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Essential norms of composition operators and multipliers acting between different Hardy spaces

Frédéric Bayart

Abstract. We compute the essential norms of inclusion operators, composition operators and multipliers acting from a closed subspace of some L^p-space into a subspace of some L^q-space, with $p > q$.

Eri Hardyn avaruuksien välillä kuvaavien yhdistely- ja tulo-operaattoreiden oleelliset normit

Tiivistelmä. Tässä työssä lasketaan avaruuden L^p suljetun aliavaruuden toisen avaruuden L^q aliavaruuteen kuvaavien sisältymis-, yhdistely- ja tulo-operaattoreiden oleellinen normi, kun $p > q$.

1. Introduction

1.1. General context. Let $(\Omega_1, \mathcal{E}, \mu)$ and $(\Omega_2, \mathcal{F}, \nu)$ be two measure spaces, let $p, q \in [1, +\infty]$, let X_p, Y_q be two closed subspaces of $L^p(\Omega_1)$ (resp. $L^q(\Omega_2)$) and let $T_\varphi \colon X_p \to Y_q$ be a linear map depending on some "symbol" φ. Our aim in this paper is to obtain estimates of the essential norm of T_φ by quantities depending only on the symbol φ. To emphasize that we work with different values of p and q, we will denote $\|T\|_{p\to q}$ (resp. $\|T\|_{e,p\to q}$) the norm (resp. the essential norm) of any operator $T \colon X_p \to Y_q$. In particular, we will be concerned with composition operators and multiplication operators.

1.2. Composition operators. Let φ be a holomorphic self-map of the unit disc $\mathbb{D}$ and let $C_\varphi(f) = f \circ \varphi$ be the associated composition operator. Let also $p, q \in [1, +\infty]$. The characterization of compact composition operators $C_\varphi \colon H^p(\mathbb{D}) \to H^q(\mathbb{D})$ and the computation of the essential norm $\|C_\varphi\|_{e,p\to q}$ have been investigated by many mathematicians (see for instance [20], [5], [12] or [7] and the references therein). In particular, the case $p \leq q$ is fairly well understood and $\|C_\varphi\|_{e,p\to q}$ is estimated by quantities depending only on φ and involving either Nevanlinna counting functions or Carleson measures or integrals.

The case $p > q \geq 1$ remains more mysterious. Jarchow and Gobeler have shown ([14, 11]) that C_φ is compact if and only if $E = E_\varphi = \{\xi \in \mathbb{T} \colon |\varphi^*(\xi)| = 1\}$ has (Lebesgue) measure 0, where φ^* denotes the radial limit function of φ. Upper and lower estimates for $\|C_\varphi\|_{e,p\to q}$ have been obtained in [12] when $q > 1$ and generalized to $q = 1$ in [9] but they do not coincide.

Our first main result in this paper is to get an estimation for $\|C_\varphi\|_{e,p\to q}$ in the spirit of what has been done in the case $p \leq q$. Thus assume that $\sigma(E) > 0$ where σ is the normalized Lebesgue measure on the circle. The map $\varphi^*_{|E} \colon E \to \varphi^*(E)$ is a nonsingular transformation from (E, σ) into $(\varphi^*(E), \sigma)$ meaning that it does not collapse a set of positive measure into a set of measure 0. We shall denote by F_φ the

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Radon–Nikodym derivative of $\sigma_{|E} \circ (\varphi^*)^{-1}_{|\varphi^*(E)}$ with respect to $\sigma_{|\varphi^*(E)}$. It turns out that $\|C_\varphi\|_{e,p\to q}$ is comparable to $\|F_\varphi\|_s^{1/q}$ with $s = p/(p-q)$.

Theorem 1.1. *Let $1 \le q < p$, let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be holomorphic with $\sigma(E_\varphi) > 0$. Set $s = p/(p-q)$. Then*

$$\|F_\varphi\|_{L^s}^{1/q} \le \|C_\varphi\|_{e,p\to q} \le 2\|F_\varphi\|_{L^s}^{1/q}.$$

Moreover, when $q = 2$, $\|C_\varphi\|_{e,p\to 2} = \|F_\varphi\|_s^{1/2}$.

The proof of this theorem will be given in Section 3 in the wider context of composition operators on the Hardy spaces of the complex unit ball $\mathbb{B}_d$. It will use general results on inclusion operators inspired by [2, 18] which will be developed in Section 2.

1.3. Multipliers on Hardy spaces of Dirichlet series. We turn to our second example, multipliers on Hardy spaces of Dirichlet series. The Hardy spaces of Dirichlet series $\mathcal{H}^p$ were introduced by Hedenmalm, Lindqvist and Seip [13] for $p = 2$ and by the author [1] for the remaining cases in the range $p \in [1, +\infty]$. A way to define these spaces is to consider first the following norm in the space of Dirichlet polynomials (i.e. all finite sums $\sum_{n=1}^N a_n n^{-s}$, $a_n \in \mathbb{C}$, $N \in \mathbb{N}$):

$$\left\|\sum_{n=1}^N a_n n^{-s}\right\|_p^p = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left|\sum_{n=1}^N a_n n^{it}\right|^p dt.$$

The space $\mathcal{H}^p$, $1 \le p < +\infty$, is then defined as the completion of the Dirichlet polynomials under this norm. Functions in $\mathcal{H}^p$ are Dirichlet series which converge in the half-plane $\mathbb{C}_{1/2}$ and are holomorphic there, where for $a > 0$, $\mathbb{C}_a = \{s \in \mathbb{C}\colon \Re e(s) > a\}$. We also need to introduce $\mathcal{H}^\infty$, the space of Dirichlet series that define a bounded holomorphic function on the half-plane $\mathbb{C}_0$. It is endowed with the norm $\|D\|_\infty = \sup_{\Re e(s)>0} |D(s)|$.

The multipliers of $\mathcal{H}^p$ have been characterized in [13, 1]. A holomorphic self-map $D\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ induces a bounded map $M_D\colon \mathcal{H}^p \to \mathcal{H}^p$, $f \mapsto Df$ if and only if $D \in \mathcal{H}^\infty$. In that case, $\|M_D\|_{p\to p} = \|D\|_\infty$. Very recently, the multipliers between different Hardy spaces have been studied in [10]. In that paper, it is shown that

- there is no bounded multiplier from $\mathcal{H}^p$ into $\mathcal{H}^q$ if $1 \le p < q < +\infty$;
- for $1 \le q < p < +\infty$, D induces a bounded map from $\mathcal{H}^p$ into $\mathcal{H}^q$ if and only if $D \in \mathcal{H}^r$, with $r = pq/(p-q)$. In that case, $\|M_D\|_{p\to q} = \|D\|_r$ and
$$\|D\|_q \le \|M_D\|_{e,p\to q} \le \|D\|_r;$$
- for $p > 1$, $\|M_D\|_{e,p\to p} = \|D\|_\infty$; for $p = 1$,
$$\frac{1}{2}\|D\|_\infty \le \|M_D\|_{e,1\to 1} \le \|D\|_\infty.$$

We fully complete the picture by computing the essential norm in the remaining cases:

Theorem 1.2. (a) *Let $1 \le q < p$ and $D \in \mathcal{H}^r$ with $r = pq/(p-q)$. Then $\|M_D\|_{e,p\to q} = \|D\|_r$.*
(b) *Let $D \in \mathcal{H}^\infty$. Then $\|M_D\|_{e,1\to 1} = \|D\|_\infty$.*

Our method of proof will use the Bohr lift and harmonic analysis on the polytorus. As a consequence, we will get results corresponding to Theorem 1.2 for multipliers on Hardy spaces of the polytorus which seem new even for the circle (see [21, 10] for details in this case).

1.4. Multipliers on Lebesgue spaces. Our final example deals with multipliers on Lebesgue spaces without any extra structure. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $u\colon \Omega \to \Omega$ be measurable. It is only recently that the essential norm of the multiplier $M_u\colon f \mapsto uf$, as an operator on $L^p(\mu)$, $p \geq 1$, has been computed (see [3, 23]). We shall do the same when M_u is viewed as an operator from $L^p(\mu)$ to $L^q(\mu)$ with $1 \leq q < p$ (continuity has been characterized in [22] and is equivalent to $u \in L^r(\mu)$, $r = pq/(p-q)$ and compactness has been characterized in [15] in the more general context of weighted composition operators). In order to describe that result, we recall that the measure space can be decomposed as a disjoint union $\Omega = \Omega_d \cup \Omega_a$, where $\Omega_d, \Omega_a \in \mathcal{A}$, the restriction μ_d of μ to Ω_d is a diffuse measure and the restriction μ_a of μ to Ω_a is purely atomic. Namely,

- for any measurable subset A of Ω_d with $\mu_d(A) > 0$, for every $\alpha \in (0, \mu_d(A))$, there exists $A' \in \mathcal{A}$ with $A' \subset A$ and $\mu_d(A') = \alpha$.
- Ω_a is the disjoint union of a sequence (A_n) of atoms (any measurable subset of A_n has measure equal to 0 or $\mu_a(A_n)$).

We shall also recall that $(\Omega, \mathcal{A}, \mu)$ is a separable measure space provided there exists a sequence $(B_n) \subset \mathcal{A}$ such that, for any $B \in \mathcal{A}$, for any $\varepsilon > 0$, one may find $n \geq 1$ such that $\mu(B \Delta B_n) < \varepsilon$. Under this assumption, for any $p \in [1, +\infty)$, $L^p(\mu)$ is separable: the set of steps functions $\mathbf{1}_{B_n}$ spans a dense subspace of $L^p(\mu)$.

Theorem 1.3. *Let $1 \leq q < p$ and set $r = pq/(p-q)$. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite separable measure space and let $u \in L^r(\mu)$. Then $\|M_u\|_{e,p\to q} = \|u_{|\Omega_d}\|_r$.*

If we allow $p = +\infty$, we lose a factor $1/2$ in the estimate of the essential norm.

Theorem 1.4. *Let $q \geq 1$, let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite separable measure space and let $u \in L^q(\mu)$. Then $\frac{1}{2}\|u_{|\Omega_d}\|_q \leq \|M_u\|_{e,\infty\to q} \leq \|u_{|\Omega_d}\|_q$.*

1.5. A general argument. We shall use several times the following lemma, inspired by [4, Proposition 2.3].

Lemma 1.5. *Let X, Y be Banach spaces, let $T \in \mathcal{L}(X, Y)$ and let $\lambda > 0$.*

a) *Let $(\mathcal{R}_n) \subset \mathcal{L}(Y)$ be a sequence of bounded operators such that $\|\mathcal{R}_n\| \leq \lambda$ for all n. Assume that $(\mathcal{R}_n)$ converges pointwise to 0. Then*

$$\|T\|_{e,X\to Y} \geq \frac{1}{\lambda} \limsup_n \|\mathcal{R}_n T\|_{X\to Y}.$$

b) *Let $(\mathcal{Q}_n) \subset \mathcal{L}(X)$ be a sequence of compact operators and let $\mathcal{R}_n = \mathrm{Id}_X - \mathcal{Q}_n$. Then $\|T\|_{e,X\to Y} \leq \liminf_n \|T\mathcal{R}_n\|_{X\to Y}$.*

Proof. a) Let $K\colon X \to Y$ be compact. Then

$$\|T - K\|_{X\to Y} \geq \frac{1}{\lambda}\|\mathcal{R}_n T - \mathcal{R}_n K\|_{X\to Y} \geq \frac{1}{\lambda}\|\mathcal{R}_n T\|_{X\to Y} - \frac{1}{\lambda}\|\mathcal{R}_n K\|_{X\to Y}.$$

Now, since K is compact, $(\|\mathcal{R}_n\|)$ is bounded and $(\mathcal{R}_n)$ converges pointwise to 0; it follows from a standard compactness argument that $\|\mathcal{R}_n K\|_{X\to Y}$ tends to 0. Hence

$$\|T - K\|_{X\to Y} \geq \frac{1}{\lambda} \limsup_n \|\mathcal{R}_n T\|_{X\to Y}$$

and we conclude by taking the infimum over the compact operators $K\colon X \to Y$.

b) This is an easy consequence of

$$\|T\|_{e,X\to Y} = \|T\mathcal{R}_n + T\mathcal{Q}_n\|_{e,X\to Y} \le \|T\mathcal{R}_n\|_{X\to Y}. \qquad \square$$

Notation. Throughout this paper, we shall use the following notations. For $p \ge 1$, p^* will stand for the conjugate exponent of p. We shall denote by σ the rotation invariant probability measure on $\mathbb{S}_d$. For two points $z, w \in \mathbb{C}^d$, we write

$$\langle z, w\rangle = \sum_{j=1}^{d} z_j \overline{w_j}$$

and $|z| = \sqrt{\langle z, z\rangle}$. If we consider two functions $f\colon E \to \mathbb{R}$, we write $f \lesssim g$ is there is some $c > 0$ such that $f \le cg$ and $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

2. Inclusion operators

2.1. Some results on functions on the ball. Let $\varphi\colon \mathbb{B}_d \to \mathbb{B}_d$ be holomorphic. For almost every $\xi \in \mathbb{S}_d$, $\varphi^*(\xi) = \lim_{r\to 1} \varphi(r\xi)$ exists. Thus we may regard φ as a map of $\overline{\mathbb{B}_d}$ into $\overline{\mathbb{B}_d}$ and we will usually keep on writing φ for this map, and reserve the notation φ^* for the map from $\mathbb{S}_d$ into $\overline{\mathbb{B}_d}$ as defined above.

The existence of inner functions on $\mathbb{B}_d$ will play an essential role. In particular, we shall use the following corollary (see [16]): for every $G\colon \mathbb{S}_d \to (0, +\infty)$ continuous, one may find $f \in H^\infty(\mathbb{B}_d)$ such that $|f| = G$ a.e. on $\mathbb{S}_d$. In particular this yields the following lemma.

Lemma 2.1. *Let $1 \le q < p$ and set $s = p/(p-q)$. Then for all $F\colon \mathbb{S}^d \to [0, +\infty)$ measurable,*

$$\|F\|_{L^s(\sigma)} = \sup\left(\int_{\mathbb{S}_d} F|g|^q \, d\sigma \colon g \in B_{H^p(\mathbb{B}_d)}\right).$$

Proof. This follows from

$$\|F\|_s = \sup\left(\int_{\mathbb{S}_d} FG \, d\sigma \colon G\colon \mathbb{S}_d \to (0, +\infty) \text{ continuous, } \|G\|_{s^*} = 1\right)$$

and from $\|g\|_p = 1$ provided $g \in H^\infty(\mathbb{B}_d)$ is such that $|g| = G^{1/q}$ a.e. on $\mathbb{S}_d$ with $\|G\|_{s^*} = 1$ (here, $s^* = p/q$). $\square$

For $\xi \in \mathbb{S}_d$, the admissible approach region $\Gamma(\xi)$ is defined as

$$\Gamma(\xi) = \left\{z \in \mathbb{B}_d \colon |1 - \langle z, \xi\rangle| < 1 - |z|^2\right\}.$$

As a consequence of Fubini's theorem, one can prove (see e.g. [18, equation (2.1)]) that for all nonnegative mesurable functions f and for all positive Borel measures μ,

$$\int_{\mathbb{B}_d} f(z)\, d\mu(z) \asymp \int_{\mathbb{S}_d} \int_{\Gamma(\xi)} f(z) \frac{d\mu(z)}{(1-|z|^2)^d} \, d\sigma(\xi). \tag{2.1}$$

If f is a function on $\mathbb{B}_d$, its maximal function $\mathcal{M}f$ is defined on $\mathbb{S}_d$ by $\mathcal{M}f(\xi) = \sup_{z\in\Gamma(\xi)} |f(z)|$. The maximal function has the following L^p-boundedness property ([19, Theorem 5.4.10]): for all $p > 1$, there exists $A(p) > 0$ such that, for all $f \in H^p(\mathbb{B}_d)$, $\|\mathcal{M}f\|_{L^p(\sigma)} \le A(p)\|f\|_{H^p}$.

2.2. Essential norms of inclusion operators on $H^p(\mathbb{B}_d)$. Let μ be a positive Borel measure on $\overline{\mathbb{B}_d}$. We are intested in the inclusion operator $J_\mu\colon H^p(\mathbb{B}_d) \to L^q(\mu)$ when $p > q \ge 1$. This operator has already been investigated by Pau in [18] when μ is supported in $\mathbb{B}_d$. With this assumption, it is shown that J_μ is continuous if and only if $\hat{\mu}\colon \xi \in \mathbb{S}_d \mapsto \int_{\Gamma(\xi)} \frac{d\mu(z)}{(1-|z|^2)^d} \in L^s(\sigma)$ where $s = p/(p-q)$. For the general case, we denoty by $\mu_{\mathbb{B}}$ the restriction of μ to $\mathbb{B}_d$ and by $\mu_{\mathbb{S}}$ its restriction to $\mathbb{S}_d$. Since functions in $H^p(\mathbb{B}_d)$ are only defined almost everywhere on $\mathbb{S}_d$, we restrict ourselves to the case where $\mu_{\mathbb{S}} = F d\sigma$ for some nonnegative $F \in L^1(\sigma)$. Under these assumptions, we can characterize the continuity of J_μ and compute its essential norm.

Theorem 2.2. *Let $p > q \geq 1$, let $s = p/(p-q)$ and let $\mu = \mu_{\mathbb{B}} + F d\sigma$ be a positive Borel measure on $\overline{\mathbb{B}_d}$. Then $J_\mu\colon H^p(\mathbb{B}_d) \to L^q(\mu)$ is bounded if and only if*

a) $\hat{\mu}\colon \xi \in \mathbb{S}_d \mapsto \displaystyle\int_{\Gamma(\xi)} \frac{d\mu(z)}{(1-|z|^2)^d}$ *belongs to* $L^s(\sigma)$,

b) F *belongs to* $L^s(\sigma)$.

Moreover, provided the above assumptions are satisfied, $\|J_\mu\|_{e,p\to q} = \|F\|_s^{1/q}$.

Proof. That a) and b) imply the continuity of J_μ follows partly from Pau's result and partly from Hölder's inequality. Indeed, for $f \in H^p(\mathbb{B}_d)$,

$$\int_{\mathbb{S}_d} |f|^q F\, d\sigma \leq \left(\int_{\mathbb{S}_d} |f|^p\, d\sigma\right)^{q/p} \left(\int_{\mathbb{S}_d} F^s\, d\sigma\right)^{1/s}$$

where we have used Hölder's inequality with the exponents p/q and s. Conversely assume that J_μ is bounded. Again, a) follows from Pau's result. To prove b) we observe that for all $g \in H^p(\mathbb{B}_d)$ with norm 1,

$$\int_{\mathbb{S}_d} F|g|^q\, d\sigma \leq \|J_\mu\|_{p\to q}^q$$

and we conclude by Lemma 2.1.

Let us now compute the essential norm. Our first task is to show that if μ is supported in $\mathbb{B}_d$, then J_μ is compact. Since $H^p(\mathbb{B}_d)$ is reflexive (recall that $p > 1$) we only have to show that J_μ is completely continuous. Let (f_n) be a weakly-null sequence in $H^p(\mathbb{B}_d)$. Using (2.1), we have to prove that

$$\int_{\mathbb{S}_d} \int_{\Gamma(\xi)} |f_n(z)|^q \frac{d\mu(z)}{(1-|z|^2)^d}\, d\sigma(\xi) \to 0. \tag{2.2}$$

Let $\varepsilon > 0$, let $r \in (0,1)$ and let us set $\Gamma_r(\xi) = \{z \in \Gamma(\xi)\colon |z| \leq r\}$. On the one hand,

$$\begin{aligned}\int_{\mathbb{S}_d} \int_{\Gamma(\xi)\setminus\Gamma_r(\xi)} |f_n(z)|^q \frac{d\mu(z)}{(1-|z|^2)^d}\, d\sigma(\xi) &\leq \int_{\mathbb{S}_d} |\mathcal{M} f_n(\xi)|^q \int_{\Gamma(\xi)\setminus\Gamma_r(\xi)} \frac{d\mu(z)}{(1-|z|^2)^d}\, d\sigma(\xi) \\ &\lesssim \|f_n\|_p^q \left(\int_{\mathbb{S}_d} |\widehat{\mu_r}(\xi)|^s\, d\sigma(\xi)\right)^{1/s} \end{aligned} \tag{2.3}$$

where we have used Hölder's inequality with exponents s and $s^* = p/q$ and we have set μ_r the restriction of μ to $\Gamma(\xi)\setminus\Gamma_r(\xi)$. Observe that, for all $r \in (0,1)$ and all $\xi \in \mathbb{S}_d$, $\widehat{\mu_r}(\xi) \leq \widehat{\mu}(\xi)$ and $\hat{\mu} \in L^s(\sigma)$. We prove that $\widehat{\mu_r}(\xi)$ tends to 0 as r tends to 1 for almost all $\xi \in \mathbb{S}_d$. Write

$$\widehat{\mu_r}(\xi) = \int_{\Gamma(\xi)} F_r(z)\, d\mu(z)$$

with $F_r(z) = \mathbf{1}_{\overline{\mathbb{B}_d}\setminus r\overline{\mathbb{B}_d}}(z) \frac{1}{(1-|z|^2)^d}$, $z \in \Gamma(\xi)$. Let ξ be such that $\widehat{\mu}(\xi) < +\infty$ (this holds for a.e. ξ since $\widehat{\mu} \in L^s(\sigma)$). Then $1/(1-|z|^2)^d \in L^1(\Gamma(\xi), \mu)$ and $F_r(z) \to 0$ as $r \to 1$. Lebesgue's theorem implies that $\widehat{\mu_r}(\xi)$ tends to zero. Therefore, by a second application of Lebesgue's theorem, $\|\widehat{\mu_r}\|_s$ tends to zero and it follows, since $(\|f_n\|_p)$ is bounded, that for r sufficiently close to 1 and for all $n \geq 1$,

$$\int_{\mathbb{S}_d}\int_{\Gamma(\xi)\backslash\Gamma_r(\xi)} |f_n(z)|^q \frac{d\mu(z)}{(1-|z|^2)^d}\, d\sigma(\xi) \leq \varepsilon.$$

Such a value of r being fixed, we now observe that (f_n) converges uniformly to 0 in $r\overline{\mathbb{B}_d}$. This implies easily that, for n large enough,

$$\int_{\mathbb{S}_d}\int_{\Gamma_r(\xi)} |f_n(z)|^q \frac{d\mu(z)}{(1-|z|^2)^d}\, d\sigma(\xi) \leq \varepsilon.$$

Hence (2.2) is proved and J_μ is completely continuous.

Let us return now to the general case. Let $T\colon H^p(\mathbb{B}_d) \to L^q(\mu),\ f \mapsto f\mathbf{1}_{\mathbb{B}_d}$. Then $T = i \circ J_{\mu_{\mathbb{B}}}$ where i is the inclusion operator $L^q(\mu_{\mathbb{B}}) \to L^q(\mu)$. Since $J_{\mu_{\mathbb{B}}}$ is compact, T is also compact. Moreover, for $f \in H^p(\mathbb{B}_d)$,

$$\|(J_\mu - T)f\|_{L^q(\mu)} = \left(\int_{\mathbb{S}_d} |f|^q\, d\mu_{\mathbb{S}}\right)^{1/q} = \left(\int_{\mathbb{S}_d} |f|^q F\, d\sigma\right)^{1/q} \leq \|f\|_p \cdot \|F\|_s^{1/q}$$

by Hölder's inequality applied to the conjugate exponents p/q and s.

Conversely, let $\varepsilon > 0$ and let $g \in B_{H^p}$ be such that

$$\int_{\mathbb{S}_d} F|g|^q\, d\sigma \geq \|F\|_s - \varepsilon.$$

Let also I be an inner function on $\mathbb{B}_d$ with $I(0) = 0$ and let us consider $g_k = I^k g$. Then $(g_k)_k$ converges weakly to 0 (see [12, p. 37]). Therefore,

$$\|J_\mu\|_{e,p\to q} \geq \limsup_{k\to+\infty} \frac{\|J_\mu(g_k)\|_q}{\|g_k\|_p}.$$

Now, $\|g_k\|_p = \|g\|_p = 1$ whereas

$$\|J_\mu(g_k)\|_q \geq \left(\int_{\mathbb{S}_d} |g|^q F\, d\sigma\right)^{1/q} \geq (\|F\|_s - \varepsilon)^{1/q}.$$

Since $\varepsilon > 0$ is arbitrary, we get the lower inequality $\|J_\mu\|_{e,p\to q} \geq \|F\|_s^{1/q}$. □

Remark 2.3. By the above theorem, we observe that if μ has support in $\mathbb{B}_d$, then J_μ is continuous if and only if J_μ is compact.

3. Composition operators

3.1. Composition operators on Hardy spaces of the ball. We turn to composition operators on the Hardy spaces of the ball $\mathbb{B}_d$. We fix φ a holomorphic self-map of $\mathbb{B}_d$ such that C_φ induces a bounded composition operator on some (therefore, on all) Hardy spaces $H^p(\mathbb{B}_d)$. In particular, this implies the following facts which we shall use repeatedly:

(H1) no set of positive measure in $\mathbb{S}_d$ is mapped by φ onto a set of measure 0 in $\mathbb{S}_d$ (see Corollary 3.38 of [6]);

(H2) for any $f \in H^p(\mathbb{B}_d)$, $(f\circ\varphi)^*(\xi) = f(\varphi^*(\xi))$ for a.e. $\xi \in \mathbb{S}_d$ (see Lemma 1.6 of [17]).

We point out that, in what follows, we could replace the assumption C_φ *is continuous on* H^p by these two assumptions. They are satisfied for any holomorphic self-map of $\mathbb{B}_d$ when $d=1$.

Suppose now that $\sigma(E)>0$, where $E=E_\varphi=\{\xi\in\mathbb{S}_d\colon |\varphi(\xi)|=1\}$. Then $\varphi^*_{|E}$ induces a nonsingular transformation from (E,σ) into $(\varphi^*(E),\sigma)$. Let F_φ be the Radon–Nikodym derivative of $\sigma_{|E}\circ(\varphi^*)^{-1}_{|\varphi^*(E)}$ with respect to $\sigma_{|\varphi^*(E)}$. We extend F_φ on $\mathbb{S}_d$ outside $\varphi^*(E)$ by setting it equal to 0. If $\sigma(E)=0$, we just set $F_\varphi=0$ on $\mathbb{S}_d$. We intend to prove the following general version of Theorem 1.1.

Theorem 3.1. *Let $1\le q<p$ and set $s=p/(p-q)$. For all analytic maps $\varphi\colon\mathbb{B}_d\to\mathbb{B}_d$ inducing a bounded operator $C_\varphi\colon H^p(\mathbb{B}_d)\to H^p(\mathbb{B}_d)$,*

$$\|F_\varphi\|_s^{1/q}\le\|C_\varphi\|_{e,p\to q}\le\min(2,\|P_q\|)\|F_\varphi\|_s^{1/q}$$

where $P_q\colon L^q(\sigma)\to H^q(\mathbb{B}_d)$ is the Szegö projection.

In particular, for $q=2$, we get $\|C_\varphi\|_{e,p\to q}=\|F_\varphi\|_s^{1/2}$.

Proof. If $\sigma(E)=0$, then C_φ is compact by [12, Corollary 2] and there is nothing to prove. Therefore we will assume $\sigma(E)>0$. Let $\mu_\varphi=\sigma\circ(\varphi^*)^{-1}$ be the pullback measure of σ by φ^*, which is a measure on $\overline{\mathbb{B}_d}$. Its restriction to $\mathbb{S}_d$ is $F_\varphi d\sigma$. The change of variables formula shows that, for any $f\in H^p(\mathbb{B}_d)$

$$\|C_\varphi(f)\|_q=\|J_{\mu_\varphi}(f)\|_{L^q(\mu_\varphi)}.$$

However, without any extra work, this does not rely directly $\|C_\varphi\|_{e,p\to q}$ to $\|J_{\mu_\varphi}\|_{e,p\to q}$. We first give a proof working for $q>1$ (recall that the Szegö projection is bounded if and only if $q>1$).

Let us introduce

$$\begin{array}{rcl} W_q\colon L^q(\mu_\varphi) & \to & L^q(\sigma),\\ g & \mapsto & g\circ\varphi^*,\end{array}\qquad \begin{array}{rcl} R_q\colon H^q(\mathbb{B}_d) & \to & L^q(\sigma),\\ f & \mapsto & f^*.\end{array}$$

The maps W_q and R_q are both isometries and from $(f\circ\varphi)^*=f\circ\varphi^*$, we deduce that

$$R_q\circ C_\varphi=W_q\circ J_{\mu_\varphi}\colon H^p(\mathbb{B}_d)\to L^q(\sigma).$$

Observe also that $P_qR_q=\mathrm{Id}_{H^q}$. Let finally $K\colon H^p(\mathbb{B}_d)\to L^q(\mu_\varphi)$ be a compact operator. Then

$$\begin{aligned}\|J_{\mu_\varphi}-K\|_{p\to q}&\ge\|W_qJ_{\mu_\varphi}-W_qK\|_{p\to q}\ge\|R_qC_\varphi-W_qK\|_{p\to q}\\&\ge\|P_q\|^{-1}\|C_\varphi-PW_qK\|_{p\to q}\ge\|P_q\|^{-1}\|C_\varphi\|_{e,p\to q}\end{aligned}$$

which shows that $\|C_\varphi\|_{e,p\to q}\le\|P_q\|\cdot\|J_{\mu_\varphi}\|_{e,p\to q}$. Conversely, let us define $V_q\colon L^q(\sigma)\to L^q(\mu_\varphi)$ by duality: for $f\in L^q(\sigma)$ and $g\in L^{q^*}(\mu_\varphi)$,

$$\int_{\overline{\mathbb{B}_d}}V_qf\bar g\,d\mu_\varphi=\int_{\mathbb{S}_d}f\overline{W_{q^*}g}\,d\sigma.$$

In particular, $\|V_q\|\le1$ since W_{q^*} is an isometry. Observe also that $V_qW_q=\mathrm{Id}_{L^q(\mu_\varphi)}$ since for $(f,g)\in L^q(\mu_\varphi)\times L^{q^*}(\mu_\varphi)$,

$$\int_{\overline{\mathbb{B}_d}}V_qW_q(f)\bar g\,d\mu_\varphi=\int_{\mathbb{S}_d}W_q(f)\overline{W_{q^*}(g)}\,d\sigma=\int_{\overline{\mathbb{B}_d}}f\bar g\,d\mu_\varphi$$

by the change of variables formula. Now, let $K\colon H^p(\mathbb{B}_d)\to H^q(\mathbb{B}_d)$ be a compact operator. Then

$$\begin{aligned}\|C_\varphi-K\|_{p\to q}&\geq \|R_qC_\varphi-R_qK\|_{p\to q}\geq \|W_qJ_{\mu_\varphi}-R_qK\|_{p\to q}\\&\geq \|J_{\mu_\varphi}-V_qR_qK\|_{p\to q}\geq \|J_{\mu_\varphi}\|_{e,p\to q}\end{aligned}$$

which shows the reverse inequality, $\|J_{\mu_\varphi}\|_{e,p\to q}\leq \|C_\varphi\|_{e,p\to q}$. Now we conclude because $F_{\mu_\varphi}=F_\varphi$ by definition.

We now prove the upper inequality for all values of $q\geq 1$ and for the constant 2. Let $n\geq 1$ and let $\mathcal{Q}_n\colon H^p\to H^p,\ f\mapsto f\left(\left(1-\frac{1}{n}\right)\cdot\right)$. Then $\mathcal{Q}_n$ is a compact operator with norm less than 1. Let $\mathcal{R}_n=I-\mathcal{Q}_n$, $\|\mathcal{R}_n\|\leq 2$. Then for all $n\geq 1$,

$$\|C_\varphi\|_{e,p\to q}\leq \liminf_n \|C_\varphi\mathcal{R}_n\|_{p\to q}.$$

Let $f\in H^p(\mathbb{B}_d)$ with $\|f\|\leq 1$ and let $r\in(0,1)$.

$$\begin{aligned}\|C_\varphi\mathcal{R}_n(f)\|^q&=\int_{\mathbb{S}_d}|\mathcal{R}_n(f)\circ\varphi|^q\,d\sigma=\int_{r\overline{\mathbb{B}_d}}|\mathcal{R}_n(f)|^q\,d\mu_\varphi+\int_{\overline{\mathbb{B}_d}\backslash r\overline{\mathbb{B}_d}}|\mathcal{R}_n(f)|^q\,d\mu_\varphi\\&=: I_{1,n}(r)+I_{2,n}(r).\end{aligned}$$

By Cauchy integral formula and by [24, Theorem 4.17], for any $z\in r\overline{\mathbb{B}_d}$,

$$|\mathcal{R}_n(f)(z)|\leq \frac{1}{n}\sup_{w\in r\overline{\mathbb{B}_d}}|f'(w)|\leq \frac{C(r,d)}{n}\|f\|_p$$

so that $I_{1,n}(r)\leq C(\varphi,r,d)/n^q$ where $C(\varphi,r,d)$ only depends on φ, r and d.

Let us turn to $I_{2,n}(r)$ and let us denote by μ_r the restriction of μ_φ to $\overline{\mathbb{B}_d}\backslash\mathbb{B}_d$. Then by combining Pau's argument (see inequality (2.3)) and the proof of Theorem 2.2, we get that

$$\begin{aligned}I_{2,n}(r)&=\int_{\overline{\mathbb{B}_d}}|\mathcal{R}_n(f)|^q\,d\mu_r\leq \left(\|F_\varphi\|_s+A(p)\|\widehat{\mu_r}\|_s\right)\|\mathcal{R}_n(f)\|_p^q\\&\leq 2^q\|F_\varphi\|_s+2^qA(p)\|\widehat{\mu_r}\|_s\end{aligned}$$

where $A(p)$ only depends on p. But as in the proof of Theorem 2.2, we get that $\|\widehat{\mu_r}\|_s\to 0$ as $r\to 1$. Putting everything together, we finally get

$$\liminf_{n\to+\infty}\|C_\varphi\mathcal{R}_n\|_{p\to q}\leq 2\|F_\varphi\|_s^{1/q},$$

which achieves the proof of the upper estimate.

We conclude by providing a proof for the lower estimate. Let M_E be the operator of multiplication by $\mathbf{1}_E$ from $H^q(\mathbb{B}_d)$ to $L^q(\sigma)$. It is shown in [12] that

$$\|C_\varphi\|_{e,p\to q}\geq \|M_EC_\varphi\|_{p\to q}.$$

Now,

$$\begin{aligned}\|M_EC_\varphi\|_{p\to q}^q&=\sup_{g\in B_{H^p}}\int_E|g\circ\varphi|^q\,d\sigma=\sup_{g\in B_{H^p}}\int_{\mathbb{S}_d}|g|^q\,d\mu_\varphi\\&=\sup_{g\in B_{H^p}}\int_{\mathbb{S}_d}|g|^qF_\varphi\,d\sigma=\|F_\varphi\|_s\end{aligned}$$

by Lemma 2.1. □

3.2. Weighted composition operators. Without extra-work, we can also give an estimate of the essential norm of weighted composition operators. Let $u\colon\mathbb{B}_d\to\mathbb{C}$ and $\varphi\colon\mathbb{B}_d\to\mathbb{B}_d$ be holomorphic. Then the weighted composition operator uC_φ is defined by $(uC_\varphi)(f) = u \cdot (f \circ \varphi)$. Again, we assume that C_φ induces a bounded operator on $H^p(\mathbb{B}_d)$. If $\sigma(E) > 0$, then $\varphi^*_{|E}$ induces a nonsingular transformation from (E, σ) into $(\varphi^*(E), \sigma)$. Let $F_{u,\varphi}$ be the Radon–Nikodym derivative of $|u|^q \sigma_{|E} \circ (\varphi^*)^{-1}_{|\varphi^*(E)}$ with respect to $\sigma_{|\varphi^*(E)}$.

Corollary 3.2. *Let $1 \le q < p$ and set $s = p/(p-q)$. For all analytic maps $\varphi\colon \mathbb{B}_d \to \mathbb{B}_d$ and $u\colon \mathbb{B}_d \to \mathbb{C}^d$ such that $C_\varphi\colon H^p(\mathbb{B}_d) \to H^p(\mathbb{B}_d)$ and $uC_\varphi\colon H^p(\mathbb{B}_d) \to H^q(\mathbb{B}_d)$ are bounded,*

$$\|F_{u,\varphi}\|_s^{1/q} \le \|uC_\varphi\|_{e,p\to q} \le \min(2, \|P_q\|)\|F_{u,\varphi}\|_s^{1/q}$$

where $P_q\colon L^q(\sigma) \to H^q(\mathbb{B}_d)$ is the Szegö projection.

Proof. Let $\mu_\varphi = (|u|^q\sigma) \circ (\varphi^*)^{-1}$ be the pullback measure of $|u|^q\, d\sigma$ by φ, which is a measure on $\overline{\mathbb{B}_d}$. The change of variables formula now writes for any $f \in H^p(\mathbb{B}_d)$,

$$\|C_\varphi(f)\|_q = \|J_{\mu_\varphi}(f)\|_{L^q(\mu_\varphi)}.$$

The proof of the upper estimate follows exactly that of Theorem 3.1. For the lower estimate, we can do exactly the same proof provided we show that

$$\|uC_\varphi\|_{e,p\to q} \ge \|(\mathbf{1}_E u)C_\varphi\|_{p\to q}.$$

Let K be a compact operator and let I be an inner function on $\mathbb{B}_d$ such that $I(0) = 0$. Then for any f in the unit ball of $H^p(\mathbb{B}_d)$,

$$\|uC_\varphi - K\|_{p\to q} \ge \|uC_\varphi(I^n f)\|_q - \|K(I^n f)\|_q.$$

Since $(I^n f)$ goes weakly to zero, and since

$$\|uC_\varphi(I^n f)\|_q^q = \int_{\mathbb{S}_d} |u(\xi)|^q |I^n \circ \varphi^*|^q |f \circ \varphi^*|^q\, d\sigma \to \int_E |u(\xi)|^q |f \circ \varphi^*|^q\, d\sigma,$$

we get the result. Observe that the last part of the proof uses (H1) and (H2) to ensure the a.e. convergence of $|I^n \circ \varphi^*|$ to $\mathbf{1}_E$. □

The most important case in the previous theorem happens when $\varphi(z) = z$. Then uC_φ is the multiplication operator M_u. In the setting, one can say more, since $\|M_u\|_{e,p\to q} \le \|M_u\|_{p\to q} = \|u\|_r$ where $r = pq/(p-q)$. Since moreover $F_{u,\varphi} = |u|^q$, we get $\|M_u\|_{e,p\to q} = \|u\|_r$. The extension of this result to Hardy spaces of the polydisc and to Dirichlet series will be the subject of the next section.

4. Multipliers on spaces of Dirichlet series

4.1. Some facts on Hardy spaces of Dirichlet series. We shall need the following facts on Dirichlet series. We refer to [10] and the references therein for details. Let $N \ge 1$ and let $D(s) = \sum_{n\ge1} a_n n^{-s}$ be a Dirichlet series. We denote by D_N the restriction of D to the first N prime numbers: $D_N(s) = \sum_{\mathrm{gpd}(n)\le p_N} a_n n^{-s}$ where $\mathrm{gpd}(n)$ denotes the biggest prime divisor of n and $(p_n)_{n\ge1}$ is the increasing family of prime numbers. Then the map $\mathcal{P}_N\colon \mathcal{H}^p \to \mathcal{H}^p$, $D \mapsto D_N$ is a contraction for any $p \in [1, +\infty]$ and when $p \in [1, +\infty)$, $\mathcal{P}_N(f) \to f$ in $\mathcal{H}^p$ as $N \to +\infty$. If $p = +\infty$, the convergence holds in the weak-star topology and it is still true that $\|\mathcal{P}_N f\|_\infty \to \|f\|_\infty$ as $N \to +\infty$ (see [8, Chapter 5]). In the following, we will set $\mathcal{H}^p_N = \mathcal{P}_N(\mathcal{H}^p)$.

Hardy spaces of the infinite polytorus and of Dirichlet series are linked by the Bohr point of view. Let $f(s) = \sum_{n=1}^N a_n n^{-s}$ be a Dirichlet polynomial. Any integer

n factorizes as $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. We define the Bohr lift of f by

$$\mathcal{L}(f) = \sum_{n=1}^{N} a_n z^{\alpha(n)}$$

where $\alpha(n) = (\alpha_1, \dots, \alpha_r, 0, \dots)$ provided $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Then $\mathcal{L}$ induces an isometric isomorphism between $\mathcal{H}^p$ and $H^p(\mathbb{T}^\infty)$ for all $p \in [1, +\infty]$. Its inverse will be denoted by $\mathcal{B}$ and will be called the Bohr transform. Observe that $\mathcal{L}$ induces an isometric isomorphism between $\mathcal{H}^p_N$ and $H^p(\mathbb{T}^N)$.

4.2. Essential norm of multipliers. This subsection is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2, part (a). By [10, Theorem 9] we only need to prove the lower bound. Let $K\colon \mathcal{H}^p \to \mathcal{H}^q$ be a compact operator, let $N \geq 1$ and let $\mathcal{P}_N\colon \mathcal{H}^p \to \mathcal{H}^p_N$ be the canonical projection. We set $D_N = \mathcal{P}_N(D)$ and $K_N = \mathcal{P}_N K \mathcal{P}_N$. Then D_N induces a multiplier $M_{D_N}\colon \mathcal{H}^p_N \to \mathcal{H}^q_N$, K_N is a compact operator from $\mathcal{H}^p_N$ to $\mathcal{H}^q_N$ and

$$\|M_D - K\|_{p\to q} \geq \|\mathcal{P}_N M_D \mathcal{P}_N - \mathcal{P}_N K \mathcal{P}_N\|_{p\to q} = \|M_{D_N} - K_N\|_{p\to q}.$$

We move to the polydisc $\mathbb{T}^N$ by considering $F_N = \mathcal{L}(D_N)$ and we still denote $K_N = \mathcal{L} \circ K_N \circ \mathcal{B}$. We intend to show that

$$\|M_{F_N} - K_N\|_{p\to q} \geq \|F_N\|_{H^r(\mathbb{T}^N)} = \|D_N\|_r.$$

Letting N to $+\infty$ will yield the result, since $\|D_N\|_r \to \|D\|_r$.

We set $t = q/(p-q)$ and $G = |F_N|^t$. Then $G \in L^p(\mathbb{T}^N)$ and $\|G\|_p^p = \|F_N\|_r^r$. There exists a sequence of trigonometric polynomials (Q_n) such that $\|Q_n - G\|_p \to 0$ and $Q_n \to G$ a.e. on $\mathbb{T}^N$. For a fixed $n \geq 1$, let $P_n = \prod_{j=1}^N z_j^d$ where $d \geq 0$ is sufficiently large so that $P_n Q_n \in H^p(\mathbb{T}^N)$ and let for $k \geq 1$ $E_{k,n} = z_1^k P_n Q_n$. Then $E_{k,n}$ belongs to $H^p(\mathbb{T}^N)$, $|E_{k,n}| = |Q_n|$ on $\mathbb{T}^N$ and $E_{k,n}(z) \to 0$ as $k \to +\infty$ for any $z \in \mathbb{D}^N$. Therefore, by [10, Lemma 13], $(E_{k,n})_k$ converges to 0 in the weak-star topology of $H^p(\mathbb{T}^N)$, therefore in its weak topology since H^p is reflexive. Now,

$$\begin{aligned}\|M_{F_N}(E_{k,n})\|_q &= \left(\int_{\mathbb{T}^N} |Q_n|^q |F_N|^q\right)^{1/q} \\ &\geq \left(\int_{\mathbb{T}^N} |G|^q |F_N|^q\right)^{1/q} - \left(\int_{\mathbb{T}^N} |Q_n - G|^q |F_N|^q\right)^{1/q}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|M_{F_N} - K_N\|_{p\to q} &\geq \limsup_{k\to+\infty} \frac{\|M_{F_N}(E_{k,n}) - K_N(E_{k,n})\|_q}{\|E_{k,n}\|_p} \\ &\geq \frac{1}{\|Q_n\|_p}\left(\left(\int_{\mathbb{T}^N} |G|^q |F_N|^q\right)^{1/q} - \left(\int_{\mathbb{T}^N} |Q_n - G|^q |F_N|^q\right)^{1/q}\right) \\ &\geq \frac{1}{\|Q_n\|_p}\left(\|F_N\|_r^{r/q} - \|Q_n - G\|_p^{1/q} \|F_N\|_r^{1/q}\right)\end{aligned}$$

by Hölder's inequality applied to the pair of conjugated exponents p/q and $p/(p-q)$. We let n to $+\infty$ to get

$$\|M_{F_N} - K_N\|_{p\to q} \geq \frac{\|F_N\|_r^{r/q}}{\|F_N\|_r^t} = \|F_N\|_r. \qquad \square$$

Remark 4.1. Observe that the above proof is based on two arguments similar to those introduced in the previous sections: we use that we can compute the norm of an element in $L^r(\mathbb{T}^N)$ using only functions in B_{H^p} and we use the existence of inner functions on the polydisc to get a sequence going weakly to zero with prescribed modulus at the distinguished boundary. Here Fourier analysis arguments simplify the proofs.

Proof of Theorem 1.2, part (b). Arguing as above, it is sufficient to prove that, for each $N \geq 1$, for each $F \in H^\infty(\mathbb{T}^N)$, $F \neq 0$, $\|M_F\|_{e,1\to 1} \geq \|F\|_\infty$. The main difficulty we are facing is that $\mathcal{H}^1$ is no longer reflexive and it is more difficult to exhibit sequences converging weakly to 0. Our strategy (inspired by [23]) will be, given $\varepsilon > 0$, to construct a bounded sequence (R_n) in $H^1(\mathbb{T}^N)$ so that, for all $m > n$, $\int_{\mathbb{T}^N} |F| \cdot |R_n - R_m| \geq (\|F\|_\infty - \varepsilon)\|R_n - R_m\|_1$. This construction will be achieved by regularizing functions peaking around $\{z \in \mathbb{T}^N : |F(z)| \geq \|F\|_\infty - \varepsilon\}$.

Thus let $\varepsilon > 0$, $\varepsilon < \min(1/4, \|F\|_\infty)$ and let us denote by μ the Haar measure on $\mathbb{T}^N$. There exists a decreasing sequence of measurable subsets (A_n) of $\mathbb{T}^N$ such that

$$\begin{cases} |F(x)| \geq \|F\|_\infty - \varepsilon & \text{for all } x \in A_n, \\ \mu(A_{n+1}) \leq \frac{1}{4}\mu(A_n). \end{cases}$$

If we take the convolution product of the nonnegative functions $\frac{1}{\mu(A_n)}\mathbf{1}_{A_n}$ with the Féjer kernel, we get for each $n \geq 1$ a sequence of trigonometric polynomials $(G_{n,k})_k$ such that

$$G_{n,k} \xrightarrow{k\to+\infty} \frac{1}{\mu(A_n)}\mathbf{1}_{A_n} \text{ a.e.},$$
$$\forall n,k \geq 1,\ 0 \leq G_{n,k} \leq \frac{1}{\mu(A_n)},$$
$$\forall n,k \geq 1,\ \|G_{n,k}\|_1 \leq 1.$$

Using Egorov's theorem, we obtain for each $n \geq 1$ a trigonometric polynomial Q_n and a measurable set $B_n \subset \mathbb{T}^N$ such that

$$\mu(\mathbb{T}^N \backslash B_n) \leq \varepsilon\mu(A_n), \qquad \left|Q_n - \frac{1}{\mu(A_n)}\mathbf{1}_{A_n}\right| \leq \varepsilon \text{ on } B_n,$$
$$0 \leq Q_n \leq \frac{1}{\mu(A_n)}, \qquad \|Q_n\|_1 \leq 1.$$

We then multiply Q_n by some unimodular polynomial $P_n = \prod_{j=1}^N z_j^d$ to get a holomorphic polynomial R_n with the same modulus as Q_n. We claim that the following fact is true.

Fact. For any $m > n \geq 1$,

$$\int_{\mathbb{T}^N \backslash A_n} |R_n - R_m| < 4\varepsilon \quad \text{and} \quad \int_{A_n} |R_n - R_m| \geq \frac{1}{8}.$$

Let us admit the fact for a while to achieve the proof of Theorem 1.2. The sequence (R_n) is a bounded sequence of $H^1(\mathbb{T}^N)$. Let $K\colon H^1(\mathbb{T}^N) \to H^1(\mathbb{T}^N)$ be compact. Extracting if necessary, we may assume that $(K(R_n))$ converges. Let

$m > n$ be such that $\|KR_m - KR_n\| \leq \varepsilon$. Then

$$\begin{aligned}\|(M_F - K)(R_n - R_m)\|_1 &\geq \|M_F(R_n - R_m)\|_1 - \varepsilon \geq \int_{A_n} |F| \cdot |R_n - R_m| - \varepsilon\\ &\geq (\|F\|_\infty - \varepsilon) \int_{A_n} |R_n - R_m| - \varepsilon.\end{aligned}$$

By the fact,

$$\int_{\mathbb{T}^N} |R_n - R_m| \leq \int_{A_n} |R_n - R_m| + \int_{\mathbb{T}^N \backslash A_n} |R_n - R_n| \leq (1 + 32\varepsilon) \int_{A_n} |R_n - R_m|$$

so that

$$\|(M_F - K)(R_n - R_m)\|_1 \geq \frac{\|F\|_\infty - \varepsilon}{1 + 32\varepsilon} \|R_n - R_m\|_1 - 8\varepsilon \|R_n - R_m\|_1.$$

Since $\varepsilon > 0$ is arbitrary, we get $\|M_F\|_{e,1\to 1} \geq \|F\|_\infty$.

It remains to prove the fact. We first observe that

$$\int_{A_n} |R_n - R_m| \geq \int_{A_n \cap B_n \cap B_m \backslash A_m} |P_n Q_n - P_m Q_m|.$$

Now, provided $z \in A_n \cap B_n \cap B_m \backslash A_m$,

$$|P_n Q_n(z)| \geq \frac{1}{\mu(A_n)} - \varepsilon \text{ and } |P_m Q_m(z)| \leq \varepsilon$$

so that

$$\begin{aligned}\int_{A_n} |R_n - R_m| &\geq \mu(A_n \cap B_n \cap B_m \backslash A_m) \left(\frac{1}{\mu(A_n)} - 2\varepsilon\right)\\ &\geq \big(\mu(A_n \backslash A_m) - \mu(\mathbb{T}^N \backslash B_n) - \mu(\mathbb{T}^N \backslash B_m)\big) \left(\frac{1}{\mu(A_n)} - 2\varepsilon\right)\\ &\geq \frac{1}{4}\mu(A_n) \cdot \left(\frac{1}{\mu(A_n)} - 2\varepsilon\right) \geq \frac{1}{4} - \frac{\varepsilon\mu(A_n)}{2} \geq \frac{1}{8}\end{aligned}$$

since $\varepsilon < 1/4$. Furthermore,

$$\int_{\mathbb{T}^N \backslash A_n} |R_n - R_m| \leq \int_{\mathbb{T}^N \backslash A_n} |R_n| + \int_{\mathbb{T}^N \backslash A_n} |R_m| \leq \int_{\mathbb{T}^N \backslash A_n} |R_n| + \int_{\mathbb{T}^N \backslash A_m} |R_m|.$$

We just need to study

$$\int_{\mathbb{T}^N \backslash A_n} |R_n| \leq \int_{(\mathbb{T}^N \cap B_n) \backslash A_n} |R_n| + \int_{\mathbb{T}^N \backslash B_n} |R_n| \leq \varepsilon + \mu(\mathbb{T}^N \backslash B_n) \times \frac{1}{\mu(A_n)} \leq 2\varepsilon. \ \square$$

A corollary of our proof is the following result.

Corollary 4.2. *Let $N \in \mathbb{N} \cup \{+\infty\}$, let $1 \leq q < p$ and let $u \in H^r(\mathbb{T}^N)$ with $r = pq/(p - q)$. Then $\|M_u\|_{e,p\to q} = \|u\|_r$.*

It remains one case studied in [10] where we are not able to give a formula for the essential norm: for $q \geq 1$ and $D \in \mathcal{H}^q$, it is shown in [10] that

$$\frac{1}{2}\|D\|_q \leq \|M_D\|_{e,\infty\to q} \leq \|D\|_q.$$

We can at least improve this for $q = 2$.

Proposition 4.3. *Let $D \in \mathcal{H}^2$. Then $\|M_D\|_{e,\infty\to 2} = \|D\|_2$.*

Proof. Let $\mathcal{Q}_N$ be the orthogonal projection of $\mathcal{H}^2$ onto $\mathrm{span}(1, 2^{-2}, \cdots, N^{-s})$ and let $\mathcal{R}_N = \mathrm{Id} - \mathcal{Q}_N$ which has norm 1. By Lemma 1.5,

$$\|M_D\|_{e,\infty\to 2} \geq \limsup_{N\to+\infty} \|\mathcal{R}_N M_D\|.$$

Now, let us fix $N \geq 1$ and $n \geq 1$ such that $2^n > N$. Then

$$\|\mathcal{R}_N M_D\|_{\infty\to 2} \geq \|\mathcal{R}_N M_D(2^{-ns})\|_2 \geq \|M_D(2^{-ns})\|_2 - \|\mathcal{Q}_N M_D(2^{-ns})\|_2 \geq \|D\|_2$$

since $\mathcal{Q}_N M_D(2^{-ns}) = 0$. □

4.3. Spectrum of multipliers. We end up this section by improving a result of [10] regarding the spectrum of multipliers.

Theorem 4.4. *Let $D \in \mathcal{H}^\infty$ be a non zero Dirichlet series with associated multiplication operator $M_D \in \mathcal{L}(\mathcal{H}^p)$, $p \in [1, +\infty)$. Then $\sigma_c(M_D) \subset \overline{D(\mathbb{C}_0)} \backslash D(\mathbb{C}_0)$.*

Here, $\sigma_c(M_d)$ denotes the continuous spectrum of M_D, namely the set of complex numbers λ such that $M_D - \lambda$ is injective and has dense but not closed range. In [10], it was only shown that $\sigma_c(M_D) \subset \overline{D(\mathbb{C}_0)} \backslash D(\mathbb{C}_{1/2})$.

Proof. Since $M_D - \lambda = M_{D-\lambda}$, it is sufficient to show that if M_D has dense range, then D does not vanish on $\mathbb{C}_0$ (it is easy to show that $\sigma(M_D) \subset \overline{D(\mathbb{C}_0)}$, see [10] for details). Let $N \geq 1$. If M_D has dense range, then $M_{D_N} : \mathcal{H}^p_N \to \mathcal{H}^p_N$ has dense range too. Assume that $D_N(s_0) = 0$ for some $s_0 \in \mathbb{C}$. Since pointwise evaluation at $s_0 \in \mathbb{C}_0$ is continuous on $\mathcal{H}^p_N$, $M_{D_N}(\mathcal{H}^p_N) \subset \{E \in \mathcal{H}^p_N : E(s_0) = 0\}$ cannot be dense, a contradiction.

Therefore, for all $N \geq 1$, D_N do not vanish on $\mathbb{C}_0$. Now, $\|D_N\|_\infty \leq \|D\|_\infty$ and by Montel's theorem in $\mathcal{H}^\infty$, upon taking a subsequence, there exists $\tilde{D} \in \mathcal{H}^\infty$ such that (D_{N_j}) converges uniformly to $\tilde{D}$ on any half-plane $\mathbb{C}_\sigma$, for all $\sigma > 0$. Now, since the Dirichlet series D converges absolutely in $\mathbb{C}_{1/2}$, $(D_{N_j}(s))$ converges to $D(s)$ for any $s \in \mathbb{C}_{1/2}$. Hence $D = \tilde{D}$ on $\mathbb{C}_{1/2}$, therefore on $\mathbb{C}_0$. We can now use Hurwitz theorem to conclude that D does not vanish on $\mathbb{C}_0$. □

5. Multipliers on Lebesgue spaces

5.1. The case $p \neq +\infty$. In this subsection we intend to prove Theorem 1.3. The main new difficulty is the construction of sequences of functions tending weakly to 0. Indeed, in this general context, we can neither use Fourier analysis tools like in the proof of Theorem 1.2 nor the existence of inner functions which helped us to construct sequences tending weakly to zero. This is this part of the proof which will require that $(\Omega, \mathcal{A}, \mu)$ is separable.

Proof of Theorem 1.3. Let $\Omega = \Omega_d \cup \Omega_a$ where $\Omega_d \cap \Omega_a = \varnothing$, $\mu_d = \mu_{|\Omega_d}$ is diffuse and $\mu_a = \mu_{|\Omega_a}$ is purely atomic. Let (A_n) be a disjoint sequence of atoms such that $\Omega_a = \bigcup_n A_n$.

We first show that $\|M_u\|_{e,p\to q} \leq \|u_{|\Omega_d}\|_r$. For $N \in \mathbb{N}$, let us define

$$u_N = \sum_{n\leq N} a_n \mathbf{1}_{A_n}$$

where $u = a_n$ a.e. on A_n and $K_N = M_{u_N} f$. Since f is a.s. constant on each A_n, K_N is a finite rank operator. Hence, it is compact. Now, for any $f \in L^p(\mu)$,

$$\begin{aligned}\|M_u f - M_{u_N} f\|_q^q &= \int_{\Omega_d} |uf|^q \, d\mu + \int_{\bigcup_{n>N} A_n} |uf|^q \, d\mu \\ &\leq \|u_{|\Omega_d}\|_r^q \|f\|_p^q + \left(\int_{\bigcup_{n>N} A_n} |u|^r \, d\mu \right)^q \|f\|_p^q\end{aligned}$$

where we have used Hölder's inequality with $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Hence, $\liminf_N \|M_u - M_{u_N}\|_{p\to q} \leq \|u_{|\Omega_d}\|_r$ which yields the first inequality.

Conversely, since $(\Omega, \mathcal{A}, \mu)$ is separable, there exists a sequence (B_n) of subsets of Ω_d and belonging to $\mathcal{A}$ such that, for any $B \in \mathcal{A}$, for any $\varepsilon > 0$, one may find $n \geq 1$ such that $\mu(B \Delta B_n) < \varepsilon$. We first construct a sequence (g_n) in $L^p(\mu)$ going weakly to 0. Let us fix for a while $n \geq 1$. For $I \subset \{1, \ldots, n\}$, $I \neq \varnothing$, let us set

$$C_I = \bigcap_{k \in I} B_k \backslash \left(\bigcup_{k \in I^c} B_k \right).$$

Then the sets C_I are paiwise disjoint. Moreover, for any $k \in \{1, \ldots, n\}$, $B_k = \bigcup_{k \in I} C_I$. If $\int_{C_I} |u|^r \, d\mu = 0$, we set $g_n = |u|^r$ on C_I. Otherwise, since $|u|^r d\mu_d$ is still a diffuse measure, we may split C_I into a partition $C_I' \cup C_I''$ such that

$$\int_{C_I'} |u|^r \, d\mu_d = \int_{C_I''} |u|^r \, d\mu_d = \frac{1}{2} \int_{C_I} |u|^r \, d\mu_d.$$

In that case, we set

$$g_n = \begin{cases} |u|^{r/p} & \text{on } C_I', \\ -|u|^{r/p} & \text{on } C_I'', \end{cases}$$

so that $\int_{C_I} g_n \, d\mu_d = 0$. We finally define g_n on $\Omega_d \backslash \bigcup_{k=1}^n B_k$ by

$$g_n = \begin{cases} |u|^{r/p} & \text{on } \Omega_d \backslash \bigcup_{k=1}^n B_k, \\ 0 & \text{on } \Omega_a. \end{cases}$$

We can observe that for any $k \leq n$, $\int_{B_k} g_n \, d\mu_d = 0$. Hence, for all $k \in \mathbb{N}$, $\int_{B_k} g_n \, d\mu_d$ goes to zero as n tends to $+\infty$. Since $(\mathbf{1}_{B_n})_{n \geq 1}$ spans a dense subspace of $L^{p^*}(\mu_d)$, and $g_n = 0$ on Ω_a, this ensures that (g_n) goes weakly to 0 in $L^p(\mu)$. Hence,

$$\|M_u\|_{e, p\to q} \geq \limsup_n \frac{\|M_u(g_n)\|_q}{\|g_n\|_p}.$$

Now, $\|g_n\|_p = \|u_{|\Omega_d}\|_r^{r/p}$ and

$$\|M_u g_n\|_q = \left(\int_{\Omega_d} |u|^{rq} |u|^q \, d\mu_d \right)^{1/q} = \|u_{|\Omega_d}\|_r^{r/q}$$

so that $\|M_u\|_{e, p\to q} \geq \|u_{|\Omega_d}\|_r$ as guessed. □

The case $p = +\infty$. The proof in this case will share some similarities with that of Proposition 4.3. The key tool will be the use of the conditional expectation. The main difference with the previous subsection is that we now work in the target space.

Proof of Theorem 1.4. The proof of the upper bound is completely similar to that of Theorem 1.3. Details are left to the reader. Regarding the lower bound, we may and shall assume that $\Omega = \Omega_d$. Indeed, if P is the canonical projection $L^q(\Omega,\mu) \to L^q(\Omega_d,\mu_d)$ and $K\colon L^\infty(\Omega) \to L^q(\Omega)$ is compact, then $\|M_u - K\|_{\infty\to q} \geq \|M_{u|\Omega_d} - PK\|_{\infty\to q}$. In the same vein we may and shall assume that $(\Omega,\mathcal{A},\mu)$ is a finite measure space. Indeed, writing $\Omega = \bigcup_n \Omega_n$ where $\Omega_n \subset \Omega_{n+1}$ and $\mu(\Omega_n) < +\infty$ for any n, a similar argument shows that $\|M_u\|_{e,\infty\to q} \geq \|M_{u|\Omega_n}\|_{e,\infty\to q}$.

Let (B_n) be a sequence in $\mathcal{A}$ such that, for any $B \in \mathcal{A}$, for any $\varepsilon > 0$, there exists $n \geq 1$ with $\mu(B\Delta B_n) < \varepsilon$. Let $\mathcal{A}_n$ be the σ-algebra generated by $B_1,\dots,B_n$ and for $f \in L^1(\mu)$, let $\mathcal{Q}_n(f) = \mathbb{E}(f|\mathcal{A}_n)$ be the conditional expectation of f given $\mathcal{A}_n$. Each $\mathcal{Q}_n$ is a contraction of $L^q(\Omega)$ and it is a compact operator. Moreover, for any $f \in L^q(\Omega)$, $\mathcal{Q}_n(f)$ goes to f: this is true if f is a linear combination of step functions and we argue by density of these functions, using $\|\mathcal{Q}_n\| \leq 1$. Let $\mathcal{R}_N = I - \mathcal{Q}_n$ which satisfies $\|\mathcal{R}_n\| \leq 2$ and $(\mathcal{R}_n)$ converges to 0 pointwise. Therefore by Lemma 1.5, one obtains

$$\|M_u\|_{e,\infty\to q} \geq \frac{1}{2}\limsup_{n\to+\infty} \|\mathcal{R}_n M_u\|_{\infty\to q}.$$

Now, for $n \geq 1$, $I \subset \{1,\dots,n\}$, $I \neq \varnothing$, let us set

$$C_I = \bigcap_{k\in I} B_k \backslash \left(\bigcup_{k\in I^c} B_k\right).$$

We define a function g_n as follows. If $\int_{C_I} |u|\,d\mu = 0$, we set $g_n = 1$ on C_I. Otherwise, since $|u|d\mu$ is still a diffuse measure, we may split C_I into a partition $C_I' \cup C_I''$ such that

$$\int_{C_I'} |u|\,d\mu = \int_{C_I''} |u|\,d\mu = \frac{1}{2}\int_{C_I} |u|\,d\mu.$$

In that case, we set

$$g_n = \begin{cases} 1 & \text{on } C_I', \\ -1 & \text{on } C_I''. \end{cases}$$

We finally define g_n on $\Omega\backslash \bigcup_{k=1}^n B_k$ by $g_n = 1$. This construction ensures that, for all $A \in \mathcal{A}_n$, $\int_A u g_n\,d\mu = 0$. This yields $\mathcal{Q}_n M_u g_n = 0$. Now,

$$\|\mathcal{R}_n M_u\|_{\infty\to q} \geq \|M_u g_n\|_{\infty\to q} - \|\mathcal{Q}_n M_u g_n\|_{\infty\to q} \geq \|M_u g_n\|_{\infty\to q} \geq \left(\int_\Omega |u|^q\right)^{1/q}.$$

This finishes the proof of the lower bound $\|M_u\|_{e,\infty\to q} \geq \frac{1}{2}\|u\|_q$. □

When $q = 2$, $\mathcal{Q}_n$ is an orthogonal projection and $\|\mathcal{R}_n\| \leq 1$ for all $n \geq 1$. Therefore we obtain the following corollary:

Corollary 5.1. *Let $(\Omega,\mathcal{A},\mu)$ be a σ-finite separable measure space and let $u \in L^2(\mu)$. Then $\|M_u\|_{e,\infty\to 2} = \|u_{|\Omega_d}\|_2$.*

5.3. The case $1 \leq p < q$. Our method also gives the essential norm of $\|M_u\|_{e,p\to q}$ when $1 \leq p < q$. The situation here is easier. Indeed, for any $u\colon \Omega \to \Omega$ measurable, $M_u \in \mathcal{L}(L^p, L^q)$ if and only if $u_{|\Omega_d} = 0$ and $\sup_n |u(A_n)|/\mu(A_n)^{1/r} < +\infty$ where $r = pq/(p-q)$ and u is a.e. equal to $u(A_n)$ on A_n (see [22]).

Proposition 5.2. *Let $1 \leq p < q$ and set $r = pq/(p-q)$. Let $(\Omega,\mathcal{A},\mu)$ be a σ-finite measure space and let $u\colon \Omega \to \Omega$ be measurable such that $u_{|\Omega_d} = 0$ and $\sup_n |u(A_n)|/\mu(A_n)^{1/r} < +\infty$. Then*

$$\|M_u\|_{e,p\to q} = \limsup_{n\to+\infty} \frac{|u(A_n)|}{\mu(A_n)^{1/r}}.$$

Proof. Without loss of generality, we can assume that the sequence (A_n) is infinite and $\mu(A_n)\neq 0$ for all n (otherwise, M_u is always compact since it has finite rank). For $s\in\{p,q\}$, denote $\mathcal{Q}_n^s f=\sum_{k=1}^n \mathbf{1}_{A_k} f\in L^s(\mu)$ and $\mathcal{R}_n^s=\mathrm{Id}_{L^s}-\mathcal{Q}_n$. Then $\|\mathcal{R}_n^s\|=1$ and by Lemma 1.5,

$$\limsup_n \|\mathcal{R}_n^q M_u\|\leq \|M_u\|_{e,p\to q}\leq \liminf_n \|M_u\mathcal{R}_n^p\|.$$

Now, for any $f\in L^p$, $\mathcal{R}_n^q M_u f=M_u\mathcal{R}_n^p f=\sum_{k=n+1}^{+\infty} u(A_k)\mathbf{1}_{A_k} f=: T_n f$. We conclude by [22, Theorem 1.4] that

$$\|T_n\|_{p\to q}=\sup_{k\geq n}\frac{|u(A_n)|}{\mu(A_n)^{1/r}}. \qquad \square$$

5.4. Weighted composition operators. In the spirit of [15] or of Section 3.2 of the present paper, our method of proof has applications to weighted composition operators. Let $(\Omega_1,\mathcal{A},\mu)$ and $(\Omega_2,\mathcal{B},\nu)$ be two σ-finite measure spaces, let $u\colon\Omega_2\to\mathbb{C}$ be measurable and let $\varphi\colon\Omega_1\to\Omega_2$ be measurable and nonsingular. The weighted composition operator uC_φ is defined for $f\in L^p(\mu)$ by

$$uC_\varphi f(x)=u(x)\cdot f\circ\varphi(x),\quad x\in\Omega_2.$$

For $q\geq 1$, the measure μ_q defined for any $A\in\mathcal{A}$ by

$$\mu_q(A)=\int_{\varphi^{-1}(A)}|u|^q\,d\nu$$

is absolutely continuous with respect to μ. Its Radon–Nikodym derivative will be denoted by $d\mu_q/d\mu$. It satisfies the important property

$$\|uC_\varphi f\|_{L^q(\nu)}=\|M_{F_{q,u,\varphi}}f\|_{L^q(\mu)}$$

where $F_{q,u,\varphi}=(d\mu_q/d\mu)^{1/q}$. Then Theorem 1.3 and its proofs yields the following statement.

Theorem 5.3. *Let $(\Omega_1,\mathcal{A},\mu)$ and $(\Omega_2,\mathcal{B},\nu)$ be two σ-finite measure spaces with Ω_1 separable, let $u\colon\Omega_2\to\mathbb{C}$ be measurable and let $\varphi\colon\Omega_1\to\Omega_2$ be measurable and nonsingular. Let finally $p>q\geq 1$. Then $\|uC_\varphi\|_{e,p\to q}=\|F_{q,u,\varphi|\Omega_{1,d}}\|_r$ where $r=pq/(p-q)$ and $\Omega_{1,d}$ is the diffuse part of Ω_1.*

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Frédéric Bayart
Université Clermont Auvergne, Campus Universitaire des Cézeaux
Laboratoire de Mathématiques Blaise Pascal UMR 6620 CNRS
3 place Vasarely, 63178 Aubière Cedex, France
frederic.bayart@uca.fr