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Gagliardo–Nirenberg–Sobolev inequalities in John domains

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Abstract. We build up a Gagliardo–Nirenberg–Sobolev inequality in John domains and, conversely, under an extra separation property, we show that a bounded domain supporting such a Gagliardo–Nirenberg–Sobolev inequality should be a John domain.

Gagliardon–Nirenbergin–Sobolevin epäyhtälöt Johnin alueissa

Tiivistelmä. Tässä työssä rakennetaan Gagliardon–Nirenbergin–Sobolevin epäyhtälö Johnin alueissa. Kun lisäksi oletetaan sopiva irrallisuusominaisuus, osoitetaan käänteisesti, että rajallisen alueen, jossa Gagliardon–Nirenbergin–Sobolevin epäyhtälö on voimassa, täytyy olla Johnin alue.

1. Introduction

In the Euclidean space $\mathbb{R}^n$ with dimension $n \geqslant 2$, let $(p,s,q,\theta) \in [1,\infty]^2 \times [1,\infty) \times (0,1]$ be an *admissible quadruple*, that is, (p,s,q,θ) satisfies that

$$\frac{1}{q} = \theta\left(\frac{1}{p} - \frac{1}{n}\right) + \frac{1-\theta}{s}, \tag{1.1}$$

where $1/\infty = 0$, and also that $\theta \neq 1$ whenever $p = n$. The corresponding (p,s,q,θ)-*Gagliardo–Nirenberg–Sobolev* (for short, (p,s,q,θ)-*GNS*) *inequality* in whole $\mathbb{R}^n$ says that there exists a positive constant $C = C(n,p,s,\theta)$ such that, for any $f \in \dot W^{1,p}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$,

$$\|f\|_{L^q(\mathbb{R}^n)} \leqslant C\|\nabla f\|^{\theta}_{L^p(\mathbb{R}^n)}\|f\|^{1-\theta}_{L^s(\mathbb{R}^n)}, \tag{1.2}$$

where, when $s = \infty$ and $p < n$, either f vanishes at ∞ or $f \in L^m(\mathbb{R}^n)$ for some $m \in [1,\infty)$ in addition. Here and thereafter, for any $p \in [1,\infty]$ and any domain $\Omega \subset \mathbb{R}^n$, the *homogeneous Sobolev space* $\dot W^{1,p}(\Omega)$ is the collection of all functions $f \in L^1_{\rm loc}(\Omega)$ whose distributional derivatives $\nabla f = (\partial_{x_i} f)_{1\leqslant i\leqslant n}$ belong to $L^p(\Omega)$. The inequality (1.2) originates from Sobolev [25], Gagliardo [9], and Nirenberg [23]. Then it has been extensively studied and used in partial differential equations in the literature; see, for instance, [3, 2, 8].

We are interested in (p,s,q,θ)-GNS inequalities in bounded domains. A bounded domain Ω of $\mathbb{R}^n$ is said to *support the* (p,s,q,θ)-*GNS inequality*, if there exists

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a positive constant C, such that for some admissible (p,s,q,θ) and for any $f \in \dot{W}^{1,p}(\Omega) \cap L^s(\Omega)$,

$$\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} \leqslant C \|\nabla f\|_{L^p(\Omega)}^{\theta} \left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^s(\Omega)}^{1-\theta}. \tag{1.3}$$

Here and thereafter, we write $\operatorname{ave}_\Omega f$ as the average of the locally integrable function f on Ω, that is,

$$\operatorname*{ave}_{\Omega} f := \fint_{\Omega} f(x)\,\mathrm{d}x := \frac{1}{|\Omega|}\int_{\Omega} f(x)\,\mathrm{d}x.$$

It is a very natural question to ask which kinds of domains support the (p,s,q,θ)-GNS inequality (1.3), in particular, how to characterize geometrically bounded domains supporting (1.3).

Thanks to the Sobolev extension theory built up in [12, 15, 17], if Ω is a bounded (ε,δ)-uniform domain (including Lipschitz domains), one may deduce (1.3) from (1.2); see Appendix of this article for the details. Moreover, it was proven by Adams and Fournier [1] that, if a bounded domain satisfies the so-called weak cone condition, then it supports the GNS inequality (1.3) with $p=s$ as well as an analogue involving higher derivatives.

Beyond Sobolev extension domains and domains satisfying the weak cone condition, there are other bounded domains supporting (1.3) with a special admissible quadruple $(p,s,\frac{np}{n-p},1)$. To be precise, for any $p\in[1,n)$, it was shown by Reshetnyak [24], Martio [19], and Bojarski [4] that John domains always *support the* $(\frac{np}{n-p},p)$-*Poincaré inequality* (the imbedding of the homogeneous Sobolev space $\dot{W}^{1,p}(\Omega)$ into $L^{\frac{np}{n-p}}(\Omega)$), that is, there exists a positive constant C such that, for any $f\in\dot{W}^{1,p}(\Omega)$,

$$\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^{\frac{np}{n-p}}(\Omega)} \leqslant C\|\nabla f\|_{L^p(\Omega)}. \tag{1.4}$$

Recall from [6] that a bounded domain Ω is called as a *John domain* provided that there exist a distinguished point $x_0\in\Omega$ and a constant $C_J\in(0,1]$ such that, for any $x\in\Omega$, there exists a curve $\gamma\colon[0,l]\to\Omega$ parameterized by the arclength $l\in(0,\infty)$ with $\gamma(0)=x$ and $\gamma(l)=x_0$ such that

$$\operatorname{dist}(\gamma(t),\partial\Omega)\geqslant C_J t,\quad \forall\, t\in[0,l].$$

Roughly speaking, a John domain satisfies the twisted cone condition. Observe that (1.4) coincides with (1.3) with $q=\frac{np}{n-p}$ and $\theta=1$, where $(p,s,\frac{np}{n-p},1)$ is admissible.

Conversely, under the separation property, a bounded domain supporting (1.4) for some $p\in[1,n)$ was shown by Buckley and Koskela [6] to be a John domain. A domain Ω is said to have the *separation property* if there exist a distinguished point $x_0\in\Omega$ and a constant $C_S\in[1,\infty)$ such that, for any $x\in\Omega$, there exists a curve $\gamma\colon[0,1]\to\Omega$ with $\gamma(0)=x$ and $\gamma(1)=x_0$ such that, for any $t\in[0,1]$, either

$$\gamma([0,t])\subset B_{\gamma(t)} := B(\gamma(t), C_S\operatorname{dist}(\gamma(t),\partial\Omega))$$

or, for each $y\in\gamma([0,t])\setminus B_{\gamma(t)}$, y belongs to a different connected component of $\Omega\setminus\partial B_{\gamma(t)}$ that includes x_0. Notice that, in dimension $n=2$, a simply connected domain automatically has the separation property; in dimension $n\geqslant 3$, any domain in $\mathbb{R}^n$ that is quasiconformally equivalent to a uniform domain has the separation property. For more details, we refer to [6].

In this article, for any general admissible quadruple (p,s,q,θ), we prove that John domains also support the (p,s,q,θ)-GNS inequality (1.3) and, moreover, under the extra separation property, the converse holds.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$ be a bounded domain, and let (p, s, q, θ) be any admissible quadruple.*

(i) *If Ω is a John domain, then Ω supports* (1.3) *for some positive constant $C := C(n, p, s, \theta, C_J)$.*

(ii) *Suppose that Ω has the separation property. If Ω supports* (1.3), *then Ω is a John domain.*

In order to prove Theorem 1.1(i), we adapt the local-to-global argument by Bojarski [4]. Precisely, we first derive the GNS inequality in any cube from (1.2), where the constant is uniform in all cubes. Recall that, as shown in [5] (see Lemma 2.2), John domains always satisfy the Boman chain condition as in [5] (see also Definition 2.1). We are able to transfer the GNS inequality from cubes to domains. Indeed, denoting by Q_0 the central cube in the Boman chain condition, it suffices to bound $\int_\Omega |f - \mathrm{ave}_{Q_0} f|^q \, \mathrm{d}x$. Covering Ω by the cube family $\{Q\}_{\mathcal{C}}$ in the Boman chain condition, we are only need to bound

$$\mathrm{I}_1 = \sum_{Q\in\mathcal{C}} \int_Q \left| f(x) - \operatorname*{ave}_Q f \right|^q \mathrm{d}x \quad \text{and} \quad \mathrm{I}_2 = \sum_{Q\in\mathcal{C}} \int_Q \left| \operatorname*{ave}_Q f - \operatorname*{ave}_{Q_0} f \right|^q \mathrm{d}x.$$

On I_1, applying the GNS inequality in cubes and using the inequality in Lemmas 2.4 and 2.3, we obtain the desired upper bound. On I_2, we need to use the Boman chain condition to bound $|\mathrm{ave}_Q f - \mathrm{ave}_{Q_0} f|$ for each cube Q. Using Lemma 2.5, we also obtain the desired bound for I_2; see Section 3 for the details.

We point out that, in the case $p \in [1, n)$, (1.3) follows from (1.4) and Hölder's inequality; see Remark 3.4 for the details. But, when $p \in [n, \infty]$, we cannot obtain this from (1.4) and Hölder's inequality.

We prove Theorem 1.1(ii) in Section 4 by borrowing some ideas from [6]. The key is to bound the diameter of any connected component T of $\Omega \setminus B(z, d)$ which has empty intersection with some ball B_0 a priori; see Lemma 4.1. To this end, we apply (1.3) to some Lipschitz function which distinguishes the component T. Using this bound, we are able to show that the curve appearing the separation property satisfies

$$\operatorname{diam}(\gamma([0, t])) \leqslant C \operatorname{dist}(\gamma(t), \partial\Omega), \quad \forall\, t \in (0, 1).$$

After some appropriate modification one could obtain the desired John curve; see Section 4 for the details. Later, we provide several examples of domains that satisfy or do not satisfy the separation property.

Finally, we make some conventions on notation. Throughout this article, let

$$\mathbb{Z}_+ := \{1, 2, \ldots\} \quad \text{and} \quad \mathbb{N} := \{0, 1, 2, \ldots\}.$$

For any subset Ω of $\mathbb{R}^n$, we denote by $\mathbf{1}_\Omega$ its *characteristic function*, $\partial\Omega$ its *boundary*, $\overline{\Omega}$ its *closure*, $\Omega^\complement$ its *complement* in $\mathbb{R}^n$, and $|\Omega|$ its *Lebesgue measure*. If Ω is a bounded set, we denote by $\operatorname{diam}(\Omega)$ its *diameter*, that is,

$$\operatorname{diam}(\Omega) := \sup\{|x - y| \colon x, y \in \Omega\}.$$

We use C to denote a positive constant which is independent of the main parameters involved, but it may vary from line to line. We use the notation $A_1 \lesssim A_2$ if there exists a positive constant C, which is independent of A_1 and A_2, such that $A_1 \leqslant C A_2$. If $A_1 \lesssim A_2$ and $A_2 \lesssim A_1$, then we denote $A_1 \approx A_2$. By Q we denote an open *cube* in $\mathbb{R}^n$ whose edges parallel to the coordinate axes, and by l_Q we denote its *edge length*.

For any $\sigma \in (0,\infty)$ and any cube Q, we denote by σQ the cube concentric with Q having the edge length σl_Q. For any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, the set

$$B(x,r) := \{y \in \mathbb{R}^n \colon |y-x| < r\}$$

is called a *ball* with *center* x and *radius* r. If we don't really care about the center and radius of the ball, we simply write $B(x,r)$ as B. We use the symbol "dist" to denote the Euclidean distance between a point and a set or between two different sets, for instance,

$$\operatorname{dist}(x,\Omega) = \inf\{|x-y| \colon y \in \Omega\}, \quad \forall\, x \in \mathbb{R}^n,$$
$$\operatorname{dist}(A,\Omega) = \inf\{|x-y| \colon x \in A,\ y \in \Omega\}.$$

2. Preliminaries

In this section we recall several results which are used later. We begin with the following Boman chain condition.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. Then Ω is said to satisfy the (σ,τ,K)-*Boman chain condition* for some $\sigma,\tau,K \in [1,\infty)$ if there exists a covering $\mathcal{C}$ of Ω, consisting of open cubes of Ω, such that

(i) for any $x \in \mathbb{R}^n$, $\sum_{Q\in\mathcal{C}} \mathbf{1}_{\sigma Q}(x) \leqslant \tau \mathbf{1}_\Omega(x)$,

(ii) for some fixed cube $Q_0 \in \mathcal{C}$, called the *central cube*, and for any $Q \in \mathcal{C}$, there exists a chain $Q_0, Q_1, \ldots, Q_N = Q$ of cubes from $\mathcal{C}$ such that

$$Q \subset \tau Q_i, \quad i \in \{0,1,\ldots,N\},$$

(iii) the consecutive cubes of the connecting chain are comparable in size and overlap in some uniform way:

$$\max\{|Q_i|,|Q_{i+1}|\} \leqslant K|Q_i \cap Q_{i+1}|, \quad i \in \{0,1,\ldots,N-1\}. \tag{2.1}$$

It was proved as below by Boman [5] that John domains satisfy the aforementioned chain condition. A converse result was established by Buckley, Koskela and Lu [7].

Lemma 2.2. *Let Ω be a John domain. For any $\sigma \in [2,\infty)$, there exist $\tau,K \in [2,\infty)$, depending on C_J, such that Ω satisfies the (σ,τ,K)-Boman chain condition.*

The following inequality is well known.

Lemma 2.3. *For any $\{a_i\}_{i=1}^\infty \subset [0,\infty)$, if $p \in (0,1]$, then*

$$\left(\sum_{i=1}^\infty a_i\right)^p \leqslant \sum_{i=1}^\infty a_i^p \tag{2.2}$$

and, if $p \in [1,\infty)$, then

$$\left(\sum_{i=1}^\infty a_i\right)^p \geqslant \sum_{i=1}^\infty a_i^p. \tag{2.3}$$

As a consequence of this and Hölder's inequality, one has the following.

Lemma 2.4. *Let $p_1 \in (0,\infty)$, $p_2 \in (0,\infty)$, and $\varpi := \frac{1}{p_1}+\frac{1}{p_2}$. If $\varpi \in [1,\infty)$, $p_1\varpi \in (1,\infty)$, and $p_2\varpi \in (1,\infty)$, then, for any $\{a_i\}_{i=1}^\infty, \{b_i\}_{i=1}^\infty \subset [0,\infty)$,*

$$\sum_{i=1}^\infty a_i^{\frac{1}{p_1}} b_i^{\frac{1}{p_2}} \leqslant \left(\sum_{i=1}^\infty a_i\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^\infty b_i\right)^{\frac{1}{p_2}}. \tag{2.4}$$

Proof. Due to $\frac{1}{p_1}+\frac{1}{p_2}=\varpi\in[1,\infty)$, we obtain $\frac{1}{p_1\varpi}+\frac{1}{p_2\varpi}=1$. From (2.3) and Hölder's inequality, it follows that

$$\begin{aligned}\sum_{i=1}^{\infty} a_i^{\frac{1}{p_1}} b_i^{\frac{1}{p_2}} &= \sum_{i=1}^{\infty}\left(a_i^{\frac{1}{p_1\varpi}} b_i^{\frac{1}{p_2\varpi}}\right)^{\varpi} \leqslant \left(\sum_{i=1}^{\infty} a_i^{\frac{1}{p_1\varpi}} b_i^{\frac{1}{p_2\varpi}}\right)^{\varpi}\\ &\leqslant \left[\left(\sum_{i=1}^{\infty} a_i\right)^{\frac{1}{p_1\varpi}}\left(\sum_{i=1}^{\infty} b_i\right)^{\frac{1}{p_2\varpi}}\right]^{\varpi} = \left(\sum_{i=1}^{\infty} a_i\right)^{\frac{1}{p_1}}\left(\sum_{i=1}^{\infty} b_i\right)^{\frac{1}{p_2}},\end{aligned}$$

which completes the proof of Lemma 2.4. □

We refer to [5] and [4] for the following lemma.

Lemma 2.5. *Let $p\in[1,\infty)$. Then there exists a positive constant $C:=C(n,p)$ such that, for any $\tau\in[1,\infty)$, any family $\{Q_\alpha\}_\alpha$ of cubes in $\mathbb{R}^n$, and any family $\{a_\alpha\}_\alpha$ of non-negative numbers,*

$$\left\|\sum_{\alpha} a_\alpha \mathbf{1}_{\tau Q_\alpha}\right\|_{L^p(\mathbb{R}^n)} \leqslant C\tau^n \left\|\sum_{\alpha} a_\alpha \mathbf{1}_{Q_\alpha}\right\|_{L^p(\mathbb{R}^n)}. \tag{2.5}$$

The following (p,p)-Poincaré inequality is standard; see, for instance, [18].

Lemma 2.6. *Let $p\in[1,\infty]$. Then there exists a positive constant $C:=C(n,p)$ such that, for any cube $Q\subset\mathbb{R}^n$ with edge length l_Q and for any $f\in\dot W^{1,p}(Q)$, $f\in W^{1,p}(Q)$ and*

$$\left\|f-\operatorname*{ave}_Q f\right\|_{L^p(Q)} \leqslant Cl_Q\|\nabla f\|_{L^p(Q)}. \tag{2.6}$$

3. Proof of Theorem 1.1(i)

In this section, without special mention, we always assume that the quadruple (p,s,q,θ) is admissible. First we need the following (p,s,q,θ)-GNS inequality in cubes, where the positive constants are uniform in all cubes.

Lemma 3.1. *There exists a positive constant $C:=C(n,p,s,\theta)$ such that, for any cube Q, any $\sigma\in(1,\infty)$, and any $f\in\dot W^{1,p}(\sigma Q)\cap L^s(\sigma Q)$,*

$$\left\|f-\operatorname*{ave}_Q f\right\|_{L^q(Q)} \leqslant C\left[\frac{\sigma^n}{(\sigma-1)^\theta}+\sigma^{(1-\theta)n}\right]\|\nabla f\|_{L^p(\sigma Q)}^{\theta}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{1-\theta}. \tag{3.1}$$

Proof. Let $l_Q\in(0,\infty)$ be the edge length of Q. Let $\eta\in C_c^\infty(\mathbb{R}^n)$ be a cutoff function such that

$$0\leqslant\eta\leqslant1,\quad |\nabla\eta|\leqslant\frac{C(n)}{(\sigma-1)l_Q},\quad \eta=1 \text{ on } \overline{Q},\quad\text{and}\quad \operatorname{supp}\eta\subset\sigma Q.$$

It is easy to show that $(f-\operatorname{ave}_Q f)\eta\in\dot W^{1,p}(\mathbb{R}^n)\cap L^s(\mathbb{R}^n)$. According to (1.2), we obtain

$$\begin{aligned}\left\|f-\operatorname*{ave}_Q f\right\|_{L^q(Q)} &\leqslant \left\|\left(f-\operatorname*{ave}_Q f\right)\eta\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|\left|\nabla\left[\left(f-\operatorname*{ave}_Q f\right)\eta\right]\right|\right\|_{L^p(\mathbb{R}^n)}^{\theta}\left\|\left(f-\operatorname*{ave}_Q f\right)\eta\right\|_{L^s(\mathbb{R}^n)}^{1-\theta}.\end{aligned} \tag{3.2}$$

On the one hand, using Minkowski's inequality and (2.2), we conclude that

$$\begin{aligned}\left\|\left|\nabla\left[\left(f-\operatorname*{ave}_Q f\right)\eta\right]\right|\right\|_{L^p(\mathbb{R}^n)}^{\theta} &\leqslant \|(\nabla f)\eta\|_{L^p(\mathbb{R}^n)}^{\theta}+\left\|\left(f-\operatorname*{ave}_Q f\right)\nabla\eta\right\|_{L^p(\mathbb{R}^n)}^{\theta}\\ &\leqslant \|\nabla f\|_{L^p(\sigma Q)}^{\theta}+\left[\frac{C(n)}{(\sigma-1)l_Q}\right]^{\theta}\left\|f-\operatorname*{ave}_Q f\right\|_{L^p(\sigma Q)}^{\theta}. \end{aligned} \tag{3.3}$$

By (2.6), we find that

$$\begin{aligned}\left\|f-\operatorname*{ave}_Q f\right\|_{L^p(\sigma Q)}^{\theta} &\leqslant \left[\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^p(\sigma Q)}+\left\|\operatorname*{ave}_{\sigma Q} f-\operatorname*{ave}_Q f\right\|_{L^p(\sigma Q)}\right]^{\theta}\\ &=\left(1+\sigma^{\frac{n}{p}}\right)^{\theta}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^p(\sigma Q)}^{\theta}\\ &\lesssim \sigma^{\theta n}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^p(\sigma Q)}^{\theta}\\ &\lesssim \sigma^{\theta n} l_Q^{\theta}\,\|\nabla f\|_{L^p(\sigma Q)}^{\theta}. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned}\left\|\left|\nabla\left[\left(f-\operatorname*{ave}_Q f\right)\eta\right]\right|\right\|_{L^p(\mathbb{R}^n)}^{\theta} &\leqslant \|\nabla f\|_{L^p(\sigma Q)}^{\theta}+\frac{C(n,\theta)}{(\sigma-1)^{\theta} l^{\theta}}\left\|f-\operatorname*{ave}_Q f\right\|_{L^p(\sigma Q)}^{\theta}\\ &\leqslant \|\nabla f\|_{L^p(\sigma Q)}^{\theta}+\frac{C(n,\theta)}{(\sigma-1)^{\theta} l_Q^{\theta}}C(n,p,\theta)\sigma^{\theta n} l_Q^{\theta}\,\|\nabla f\|_{L^p(\sigma Q)}^{\theta}\\ &\approx \left[1+\left(\frac{\sigma^n}{\sigma-1}\right)^{\theta}\right]\|\nabla f\|_{L^p(\sigma Q)}^{\theta}. \end{aligned} \tag{3.5}$$

On the other hand, one has

$$\begin{aligned}\left\|\left(f-\operatorname*{ave}_Q f\right)\eta\right\|_{L^s(\mathbb{R}^n)}^{1-\theta} &\leqslant \left\|f-\operatorname*{ave}_Q f\right\|_{L^s(\sigma Q)}^{1-\theta}\\ &\lesssim \sigma^{(1-\theta)n}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{1-\theta}. \end{aligned} \tag{3.6}$$

Consequently, combining (3.2), (3.5), and (3.6), we derive

$$\begin{aligned}\left\|f-\operatorname*{ave}_Q f\right\|_{L^q(Q)} &\lesssim \left[1+\left(\frac{\sigma^n}{\sigma-1}\right)^{\theta}\right]\|\nabla f\|_{L^p(\sigma Q)}^{\theta}\,\sigma^{(1-\theta)n}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{1-\theta}\\ &\lesssim \left[\frac{\sigma^n}{(\sigma-1)^{\theta}}+\sigma^{(1-\theta)n}\right]\|\nabla f\|_{L^p(\sigma Q)}^{\theta}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{1-\theta}. \end{aligned}$$

This finishes the proof of Lemma 3.1. □

Remark 3.2. There exists a positive constant $C := C(n,p,s,\theta)$ such that, for any cube Q and any $f \in \dot{W}^{1,p}(Q)\cap L^s(Q)$,

$$\left\|f-\operatorname*{ave}_Q f\right\|_{L^q(Q)} \leqslant C\,\|\nabla f\|_{L^p(Q)}^{\theta}\left\|f-\operatorname*{ave}_Q f\right\|_{L^s(Q)}^{1-\theta}.$$

Next we prove Theorem 1.1(i). Since the case $\theta = 1$ follows from (1.4) in an obvious way, here we assume $\theta \in (0, 1)$.

Proof of Theorem 1.1(i). Assume that Ω is a John domain, we are going to show (1.3), that is, for any $f \in \dot{W}^{1,p}(\Omega) \cap L^s(\Omega)$,

$$\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} \leqslant C \left\| \nabla f \right\|^{\theta}_{L^p(\Omega)} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{1-\theta}_{L^s(\Omega)}, \tag{3.7}$$

where $C := C(n, p, s, \theta, C_J)$ is a positive constant. Notice that C_J depends only on τ and K and it does not depend on σ.

First, by a standard truncation approximation, we only need to prove that (3.7) holds for any $f \in \dot{W}^{1,p}(\Omega) \cap L^\infty(\Omega)$. For the convenience of the reader, we give the details here. Indeed, if $s = \infty$, then there is nothing to show, hence we assume $s \in [1, \infty)$. Given any $f \in \dot{W}^{1,p}(\Omega) \cap L^s(\Omega)$, write

$$f_i := \min \left\{ \max\{f, -i\}, i \right\}, \quad \forall\, i \in \mathbb{Z}_+.$$

It is clear that $f_i \in \dot{W}^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $|\nabla f_i| = |\nabla f| \mathbf{1}_{\{-i \leqslant f < i\}}$ and hence

$$\left\| \nabla f_i \right\|_{L^p(\Omega)} \leqslant \left\| \nabla f \right\|_{L^p(\Omega)}.$$

Moreover, by the dominated convergence theorem, we find that

$$\lim_{i \to \infty} \left\| f_i - \operatorname*{ave}_{\Omega} f_i \right\|_{L^s(\Omega)} = \left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^s(\Omega)}.$$

Assume that (3.7) holds for f_i, that is,

$$\left\| f_i - \operatorname*{ave}_{\Omega} f_i \right\|_{L^q(\Omega)} \lesssim \left\| \nabla f_i \right\|^{\theta}_{L^p(\Omega)} \left\| f_i - \operatorname*{ave}_{\Omega} f_i \right\|^{1-\theta}_{L^s(\Omega)}.$$

Then, by Fatou's lemma, we conclude that

$$\begin{aligned}
\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} &= \left\| \lim_{i \to \infty} \left(f_i - \operatorname*{ave}_{\Omega} f_i \right) \right\|_{L^q(\Omega)} \leqslant \varliminf_{i \to \infty} \left\| f_i - \operatorname*{ave}_{\Omega} f_i \right\|_{L^q(\Omega)} \\
&\lesssim \varliminf_{i \to \infty} \left\| \nabla f_i \right\|^{\theta}_{L^p(\Omega)} \left\| f_i - \operatorname*{ave}_{\Omega} f_i \right\|^{1-\theta}_{L^s(\Omega)} \\
&\leqslant \left\| \nabla f \right\|^{\theta}_{L^p(\Omega)} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{1-\theta}_{L^s(\Omega)}.
\end{aligned}$$

Below we assume that $f \in \dot{W}^{1,p}(\Omega) \cap L^\infty(\Omega)$. As a direct consequence, we find $f \in L^q(\Omega)$. Since Ω is a John domain, from Lemma 2.2, we infer that Ω satisfies the (σ, τ, K)-Boman chain condition for some fixed $\sigma, \tau, K \in [2, \infty)$. Denote by $\mathcal{C}$ the corresponding cover of Ω. Now we choose a central cube $Q_0 \in \mathcal{C}$. According to Minkowski's inequality, we have

$$\begin{aligned}
\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} &\leqslant \left\| f - \operatorname*{ave}_{Q_0} f \right\|_{L^q(\Omega)} + \left\| \operatorname*{ave}_{Q_0} f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} \\
&\lesssim \left[\int_{\Omega} \left| f(x) - \operatorname*{ave}_{Q_0} f \right|^q \mathrm{d}x \right]^{\frac{1}{q}}.
\end{aligned}$$

Since $\mathcal{C}$ is the cover of Ω, we deduce that

$$\begin{aligned}\int_\Omega \left|f(x)-\operatorname*{ave}_{Q_0} f\right|^q \mathrm{d}x &\leqslant \sum_{Q\in\mathcal{C}}\int_Q \left|f(x)-\operatorname*{ave}_{Q_0} f\right|^q \mathrm{d}x\\ &\lesssim \sum_{Q\in\mathcal{C}}\int_Q \left|f(x)-\operatorname*{ave}_{Q} f\right|^q \mathrm{d}x+\sum_{Q\in\mathcal{C}}\int_Q \left|\operatorname*{ave}_{Q} f-\operatorname*{ave}_{Q_0} f\right|^q \mathrm{d}x\\ &=: \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

To show (3.7), it then suffices to prove

$$\mathrm{I}_1+\mathrm{I}_2 \lesssim \|\nabla f\|_{L^p(\Omega)}^{\theta q}\left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^s(\Omega)}^{(1-\theta)q}.$$

We bound I_1 and I_2 from above separately.

Estimate for $\mathbf{I_1}$. Applying (3.1) to each $Q\in\mathcal{C}$, we obtain

$$\begin{aligned}\mathrm{I}_1 &= \sum_{Q\in\mathcal{C}}\left\|f-\operatorname*{ave}_{Q} f\right\|_{L^q(Q)}^q\\ &\lesssim \sum_{Q\in\mathcal{C}}\left[\|\nabla f\|_{L^p(\sigma Q)}^{\theta}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{1-\theta}\right]^q\\ &= \sum_{Q\in\mathcal{C}}\|\nabla f\|_{L^p(\sigma Q)}^{\theta q}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{(1-\theta)q}.\end{aligned}$$

To bound I_1 from above, it suffices to show

$$\sum_{Q\in\mathcal{C}}\|\nabla f\|_{L^p(\sigma Q)}^{\theta q}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{(1-\theta)q} \lesssim \|\nabla f\|_{L^p(\Omega)}^{\theta q}\left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^s(\Omega)}^{(1-\theta)q}. \tag{3.8}$$

We next consider three cases: $p,s\in[1,\infty)$; $p=\infty$ and $s\in[1,\infty)$; $p\in[1,\infty)$ and $s=\infty$. Notice that it will not happen that $p=s=\infty$.

Case 1: $p,s\in[1,\infty)$. In this case, write

$$\begin{aligned}&\sum_{Q\in\mathcal{C}}\|\nabla f\|_{L^p(\sigma Q)}^{\theta q}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{(1-\theta)q}\\ &= \sum_{Q\in\mathcal{C}}\left(\int_{\sigma Q}|\nabla f|^p\,\mathrm{d}x\right)^{\frac{\theta q}{p}}\left[\int_{\sigma Q}\left|f(x)-\operatorname*{ave}_{\sigma Q} f\right|^s \mathrm{d}x\right]^{\frac{(1-\theta)q}{s}}.\end{aligned}$$

According to (1.1), we find that

$$\varpi := \frac{\theta q}{p}+\frac{(1-\theta)q}{s}=1+\frac{\theta q}{n}\in(1,\infty).$$

Obviously,

$$\frac{\theta q}{p\varpi}+\frac{(1-\theta)q}{s\varpi}=1,\quad \frac{p}{\theta q}\varpi\in(1,\infty),\quad\text{and}\quad \frac{s}{(1-\theta)q}\varpi\in(1,\infty).$$

By (2.4), we have

$$\sum_{Q\in\mathcal{C}} \|\nabla f\|^{\theta q}_{L^p(\sigma Q)} \left\| f - \operatorname*{ave}_{\sigma Q} f \right\|^{(1-\theta)q}_{L^s(\sigma Q)} \leqslant \left(\sum_{Q\in\mathcal{C}} \int_{\sigma Q} |\nabla f|^p \,\mathrm{d}x \right)^{\frac{\theta q}{p}} \left[\sum_{Q\in\mathcal{C}} \int_{\sigma Q} \left| f(x) - \operatorname*{ave}_{\sigma Q} f \right|^s \mathrm{d}x \right]^{\frac{(1-\theta)q}{s}}.$$

Since

$$\begin{aligned}\int_{\sigma Q} |f(x) - \operatorname*{ave}_{\sigma Q} f|^s \,\mathrm{d}x &\lesssim \int_{\sigma Q} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^s \mathrm{d}x + \int_{\sigma Q} \left| \operatorname*{ave}_{\Omega} f - \operatorname*{ave}_{\sigma Q} f \right|^s \mathrm{d}x \\ &\lesssim \int_{\sigma Q} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^s \mathrm{d}x,\end{aligned}$$

we infer that

$$\sum_{Q\in\mathcal{C}} \|\nabla f\|^{\theta q}_{L^p(\sigma Q)} \left\| f - \operatorname*{ave}_{\sigma Q} f \right\|^{(1-\theta)q}_{L^s(\sigma Q)} \lesssim \left(\sum_{Q\in\mathcal{C}} \int_{\sigma Q} |\nabla f|^p \,\mathrm{d}x \right)^{\frac{\theta q}{p}} \left[\sum_{Q\in\mathcal{C}} \int_{\sigma Q} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^s \mathrm{d}x \right]^{\frac{(1-\theta)q}{s}}.$$

Using Definition 2.1(i), we find that

$$\sum_{Q\in\mathcal{C}} \int_{\sigma Q} |\nabla f|^p \,\mathrm{d}x = \int_{\mathbb{R}^n} |\nabla f|^p \sum_{Q\in\mathcal{C}} \mathbf{1}_{\sigma Q}(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^n} |\nabla f|^p \tau \mathbf{1}_{\Omega}(x)\,\mathrm{d}x = \tau \int_{\Omega} |\nabla f|^p \,\mathrm{d}x$$

and, similarly,

$$\sum_{Q\in\mathcal{C}} \int_{\sigma Q} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^s \mathrm{d}x \leqslant \tau \int_{\Omega} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^s \mathrm{d}x.$$

Thus,

$$\begin{aligned}\sum_{Q\in\mathcal{C}} \|\nabla f\|^{\theta q}_{L^p(\sigma Q)} \left\| f - \operatorname*{ave}_{\sigma Q} f \right\|^{(1-\theta)q}_{L^s(\sigma Q)} &\lesssim \left(\tau \int_{\Omega} |\nabla f|^p \,\mathrm{d}x \right)^{\frac{\theta q}{p}} \left[\tau \int_{\Omega} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^s \mathrm{d}x \right]^{\frac{(1-\theta)q}{s}} \\ &\lesssim \|\nabla f\|^{\theta q}_{L^p(\Omega)} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{(1-\theta)q}_{L^s(\Omega)},\end{aligned}$$

which is the desired inequality.

Case 2: $p = \infty$ and $s \in [1, \infty)$. In this case, noticing

$$\|\nabla f\|_{L^\infty(\sigma Q)} \leqslant \|\nabla f\|_{L^\infty(\Omega)},$$

we have

$$\begin{aligned}\sum_{Q\in\mathcal{C}} \|\nabla f\|^{\theta q}_{L^\infty(\sigma Q)} \left\| f - \operatorname*{ave}_{\sigma Q} f \right\|^{(1-\theta)q}_{L^s(\sigma Q)} &\leqslant \sum_{Q\in\mathcal{C}} \|\nabla f\|^{\theta q}_{L^\infty(\Omega)} \left\| f - \operatorname*{ave}_{\sigma Q} f \right\|^{(1-\theta)q}_{L^s(\sigma Q)} \\ &= \|\nabla f\|^{\theta q}_{L^\infty(\Omega)} \sum_{Q\in\mathcal{C}} \left[\int_{\sigma Q} \left| f(x) - \operatorname*{ave}_{\sigma Q} f \right|^s \mathrm{d}x \right]^{\frac{(1-\theta)q}{s}}.\end{aligned}$$

Since $p=\infty$ implies $\frac{1}{p}=0$, then, from (1.1), we deduce that

$$\frac{(1-\theta)q}{s}=1+\frac{\theta q}{n}\in(1,\infty).$$

Using (2.3) and Definition 2.1(i), we conclude that

$$\begin{aligned}
\sum_{Q\in\mathcal{C}}\|\nabla f\|_{L^\infty(\sigma Q)}^{\theta q}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^s(\sigma Q)}^{(1-\theta)q} &\lesssim \|\nabla f\|_{L^\infty(\Omega)}^{\theta q}\left[\sum_{Q\in\mathcal{C}}\int_{\sigma Q}\left|f(x)-\operatorname*{ave}_{\sigma Q} f\right|^s\,\mathrm{d}x\right]^{\frac{(1-\theta)q}{s}}\\
&\lesssim \|\nabla f\|_{L^\infty(\Omega)}^{\theta q}\left[\int_{\Omega}\left|f(x)-\operatorname*{ave}_{\Omega} f\right|^s\,\mathrm{d}x\right]^{\frac{(1-\theta)q}{s}}\\
&= \|\nabla f\|_{L^\infty(\Omega)}^{\theta q}\left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^s(\Omega)}^{(1-\theta)q}.
\end{aligned}$$

Case 3: $p\in[1,\infty)$ and $s=\infty$. In this case, noticing that

$$\begin{aligned}
\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^\infty(\sigma Q)} &\leqslant \left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^\infty(\Omega)}+\left|\operatorname*{ave}_{\Omega} f-\operatorname*{ave}_{\sigma Q} f\right|\\
&\lesssim \left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^\infty(\Omega)},
\end{aligned}$$

we derive

$$\begin{aligned}
\sum_{Q\in\mathcal{C}}\|\nabla f\|_{L^p(\sigma Q)}^{\theta q}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^\infty(\sigma Q)}^{(1-\theta)q} &\lesssim \left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^\infty(\Omega)}^{(1-\theta)q}\sum_{Q\in\mathcal{C}}\|\nabla f\|_{L^p(\Omega)}^{\theta q}\\
&= \left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^\infty(\Omega)}^{(1-\theta)q}\sum_{Q\in\mathcal{C}}\left(\int_{\sigma Q}|\nabla f|^p\,\mathrm{d}x\right)^{\frac{\theta q}{p}}.
\end{aligned}$$

Since $s=\infty$ implies $\frac{1}{s}=0$, then, from (1.1), we infer that

$$\frac{\theta q}{p}=1+\frac{\theta q}{n}\in(1,\infty).$$

Using (2.3) and Definition 2.1(i), we find that

$$\begin{aligned}
\sum_{Q\in\mathcal{C}}\|\nabla f\|_{L^p(\sigma Q)}^{\theta q}\left\|f-\operatorname*{ave}_{\sigma Q} f\right\|_{L^\infty(\sigma Q)}^{(1-\theta)q} &\lesssim \left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^\infty(\Omega)}^{(1-\theta)q}\left(\sum_{Q\in\mathcal{C}}\int_{\sigma Q}|\nabla f|^p\,\mathrm{d}x\right)^{\frac{\theta q}{p}}\\
&\lesssim \left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^\infty(\Omega)}^{(1-\theta)q}\left(\int_{\Omega}|\nabla f|^p\,\mathrm{d}x\right)^{\frac{\theta q}{p}}\\
&= \|\nabla f\|_{L^p(\Omega)}^{\theta q}\left\|f-\operatorname*{ave}_{\Omega} f\right\|_{L^\infty(\Omega)}^{(1-\theta)q}
\end{aligned}$$

as desired.

Estimate for I_2. Given any $Q\in\mathcal{C}$, then, by Definition 2.1(ii), we find that there exists a connecting chain $\{Q_i\}_{i=0}^N\subset\mathcal{C}$ for the pair (Q,Q_0). Write

$$\begin{aligned}\left|\operatorname*{ave}_{Q} f-\operatorname*{ave}_{Q_0} f\right| &= \left|\operatorname*{ave}_{Q_N} f-\operatorname*{ave}_{Q_0} f\right|\\ &=\left|\sum_{i=0}^{N-1}\left(\operatorname*{ave}_{Q_{i+1}} f-\operatorname*{ave}_{Q_i} f\right)\right|\leqslant\sum_{i=0}^{N-1}\left|\operatorname*{ave}_{Q_{i+1}} f-\operatorname*{ave}_{Q_i} f\right|\\ &\leqslant\sum_{i=0}^{N-1}\left(\left|\operatorname*{ave}_{Q_{i+1}} f-\operatorname*{ave}_{Q_{i+1}\cap Q_i} f\right|+\left|\operatorname*{ave}_{Q_{i+1}\cap Q_i} f-\operatorname*{ave}_{Q_i} f\right|\right).\end{aligned}\tag{3.9}$$

From (2.1), it follows that

$$\begin{aligned}\left|\operatorname*{ave}_{Q_{i+1}} f-\operatorname*{ave}_{Q_{i+1}\cap Q_i} f\right| &=\left|\operatorname*{ave}_{Q_{i+1}} f-\frac{1}{|Q_{i+1}\cap Q_i|}\int_{Q_{i+1}\cap Q_i} f(x)\,\mathrm{d}x\right|\\ &\leqslant\frac{1}{|Q_{i+1}\cap Q_i|}\int_{Q_{i+1}\cap Q_i}\left|f(x)-\operatorname*{ave}_{Q_{i+1}} f\right|\mathrm{d}x\\ &\leqslant\frac{1}{K|Q_{i+1}|}\int_{Q_{i+1}}\left|f(x)-\operatorname*{ave}_{Q_{i+1}} f\right|\mathrm{d}x\\ &\leqslant\frac{1}{K}\left[\fint_{Q_{i+1}}\left|f(x)-\operatorname*{ave}_{Q_{i+1}} f\right|^q\mathrm{d}x\right]^{\frac1q}.\end{aligned}\tag{3.10}$$

Similarly,

$$\left|\operatorname*{ave}_{Q_{i+1}\cap Q_i} f-\operatorname*{ave}_{Q_i} f\right|\leqslant\frac{1}{K}\left[\fint_{Q_i}\left|f(x)-\operatorname*{ave}_{Q_i} f\right|^q\mathrm{d}x\right]^{\frac1q}.\tag{3.11}$$

Combining (3.9), (3.10), and (3.11), we obtain

$$\left|\operatorname*{ave}_{Q} f-\operatorname*{ave}_{Q_0} f\right|\lesssim\sum_{i=0}^{N}\left[\fint_{Q_i}\left|f(x)-\operatorname*{ave}_{Q_i} f\right|^q\mathrm{d}x\right]^{\frac1q}.$$

From (3.1), it follows that

$$\left|\operatorname*{ave}_{Q} f-\operatorname*{ave}_{Q_0} f\right|\lesssim\sum_{i=0}^{N}\frac{1}{|Q_i|^{\frac1q}}\,\|\nabla f\|_{L^p(\sigma Q_i)}^{\theta}\left\|f-\operatorname*{ave}_{\sigma Q_i} f\right\|_{L^s(\sigma Q_i)}^{1-\theta}.$$

Now, denote by $\mathcal{W}_Q$ the set of all τ-neighbors of Q in $\mathcal{C}$, that is,

$$\mathcal{W}_Q:=\{P\in\mathcal{C}:\tau P\supset Q\}.$$

Definition 2.1(ii) says that, for any $i\in\{0,1,\dots,N\}$, $\tau Q_i\supset Q$ and hence $Q_i\in\mathcal{W}_Q$. Thus,

$$\left|\operatorname*{ave}_{Q} f-\operatorname*{ave}_{Q_0} f\right|\lesssim\sum_{P\in\mathcal{W}_Q}\frac{1}{|P|^{\frac1q}}\,\|\nabla f\|_{L^p(\sigma P)}^{\theta}\left\|f-\operatorname*{ave}_{\sigma P} f\right\|_{L^s(\sigma P)}^{1-\theta}=:\sum_{P\in\mathcal{W}_Q}a_P.\tag{3.12}$$

Next, by (3.12), one has

$$\mathrm{I}_2=\sum_{Q\in\mathcal{C}}\int_Q\left|\operatorname*{ave}_{Q} f-\operatorname*{ave}_{Q_0} f\right|^q\mathrm{d}x\lesssim\sum_{Q\in\mathcal{C}}\int_Q\left|\sum_{P\in\mathcal{W}_Q}a_P\right|^q\mathrm{d}x.$$

Given any $Q \in \mathcal{C}$, since for any $P \in \mathcal{W}_Q$, one has

$$\mathbf{1}_{\tau P}(x) = 1, \quad \forall\, x \in Q,$$

we deduce that

$$\sum_{P\in\mathcal{W}_Q} a_P = \sum_{P\in\mathcal{W}_Q} a_P \mathbf{1}_{\tau P}(x) \leqslant \sum_{P\in\mathcal{C}} a_P \mathbf{1}_{\tau P}(x), \quad \forall\, x \in Q.$$

Thus, by Definition 2.1(i), we obtain

$$\mathrm{I}_2 \lesssim \sum_{Q\in\mathcal{C}} \int_Q \left|\sum_{P\in\mathcal{C}} a_P \mathbf{1}_{\tau P}(x)\right|^q \mathrm{d}x \lesssim \int_{\mathbb{R}^n} \left|\sum_{P\in\mathcal{C}} a_P \mathbf{1}_{\tau P}(x)\right|^q \mathrm{d}x.$$

From (2.5), it follows that

$$\mathrm{I}_2 \lesssim \int_{\mathbb{R}^n} \left|\sum_{P\in\mathcal{C}} a_P \mathbf{1}_{P}(x)\right|^q \mathrm{d}x.$$

By Definition 2.1(i), we find that, for any $x \in \mathbb{R}^n$, there are at most $\lceil\tau\rceil$ many P contain x, where $\lceil\tau\rceil$ denotes the smallest integer greater than τ, and hence

$$\left|\sum_{P\in\mathcal{C}} a_P \mathbf{1}_{P}(x)\right|^q \leqslant \lceil\tau\rceil^{q-1} \sum_{P\in\mathcal{C}} a_P^q \mathbf{1}_{P}(x).$$

We therefore obtain

$$\mathrm{I}_2 \lesssim \int_{\mathbb{R}^n} \sum_{P\in\mathcal{C}} a_P^q \mathbf{1}_{P}(x)\, \mathrm{d}x = \sum_{P\in\mathcal{C}} a_P^q |P|.$$

Recalling the definition of a_P, applying (3.8) we conclude that

$$\mathrm{I}_2 \lesssim \sum_{P\in\mathcal{C}} \|\nabla f\|_{L^p(\sigma P)}^{\theta q} \left\| f - \operatorname*{ave}_{\sigma P} f \right\|_{L^s(\sigma P)}^{(1-\theta)q} \lesssim \|\nabla f\|_{L^p(\Omega)}^{\theta q} \left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^s(\Omega)}^{(1-\theta)q},$$

which completes the proof of Theorem 1.1(i). □

Below we present a different approach to prove (1.4).

Remark 3.3. We recall another approach to show (1.4), which is different from Bojarski [4]. Denote by $\mathbf{I}_1$ the Riesz potential of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, that is, for any $x \in \mathbb{R}^n$,

$$\mathbf{I}_1(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\, \mathrm{d}y.$$

If Ω is a John domain and $f \in C^1(\Omega)$, then it was proved by Reshetnyak [24] and Martio [19] that, for any $x \in \Omega$,

$$\left| f(x) - \operatorname*{ave}_{\Omega} f \right| \leqslant C \int_{\Omega} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, \mathrm{d}y = C\mathbf{I}_1(|\nabla f|\mathbf{1}_{\Omega})(x). \tag{3.13}$$

For any $p \in (1, n)$, (1.4) follows directly from (3.13), the boundedness of $\mathbf{I}_1$ from $L^p(\mathbb{R}^n)$ to $L^{\frac{np}{n-p}}(\mathbb{R}^n)$, and the density of $C^1(\Omega) \cap \dot{W}^{1,p}(\Omega)$ in $\dot{W}^{1,p}(\Omega)$. For $p = 1$, (1.4) follows from (3.13), the boundedness of $\mathbf{I}_1$ from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)$, and a truncation argument; see [10, 11, 16] for more details.

When $p \in [1, n)$, it is standard to deduce Theorem 1.1(i) from (1.4) and Hölder's inequality.

Remark 3.4. If a domain Ω supports the $(\frac{np}{n-p}, p)$-Poincaré inequality (1.4) for some $p \in [1, n)$, then, for any $s \in [1, \infty]$, $q \in [1, \infty)$, $\theta \in (0, 1]$ such that (p, s, q, θ) is admissible, Ω supports the (p, s, q, θ)-GNS inequality (1.3). To see this, we first notice that the $(p, s, \frac{np}{n-p}, 1)$-GNS inequality follows directly from the $(\frac{np}{n-p}, p)$-Poincaré inequality. Next, by assuming $\theta \in (0, 1)$, we consider the following two cases.

Case 1: $s \in [1, \infty)$. According to (1.1) and $p \in [1, n)$, we have

$$1 = \theta q \left(\frac{1}{p} - \frac{1}{n}\right) + \frac{(1-\theta)q}{s}, \quad \theta q \left(\frac{1}{p} - \frac{1}{n}\right) \in (0, 1), \quad \text{and} \quad \frac{(1-\theta)q}{s} \in (0, 1).$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
&\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} \\
&= \left[\int_\Omega \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{\theta q} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{(1-\theta)q} \mathrm{d}x \right]^{\frac{1}{q}} \\
&\leqslant \left\{ \left[\int_\Omega \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{\theta q \frac{1}{\theta q (\frac{1}{p} - \frac{1}{n})}} \mathrm{d}x \right]^{\theta q (\frac{1}{p} - \frac{1}{n})} \left[\int_\Omega \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{(1-\theta) q \frac{1}{\frac{(1-\theta)q}{s}}} \mathrm{d}x \right]^{\frac{(1-\theta)q}{s}} \right\}^{\frac{1}{q}} \\
&= \left[\int_\Omega \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{\frac{np}{n-p}} \mathrm{d}x \right]^{\frac{\theta(n-p)}{np}} \left[\int_\Omega \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{s} \mathrm{d}x \right]^{\frac{1-\theta}{s}} \\
&= \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{\theta}_{L^{\frac{np}{n-p}}(\Omega)} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{1-\theta}_{L^s(\Omega)}.
\end{aligned}$$

From (1.4), it follows that

$$\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} \lesssim \|\nabla f\|^{\theta}_{L^p(\Omega)} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{1-\theta}_{L^s(\Omega)}.$$

Case 2: $s = \infty$. In view of (1.1), we have

$$\theta q \left(\frac{1}{p} - \frac{1}{n}\right) = 1.$$

Then

$$\begin{aligned}
\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} &= \left[\int_\Omega \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{\theta q} \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{(1-\theta)q} \mathrm{d}x \right]^{\frac{1}{q}} \\
&\leqslant \left[\int_\Omega \left| f(x) - \operatorname*{ave}_{\Omega} f \right|^{\theta q} \mathrm{d}x \right]^{\frac{1}{q}} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{1-\theta}_{L^\infty(\Omega)} \\
&= \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{\theta}_{L^{\frac{np}{n-p}}(\Omega)} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{1-\theta}_{L^\infty(\Omega)}.
\end{aligned}$$

On account of (1.4), we deduce that

$$\left\| f - \operatorname*{ave}_{\Omega} f \right\|_{L^q(\Omega)} \lesssim \|\nabla f\|^{\theta}_{L^p(\Omega)} \left\| f - \operatorname*{ave}_{\Omega} f \right\|^{1-\theta}_{L^\infty(\Omega)}.$$

Inspired by Remark 3.4, here we are interested in considering whether a domain supports $(\widetilde{p}, \widetilde{s}, \widetilde{q}, \widetilde{\theta})$-GNS inequalities with admissible $(\widetilde{p}, \widetilde{s}, \widetilde{q}, \widetilde{\theta})$ from supporting a (p, s, q, θ)-GNS inequality for some admissible quadruple (p, s, q, θ).

Remark 3.5. Suppose that a domain Ω supports the (p,s,q,θ)-GNS inequality (1.3) for some admissible quadruple (p,s,q,θ).

If $\theta=1$, since $(p,s,q,1)$ is admissible, one must have $q=\frac{np}{n-p}$ and $p\in[1,n)$. By the argument similar to Remark 3.4, we know that, for any $\widetilde{s}\in[1,\infty]$, $\widetilde{q}\in[1,\infty)$ and $\widetilde{\theta}\in(0,1)$ such that $(p,\widetilde{s},\widetilde{q},\widetilde{\theta})$ is admissible, Ω supports the $(p,\widetilde{s},\widetilde{q},\widetilde{\theta})$-GNS inequality.

Now we assume $\theta\in(0,1)$. In this case, for any $\widetilde{q}\in[1,\infty)$ and $\widetilde{\theta}\in(0,\theta)$ such that $(p,s,\widetilde{q},\widetilde{\theta})$ is admissible, Ω supports the $(p,s,\widetilde{q},\widetilde{\theta})$-GNS inequality. Indeed, by letting $\widehat{\theta}:=\frac{\widetilde{\theta}}{\theta}\in(0,1)$, we find

$$\frac{1}{\widetilde{q}}=\widetilde{\theta}\left(\frac{1}{p}-\frac{1}{n}\right)+\frac{1-\widetilde{\theta}}{s}=\frac{\widetilde{\theta}}{\theta}\left(\frac{1}{q}-\frac{1-\theta}{s}\right)+\frac{1-\widetilde{\theta}}{s}=\frac{\widehat{\theta}}{q}+\frac{1-\widehat{\theta}}{s}.$$

Using Hölder inequality, we obtain

$$\begin{aligned}\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^{\widetilde{q}}(\Omega)}&=\left[\int_{\Omega}\left|f-\operatorname*{ave}_{\Omega}f\right|^{\widehat{\theta}\widetilde{q}}\left|f-\operatorname*{ave}_{\Omega}f\right|^{(1-\widehat{\theta})\widetilde{q}}\,\mathrm{d}x\right]^{\frac{1}{\widetilde{q}}}\\&\leqslant\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^{q}(\Omega)}^{\widehat{\theta}}\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^{s}(\Omega)}^{1-\widehat{\theta}}.\end{aligned}$$

Applying the (p,s,q,θ)-GNS inequality as assumed, one has

$$\begin{aligned}\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^{\widetilde{q}}(\Omega)}&\lesssim\left[\|\nabla f\|_{L^p(\Omega)}^{\theta}\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^{s}(\Omega)}^{1-\theta}\right]^{\widehat{\theta}}\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^{s}(\Omega)}^{1-\widehat{\theta}}\\&=\|\nabla f\|_{L^p(\Omega)}^{\widetilde{\theta}}\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^{\widetilde{s}}(\Omega)}^{1-\widetilde{\theta}}\end{aligned}$$

as desired. Finally, it's worth mentioning that in this case we cannot deduce the $(p,s,\frac{np}{n-p},1)$-GNS inequality from the (p,s,q,θ)-GNS inequality.

Unfortunately, we don't know if there are any other $(\widetilde{p},\widetilde{s},\widetilde{q},\widetilde{\theta})$-GNS inequalities with admissible $(\widetilde{p},\widetilde{s},\widetilde{q},\widetilde{\theta})$ holds.

4. Proof of Theorem 1.1(ii)

In order to show Theorem 1.1(ii), we need the following lemma. Below we also assume $\theta\in(0,1)$ since $\theta=1$ was considered in [6, Theorem 2.1].

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain supporting* (1.3) *for some admissible quadruple (p,s,q,θ). Fix a ball $B_0\subset\Omega$. Then there exists a positive constant $C:=C(C_0,n,p,s,\theta,\Omega,B_0)$, where C_0 denotes the positive constant C appearing in* (1.3), *such that*

$$\operatorname{diam}(T)\leqslant Cd,$$

whenever T is a connected component of $\Omega\setminus B(z,d)$ for some $z\in\Omega$ and $d\in(0,\infty)$ and that $T\cap B_0=\emptyset$.

Proof. Let T be any given connected component of $\Omega\setminus B(z,d)$ for some $z\in\Omega$ and $d\in(0,\infty)$ and let $T\cap B_0=\emptyset$. Notice that $d\geqslant\operatorname{dist}(z,\partial\Omega)$ and $T\cap B(z,d)=\emptyset$.

For any $\rho\geqslant d$, let

$$T(\rho):=T\setminus B(z,\rho).$$

Notice that $T(d)=T$. For any $\rho_2>\rho_1\geqslant d$, write

$$A(\rho_1,\rho_2):=T(\rho_1)\setminus T(\rho_2)=T\cap B(z,\rho_2)\setminus B(z,\rho_1).$$

Given any r,ρ with $T(r)\neq\emptyset$ and $r>\rho\geqslant d$, let

$$f(x)=\begin{cases}0 & \text{if } x\in\Omega\setminus T(\rho),\\ \dfrac{|x-z|-\rho}{r-\rho} & \text{if } x\in A(\rho,r),\\ 1 & \text{if } x\in T(r).\end{cases}$$

By a direct calculation, one has, for any $x,y\in\Omega$,

$$|f(x)-f(y)|\leqslant\frac{|x-y|}{r-\rho},$$

which further implies that f is a Lipschitz function on Ω. According to Rademacher's theorem, we find that $f\in W^{1,\infty}(\Omega)$. Moreover, we obtain

$$|\nabla f|(x)=\begin{cases}0 & \text{if } x\in\Omega\setminus T(\rho),\\ \dfrac{1}{r-\rho} & \text{if } x\in A(\rho,r),\\ 0 & \text{if } x\in T(r).\end{cases}$$

Notice that

$$|T(r)|=\int_{T(r)}\mathrm{d}x=\int_{T(r)}|f(x)|^q\,\mathrm{d}x\leqslant\|f\|^q_{L^q(\Omega)}. \tag{4.1}$$

Since f vanishes in $B_0\subset\Omega\setminus T$, we infer that

$$\begin{aligned}\|f\|_{L^q(\Omega)}&=\left\|f\mathbf{1}_{\Omega\setminus B_0}\right\|_{L^q(\Omega)}\\ &\leqslant\left\|\left(f-\operatorname*{ave}_{\Omega}f\right)\mathbf{1}_{\Omega\setminus B_0}\right\|_{L^q(\Omega)}+|\Omega\setminus B_0|^{\frac1q}\left|\operatorname*{ave}_{\Omega}f\right|\\ &\leqslant\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^q(\Omega)}+|\Omega\setminus B_0|^{\frac1q}\fint_{\Omega}|f(x)|\,\mathrm{d}x\\ &\leqslant\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^q(\Omega)}+\left(\frac{|\Omega\setminus B_0|}{|\Omega|}\right)^{\frac1q}\|f\|_{L^q(\Omega)}.\end{aligned}$$

Let $\gamma:=\left(\frac{|\Omega\setminus B_0|}{|\Omega|}\right)^{\frac1q}$. Noticing $\gamma\in(0,1)$, we can absorb $\gamma\|f\|_{L^q(\Omega)}$ to the left side and then obtain

$$\|f\|_{L^q(\Omega)}\leqslant(1-\gamma)^{-1}\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^q(\Omega)}=C(q,\Omega,B_0)\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^q(\Omega)}.$$

Therefore, applying (1.3), we conclude that

$$\begin{aligned}\|f\|_{L^q(\Omega)}&\lesssim\left\|f-\operatorname*{ave}_{\Omega}f\right\|_{L^q(\Omega)}\leqslant C_0\|\nabla f\|^\theta_{L^p(\Omega)}\left\|f-\operatorname*{ave}_{\Omega}f\right\|^{1-\theta}_{L^s(\Omega)}\\ &\lesssim\|\nabla f\|^\theta_{L^p(\Omega)}\|f\|^{1-\theta}_{L^s(\Omega)}.\end{aligned}\tag{4.2}$$

Below we also consider three cases.

Case 1: $p,s\in[1,\infty)$. In this case, since

$$\|\nabla f\|_{L^p(\Omega)}\leqslant\left[\int_{A(\rho,r)}\frac{1}{(r-\rho)^p}\,\mathrm{d}x\right]^{\frac1p}=\frac{|A(\rho,r)|^{\frac1p}}{r-\rho}$$

and

$$\|f\|_{L^s(\Omega)}\leqslant\left[\int_{A(\rho,r)\cup T(r)}\mathrm{d}x\right]^{\frac1s}=|T(\rho)|^{\frac1s},$$

from (4.1) and (4.2), we deduce that

$$|T(r)|^{\frac{1}{q}} \lesssim \frac{|A(\rho,r)|^{\frac{\theta}{p}}|T(\rho)|^{\frac{1-\theta}{s}}}{(r-\rho)^{\theta}},$$

which further gives that

$$r-\rho \lesssim \frac{|A(\rho,r)|^{\frac{1}{p}}|T(\rho)|^{\frac{1-\theta}{\theta s}}}{|T(r)|^{\frac{1}{\theta q}}}. \tag{4.3}$$

Write $r_0 := d$. Then, for any $i \in \mathbb{Z}_+$, choose r_i such that $r_i > r_{i-1}$ and

$$|A(r_{i-1},r_i)| = |T(r_{i-1}) \setminus T(r_i)| = 2^{-i}|T|.$$

Obviously,

$$|T(r_1)| = |T(r_0) \setminus A(r_0,r_1)| = |T| - 2^{-1}|T| = 2^{-1}|T|$$

and, for any $i \in \mathbb{Z}_+$,

$$|T(r_i)| = |T(r_{i-1}) \setminus A(r_{i-1},r_i)| = |T(r_{i-1})| - 2^{-i}|T|.$$

Thus, $|T(r_i)| = 2^{-i}|T|$ for any $i \in \mathbb{N}$. By (4.3), one then has, for any $i \in \mathbb{Z}_+$,

$$r_i - r_{i-1} \lesssim \frac{|A(r_{i-1},r_i)|^{\frac{1}{p}}|T(r_{i-1})|^{\frac{1-\theta}{\theta s}}}{|T(r_i)|^{\frac{1}{\theta q}}} \lesssim \left(2^{-i}|T|\right)^{\frac{1}{p}+\frac{1-\theta}{\theta s}-\frac{1}{\theta q}}.$$

Since (1.1) leads to

$$\frac{1}{p} + \frac{1-\theta}{\theta s} - \frac{1}{\theta q} = \frac{1}{n},$$

we infer that

$$r_i - r_{i-1} \lesssim \left(2^{-i}|T|\right)^{\frac{1}{n}}$$

and hence

$$\sum_{i=1}^{\infty}(r_i - r_{i-1}) \lesssim \sum_{i=1}^{\infty}\left(2^{-i}|T|\right)^{\frac{1}{n}} \lesssim |T|^{\frac{1}{n}}. \tag{4.4}$$

Case 2: $p = \infty$ and $s \in [1,\infty)$. In this case, noticing

$$\|\nabla f\|_{L^\infty(\Omega)} \leqslant \frac{1}{r-p} \text{ and } \|f\|_{L^s(\Omega)} \leqslant |T(\rho)|^{\frac{1}{s}}$$

and then using (4.1) and (4.2), we find that

$$r-p \lesssim \frac{|T(\rho)|^{\frac{1-\theta}{\theta s}}}{|T(r)|^{\frac{1}{\theta q}}}.$$

By a similar construction of $\{r_i\}_{i=0}^{\infty}$ in Case 1 and by (1.1), we find that, for any $i \in \mathbb{Z}_+$,

$$r_i - r_{i-1} \lesssim \frac{|T(r_{i-1})|^{\frac{1-\theta}{\theta s}}}{|T(r_i)|^{\frac{1}{\theta q}}} \lesssim \left(2^{-i}|T|\right)^{\frac{1-\theta}{\theta s}-\frac{1}{\theta q}} = \left(2^{-i}|T|\right)^{\frac{1}{n}}.$$

This further implies that (4.4) also holds in this case.

Case 3: $p \in [1,\infty)$ and $s = \infty$. In this case, noticing

$$\|\nabla f\|_{L^p(\Omega)} \leqslant \frac{|A(\rho,r)|^{\frac{1}{p}}}{r-\rho} \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} \leqslant 1$$

and then using (4.1) and (4.2), we find

$$r-\rho \lesssim \frac{|A(\rho,r)|^{\frac{1}{p}}}{|T(r)|^{\frac{1}{\theta q}}}.$$

Both a similar construction of $\{r_i\}_{i=0}^{\infty}$ to the one in Case 1 and (1.1) lead to, for any $i\in\mathbb{Z}_+$,

$$r_i-r_{i-1} \lesssim \frac{|A(r_{i-1},r_i)|^{\frac{1}{p}}}{|T(r_i)|^{\frac{1}{\theta q}}} \lesssim \left(2^{-i}|T|\right)^{\frac{1}{p}-\frac{1}{\theta q}} = \left(2^{-i}|T|\right)^{\frac{1}{n}}.$$

By this, we also obtain (4.4).

Notice that $T=\bigcup_{i=1}^{\infty}A(r_{i-1},r_i)$. Otherwise, there exists a point $x\in T$ but $x\notin\bigcup_{i=1}^{\infty}A(r_{i-1},r_i)$. One then has

$$|x-z|\geqslant r_0+\sum_{i=1}^{\infty}(r_i-r_{i-1})$$

and hence $|x-z|>r_j$ for any $j\in\mathbb{N}$. Choose a ball with center x and radius $r_x\in(0,\infty)$ such that $B(x,r_x)\subset\Omega$. Since T is a connected component and $B(x,r_x)\setminus B(z,r_j)$ is connected, it follows that, for any $j\in\mathbb{N}$,

$$B(x,r_x)\setminus B(z,r_j)\subset T,$$

which further implies that

$$B(x,r_x)\setminus B(z,r_j)\subset T\setminus B(z,r_j)=T(r_j).$$

By $|T(r_j)|=2^{-j}|T|$ and $x\notin B(z,r_j)$, we conclude that, for any $i\in\mathbb{N}$,

$$2^{-j}|T|=|T(r_j)|\geqslant|B(x,r_x)\setminus B(z,r_j)|\geqslant\frac{1}{2}|B(x,r_x)|,$$

which is impossible when j is largely enough.

Therefore,

$$\operatorname{diam}(T)\leqslant 2d+\sum_{i=1}^{\infty}2(r_i-r_{i-1})\lesssim d+|T|^{\frac{1}{n}}.$$

Since Ω is a bounded set, we deduce that there exists a constant k_0, depending on Ω, such that

$$T\subset\Omega\subset B(z,k_0d),$$

which means that

$$|T|\leqslant|B(z,k_0d)|\approx d^n.$$

Consequently, we derive

$$\operatorname{diam}(T)\lesssim d+|T|^{\frac{1}{n}}\lesssim d,$$

which completes the proof of Lemma 4.1. □

We now turn to prove Theorem 1.1(ii). We employ some ideas from the proof of [14, Theorem 2.1] (originally from the proof of [6, Theorem 1.1]) for the sake of completeness.

Proof of Theorem 1.1(ii). Given $x_0\in\Omega$, then, for any $x\in\Omega$, pick a curve $\gamma\colon[0,1]\to\Omega$ with $\gamma(0)=x$ and $\gamma(1)=x_0$ as in the definition of the separation property. We show that

$$\operatorname{diam}(\gamma([0,t]))\leqslant C\operatorname{dist}(\gamma(t),\partial\Omega),\quad\forall\, t\in(0,1)\tag{4.5}$$

for some constant C independent of x and t. This condition guarantees that γ can be modified to obtain a John curve for x; see [20, pp. 385–386] and [22, pp. 7–8].

To prove (4.5), it suffices to show that one has

$$\gamma([0,t]) \subset B\left(\gamma(t), C\operatorname{dist}(\gamma(t),\partial\Omega)\right), \quad \forall\, t\in(0,1) \tag{4.6}$$

for some constant C independent of x and t.

Given any $t\in(0,1)$, write

$$B_{\gamma(t)} := B\left(\gamma(t), C_S\operatorname{dist}\left(\gamma(t),\partial\Omega\right)\right),$$

where C_S is the same constant as in the definition of the separation property. Below we may assume that $\gamma\left([0,t]\right)\not\subset B_{\gamma(t)}$; otherwise (4.6) holds with $C:=C_S$. Let

$$B_0 := B\left(x_0, \frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)\right).$$

If $B_{\gamma(t)}\cap B_0\neq\emptyset$, then take $z\in B_{\gamma(t)}\cap B_0$. Noticing that $\partial B_{\gamma(t)}\cap\partial\Omega$ is not empty and hence it includes some point w, we have

$$\operatorname{diam}(B_{\gamma(t)}) \geqslant |z-w| \geqslant \operatorname{dist}(B_0,\partial\Omega) = \frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$$

and hence

$$\frac{4C_S\operatorname{dist}(\gamma(t),\partial\Omega)}{\operatorname{dist}(x_0,\partial\Omega)} \geqslant 1.$$

Therefore,

$$\gamma\left([0,t]\right)\subset\Omega\subset B\left(\gamma(t),\operatorname{diam}(\Omega)\right)\subset B\left(\gamma(t), \frac{4\operatorname{diam}(\Omega)}{\operatorname{dist}(x_0,\partial\Omega)}C_S\operatorname{dist}(\gamma(t),\partial\Omega)\right),$$

which gives (4.6) by taking $C:=\frac{4\operatorname{diam}(\Omega)}{\operatorname{dist}(x_0,\partial\Omega)}C_S$.

If $B_{\gamma(t)}\cap B_0=\emptyset$, then denote by U_0 the connected component of $\Omega\setminus\partial B_{\gamma(t)}$ that includes x_0. It follows that $B_0\subset U_0$. Let T be any connected component of the set $\gamma([0,t])\setminus B_{\gamma(t)}$. According to the definition of the separation property, T is contained in some connected component of $\Omega\setminus B_{\gamma(t)}$ different from U_0, that is, $T\cap U_0=\emptyset$. Therefore, $T\cap B_0=\emptyset$. By Lemma 4.1, we find that

$$\operatorname{diam}(T)\leqslant C'C_S\operatorname{dist}(\gamma(t),\partial\Omega),$$

where $C':=(C_0,n,p,s,\theta,\Omega,B_0)$ denotes the positive constant in Lemma 4.1. Let x_T be any point satisfying $x_T\in T\cap\partial B_{\gamma(t)}$. Then

$$T\subset B\left(x_T, 2\operatorname{diam}(T)\right)\subset B\left(x_T, 2C'C_S\operatorname{dist}(\gamma(t),\partial\Omega)\right)\subset B\left(\gamma(t), C\operatorname{dist}(\gamma(t),\partial\Omega)\right),$$

where now $C:=C_S+2C'C_S$. As a result, we find

$$\gamma\left([0,t]\right)\subset B\left(\gamma(t), C\operatorname{dist}(\gamma(t),\partial\Omega)\right)$$

as desired. This finishes the proof of Theorem 1.1(ii) and hence Theorem 1.1. □

Remark 4.2. As we have seen in Remark 3.5, under Hölder's inequality, there are limited results to infer that Ω supports the $(\widetilde{p},\widetilde{s},\widetilde{q},\widetilde{\theta})$-GNS inequalities with admissible $(\widetilde{p},\widetilde{s},\widetilde{q},\widetilde{\theta})$ from supporting (p,s,q,θ)-GNS inequalities with admissible (p,s,q,θ). However, after using the separation property, the situation has changed significantly. Under the separation property, if a domain Ω supports the (p,s,q,θ)-GNS inequality for some admissible quadruple (p,s,q,θ), then by Theorem 1.1(ii), we know that Ω is a John domain. As a result, by Theorem 1.1(i) we know that Ω supports the $(\widetilde{p},\widetilde{s},\widetilde{q},\widetilde{\theta})$-GNS inequality for all admissible quadruples $(\widetilde{p},\widetilde{s},\widetilde{q},\widetilde{\theta})$.

Below are some comments on the additional separation property assumed in Theorem 1.1(ii). Notice that by definitions a John domain always enjoys the separation property, but the converse is necessarily not true as witted by the planar cusp domain

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \colon -x_1^2 < x_2 < x_1^2,\ 0 < x_1 < 1\right\},$$

which satisfies the separation property but is not a John domain.

Remark 4.3. It is a natural question to classify domains, which have or do not have the separation property. It has been shown in [6, Lemma 3.3] that any domain which is quasiconformally equivalent to a uniform domain must have the separation property. In particular, each simply connected planar domain has the separation property. Moreover, any finitely connected planar domain has the separation property; see [13, Corollary 6.2] for a proof. However, an infinitely connected domain may have or not have the separation property. For instance, the domain

$$B(0,1) \setminus \bigcup_{k\in\mathbb{Z}_+} \{(1-2^{-k}, 0)\} \subset \mathbb{R}^2$$

is a John domain, hence it has the separation property. In contrast, following [6] (see also [13, Example 1.7]), we set

$$\Omega_* := B(0,1) \setminus \bigcup_{k\in\mathbb{Z}_+} \{x_{k,j}\}_{j=1}^{k!} \subset \mathbb{R}^2,$$

where for each $k \in \mathbb{Z}_+$, $\{x_{k,j}\}_{j=1}^{k!}$ are equally spaced on the circle $\partial B(0, 1-2^{-k}) \subset \mathbb{R}^2$ and $k!$ stands for the factorial of k. Obviously, Ω_* is an infinitely connected planar domain. However, Ω_* is not a John domain as indicated by [6] and also [13, Example 1.7]. From the argument in [13, Example 1.7] with some modifications, one further see that Ω_* does not have the separation property. Here we omit the details.

Remark 4.4. There exist domains which support the (p, s, q, θ)-GNS inequality, but they are neither John domains nor enjoying the separation property. Indeed, the domain Ω_* in Remark 4.3 plays such a role.

Since $E := \bigcup_{k\in\mathbb{Z}_+}\{x_{k,j}\}_{j=1}^{k!}$ is a relatively closed subset of $B := B(0,1)$ with $\mathcal{H}^{n-1}(E) = 0$, where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure, by [21, Theorem 1.1.18] (also see [18, Exercise 11.10]) we find $\dot{W}^{1,p}(\Omega_*) = \dot{W}^{1,p}(B)$ for all $p \in [1, \infty]$. Recall the ball B supports the (p, s, q, θ)-GNS inequality for all admissible quadruples. One then gets

$$\begin{aligned}\left\|f - \operatorname*{ave}_{\Omega_*} f\right\|_{L^q(\Omega_*)} &= \left\|f - \operatorname*{ave}_{B} f\right\|_{L^q(B)} \\ &\lesssim \|\nabla f\|_{L^p(B)}^{\theta} \left\|f - \operatorname*{ave}_{B} f\right\|_{L^s(B)}^{1-\theta} \\ &= \|\nabla f\|_{L^p(\Omega_*)}^{\theta} \left\|f - \operatorname*{ave}_{\Omega_*} f\right\|_{L^s(\Omega_*)}^{1-\theta}\end{aligned}$$

for all suitable f. That is, Ω_* supports the (p, s, q, θ)-GNS inequality for all admissible quadruples.

5. Appendix: GNS inequalities in Sobolev extension domains

Let (p, s, q, θ) be admissible and $\Omega \subset \mathbb{R}^n$ a bounded domain. Assume that Ω has the $\dot{W}^{1,p} \cap L^s$-extension property in the sense that, for any $f \in \dot{W}^{1,p}(\Omega) \cap L^s(\Omega)$

with $\int_\Omega f(x)\,\mathrm{d}x = 0$, there exist $\tilde{f} \in \dot{W}^{1,p}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ and a positive constant C, independent of f and $\tilde{f}$, such that

$$(5.1)\quad \tilde{f}\big|_\Omega = f \text{ a.e.}, \quad \left\|\nabla\tilde{f}\right\|_{L^p(\mathbb{R}^n)} \leqslant C\,\|\nabla f\|_{L^p(\Omega)}, \quad \text{and} \quad \left\|\tilde{f}\right\|_{L^s(\mathbb{R}^n)} \leqslant C\,\|f\|_{L^s(\Omega)}.$$

Then (1.3) follows from (1.2). Indeed, for any $g \in \dot{W}^{1,p}(\Omega)\cap L^s(\Omega)$, let $f = g - \mathrm{ave}_\Omega\, g$. By the above assumption, there exists a function $\tilde{f}$ satisfying (5.1). Obviously,

$$\left\|g - \operatorname*{ave}_{\Omega} g\right\|_{L^q(\Omega)} = \|f\|_{L^q(\Omega)} = \left\|\tilde{f}\right\|_{L^q(\Omega)} \leqslant \left\|\tilde{f}\right\|_{L^q(\mathbb{R}^n)}.$$

Applying (1.2) to $\tilde{f}$, we obtain

$$\left\|\tilde{f}\right\|_{L^q(\mathbb{R}^n)} \lesssim \left\|\nabla\tilde{f}\right\|^{\theta}_{L^p(\mathbb{R}^n)} \left\|\tilde{f}\right\|^{1-\theta}_{L^s(\mathbb{R}^n)}.$$

By (5.1), one has

$$\left\|\nabla\tilde{f}\right\|^{\theta}_{L^p(\mathbb{R}^n)} \left\|\tilde{f}\right\|^{1-\theta}_{L^s(\mathbb{R}^n)} \lesssim \|\nabla f\|^{\theta}_{L^p(\Omega)}\, \|f\|^{1-\theta}_{L^s(\Omega)}.$$

Combining these we obtain

$$\left\|g - \operatorname*{ave}_{\Omega} g\right\|_{L^q(\Omega)} \lesssim \|\nabla g\|^{\theta}_{L^p(\Omega)} \left\|g - \operatorname*{ave}_{\Omega} g\right\|^{1-\theta}_{L^s(\Omega)}$$

as desired.

Next, a bounded (ε,δ)-uniform domain has the above $\dot{W}^{1,p}\cap L^s$-extension property, which was essentially given in [15, 12]. Recall that a domain Ω is called an (ε,δ)-*uniform domain* if, for some $\varepsilon,\delta \in (0,\infty)$ and any pair of points, $x, y \in \Omega$ with $|x-y| < \delta$, there exists a rectifiable arc $\gamma \subset \Omega$ joining x to y and satisfying

$$l(\gamma) \leqslant \frac{1}{\varepsilon}|x-y|$$

and

$$\mathrm{dist}(z,\partial\Omega) \geqslant \frac{\varepsilon|z-x||z-y|}{|x-y|}, \quad \forall\, z \in \gamma,$$

where $l(\gamma)$ stands for the arclength of γ. Given any $f \in \dot{W}^{1,p}(\Omega)\cap L^s(\Omega)$, we sketch the construction of its extension $\tilde{f}$ by [15] with a slight modification as below (see also [12, 17]). Denote by $\mathcal{W}_1 := \{Q_i\}_i$ the Whitney decomposition of Ω and $\mathcal{W}_2 := \{Q_j\}_j$ the Whitney decomposition of $(\overline{\Omega})^{\complement}$. Let

$$\mathcal{W}_3 := \left\{Q \in \mathcal{W}_2 : l_Q \leqslant \frac{\varepsilon\delta}{16n}\right\}.$$

For any cube $Q \in \mathcal{W}_3$, by [15] there is a reflection cube $Q^* \in \mathcal{W}_1$ such that

$$1 \leqslant \frac{l_{Q^*}}{l_Q} \leqslant 4 \text{ and } \mathrm{dist}(Q,Q^*) \leqslant Cl_Q,$$

where C is a positive constant depending on n and ε. For any $Q \in \mathcal{W}_2 \setminus \mathcal{W}_3$, we write $Q^* = \Omega$. Denote by $\{\phi_Q\}_{Q\in\mathcal{W}_2}$ a partition of unity associated to $\mathcal{W}_2$ such that

$\operatorname{supp}\phi_Q \subset \frac{17}{16}Q$. Define

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega, \\ \lim_{r\to 0} \fint_{B(x,r)\cap\Omega} f(y)\,\mathrm{d}y & \text{if } x \in \partial\Omega, \\ \sum_{Q\in\mathcal{W}_2} \left[\fint_{Q^*} f(y)\,\mathrm{d}y\right]\phi_Q & \text{if } x \in (\overline{\Omega})^{\complement}. \end{cases}$$

Following [15] and [12, 17], one has (5.1). Here we omit the details.

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