\right)^{1/2}.$$

This implies

$$I(E', \mathbb{T}^h_j) \lesssim \delta^{-O(\eta+\varepsilon)}\left((\#\mathbb{T}^h_j)^{1/2}(\delta^{-s}2^{-j}\#F)^{1/2} + \#\mathbb{T}^h_j\right).$$

Comparing with the lower bound (69), when we choose η, ε sufficiently small compared with τ_0, we obtain

$$\#\mathbb{T}^h_j \lesssim \delta^{-O(\eta+\varepsilon)+\tau_0/2}\delta^{-s}2^{-j}\#F,$$

which contradicts (68), since η, ε can be chosen much smaller than τ_0.

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