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Ricci curvature bounded below and uniform rectifiability

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Abstract. We prove that Ahlfors-regular RCD spaces are uniformly rectifiable and satisfy the Bilateral Weak Geometric Lemma with Euclidean tangents—a quantitative flatness condition. The same is shown for Ahlfors regular boundaries of non-collapsed RCD spaces. As an application we deduce a type of quantitative differentiation for Lipschitz functions on these spaces.

Alarajallinen Riccin kaarevuus ja tasainen suoristuvuus

Tiivistelmä. Tässä työssä osoitetaan, että Ahlforsin-säännölliset ja Riemannin kaarevuus–ulottuvuusehdon (RCD) toteuttavat avaruudet ovat tasaisesti suoristuvia, ja että niissä pätee kaksisuuntainen heikko geometrinen lemma euklidisin tangentein, joka on laakeutta mittaava ehto. Sama osoitetaan luhistumattomien RCD-avaruuksien Ahlforsin-säännöllisille reunoille. Sovelluksena johdetaan Lipschitzin funktioiden karkean mittakaavan derivoituvuus näissä avaruuksissa.

1. Introduction

The aim of this note is to provide some concrete examples of *uniformly rectifiable* metric spaces. More precisely, we show that a vast class of RCD spaces, are uniformly rectifiable (UR).

1.1. Uniform rectifiability. Uniform rectifiability is a quantitative strengthening of the qualitative property of being rectifiable. Recall that if $E \subset \mathbb{R}^n$ has finite $\mathcal{H}^k$ measure then one says that E is k-rectifiable if there are Lipschitz maps $f_i \colon \mathbb{R}^k \to \mathbb{R}^n$, $i = 1, 2, \dots$ so that

$$\mathcal{H}^k\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)\right) = 0.$$

A k-rectifiable set $E \subset \mathbb{R}^n$ looks like $\mathbb{R}^k$ asymptotically, but we cannot say anything at any definite scale. On the other hand, E being uniformly k-rectifiable tells us that the scales at which E is non-flat, that is, very far from looking like $\mathbb{R}^k$, are just a few. A different but equivalent way to put this is to say: if we look at a k-rectifiable set E in a ball $B_r(x)$, $x \in E$, then we are guaranteed that $\mathcal{H}^k(B_r(x) \cap E \cap f(\mathbb{R}^k)) > 0$, where $f \colon \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz—but no more. If E is k-UR, however, we know that $\mathcal{H}^k(B_r(x) \cap E \cap f(\mathbb{R}^k)) \geq cr^k$, and c is uniform in x and r. These

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'quantifications' (that can be traced back to the landmark [48] and [28, 29]) have had far reaching consequences, at least as far as sets and measures in Euclidean space are concerned: the solution of the Painlevè problem [56, 67]; the partial one of Vitushkin's conjecture [27, 19, 26]; the (also partial) solution of the David–Semmes problem on the boundedness of the Riesz transform [60]; that of Bishop conjecture on harmonic measure [7]; that of the Dirichlet [8] and regularity [59] problems in elliptic PDEs—all these rest upon the quantifications mentioned above. We refer the reader to Mattila's survey [55, Section 6]. Very recently, the first author together with Bate and Schul [9] generalised some aspects of this theory to metric spaces[1]. This note confirms that there are plenty of (relevant) UR metric spaces.

1.2. Spaces with lower bounds on the Ricci curvature. The metric spaces which are the focus of this note are *RCD spaces*. Roughly speaking, RCD spaces are a class of metric measure spaces (m.m.s) which have the defining property of having lower bounds on (a synthetic notion of) Ricci curvature. This condition originated in the study of Riemannian manifolds and from the fundamental question of how lower bounds on curvature (be it Ricci, sectional, scalar) affect their global and local geometry. Restricting now the discussion to manifolds with Ricci lower bounds, in order to study their local properties it comes natural to take sequences of such manifolds, in some appropriate sense, and study whatever limiting object (a *Ricci limit*) is found. In this sense, it was observed by Gromov [41] that the family of n-dimensional Riemannian manifolds having Ricci curvature bounded below by $K \in \mathbb{R}$ and diameter bounded above by $D < \infty$, is pre-compact in the Gromov–Hausdorff topology. The study of *Ricci limits* went through a major development in a series of works by Cheeger and Colding in the nineties [22, 23, 24, 21] (see also the survey [68]). By then, however, it was still unclear whether a notion of *intrinsic* Ricci curvature lower bound could be defined, so as to impose it on a general m.m.s. *a priori*—without having to rely on the lower bound of limiting sequences of Riemannian manifolds (see [23, Appendix 2]). To this end, Lott–Villani and Sturm [66, 65, 54] independently introduced the so called *Curvature Dimension* (CD) condition, which is, to all effect, a satisfactory synthetic notion of Ricci curvature lower bound in the setting of metric measure spaces. The CD condition is compatible with the smooth case, i.e. it coincides with the classical definition in the case of Riemannian manifolds and it is strong enough to obtain *interesting* theorems. It is also sufficiently weak to be *stable* under (measure) Gromov Hausdorff convergence and, in particular, includes Ricci limits. On the other hand, it is satisfied by spaces which are quite far from being Riemannian, for example Finsler manifolds such as $(\mathbb{R}^n, \|\cdot\|, \mathcal{H}^n)$ for any norm $\|\cdot\|$. In the last decade, a stronger condition has become rather prominent and much studied—the so called *Riemannian Curvature Dimension* (RCD) condition (see the surveys [1, 37] for more details and historical notes). Notable examples of RCD spaces are weighted Riemannian manifolds, Ricci limits, Alexandrov spaces with (sectional) curvature bounded below [62, 69] and stratified spaces [10]. Non-Riemannian Finsler

[1] We recall, however, that there is a vast and expanding literature about quantitative and qualitative rectifiability both for specific model spaces and for the general setting. Heisenberg groups and parabolic spaces are two of the most studied models. We refer to the introduction of [35] for a rather thorough review of the literature. It should be remarked that rectifiability in those contexts is understood in terms of intrinsic objects (e.g. *intrinsic* Lipschitz graph) which are unrectifiable from the Euclidean point of view. Here, on the other hand, we are interested in to what extent a metric space looks *Euclidean*. For one-dimensional metric spaces, the theory is rather well developed, see [35] and [55].

spaces, on the other hand, are ruled out by the RCD condition. Nonetheless, we stress that RCD spaces (and even Ricci limits) are singular spaces, for instance they are not necessarily locally Euclidean, can have both non-unique and non Euclidean tangent spaces (see e.g. [25]) and admit conical singularities [52]. The standard notation for the class of these spaces is $\mathrm{RCD}(K,N)$, where $K \in \mathbb{R}$ represents the (synthetic notion of) Ricci lower bound, while $N \in [1,\infty]$ is the upper bound for the dimension.

Besides the conceptual importance to develop a theory of Ricci curvature bounded below in the non-smooth setting, RCD spaces have proven useful and even necessary in answering questions about smooth Riemannian manifolds. For example, they have been used to show existence of isoperimetric sets and sharp concavity estimates of the isoperimetric profile in non-compact Riemannian manifolds [5, 4], to prove stability of sharp Sobolev inequalities under non-negative Ricci curvature and Euclidean volume growth [61], to show lack of uniform C^1-estimates for harmonic functions assuming only Ricci or sectional lower bounds [34]. Furthermore, several new almost-rigidity results for functional and geometric inequalities in Riemannian manifolds have been obtained by means of the RCD theory (see, e.g., [18, 58, 44, 53]). We refer to [37, Section 7] for more details and examples. Notably RCD spaces found application also in the gravitational fields theory from physics, where they have been exploited to obtain eigenvalues bounds in some singular weighted manifolds [30, 31].

The rectifiability of RCD spaces, in the case $N < \infty$, was proved in [57]. This was further developed in the independent works [40, 51, 33], where it was shown that RCD spaces are rectifiable as metric measure spaces—that is to say, the reference measure is absolutely continuous with respect to the appropriate Hausdorff measure. In fact, it follows from [57] that RCD spaces are *strongly rectifiable*, in the sense that they can be covered up to a measure zero set by $(1+\varepsilon)$-biLipschitz images of the Euclidean space, where $\varepsilon > 0$ can be taken arbitrarily small.

In our main result we show that in the case of bounded Ahlfors regular RCD spaces, rectifiability can be upgraded to uniform rectifiability (see Definition 2.1). Roughly speaking, this says (using Remark 2.2 below) that any ball of an Ahlfors regular RCD space has a large portion which is bi-Lipschitz equivalent to a subset of the Euclidean space.

Theorem 1.1. *Every bounded Ahlfors k-regular* $\mathrm{RCD}(K,N)$ *space, $N < \infty$, is uniformly k-rectifiable. In particular any bounded non-collapsed* $\mathrm{RCD}(K,N)$ *space is uniformly N-rectifiable.*

Non-collapsed $\mathrm{RCD}(K,N)$ spaces, with $N < \infty$, are the ones having the Hausdorff measure $\mathcal{H}^N$ as reference measure (see Section 2). Note that Ahlfors regularity is part of the definition of UR, hence it is a not a restrictive assumption. We recall also that there are non-Ahlfors regular $\mathrm{RCD}(K,N)$ spaces, e.g. the $\mathrm{RCD}(0,N)$ space $([0,1], t^{N-1}\,\mathrm{d}t, \mathsf{d}_{\mathrm{Eucl}})$.

Concerning the unbounded case we can prove uniform rectifiability under non-negative Ricci curvature lower bound, while in the general case we can still obtain a local version of Theorem 1.1, stated below. Unbounded RCD spaces are of considerable interest and are often studied as they naturally appears for example when taking blow-ups of RCD spaces or blow-downs of Riemannian manifolds, therefore in this setting it is relevant to consider uniform rectifiability also in the unbounded case.

Theorem 1.2. *A locally Ahlfors k-regular* $\mathrm{RCD}(K,N)$ *space,* $N<\infty$, *is locally uniformly k-rectifiable. Additionally any Ahlfors k-regular* $\mathrm{RCD}(0,N)$ *space is uniformly k-rectifiable.*

We do not expect in general (global) uniform rectifiability to hold, indeed typically for K negative the constants in most functional and geometric inequalities degenerate at large scales, e.g. in the Poincaré inequality and Bishop–Gromov volume comparison [66, 63].

As a second result we obtain the uniform rectifiability of the boundary $\partial\mathrm{X}$ of non-collapsed $\mathrm{RCD}(K,N)$ spaces which, roughly speaking, is the closure of the points having the half space $\mathbb{R}^N_+$ as a tangent (see Section 2 for details).

Theorem 1.3. *Let* $(\mathrm{X},\mathsf{d},\mathcal{H}^N)$ *be an* $\mathrm{RCD}(-(N-1),N)$ *space such that the boundary* $\partial\mathrm{X}$ *is bounded (resp. unbounded) and locally* $(N-1)$*-Ahlfors regular. Then* $\partial\mathrm{X}$ *is uniformly* $(N-1)$*-rectifiable (resp. locally uniformly* $(N-1)$*-rectifiable).*

The previous result heavily relies on the regularity results for the boundary obtained recently in [12] where, among other things, it is shown that $\partial\mathrm{X}$ is $(N-1)$-rectifiable. It was also conjectured in [12] and proved there in the case of Ricci limits and Alexandrov spaces with curvature bounded below, that $\partial\mathrm{X}$ is in fact always locally $(N-1)$-Ahlfors regular, a property which could then be removed as an assumption from Theorem 1.3 (see also Remark 3.6).

It follows from our main results and the results in [9] that the various spaces considered in Theorems 1.1–1.3 satisfy the so called *Bilateral Weak Geometric Lemma* (BWGL) (see [9, Definition 3.1.5]). This means that, in most balls, the space is well-approximated by a ball of the same radius in some k-dimensional Banach space. However, in Section 3.3 we prove directly a stronger version of the BWGL for these spaces in which each approximating ball is Euclidean (see Proposition 3.8). Also is Section 3.3, we deduce a statement about quantitative differentiation of Lipschitz functions on bounded RCD spaces i.e. we obtain information about how well Lipschitz functions are approximated at coarse scales, not just infinitesimally. We show, for any given Lipschitz $f\colon \mathrm{X}\to\mathbb{R}$, that most balls in X are well-approximate by a Euclidean ball of the same radius *and* that f is well-approximated by an affine function on the corresponding tangent space (see Corollary 3.14). This is similar in spirit to a result of Jones [47] for Lipschitz functions defined on $\mathbb{R}^n$. A different notion of quantitative differentiability on spaces admitting a Poincaré inequality (in particular, RCD spaces) has also been investigated in [20].

2. Preliminaries

A *metric measure space* (m.m.s.) is a triple $(\mathrm{X},\mathsf{d},\mathfrak{m})$, where (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a positive Borel measure finite on bounded set, called *reference measure*. We will always assume that $\mathrm{supp}(\mathfrak{m})=\mathrm{X}$. We will denote by $\mathcal{H}^\alpha$ and $\mathcal{H}^\alpha_\infty$ the α-dimensional Hausdorff measure and Hausdorff content in (X,d). For all $x\in\mathrm{X}$ and $r>0$ we also set $B^{\mathrm{X}}_r(x):=\{y\in\mathrm{X}\colon \mathsf{d}(x,y)<r\}$, omitting the superscript X when there is no confusion in doing so. We will say that a subset E of a metric space (X,d) is $(L$-)*biLipschitz equivalent* to a subset $\mathbb{R}^n$ if there exists an (L)-biLipschitz map $f\colon E\to\mathbb{R}^n$.

Given $\alpha>0$ we say that a closed subset $E\subset\mathrm{X}$ of a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is *locally Ahlfors α-regular* if for every bounded set $B\subset\mathrm{X}$ there exists a constant $C_B \geq 1$ and a radius $R_B > 0$ such that

$$C_B^{-1} r^\alpha \leq \mathfrak{m}(B_r(x) \cap E) \leq C_B r^\alpha, \quad \forall\, x \in E \cap B, \quad \forall\, 0 < r < R_B. \tag{2.1}$$

If $C_B \equiv C$ can be taken independent of B and $R_B = \operatorname{diam}(E)$ for all B we say that E is Ahlfors α-regular and we call the minimal such C the Ahlfors regularity constant of E. Our definition of local Ahlfors regularity, where the constant might depend on the location of the space, is motivated by the fact that any non-collapsed $\mathrm{RCD}(K,N)$ space is automatically locally Ahlfors N-regular (see (2.21) below). By standard facts about differentiation of measures (see [3, Theorem 2.4.3]), if E is locally Ahlfors α-regular then for all bounded sets $B \subset \mathrm{X}$

$$(c_{\alpha,B})^{-1} \mathcal{H}^\alpha|_{E\cap B} \leq \mathfrak{m}|_{E\cap B} \leq c_{\alpha,B} \mathcal{H}^\alpha|_{E\cap B}, \tag{2.2}$$

where $c_{\alpha,B} \geq 1$ is a constant depending only on C_B and α. In particular E is (locally) Ahlfors regular in the space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ if and only if it is so in $(\mathrm{X}, \mathsf{d}, \mathcal{H}^\alpha)$.

Definition 2.1. (Uniform rectifiability) A closed subset $E \subset \mathrm{X}$ of a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$[2] is said to be *locally uniformly k-rectifiable (locally UR)* if it is locally Ahlfors k-regular and has *locally Big Pieces of Lipschitz Images (locally BPLI) of* $\mathbb{R}^k$ i.e. for every bounded set $B \subset \mathrm{X}$ there exist constants $\theta_B > 0, L_B \geq 0$ such that for each $x \in E \cap B$ and $0 < r < R_B$ there is a set $F \subset B_r^{\mathbb{R}^k}(0)$ and an L_B-Lipschitz map $f\colon F \to E$ such that

$$\mathfrak{m}(B_r(x) \cap f(F)) \geq \theta_B r^k. \tag{2.3}$$

Moreover if E is Ahlfors k-regular and has Big Pieces of Lipschitz Images (BPLI) of $\mathbb{R}^k$, that is to say it has locally BPLI of $\mathbb{R}^k$ but we can take the constants θ_B, L_B to be independent of B and $R_B = \operatorname{diam}(E)$ for all B, we say that E is uniformly k-rectifiable (UR).

Note that if E is locally uniformly k-rectifiable, then for any B we can take R_B arbitrarily large, up to decreasing the constant θ_B. In particular any bounded Ahlfors k-regular set which is locally uniformly k-rectifiable is in fact uniformly k-rectifiable.

Remark 2.2. It is proved in [64] that in Definition 2.1 it is equivalent to require that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ has Big Pieces of *biLipschitz* Images (BPBI) of $\mathbb{R}^k$, i.e. that the function f is L_B-biLipschitz.

We assume the reader to be familiar with the notion of *Gromov–Hausdorff distance*, d_{GH}, together with *pointed measure Gromov–Hausdorff* (pmGH) *convergence and distance*, $\mathsf{d}_{\mathrm{pmGH}}$, referring to [38] for the relevant background. We will call a map $g\colon (\mathrm{X}_1, \mathsf{d}_1) \to (\mathrm{X}_2, \mathsf{d}_2)$ between two metric spaces an *ε-isometry* for some $\varepsilon > 0$ if

$$|\mathsf{d}_2(g(x), g(y)) - \mathsf{d}_1(x,y)| \leq \varepsilon, \quad \forall x, y \in \mathrm{X}_1.$$

If g is both an ε-isometry and $g(\mathrm{X}_1)$ is ε-dense in X_2, we say that g is a *ε-GH isometry.*

The following definition is taken from [9, Section 3.1].

Definition 2.3. Let (X, d) be a metric space, $L \geq 1$ and $f\colon \mathrm{X} \to \mathbb{R}$. For a point $x \in \mathrm{X}$, a radius $0 < r < \operatorname{diam}(\mathrm{X})$, a norm $\|\cdot\|$ on $\mathbb{R}^k$, and a map $u\colon B_r(x) \to$

[2] Recalling (2.2), in Definition 2.1 it is enough to consider $\mathcal{H}^k$ in (2.3) in place of $\mathfrak{m}$. In particular our notion of UR is indeed equivalent to the one given in [9, Def. 1.2.1].

$B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)$, define

$$\zeta(x,r,\|\cdot\|,u) = \frac{1}{r} \sup_{y,z\in B_r(x)} \left|\mathsf{d}(y,z) - |u(y)-u(z)|\right|, \tag{2.4}$$

$$\eta(x,r,\|\cdot\|,u) = \frac{1}{r} \sup_{y\in B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)} \mathsf{dist}_{\|\cdot\|}(y, u(B_r(x))), \tag{2.5}$$

$$\Omega_f^L(x,r,\|\cdot\|,u) = \frac{1}{r} \inf_A \sup_{y\in B_r(x)} |f(y) - A(u(y))|, \tag{2.6}$$

where the infimum is taken over all affine mappings $A : \mathbb{R}^k \to \mathbb{R}^k$ with $\mathrm{Lip}(A) \le L$ when viewed as a function from $(\mathbb{R}^k, \|\cdot\|)$ to $\mathbb{R}$. Then, define

$$\gamma_f^L(x,r,\|\cdot\|) = \inf_u [\zeta(x,r,\|\cdot\|,u) + \eta(x,r,\|\cdot\|,u) + \Omega_f^L(x,r,\|\cdot\|,u)],$$

where the infimum is taken over all maps $u\colon B_r(x) \to B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)$, and

$$\gamma_f^L(x,r) = \inf_{\|\cdot\|} \gamma_f^L(x,r,\|\cdot\|),$$

where the infimum is taken over all norms $\|\cdot\|$ on $\mathbb{R}^k$.

Remark 2.4. In words, if $\varepsilon > 0$ and $\gamma_f^L(x,r) < \varepsilon$, there exist a norm $\|\cdot\|$, an εr-GH isometry $u\colon B_r(x) \to B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)$, and an L-Lipschitz affine map on $u(B_r(x))$ well-approximates f up to an error εr. Since u is an εr-GH isometry, one may like to think of A as being approximately an "affine map on $B_r(x)$". Thus, if $\gamma_f^L(x,r)$ is small, then f is well-approximated by an "affine function on $B_r(x)$".

We will also need the following simple lemma, which allows to replace L in the coefficient γ_f^L with a constant arbitrarily close to one, up to paying a constant factor.

Lemma 2.5. *Let (X,d) be a metric space, $L \ge 1$ and $f\colon \mathrm{X} \to \mathbb{R}$ be 1-Lipschitz. For each $\alpha > 0$, $x \in \mathrm{X}$ and $0 < r < \mathrm{diam}(\mathrm{X})$, we have*

$$\gamma_f^{1+\alpha}(x,r) \le C_{\alpha,L}\gamma_f^L(x,r), \tag{2.7}$$

where $C_{\alpha,L}$ is a constant depending only on α and L.

Proof. Fix $\alpha > 0$, $x \in \mathrm{X}$ and $0 < r < \mathrm{diam}(\mathrm{X})$. If $L \le 1+\alpha$ it follows immediately from Definition 2.3 that $\gamma_f^{1+\alpha}(x,r) \le \gamma_f^L(x,r)$, so we will suppose instead that $L > 1+\alpha$. Let $\varepsilon > 0$, to be chosen small enough depending on α and L, and suppose without loss of generality that

$$\gamma_f^L(x,r) < \varepsilon. \tag{2.8}$$

Then, there exists a norm $\|\cdot\|$ on $\mathbb{R}^k$, an εr-GH isometry $u\colon B_r(x) \to B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)$, and an affine map $A\colon (\mathbb{R}^k,\|\cdot\|) \to \mathbb{R}$ which satisfy $\mathrm{Lip}(A) \le L$ and

$$\sup_{y\in B_r(x)} |f(y) - A(u(y))| \le \varepsilon r. \tag{2.9}$$

It suffices to show that $\mathrm{Lip}(A) \le 1+\alpha$, which will be the case so long as ε is chosen sufficiently small. Indeed, let $y, z \in \mathbb{R}^k$ and set

$$w = \frac{r(y-z)}{\|y-z\|} \in B_r^{(\mathbb{R}^k,\|\cdot\|)}(0). \tag{2.10}$$

Since u is an εr-GH isometry, there exists $a, b \in B_r(x)$ such that $\|w - u(a)\| \le \varepsilon r$, $\|u(b)\| \le \varepsilon r$, and $|\mathsf{d}(a,b) - \|u(a) - u(b)\|| \le \varepsilon r$. Using these estimates, along with (2.9) and the fact that A is L-Lipschitz, we get

$$|A(w) - A(0)| \le |A(w) - A(u(a))| + |A(0) - A(u(b))| + |A(u(a)) - A(u(b))| \tag{2.11}$$

$$\le |f(a) - f(b)| + (2L+2)\varepsilon r \le \mathsf{d}(a,b) + (2L+2)\varepsilon r \tag{2.12}$$

$$\le \|u(a) - u(b)\| + (2L+3)\varepsilon r \le \|w\| + (2L+5)\varepsilon r. \tag{2.13}$$

Choosing $\varepsilon = \alpha/(2L+5)$, using that A is affine and $\|w\| = r$, this gives

$$\begin{aligned} |A(y) - A(z)| &= \frac{\|y - z\|}{r}|A(w) - A(0)| \\ &\le (1 + (2L+5)\varepsilon)\|y - z\| \le (1+\alpha)\|y - z\|. \end{aligned} \tag{2.14}$$

Since $y, z \in \mathbb{R}^k$ were arbitrary, we conclude that A is $(1+\alpha)$-Lipschitz, as required. $\square$

We say that the pointed m.m.s. $(Y, \mathsf{d}_Y, \mu, y)$, $y \in \mathrm{Y}$, is a *tangent* to $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ at $x \in \mathrm{X}$ if there exists a sequence $r_n \to 0^+$ such that $(\mathrm{X}, r_n^{-1}\mathsf{d}, (c^x_{r_n})^{-1}\mathfrak{m}, x)$ pmGH-converges to $(Y, \mathsf{d}_Y, \mu, y)$, where

$$c^x_r := \int_{B_r(x)} 1 - \frac{\mathsf{d}(\cdot, x)}{r}\,\mathrm{d}\mathfrak{m}. \tag{2.15}$$

Tangents are not necessarily unique, and we denote by $\mathrm{Tan}(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x)$ the class of all tangents to $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ at $x \in \mathrm{X}$. The *k-dimensional regular set* $\mathcal{R}_k$ is given by

$$\mathcal{R}_k := \left\{ x \in \mathrm{X} \colon \mathrm{Tan}(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x) = \{(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k\mathcal{H}^k, 0^k)\} \right\},$$

where $c_k := \int_{B_1(0^k)} (1 - |y|)\,\mathrm{d}\mathcal{H}^k(y)$. A pointed metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x)$ satisfying $c^x_1 = 1$ is called *normalised.*

For an introduction to the theory of $\mathrm{RCD}(K, N)$ spaces we refer to the surveys [1, 37] and references therein, limiting ourselves here to recall the main results that we need.

Remark 2.6. A key property of RCD spaces is compactness in the pmGH-topology, i.e. a sequence $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ of pointed $\mathrm{RCD}(K, N)$ spaces satisfying $\mathfrak{m}_n(B_1(x_n)) \in [v^{-1}, v]$ for some $v \ge 1$, admits a subsequence converging in the pmGH-sense to a limit $\mathrm{RCD}(K, N)$ space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$.

A basic scaling property is:

$$\begin{aligned} &(\mathrm{X}, \mathsf{d}, \mathfrak{m}) \text{ is an } \mathrm{RCD}(K, N) \text{ space } \Longrightarrow \\ &(\mathrm{X}, r^{-1}\mathsf{d}, \mathfrak{m}) \text{ is an } \mathrm{RCD}(r^2K, N) \text{ space.} \end{aligned} \tag{2.16}$$

The following is part of the now well established structural and rectifiability properties of RCD spaces.

Theorem 2.7. [57, 2] *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space with* $N < \infty$. *Then*

$$\mathfrak{m}\left(\mathrm{X} \setminus \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k\right) = 0$$

and $\lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{r^k} \in (0, \infty)$ *for* $\mathfrak{m}$-a.e. $x \in \mathcal{R}_k$, *for all* $k = 1, \ldots, \lfloor N \rfloor$.

Actually, as shown in [15], $\mathfrak{m}(\mathcal{R}_k) = 0$ for all except one k, however Theorem 2.7 will suffice for our purposes.

According to the notation introduced in [32], an $\mathrm{RCD}(K, N)$ space, with $N < \infty$, endowed with the reference measure $\mathfrak{m} = \mathcal{H}^N$ is said to be *non-collapsed.* It is also shown in [32] that N must be an integer.

Remark 2.8. It is worth to mention that if $(\mathrm{X}, \mathsf{d}, \mathcal{H}^n)$ is an $\mathrm{RCD}(K, N)$ space, with $N < \infty$, which is also n-locally Ahlfors regular (as usually assumed here) then it is in fact an $\mathrm{RCD}(K, n)$ space and in particular it is non-collapsed in the terminology above (see [43, 11]). It is conjectured in [42] that this actually holds without assuming local Ahlfors regularity.

For a non-collapsed $\mathrm{RCD}(K, N)$ space $(\mathrm{X}, \mathsf{d}, \mathcal{H}^N)$ we also recall the notion of *k-singular set*, for all $0 \le k \le N - 1$,

$$\mathcal{S}^k := \{x \in \mathrm{X} \colon \text{no element of } \mathrm{Tan}(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x) \text{ splits off } \mathbb{R}^{k+1} \text{ isometrically}\}.$$

The k-singular sets are nested and induce the following stratification

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{N-1} = \mathrm{X} \setminus \mathcal{R}_N.$$

It was proved in [32] that

$$\dim_H(\mathcal{S}^k) \le k, \quad \text{for all } 0 \le k \le N - 1. \tag{2.17}$$

The *boundary* of a non-collapsed $\mathrm{RCD}(K, N)$ space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ was introduced in [32] as

$$\partial\mathrm{X} := \overline{\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}}.$$

It easily follows from the definition that $\partial\mathrm{X} \setminus (\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}) \subset \mathcal{S}^{N-2}$ (see e.g. [12, (1.10)]), hence it also follows that

$$\dim_H \left(\partial\mathrm{X} \setminus (\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2})\right) \le N - 2. \tag{2.18}$$

As observed in [49, Lemma 4.6] it holds

$$(\mathbb{R}^N_+, \mathsf{d}_{\mathrm{Eucl}}, c_N/2\mathcal{H}^N, 0^N) \in \mathrm{Tan}(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x), \quad \text{for all } x \in \mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}. \tag{2.19}$$

In particular by the volume convergence theorem [32, Theorem 1.3] we have both

$$\begin{aligned} &\theta_N(x) = 1, \quad \forall x \in \mathcal{R}_N, \\ &\theta_N(x) = \frac{1}{2}, \quad \forall x \in \mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}. \end{aligned} \tag{2.20}$$

where $\theta_N(x) := \lim_{r \to 0^+} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N}$, which exists by the Bishop–Gromov inequality. By standard measure theory (see [3, Theorem 2.4.3]) it holds that $\theta_N(x) \le 1$ for all $x \in \mathrm{X}$. This combined with the Bishop–Gromov inequality [66] shows that

(2.21) a non-collapsed $\mathrm{RCD}(K, N)$ space $(\mathrm{X}, \mathsf{d}, \mathcal{H}^N)$ is locally Ahlfors N-regular.

The key tool that we will use in the sequel is the one of harmonic δ-splitting maps, which play the role of coordinate-functions. These type of maps were introduced in [21] in the context of Ricci limit spaces (see also [22, 23, 24]) and have been extended to the RCD setting in [14] (see also [13, 12] for further developments).

Definition 2.9. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(-(N-1), N)$ space. Let $x \in \mathrm{X}$ and $\delta > 0$ be given. Then a map $u = (u_1, \ldots, u_k) \colon B_r(x) \to \mathbb{R}^k$ is said to be a *δ-splitting map* provided:

i) $u_a \colon B_r(x) \to \mathbb{R}$ is harmonic and C_N-Lipschitz for every $a = 1, \ldots, k$,
ii) $r^2 \fint_{B_r(x)} \left|\mathrm{Hess}(u_a)\right|^2 \mathrm{d}\mathfrak{m} \le \delta$ for every $a = 1, \ldots, k$,

iii) $\fint_{B_r(x)} |\nabla_{\mathfrak{m}} u_a \cdot \nabla_{\mathfrak{m}} u_b - \delta_{ab}| \, \mathrm{d}\mathfrak{m} \le \delta$ for every $a, b = 1, \dots, k$.

In the above definition $\nabla_{\mathfrak{m}}$ and Hess denote respectively the gradient and Hessian operator in the space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. We will never explicitly use the above definition of splitting maps, but we will instead exploit some of their properties listed in the result below. For this reason we will avoid introducing the notion of gradient and Hessian in the metric setting and refer to [39] for details.

Theorem 2.10. (Properties of δ-splitting maps) *For every $N \in [1, \infty)$, $C \ge 1$ and $\varepsilon \in (0, 1/2)$ there exists $\delta = \delta(N, \varepsilon, C) \in (0, 1)$ such that the following hold. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(-\delta, N)$ space. Then for all $x \in \mathrm{X}$ and $r \in (0, 1]$ it holds:*

i) *if $u\colon B_{2r}(x) \to \mathbb{R}^k$ is an η-splitting map, with $\eta \in (0, 1)$, then there exists a Borel set $G \subseteq B_r(x)$ such that $\mathfrak{m}\big(B_r(x) \setminus G\big) \le C_N \sqrt{\eta}\, \mathfrak{m}\big(B_r(x)\big)$ and $u\colon B_s(y) \to \mathbb{R}^k$ is a $\sqrt{\eta} s$-splitting map for every $y \in G$ and $s \in (0, r)$,*

ii) *if*

$$\mathsf{d}_{\mathrm{pmGH}}((\mathrm{X}, r^{-1}\mathsf{d}, \mathfrak{m}, x)), (\mathbb{R}^k, |\cdot|, c\mathcal{H}^k, 0^k)) \le \delta, \tag{2.22}$$

for some constant $c > 0$, then there exists an εr-splitting map $u\colon B_r(x) \to \mathbb{R}^k$,

iii) *assuming furthermore that*

$$C^{-1} s^k \le \mathfrak{m}(B_s(y)) \le C s^k, \quad \text{for all } y \in B_r(x) \text{ and } s \in (0, r] \tag{2.23}$$

and that $u\colon B_{6r}(x) \to \mathbb{R}^k$ is δr-splitting map, then

$$\mathsf{d}_{\mathrm{GH}}(B_{\frac{r}{k}}(x), B^{\mathbb{R}^k}_{\frac{r}{k}}(0)) \le \varepsilon r \tag{2.24}$$

and $u\colon B_{\frac{r}{k}}(x) \to \mathbb{R}^k$ is an εr-isometry.

Proof. Item i) is just Proposition 1.6 in [13]. For items ii) and iii) by the scaling property of δ-splitting maps we can assume that $r = 1$. Item ii) is then precisely [14, Proposition 3.9]. Item iii) follows instead from *ii)* in [12, Theorem 3.8], the only difference is that therein the space is assumed to be normalised and the conclusion is that $(u, f)\colon B_{1/k}(x) \to \mathbb{R}^k \times Z$ is an ε-isometry for some (Z, d_Z) and some $f\colon B_{1/k}(x) \to Z$. We explain now how the argument in [12, Theorem 3.8] gives also the version stated here (cf. also with [12, Remark 3.10]). As in [12] by contradiction we assume the existence of a sequence $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ of $\mathrm{RCD}(-1/n, N)$ spaces such that (2.23) holds (with $x = x_n$, with $k = k_n \in \mathbb{N}$ and the same constant $C > 0$) and also of $1/n$-splitting maps $u_n\colon B_6(x_n) \to \mathbb{R}^{k_n}$ such that either $u_n\colon B_{1/k_n}(x_n) \to \mathbb{R}^{k_n}$ is not an ε-isometry or $\mathsf{d}_{\mathrm{GH}}(B_{1/k_n}(x), B^{\mathbb{R}^{k_n}}_{1/k_n}(0)) > \varepsilon$. Since $k_n \le N$ (e.g. by Theorem 2.7), up to a subsequence we can assume that $k_n \equiv k \in \mathbb{N}$. Since $\mathfrak{m}_n(B_1(x_n)) \in [C^{-1}, C]$ (which replaces the normalised assumption), up to a further subsequence, $(\mathrm{X}_n\, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ pmGH-converge to some $\mathrm{RCD}(0, N)$ space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ (recall Remark 2.6). In particular X_∞ still satisfies (2.23) with $x = x_\infty$ and $r = 1$, which implies $\dim_H(B_1(x_\infty)) \le k$ (see [3, Theorem 2.4.3]). Proceeding now verbatim as in [12] we can find a map $(u, f)\colon B_{1/k}(x_\infty) \to \mathbb{R}^k \times Z$ which is an isometry with its image (for some (Z, d_Z) and some f) and deduce for n big enough that $\mathsf{d}_{\mathrm{GH}}(B_{1/k}(x), B^{\mathbb{R}^k \times Z}_{1/k}(0)) \le \varepsilon$ and $(u_n, f_n)\colon B_{1/k}(x_n) \to \mathbb{R}^k \times Z$ is an ε-isometry for some f_n and some $z \in \mathrm{Z}$ independent of n (see respectively eq. (3.34) and (3.35) in [12]). It is therefore enough to show that $f(B_{1/k}(x_\infty)) = \{f(x_\infty)\}$, indeed we could then replace Z with $\{f(x_\infty)\}$ and get the desired contradiction. To show this we note that (u, f) is obtained in [12] applying Theorem 3.4 therein and, inspecting its proof, we see that $(u, f) := \Phi^{-1}|_{B_{1/k}(x_\infty)}$

where $\Phi\colon (-1/k,1/k)^k \times B_1(q) \to \mathrm{X}$ (for some $q \in \mathrm{Z}$) is an isometry with its image, which contains $B_{1/k}(x_\infty)$. Therefore $(u,f)(B_{1/k}(x_\infty))$ is open in $\mathbb{R}^k \times Z$. If by contradiction $f(x_\infty) \neq f(x)$ for some $x \in B_{1/k}(x_\infty)$ the Z-component of $(u,f)(\gamma)$, where γ is a geodesic from x_∞ to x (recall that X_∞ is geodesic), is itself a geodesic from $f(x_\infty)$ to $f(x)$ (see e.g. Lemma 3.6.4 in [16]). Hence $\mathcal{H}^1(B_r^Z(f(x_\infty))) > 0$ for all $r > 0$. Thus, since $(u,f)(B_{1/k}(x_\infty))$ is open, it contains $B_s^{\mathbb{R}^k}(u(x_\infty)) \times B_s^Z(f(x_\infty))$ for some $s > 0$ and thus it has positive $\mathcal{H}^{k+1}$-measure (see [36, Theorem 2.10.45]). However as observed above $\dim_H(B_{1/k}(x_\infty)) \leq k$ which contradicts the fact that (u,f) is an isometry. □

3. Proof of the results

3.1. Uniform rectifiability of Ahlfors regular RCD spaces. The proof is a combination of two results. The first (Proposition 3.1) says that in an Ahlfors k-regular RCD space every ball contains another ball of comparable size that is *almost-flat*, i.e. Gromov–Hausdorff close to an Euclidean ball in $\mathbb{R}^k$ of the same radius. The second (Proposition 3.2) shows that a big portion of an almost-flat ball is covered by a biLipschitz image of $\mathbb{R}^k$.

The result below is inspired by [6, Lemma 2.4], where a similar statement is shown for Ahlfors regular sets in the Euclidean space supporting a PoincarÃľ inequality.

Proposition 3.1. (Existence of large almost flat balls) *For every $\varepsilon > 0$, $C \geq 1$ and $N \in [1,\infty)$ there exists $\eta = \eta(\varepsilon, C, N) > 0$ such that the following holds. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(-(N-1), N)$ space and $x \in \mathrm{X}$ be such that for some $k \in \mathbb{N}$ it holds*

$$C^{-1}s^k \leq \mathfrak{m}(B_s(y)) \leq Cs^k, \quad \text{for all } y \in B_1(x) \text{ and } s \in (0,1]. \tag{3.1}$$

Then there exists $x' \in B_{\frac{1}{2}}(x)$, $r_0 \in (\eta, 1/2)$ and $c > 0$ such that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathrm{X}, {r_0}^{-1}\mathsf{d}, \mathfrak{m}, x'), (\mathbb{R}^k, |\cdot|, c\mathcal{H}^k, 0^k)) \leq \varepsilon. \tag{3.2}$$

Proof. Suppose by contradiction that there exist $\varepsilon > 0$, $C > 0$, $N \in [1,\infty)$, a sequence of pointed $\mathrm{RCD}(-(N-1), N)$ spaces $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ and integers $k_n \in \mathbb{N}$ such that (3.1) holds with $x = x_n$, $k = k_n$ and such that for all $x' \in B_{1/2}(x_n)$, all $r \in (1/n, 1)$ and all $c > 0$ it holds

$$\mathsf{d}_{\mathrm{pmGH}}((\mathrm{X}_n, r^{-1}\mathsf{d}_n, \mathfrak{m}_n, x'_n), (\mathbb{R}^k, |\cdot|, c\mathcal{H}^k, 0^k)) > \varepsilon. \tag{3.3}$$

As $k_n \leq N$ (recall Theorem 2.7) up to a subsequence we can assume that $k_n \equiv k \in \mathbb{N}$. Since by assumption it holds $\mathfrak{m}_n(B_1(x_n)) \in [C^{-1}, C]$, up to a subsequence, we have that $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converge in the pmGH-sense to an $\mathrm{RCD}(-(N-1), N)$ space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ (recall Remark 2.6). In particular X_∞ still satisfies (3.1) with $x = x_\infty$. Therefore by Theorem 2.7 we deduce that for $\mathfrak{m}_\infty$-a.e. $x \in B_1(x_\infty)$ it holds $\mathrm{Tan}(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x) = \{(\mathbb{R}^k, |\cdot|, c_k\mathcal{H}^k, 0^k)\}$. Hence we can find $x \in B_{1/4}(x_\infty)$ and $s \in (0,1)$ such that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathrm{X}_\infty, s^{-1}\mathsf{d}_\infty, \mathfrak{m}_\infty, x), (\mathbb{R}^k, |\cdot|, c_k \cdot c_s^y\mathcal{H}^k, 0^k)) \leq \frac{\varepsilon}{2}, \tag{3.4}$$

where $c_s^y > 0$ is as in (2.15). On the other hand by the pmGH-convergence we can find points $x'_n \in \mathrm{X}_n$ such that $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x'_n)$ pmGH converge to $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ (see e.g. [32, eq. (2.2)]). In particular $(\mathrm{X}_n, s^{-1}\mathsf{d}_n, \mathfrak{m}_n, x'_n)$ pmGH converge to $(\mathrm{X}_\infty, s^{-1}\mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$. However recalling (3.4), since $x'_n \in B_{1/2}(x_n)$ and $s > 1/n$ for n big enough, this gives a contradiction with (3.3). □

Note that (3.2) implies in particular that

$$\mathsf{d}_{\mathrm{GH}}(B_{r_0}(x'), B^{\mathbb{R}^k}_{r_0}(0)) \lesssim \varepsilon r_0,$$

however (3.2) takes into account also the measure and is easier to work with, in particular when applying Theorem 2.10.

The result below rests on the now well known fact that almost-splitting maps propagate well and are biLipschitz on a large sets. Similar results have appeared frequently in the theory of rectifiability of spaces with Ricci curvature lower bounds (see [57, 13, 12, 17, 24, 14, 46, 50] and also Proposition 3.7 below).

Proposition 3.2. *For every $N \in [1,\infty)$, $C \geq 1$, and $\varepsilon \in (0,1/2)$ there exist constants $\delta = \delta(N,\varepsilon,C) \in (0,1)$ and $\tilde c_N > 0$ such that the following hold. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(-\delta,N)$ space, $k \in \mathbb{N}$, $x \in \mathrm{X}$ be a point and $r \in (0,1)$ be a radius satisfying*

$$\text{(3.5)}\qquad \begin{aligned} &C^{-1}s^k \leq \mathfrak{m}(B_s(y)) \leq Cs^k, \quad \text{for all } y \in B_r(x) \text{ and } s \in (0,r],\\ &\mathsf{d}_{\mathrm{pmGH}}((\mathrm{X}, r^{-1}\mathsf{d}, \mathfrak{m}, x)), (\mathbb{R}^k, |\cdot|, c\mathcal{H}^k, 0^k)) \leq \delta, \quad \text{for some } c>0.\end{aligned}$$

Then there exists a set $U \subset B_r(x)$ with $\mathfrak{m}(U) \geq \tilde c_N \mathfrak{m}(B_r(x))$ and such that

i) *U is $(1+\varepsilon)$-biLipschitz equivalent to a subset of $\mathbb{R}^k$,*

ii)

$$\text{(3.6)}\qquad \mathsf{d}_{\mathrm{GH}}(B_s(y), B^{\mathbb{R}^k}_s(0)) \leq s\varepsilon, \quad \text{for all } s \in \big(0, \tfrac{r}{12k}\big) \text{ and } y \in U.$$

Proof. We can assume that $r = 1$, otherwise we can consider $(\mathrm{X}, r^{-1}\mathsf{d}, r^{-k}\mathfrak{m})$ which satisfy the same hypotheses with the same constants C and δ, but with $r = 1$. Fix a constant $\tau = \tau(\varepsilon, N, C) \in (0,1)$ small enough to be chosen later. If δ is chosen small enough with respect to τ, by ii) in Theorem 2.10 there exists a $\sqrt{\tau}$-splitting map $u\colon B_1(x) \to \mathbb{R}^k$. Then applying i) of the same theorem we obtain a set $G \subset B_{\frac12}(x)$ with

$$\text{(3.7)}\qquad \mathfrak{m}(G) = \mathfrak{m}(B_{\frac12}(x)) - \mathfrak{m}(B_{\frac12}(x) \setminus G) \geq (1 - C_N\sqrt{\tau})\mathfrak{m}(B_{\frac12}(x)),$$

such that for all $y \in G$ and all $s \in (0,1/2)$ the function $u\colon B_s(y) \to \mathbb{R}^k$ is an $\sqrt{\tau}s$-splitting map. Thus if τ (and thus δ) are small enough we can apply iii) in Theorem 2.10 and deduce that for all $s \in (0, \frac{1}{12k})$ it holds $\mathsf{d}_{\mathrm{GH}}(B_s(y), B^{\mathbb{R}^k}_s(0)) \leq s\varepsilon$ and $u\colon B_s(y) \to \mathbb{R}^k$ is an εs-isometry. Moreover combining (3.7) with the Bishop–Gromov inequality [66] and since $k \leq N$, eventually leads to

$$\mathfrak{m}(G \cap B_{\frac{1}{24k}}(x)) \geq \tilde c_N \mathfrak{m}(B_1(x)),$$

provided τ is small enough and where $\tilde c_N > 0$ is a constant depending only on N. We take $U \coloneqq G \cap B_{\frac{1}{24k}}(x)$. Then item ii) holds by what we said above. For i) consider $y,z \in U$ arbitrary and note that $\mathsf{d}(y,z) < \frac{1}{12k}$. Therefore the map $u\colon B_{\mathsf{d}(y,z)}(y) \to \mathbb{R}^k$ is a $\mathsf{d}(y,z)\varepsilon$-isometry, which implies

$$||u(y) - u(z)| - \mathsf{d}(y,z)| \leq \varepsilon \mathsf{d}(y,z).$$

This shows that $u\colon U \to \mathbb{R}^k$ is $(1-\varepsilon)^{-1}$-biLipschitz and so also $(1+2\varepsilon)^{-1}$-biLipschitz, which concludes the proof of i) up to choosing $\varepsilon/2$ instead of ε at the beginning. $\square$

Combining the two above results we can now obtain our main technical proposition, from which the main theorems will easily follow.

Proposition 3.3. *For every $N \in [1,\infty)$, $C \geq 1$ and $\varepsilon > 0$ there exists a constant $\delta = \delta(N,C,\varepsilon) \in (0,1)$ such that the following holds. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space, $k \in \mathbb{N}$ and $x \in \mathrm{X}$, $r \in (0,\sqrt{\delta/|K|})$ be a point and a radius satisfying*

$$C^{-1}s^k \leq \mathfrak{m}(B_s(y)) \leq Cs^k, \quad \textit{for all } y \in B_r(x) \textit{ and } s \in (0,r], \tag{3.8}$$

Then there exists a set $U \subset B_r(x)$ with $\mathfrak{m}(U) \geq \delta\mathfrak{m}(B_r(x))$ and such that U is $(1+\varepsilon)$-biLipschitz equivalent to a subset of $\mathbb{R}^k$.

Proof. We can assume that $r = 1$ and $K = -\delta$. Otherwise we can just consider the rescaled space $(\mathrm{X}, r^{-1}\mathsf{d},\mathfrak{m})$ which is an $\mathrm{RCD}(-\delta,N)$ space, thanks to the assumption $r < \sqrt{\delta/|K|}$ (recall (2.16)). Then directly combining Proposition 3.1 and Proposition 3.2 we can find a ball $B_{r_0}(x') \subset B_1(x)$ with $\eta(\varepsilon,C,N) < r_0 < 1$ and a set $U \subset B_{r_0}(x')$ satisfying $\mathfrak{m}(U) \geq \tilde{c}_N\mathfrak{m}(B_{r_0}(x'))$ and which is $(1+\varepsilon)$-biLipschitz to a subset of $\mathbb{R}^k$. Moreover by the Bishop–Gromov inequality [66] we have $\mathfrak{m}(B_{r_0}(x')) \geq C_N r_0^N \mathfrak{m}(B_1(x))$, which concludes the proof of i). □

From Proposition 3.3 we immediately obtain Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 and Theorem 1.2. It suffices to show Theorem 1.2. In deed the first part of Theorem 1.1 would follow recalling that if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is bounded, Ahlfors k-regular and locally uniformly k-rectifiable, it is in fact k-rectifiable. From this also the second part of Theorem 1.1 follows since a bounded non-collapsed $\mathrm{RCD}(K,N)$ space is Ahlfors N-regular (recall (2.21)). Let now $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a locally Ahlfors k-regular $\mathrm{RCD}(K,N)$ space and fix any $\varepsilon \in (0,1)$. Then for any bounded set $B \subset \mathrm{X}$ there exist $C_B \geq 1$ and $R_B > 0$ such that (3.8) holds for all $x \in B$ and $r < R_B$ with $C = C_B$. Denoted by $\delta = \delta(N,C_B,\varepsilon)$ the constant given by Proposition 3.3, we deduce that for all $x \in B$ and $r < r_B := \min(R_B, \sqrt{\delta/|K|})$ there exists $U \subset B_r(x)$ with $\mathfrak{m}(U) \geq \delta\mathfrak{m}(B_r(x)) \geq \delta C_B r^k$ such that U is $(1+\varepsilon)$-biLipschitz equivalent to a subset of $\mathbb{R}^k$. This shows that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ has locally BPBI of $\mathbb{R}^k$ and so it is locally uniformly k-rectifiable. This proves the first part of Theorem 1.1. For the second part note that if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is Ahlfors k-regular we can take C_B independent of B and $R_B = \mathrm{diam}(\mathrm{X})$ for all B. Thus, if also $K = 0$, we have $r_B = \mathrm{diam}(\mathrm{X})$ and so $(\mathrm{X},\mathsf{d},\mathfrak{m})$ has BPBI of $\mathbb{R}^k$ and in particular is uniformly k-rectifiable. □

Remark 3.4. As it is evident from the proof of Theorem 1.1 and Theorem 1.2, we actually prove a slightly stronger version of uniform k-rectifiability, in the sense that the space has Big Pieces of $(1+\varepsilon)$-biLipschitz Images taking $\varepsilon > 0$ arbitrarily small.

3.2. Uniform rectifiability of the boundary of RCD spaces. The overall scheme of the argument is similar to the one showing uniform rectifiability of the ambient space, presented in the previous section. The main difference is that almost flat balls are replaced by boundary-balls, i.e. balls that are Gromov–Hausdorff close to the half space $\mathbb{R}^N_+$.

Proposition 3.5. (Existence of many large boundary balls) *For every $\varepsilon > 0$, $v > 0$ and $N \in \mathbb{N}$ there exists $\eta = \eta(\varepsilon,v,N) > 0$ such that the following holds. Let $(\mathrm{X},\mathsf{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(-(N-1),N)$ space. Then for all $x \in \partial\mathrm{X}$ satisfying $\mathcal{H}^N(B_1(x)) \geq v$, $\mathcal{H}^{N-1}(\partial\mathrm{X} \cap B_{1/4}(x)) \geq v$, there exist $x' \in B_{\frac{1}{2}}(x)$ and $r \in (\eta,1/2)$ such that*

$$\mathsf{d}_{\mathrm{GH}}(B_r(x'), B_r^{\mathbb{R}^N_+}(0^N)) \leq \varepsilon r.$$

Proof. Suppose by contradiction that there exists $\varepsilon > 0$, $v > 0$, $N \in \mathbb{N}$ and a sequence of pointed $\mathrm{RCD}(-(N-1),N)$ spaces $(\mathrm{X}_n, \mathsf{d}_n, \mathcal{H}^N, x_n)$ satisfying $\mathcal{H}^N(B_1(x_n)) \geq v$ and $\mathcal{H}^{N-1}(\partial \mathrm{X}_n \cap B_{1/4}(x_n)) \geq v$, but such that for all $x' \in B_{1/2}(x_n)$ and all $r \in (1/n, 1)$ it holds

$$\mathsf{d}_{\mathrm{GH}}(B_r^{\mathrm{X}_n}(x'), B_r^{\mathbb{R}^N_+}(0^N)) > \varepsilon r. \tag{3.9}$$

By stability of non-collapsed RCD spaces [32, Theorem 1.2], up to a subsequence, $(\mathrm{X}_n, \mathsf{d}_n, \mathcal{H}^N, x_n)$ pmGH-converges to some $\mathrm{RCD}(-(N-1),N)$ space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathcal{H}^N, x_\infty)$. We claim that $\partial \mathrm{X}_\infty \cap \overline{B}_{1/4}(x_\infty) \neq \emptyset$. To show this we follow the argument in [12, Corollary 6.10]. Up to a subsequence the compact sets $C_n := \partial \mathrm{X}_n \cap \overline{B}_{1/4}(x_n)$ converge in the Hausdorff topology to a compact set $C \subset \overline{B}_{1/4}(x_\infty)$ and $\mathcal{H}^{N-1}_\infty(C) \geq \overline{\lim}_n \mathcal{H}^{N-1}_\infty(C_n) \geq v > 0$. The lower semicontinuity of the density θ_N under pmGH-convergence [32], together with (2.20), then shows that $C \subset \mathcal{S}^{N-1}$ and in particular $C \setminus \partial \mathrm{X}_\infty \subset \mathcal{S}^{N-2}$. Since $\dim_H(\mathcal{S}^{N-2}) \leq N-2$ (see (2.17)), we get $\mathcal{H}^{N-1}_\infty(C \setminus \partial \mathrm{X}_\infty) = \mathcal{H}^{N-1}(C \setminus \partial \mathrm{X}_\infty) = 0$, which gives the claim. Hence we can find $y \in B_{1/3}(x_\infty) \cap (\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2})$ and a radius $s \in (0,1)$ such that

$$\mathsf{d}_{\mathrm{GH}}(B_s^{\mathrm{X}_\infty}(y), B_s^{\mathbb{R}^N_+}(0^N)) \leq \varepsilon s/2 \tag{3.10}$$

(recall (2.19)). Since there exists a sequence $x'_n \in \mathrm{X}_n$ such that $\mathsf{d}_{\mathrm{GH}}(B_s^{\mathrm{X}_n}(x'_n), B_s^{\mathrm{X}_\infty}(y)) \to 0$ (see e.g. (2.2) in [32]) and $x'_n \in B_{1/2}(x_n)$ for n big, we obtain a contradiction with (3.9) for n big enough. □

Remark 3.6. It was conjectured in [12] that for a non-collapsed $\mathrm{RCD}(K,N)$ space $(\mathrm{X}, \mathsf{d}, \mathcal{H}^N)$ and any $x \in \partial \mathrm{X}$ it holds

$$\mathcal{H}^{N-1}(B_2(x) \cap \partial \mathrm{X}) \geq C(K)\mathcal{H}^N(B_1(x)). \tag{3.11}$$

If this was true, the assumption $\mathcal{H}^{N-1}(\partial \mathrm{X} \cap B_{1/4}(x)) \geq v$ in Proposition 3.5 could be omitted. Moreover (3.11) would also imply, by scaling, that $\partial \mathrm{X}$ is locally Ahlfors $(N-1)$-regular (recall that $\partial \mathrm{X}$ is locally *upper* Ahlfors $(N-1)$-regular by [12, Theorem 1.4]).

The following result follows directly combining [12, Theorem 8.4 -(ii)] and [12, Corollary 8.7].

Proposition 3.7. *For every $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon, N) \in (0,1)$ such that the following holds. Given any $\mathrm{RCD}(-\delta, N)$ space $(\mathrm{X}, \mathsf{d}, \mathcal{H}^N)$ and a point $x \in \mathrm{X}$ satisfying*

$$\mathsf{d}_{\mathrm{GH}}(B_1(x), B_1^{\mathbb{R}^N_+}(0^N)) \leq \delta,$$

there exists a set $U \subset B_{1/16}(x) \cap \partial \mathrm{X}$ with $\mathcal{H}^{N-1}(U) \geq \frac{1}{2}\mathcal{H}^{N-1}(B_{1/16}(x) \cap \partial \mathrm{X})$ and satisfying

i) *U is $(1+\varepsilon)$-biLipschitz equivalent to a subset of $\mathbb{R}^{N-1}$,*

ii)

$$\mathsf{d}_{\mathrm{GH}}(B_s(y) \cap \partial \mathrm{X}, B_s^{\mathbb{R}^{N-1}}(0)) \leq s\varepsilon, \quad \forall s \in (0, 1/5), \ \forall y \in U. \tag{3.12}$$

With the above two propositions at hand we can now prove our main theorem about uniform rectifiability of the boundary.

Proof of Theorem 1.3. Fix $\varepsilon \in (0,1)$ and let $\delta = \delta(\varepsilon, N)$ be the constant given by Proposition 3.7. By the local Ahlfors $(N-1)$-regularity assumption we have that for all bounded sets $B \subset \mathrm{X}$ there exist $C_B \geq 1$ and $R_B > 0$ such that $r^{-N}\mathcal{H}^{N-1}(B_r(x) \cap \partial \mathrm{X}) \in [C_B^{-1}, C_B]$ for all $x \in \partial \mathrm{X}$ with $\mathsf{d}(x, B) < 1$ and all $r < R_B$.

Moreover by Bishop–Gromov monotonicity we have $\inf_{x\in B} r^{-N}\mathcal{H}^N(B_r(x)) \geq v > 0$ for all $r \in (0,1)$ and some v depending on B. Hence by (the scaling invariant version of) Proposition 3.5 we deduce that for all $x \in B \cap \partial \mathrm{X}$, all $r < \min(R_B, 1)$ there exists $x' \in B_{r/2}(x)$ and $r' \in (\eta r, r/2)$ such that $\mathsf{d}_{\mathrm{GH}}(B_{r'}(x'), B_r^{\mathbb{R}^N_+}(0^N)) \leq \delta r'$ for some $\eta = \eta(\varepsilon, B, N) > 0$. Provided $r < \sqrt{\delta/|K|}$ we can apply the (scaled version) of Proposition 3.7 to obtain a set $U \subset B_{r'/16}(x') \cap \partial\mathrm{X} \subset B_r(x) \cap \partial\mathrm{X}$ that is $(1+\varepsilon)$-biLipschitz to a subset of $\mathbb{R}^{N-1}$ and such that

$$\mathcal{H}^{N-1}(U) \geq \frac{1}{2}\mathcal{H}^{N-1}(B_{r'/16}(x) \cap \partial\mathrm{X}) \geq \frac{C_B^{-1}}{2}(r'/16)^{N-1} \geq \frac{C_B^{-1}}{2}(\eta r/16)^{N-1},$$

having used that $\mathsf{d}(x', B) < 1$. This proves that $\partial\mathrm{X}$ has locally BPBI of $\mathbb{R}^{N-1}$ and thus concludes the proof. $\square$

3.3. BWGL and quantitative differentiation. Theorem 1.1 (resp. Theorem 1.3) tells us that that bounded Ahlfors regular RCD spaces (resp. Ahlfors regular boundaries of bounded non-collapsed RCD spaces) are UR. By applying the results in [9] we deduce that these spaces also satisfy the *Bilateral Weak Geometric Lemma* (BWGL) (see [9, Definition 3.1.5]), which roughly states that the space is uniformly approximated by Banach spaces at most scales and locations. However it is worth noting that we can deduce BWGL directly. Actually in this way we obtain a slightly stronger version of BWGL, stated below, where the comparison is made only with the Euclidean $\mathbb{R}^n$ (rather than Banach spaces).

Proposition 3.8. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an Ahlfors regular m.m.s.,* $k \in \mathbb{N}$, *with* $\mathrm{diam}(\mathrm{X}) \leq D < \infty$ *satisfying one of the following*

i) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K,N)$ *space with* $N < \infty$,
ii) $(\mathrm{X}, \mathsf{d}, \mathfrak{m}) = (\partial Y, \mathsf{d}_Y|_{\partial Y}, \mathcal{H}^k)$, *where* $(Y, \mathsf{d}_Y, \mathcal{H}^{k+1})$ *is an* $\mathrm{RCD}(K, k+1)$ *space with* $\inf_{x\in\partial Y} \mathcal{H}^{k+1}(B_1^Y(x)) \geq v > 0$.

Then for every $\varepsilon > 0$, *all* $x_0 \in \mathrm{X}$ *and* $0 < R < \mathrm{diam}(\mathrm{X})$ *it holds*

$$\int_0^R \mathcal{H}^k(\{x \in B_R^{\mathrm{X}}(x_0) \colon \mathsf{d}_{\mathrm{GH}}(B_r^{\mathrm{X}}(x), B_r^{\mathbb{R}^k}(0)) > \varepsilon r\}) \, \frac{dr}{r} \leq C R^k, \tag{3.13}$$

where C *is a constant depending only on* $\varepsilon, k, N, v, K, D$ *and the regularity constant of* X.

For the proof we need to recall the notion of Christ–David cubes for an Ahlfors regular m.m.s. We report here [9, Lemma 2.6.1] (which is based on [45]) in a simplified version, which is sufficient for our purpose.

Lemma 3.9. (Christ–David cubes) *Set* $\rho := 1/1000$ *and* $c_0 := 1/500$. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an Ahlfors regular m.m.s.. Then for all* $i \in \mathbb{Z}$ *there exists a family* $\mathcal{D}_i$ *of pairwise disjoint Borel subsets of* X, *called "cubes", satisfying the following properties:*

(1) $\bigcup_{Q\in\mathcal{D}_i} Q = \mathrm{X}$, $\forall i \in \mathbb{Z}$,
(2) *if* $Q_1, Q_2 \in \mathcal{D} := \bigcup_{i\in\mathbb{Z}} \mathcal{D}_i$ *and* $Q_1 \cap Q_2 \neq \emptyset$, *then either* $Q_1 \subset Q_2$ *or* $Q_2 \subset Q_1$,
(3) *for all* i, *denoting* $r_i := 5c_0\rho^i$ *and* $R_i := 5\rho^i$, *for all* $Q \in \mathcal{D}_i$ *there exists* $x_Q \in Q$ *such that*

$$B_Q := B_{r_i}(x_Q) \subset Q \subset B_{R_i}(x_Q). \tag{3.14}$$

We will also need a classical John–Nirenberg type lemma for metric spaces.

Lemma 3.10. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an Ahlfors regular m.m.s., let* $\mathcal{D}$ *a system of Christ–David cubes as given by Lemma 3.9 and let* $\alpha\colon \mathcal{D} \to [0, \infty)$ *be a function. Suppose that for some constants* $\eta > 0$, $M > 0$ *and a cube* $Q_0 \in \mathcal{D}$ *it holds*

$$\mathfrak{m}\Bigg(\Bigg\{x \in Q\colon \sum_{\substack{Q' \subset Q \\ Q' \ni x}} \alpha(Q') \leq M\Bigg\}\Bigg) \geq \eta \mathfrak{m}(Q), \tag{3.15}$$

for all $Q \subset Q_0$. *Then*

$$\sum_{Q \subset Q_0} \alpha(Q)\mathfrak{m}(Q) \leq C_{M,\eta}\mathfrak{m}(Q_0). \tag{3.16}$$

Proof. The argument is the same as in the Euclidean space (see [29, Section IV.1.2]). We report a brief sketch of the proof for the convenience of the reader. We define sets $F_l \subset Q_0$ and $G_l \subset Q_0$, $l \in \mathbb{N}$ inductively as follows. Set $F_1 := \{x \in Q_0\colon \sum_{Q \subset Q_0,\, Q \ni x} \alpha(Q) \leq M\}$ and $G_1 := Q_0 \setminus F_1$. Given F_l, $l \in \mathbb{N}$ and $G_l = Q_0 \setminus F_l$, we construct F_{l+1} by partitioning G_l into a family of maximal cubes $R_{j,l}$ and for each $R_{j,l}$ we define $F_{j,l} \subset R_{j,l}$ as the set of points $y \in R_{j,l}$ such that $\sum_{Q' \subset R_{j,l},\, Q' \ni y} \alpha(Q') \leq M$. By assumption (3.15) we have $\mathfrak{m}(F_{j,l}) \geq \eta \mathfrak{m}(R_{j,l})$ for all j. We take $F_{l+1} = \cup_j F_{j,l}$ and set $G_{l+1} = Q_0 \setminus F_{l+1}$. By construction $\mathfrak{m}(F_{l+1}) \geq \eta \mathfrak{m}(G_l)$ for all $l \in \mathbb{N}$, where we set $G_0 := Q_0$. Hence $\mathfrak{m}(G_l) \leq (1-\eta)^l \mathfrak{m}(Q_0)$. We claim that $\sum_{Q \subset Q_0,\, Q \ni x} \alpha(Q) \leq lM$ for all $x \in F_l$. This can be easily proved by induction. For $l = 1$ this is obvious. If now $x \in F_l$, then $x \in F_{j,l} \subset R_{j,l}$ for some j and by definition the cubes $Q \subset R_{j,l}$, $Q \ni x$ are at most M. On the other by the maximality, any cube containing $R_{j,l}$ must intersect F_{l-1} and so there are at most $(l-1)M$ of them, given the claim is true for $l-1$. This proves the claim. This implies that the function $f(x) := \sum_{Q \subset Q_0,\, Q \ni x} \alpha(Q)\mathfrak{m}(Q)$ satisfies $\mathfrak{m}(\{f > lM\}) \leq \mathfrak{m}(G_l) \leq (1-\eta)^l \mathfrak{m}(Q_0)$. Hence by the layer-cake formula $\int_{Q_0} f \,\mathrm{d}\mathfrak{m} \leq C_{M,\eta}\mathfrak{m}(Q_0)$, which is (3.16). □

We can now prove the result stated at the beginning of this section.

Proof of Proposition 3.8. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a bounded Ahlfors k-regular m.m.s., with $k \in \mathbb{N}$, Ahlfors regularity constant c_{X} and satisfying i) (resp. ii)) in the statement. Fix $\varepsilon > 0$, $\rho = 1/1000$ and let $\delta = \delta(N, \varepsilon\rho/7500, c_{\mathrm{X}})$ (resp. $\delta = \delta(\varepsilon\rho/7500, k+1)$) be the constant given by Proposition 3.2 (resp. by Proposition 3.7). We claim, for all $x_0 \in \mathrm{X}$ and $0 < R < \min\{\mathrm{diam}(\mathrm{X}), \sqrt{\delta/|K|}\}$, that

$$\int_0^R \mathcal{H}^k\big(\{y \in B_R^{\mathrm{X}}(x_0)\colon \mathsf{d}_{\mathrm{GH}}(B_s^{\mathrm{X}}(x), B_s^{\mathbb{R}^k}(0)) > \varepsilon s\}\big)\frac{\mathrm{d}s}{s} \leq \tilde{C}R^k, \tag{3.17}$$

where $\tilde{C}$ is a constant depending only on $\varepsilon, k, N, v, K, c_{\mathrm{X}}$ and D. Since X is bounded and Ahlfors k-regular, this proves (3.13) for all $x_0 \in \mathrm{X}$ and $0 < R < \mathrm{diam}(\mathrm{X})$ at the cost of increasing the constant on the right-hand side. Fix now $x_0 \in \mathrm{X}$ and $0 < R < \min\{\mathrm{diam}(\mathrm{X}), \sqrt{\delta/|K|}\}$. We may assume that $R = 1$ and $K = -\delta$. Otherwise we can just consider the rescaled space $(\mathrm{X}, R^{-1}\mathsf{d}, \mathfrak{m})$ (resp. the rescaled ambient space $(Y, R^{-1}\mathsf{d}_Y, \mathcal{H}^{k+1})$) which is an $\mathrm{RCD}(-\delta, N)$ space (resp. an $\mathrm{RCD}(-\delta, k+1)$ space), thanks to the assumption $R < \sqrt{\delta/|K|}$ (recall (2.16)). Note that, by the Bishop–Gromov inequality, after this rescaling we still have $\mathcal{H}^{k+1}(B_1^Y(x)) \geq v$ for all $x \in \mathrm{X}$, up to decreasing the constant v depending also on k, K and D. Thanks to our choice of δ we can apply (the scaled versions of) Proposition 3.1 and Proposition 3.2 (resp. Proposition 3.5 and Proposition 3.7) and obtain that for every ball $B_r^{\mathrm{X}}(x)$ with $x \in B_1^{\mathrm{X}}(x_0)$ and $r < 1$, there exists a $U \subset B_r^{\mathrm{X}}(x)$ satisfying (3.6) (resp. (3.12)) taking

$\varepsilon\rho/7500$ instead ε on the right hand side and $\mathfrak{m}(U) \geq \eta\mathfrak{m}(B^{\mathrm{X}}_r(x))$, where $\eta > 0$ is a constant depending only on $\varepsilon, k, N, v, K, c_{\mathrm{X}}$ and D. To obtain (3.17) we aim to apply Lemma 3.10. Fix $\mathcal{D}$ a system of Christ–David cubes for X. We choose $i_0 \in \mathbb{Z}$ so that $R_{i_0} = 5\rho^{i_0} \leq 1 < 5\rho^{i_0+1}$. Fix some $Q_0 \in \mathcal{D}_{i_0}$. Then $Q_0 \subset B_1(x_{Q_0})$. We now define a function $\alpha\colon \mathcal{D} \to \{0,1\}$ by setting $\alpha(Q) = 1$ if $Q \in \mathcal{D}_i$, $Q \subset Q_0$ and $\mathsf{d}_{\mathrm{GH}}(B^{\mathrm{X}}_{5\rho^i}(x), B^{\mathbb{R}^k}_{5\rho^i}(0)) > \varepsilon\rho^{i+1}/10$ for all $x \in Q$, otherwise we set $\alpha(Q) = 0$. We need to verify (3.15) for all $Q \subset Q_0 \subset B_1(x_{Q_0})$. If $Q \in \mathcal{D}_i$, by what we observed above, there exists $U \subset B_Q = B_{r_i}(x_Q)$ such that $\mathfrak{m}(U) \geq \eta\mathfrak{m}(Q)$ possibly decreasing η (where we used (3.14) and the k-Ahlfors regularity) and $\mathsf{d}_{\mathrm{GH}}(B^{\mathrm{X}}_s(x), B^{\mathbb{R}^k}_s(0)) \leq \varepsilon\rho s/7500$ for all $x \in U$ and all $s < \frac{r_i}{12k}$, where $r_i = 5c_0\rho^i$. Suppose now that $Q' \subset Q$, $Q' \in \mathcal{D}_{i+j}$ and $x \in Q'$ for some $x \in U$. Then for all $y \in Q'$ it holds $B^{\mathrm{X}}_{5\rho^{i+j}}(y) \subset B^{\mathrm{X}}_{15\rho^{i+j}}(x)$ (recall (3.14)). If $j > \log_\rho \frac{c_0}{36k}$ we have $15\rho^{i+j} < r_i/(12k)$ and since $x \in U$ we obtain

$$\begin{aligned}\mathsf{d}_{\mathrm{GH}}(B^{\mathrm{X}}_{5\rho^{i+j}}(y), B^{\mathbb{R}^k}_{5\rho^{i+j}}(0)) &\leq 50\mathsf{d}_{\mathrm{GH}}(B^{\mathrm{X}}_{15\rho^{i+j}}(x), B^{\mathbb{R}^k}_{15\rho^{i+j}}(0))\\ &\leq 50\varepsilon \cdot 15\rho^{i+j}\rho/7500 = \varepsilon\rho^{i+j+1}/10,\end{aligned}$$

where in the first step we used the almost monotonicity of the Gromov–Hausdorff distance (see e.g. [9, Lemma 2.3.6]). Hence $\alpha(Q') = 0$ if $j > \log_\rho \frac{c_0}{36k}$. Therefore, since only one cube in each $\mathcal{D}_{i+j}$ contains x, we obtain

$$\sum_{\substack{Q' \subset Q\\ Q' \ni x}} \alpha(Q) \leq \log_\rho \frac{c_0}{36k}.$$

We can thus apply Lemma 3.10 and obtain

$$\sum_{Q \subset Q_0} \alpha(Q)\mathfrak{m}(Q) \leq C_{c_0,\rho,\eta}\mathfrak{m}(Q_0). \tag{3.18}$$

We can derive an integral version of (3.18) with a standard argument:

$$\begin{aligned}&\int_0^{5\rho^{i_0+1}} \mathfrak{m}\big(\{x \in Q_0 \colon \mathsf{d}_{\mathrm{GH}}(B^{\mathrm{X}}_r(x), B^{\mathbb{R}^k}_r(0)) > \varepsilon r\}\big)\frac{\mathrm{d}r}{r}\\ &= \sum_{i=i_0+1}^{\infty} \sum_{\substack{Q \subset Q_0\\ Q \in \mathcal{D}_i}} \int_{5\rho^{i+1}}^{5\rho^i} \mathfrak{m}\big(\{x \in Q \colon \mathsf{d}_{\mathrm{GH}}(B^{\mathrm{X}}_r(x), B^{\mathbb{R}^k}_r(0)) > \varepsilon r\}\big)\frac{\mathrm{d}r}{r}\\ &\leq \sum_{i=i_0+1}^{\infty} \sum_{\substack{Q \subset Q_0\\ Q \in \mathcal{D}_i}} \frac{1-\rho}{\rho}\mathfrak{m}\big(\{x \in Q \colon \mathsf{d}_{\mathrm{GH}}(B^{\mathrm{X}}_{5\rho^i}(x), B^{\mathbb{R}^k}_{5\rho^i}(0)) > 5\varepsilon\rho^{i+1}/50\}\big)\\ &\leq \sum_{i=i_0+1}^{\infty} \sum_{\substack{Q \subset Q_0\\ Q \in \mathcal{D}_i}} \frac{1-\rho}{\rho}\alpha(Q)\mathfrak{m}(Q) = \frac{1-\rho}{\rho}\sum_{Q \subset Q_0} \alpha(Q)\mathfrak{m}(Q) \overset{(3.18)}{\leq} \frac{1-\rho}{\rho}C_{c_0,\rho,\eta}\mathfrak{m}(Q_0),\end{aligned}$$

where in the third line we used again the almost monotonicity of the Gromov–Hausdorff distance and in the first inequality of the last line the definition of $\alpha(Q)$. To obtain (3.17) for $R = 1$ from the above we simply need to recall that $5\rho^{i_0} \leq 1 < 5\rho^{i_0+1}$ and that $B_1(x_0) \subset Q_0^1 \cup \cdots \cup Q_0^m$ with $Q_0^j \in \mathcal{D}_{i_0}$ for some m depending only on c_0, ρ and the regularity constant c_{X}, as immediately follows from (3.14) and the Ahlfors regularity of X. $\square$

On these spaces, as a corollary to Theorem 1.1 and the results in [9], we obtain a statement regarding quantitative differentiability of Lipschitz functions. Before stating this, we recall a key result from [9] which states that if X is UR then the coefficients γ_f^L (see Definition 2.3) are small for *most* locations and scales.

Theorem 3.11. [9, Theorem 4.1.1] *Let* X *be an Ahlfors k-regular UR metric space. Then there exists $L \geq 1$, depending only on k and the parameters in Definition 2.1, such that for all $\varepsilon > 0$, all $x_0 \in \mathrm{X}$ and $0 < R < \mathrm{diam}(\mathrm{X})$, and all 1-Lipschitz $f\colon \mathrm{X} \to \mathbb{R}$, it holds that*

$$\int_0^R \mathcal{H}^k(\{x \in B_R(x_0)\colon \gamma_f^L(x,r) > \varepsilon\}) \,\frac{dr}{r} \leq CR^k, \tag{3.19}$$

where C is a constant depending only on ε, k, the parameters in Definition 2.1, and the regularity constant of X.

Remark 3.12. While the dependencies in [9, Theorem 4.1.1] are not explicit, the result follows by combining Propositions 4.2.1, 4.3.1 and 4.4.1 from the same paper. Inspecting those statements one observes that L depends on k and each of the parameters in Definition 2.1 and that C depends on ε, k, each of the parameters in Definition 2.1 and the regularity constant.

Remark 3.13. Thanks to Lemma 2.5, at the cost of increasing the value of C in (3.19), we can make it so that L is independent of k and the parameters in Definition 2.1.

Similar to Proposition 3.8, if X is a bounded Ahlfors regular RCD spaces or an Ahlfors regular boundary of a bounded non-collapsed RCD space, we obtain a stronger condition than (3.19) in which we can equip the tangent spaces with the Euclidean norm.

Corollary 3.14. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an Ahlfors k-regular m.m.s., $k \in \mathbb{N}$, with $\mathrm{diam}(\mathrm{X}) \leq D < \infty$ satisfying one of the following*

i) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K,N)$ *space with* $N < \infty$,
ii) $(\mathrm{X}, \mathsf{d}, \mathfrak{m}) = (\partial Y, \mathsf{d}_Y|_{\partial Y}, \mathcal{H}^k)$, *where* $(Y, \mathsf{d}_Y, \mathcal{H}^{k+1})$ *is an* $\mathrm{RCD}(K, k+1)$ *space with* $\inf_{x \in \partial Y} \mathcal{H}^{k+1}(B_1^Y(x)) \geq v > 0$.

Then there exists $L \geq 1$ such that, for all $\varepsilon > 0$, all $x_0 \in \mathrm{X}$ and $0 < R < \mathrm{diam}(\mathrm{X})$, and all 1-Lipschitz $f\colon \mathrm{X} \to \mathbb{R}$, it holds that

$$\int_0^R \mathcal{H}^k(\{x \in B_R(x_0)\colon \gamma_f^L(x,r,|\cdot|) > \varepsilon\}) \,\frac{dr}{r} \leq CR^k, \tag{3.20}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^k$ and C is a constant depending only on $\varepsilon, k, N, v, K, D, L$ and the regularity constant of X.

Proof. By Theorem 1.1 and Theorem 1.3 and their proofs, X is uniformly k-rectifiable with parameters depending only on k, N, v, K, D and the regularity constant of X (see also Proposition 3.3). Hence, by Theorem 3.11 and Remark 3.13, there exists $L' \geq 1$ and, for each $\delta > 0$, a constant $C > 1$ (depending on δ, k, N, v, K, D and the regularity constant of X) such that (3.19) holds with constant C.

Equation (3.20) follows from Proposition 3.8 and (3.19) once we establish the following claim: Let $L = 2L'$. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$\mathscr{G}_\delta := \{(x,r) \in \mathrm{X} \times (0, \mathrm{diam}(\mathrm{X}))\colon \mathsf{d}_{\mathrm{GH}}(B_r(x), B_r^{\mathbb{R}^k}(0)) \leq \delta r \text{ and } \gamma_f^{L'}(x,r) \leq \delta\} \tag{3.21}$$

then

$$\mathscr{G}_\delta \subseteq \{(x,r) \in \mathrm{X} \times (0, \mathrm{diam}(\mathrm{X}))\colon \gamma_f^L(x,r,|\cdot|) \le \varepsilon\}. \tag{3.22}$$

We now prove the claim. Let $\delta, \varepsilon > 0$ and let $(x,r) \in \mathscr{G}_\delta$. By Remark 2.4, there exists a norm $\|\cdot\|$, a δr-GH isometry $\tilde{u}\colon B_r(x) \to B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)$ and an affine function $\tilde{A}\colon \mathbb{R}^k \to \mathbb{R}$ such that $\mathrm{Lip}(\tilde{A}) \le L'$ (as a map from $(\mathbb{R}^k, \|\cdot\|)$ to $\mathbb{R}$) and

$$\sup_{y \in B_r(x)} |f(y) - \tilde{A}(\tilde{u}(y))| \le r\delta. \tag{3.23}$$

As an immediate consequence, we have that $\mathsf{d}_{\mathrm{GH}}(B_r(x), B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)) < \delta r$. In particular, recalling that $(x,r) \in \mathscr{G}_\delta$ and applying the triangle inequality for d_{GH}, we conclude that

$$\mathsf{d}_{\mathrm{GH}}(B_r^{\mathbb{R}^k}(0), B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)) < 2\delta r.$$

Hence, there exists a $2\delta r$-GH isometry $\varphi\colon B_r^{(\mathbb{R}^k,\|\cdot\|)}(0) \to B_r^{\mathbb{R}^k}(0)$. By [9, Lemma 2.3.14], for δ small enough depending on ε and L, there exists a $(1+\varepsilon)$-bi-Lipschitz affine map $T\colon B_r^{(\mathbb{R}^k,\|\cdot\|)}(0) \to B_r^{\mathbb{R}^k}(0)$ such that

$$\sup_{y \in B_r^{(\mathbb{R}^k,\|\cdot\|)}(0)} |T(y) - \varphi(y)| \le \frac{\varepsilon r}{4L}. \tag{3.24}$$

Define

$$u \coloneqq \varphi \circ \tilde{u}\colon B_r(x) \to B_r^{\mathbb{R}^k}(0) \quad \text{and} \quad A \coloneqq \tilde{A} \circ T^{-1}\colon \mathbb{R}^k \to \mathbb{R}. \tag{3.25}$$

For δ small enough, since $\tilde{u}, \varphi$ are $2\delta r$-GH isometries, it follows that u is an $\varepsilon r/2$-GH isometry. In particular, we have

$$r\zeta(x,r,u) + r\eta(x,r,u) \le \varepsilon r/2. \tag{3.26}$$

Since $0 < \varepsilon < 1$, $\tilde{A}$ is L'-Lipschitz and T^{-1} is $(1+\varepsilon)$-Lipschitz, we have $\mathrm{Lip}(A) \le 2L' = L$ (viewed as a map from $\mathbb{R}^k$ to $\mathbb{R}$). Combining this with (3.23) and (3.24) gives

$$r\Omega_f^L(x,r,u) \le \sup_{y \in B_r(x)} |f(y) - A(u(y))| \tag{3.27}$$

$$\le \sup_{y \in B_r(x)} \Big[|f(y) - \tilde{A}(\tilde{u}(y))| + |A \circ T \circ \tilde{u}(y) - A \circ \varphi \circ \tilde{u}(y)|\Big] \tag{3.28}$$

$$\le \delta r + L \sup_{y \in B_r^{\mathbb{R}^k}(0)} |T(\tilde{u}(y)) - \varphi(\tilde{u}(y))| \le \varepsilon r/2, \tag{3.29}$$

provided $\delta \le \varepsilon/4$. Equations (3.26) and (3.27) finish the proof of the claim. □

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