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Tent spaces and solutions of Weinstein type equations with $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ boundary values

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Abstract. Let $\{P_t^{[\lambda]}\}_{t>0}$ be the Poisson semigroup associated with the Bessel operator Δ_λ on $\mathbb{R}_+ := (0, \infty)$, where $\lambda > 0$ and

$$\Delta_\lambda := -x^{-2\lambda} \frac{d}{dx} x^{2\lambda} \frac{d}{dx}.$$

In this paper, the authors show that a function $u(y, t)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, has the form $u(y, t) = P_t^{[\lambda]} f(y)$ with $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$, where $dm_\lambda(x) := x^{2\lambda}\, dx$, if and only if u satisfies the Weinstein type equation

$$\mathbb{L}_\lambda u(x, t) := \frac{\partial^2 u(x, t)}{\partial t^2} - \Delta_\lambda u(x, t) = 0, \quad (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

a Carleson type condition and certain limiting conditions. For this purpose, the authors first introduce the tent spaces T_2^p with $p \in [1, \infty]$ and $T_{2,C}^\infty$ in the Bessel setting and then show that $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ has a connection with $T_{2,C}^\infty$ via $\{P_t^{[\lambda]}\}_{t>0}$. In addition, the authors obtain some boundedness results on the operator π_λ from tent spaces to some "ordinary" function spaces.

Teltta-avaruudet ja Weinsteinin-tyyppisten $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$-reuna-arvoyhtälöiden ratkaisut

Tiivistelmä. Olkoon $\{P_t^{[\lambda]}\}_{t>0}$ puolisuoran $\mathbb{R}_+ := (0, \infty)$ Besselin operaattoriin

$$\Delta_\lambda := -x^{-2\lambda} \frac{d}{dx} x^{2\lambda} \frac{d}{dx}$$

liittyvä Poissonin puoliryhmä, missä $\lambda > 0$. Tässä työssä osoitetaan, että alueessa $\mathbb{R}_+ \times \mathbb{R}_+$ määritelty funktio $u(y, t)$ voidaan esittää muodossa $u(y, t) = P_t^{[\lambda]} f(y)$, missä $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ ja $dm_\lambda(x) := x^{2\lambda}\, dx$, jos ja vain jos u toteuttaa Weinsteinin-tyyppisen yhtälön

$$\mathbb{L}_\lambda u(x, t) := \frac{\partial^2 u(x, t)}{\partial t^2} - \Delta_\lambda u(x, t) = 0, \quad (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

sekä Carlesonin-tyyppisen ehdon ja tietyt raja-arvo-ominaisuudet. Tätä varten esitellään aluksi Besselin asetelmaan sovitetut teltta-avaruudet $T_{2,C}^\infty$ ja T_2^p, missä $p \in [1, \infty]$, sekä osoitetaan avaruuksien $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ ja $T_{2,C}^\infty$ välinen yhteys puoliryhmän $\{P_t^{[\lambda]}\}_{t>0}$ kautta. Lisäksi saadaan tuloksia, jotka koskevat operaattorin π_λ rajallisuutta teltta-avaruuksista eräisiin "tavallisiin" funktioavaruuksiin.

1. Introduction

The problem of harmonic extension of a function in the space $\mathrm{BMO}(\mathbb{R}^n)$ was first studied by Fabes, Johnson and Neri in [16], based on the work of Fefferman and Stein [17]. Fabes, Johnson and Neri [16] showed that a function u on $\mathbb{R}_+^{n+1}$ can be

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represented as $u(x,t) := P_t(f)(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, for some $f \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if u is the solution to the following equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \triangle u(x,t) = 0, \quad (x,t) \in \mathbb{R}^{n+1}_+,$$

where $\triangle := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^n$, and satisfies the Carleson condition

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^n} \int_0^r \int_{B(x,r)} t\left|\nabla u(x,t)\right|^2 dx\, dt < \infty. \tag{1.1}$$

Here $\nabla := (\frac{\partial}{\partial_{x_1}}, \frac{\partial}{\partial_{x_2}}, \dots, \frac{\partial}{\partial_{x_n}}, \frac{\partial}{\partial_t})$ and $\{P_t\}_{t>0}$ is the classical Poisson semigroup. In 2014, Duong et al. [14] characterized harmonic functions whose traces belong to $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ in terms of a Carleson type condition associated with the Schrödinger operator $\mathcal{L} := -\Delta + V$, where the non-negative potential V belongs to the reverse Hölder class $RH_q(\mathbb{R}^n)$ for some $q > n$, which was further extended by Jiang and Li [22] to general metric measure spaces with improved index. Recently, Song and Wu [27] studied the Dirichlet problem for the Schrödinger equation with boundary value in $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, which is defined as the closure in the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ norm of $C_c^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support, by using the theory of classical tent spaces. For further research on this topic and applications of tent spaces, see, for example, [7, 6, 15, 20, 2, 28, 25, 21, 11, 22, 24, 23] and the references therein.

In this paper, we consider the following Weinstein type equation

$$\mathbb{L}_\lambda u(x,t) := \frac{\partial^2 u(x,t)}{\partial t^2} - \Delta_\lambda u(x,t) = 0, \quad (x,t) \in (0,\infty)\times(0,\infty), \tag{1.2}$$

where $u \in C^2((0,\infty)\times(0,\infty))$, and

$$\Delta_\lambda := -x^{-2\lambda}\frac{d}{dx}x^{2\lambda}\frac{d}{dx} = -\frac{d^2}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx}, \quad \lambda > 0,$$

is the Bessel operator on $\mathbb{R}_+ := (0,\infty)$. The operator Δ_λ has been studied by many mathematicians; see for example, [30, 31, 26, 4, 3, 2, 33, 12, 1] and the references therein.

In a previous paper, the authors [18] established a characterization of solutions of Weinstein type equations (1.2) with boundary value in $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ studied in [33]; that is, for $\lambda \geq 1/2$, a solution u belongs to $\mathrm{HMO}_\lambda(\mathbb{R}_+\times\mathbb{R}_+)$, if and only if, there exists $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ such that u can be represented as $u(x,t) = P_t^{[\lambda]}(f)(x)$, $(x,t) \in \mathbb{R}_+\times\mathbb{R}_+$; see Lemma 3.1 below. In this paper, we will further study solutions of (1.2) with boundary value in $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ introduced in [12], which is a subspace of $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$.

In the following, we recall some necessary notation and notions. We say that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, dm_\lambda)$ belongs to the space $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$, if

$$\|f\|_{*,\lambda} := \sup_{I\subset\mathbb{R}_+} \frac{1}{m_\lambda(I)} \int_I |f(y) - f_{I,\lambda}| y^{2\lambda}\, dy < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}_+$, and

$$f_{I,\lambda} := \frac{1}{m_\lambda(I)} \int_I f(y) y^{2\lambda}\, dy.$$

The space $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ is defined by the $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$-closure of $C_c^\infty(\mathbb{R}_+)$, the set of $C^\infty(\mathbb{R}_+)$ functions on $\mathbb{R}_+$ with compact support.

For a function $f \in L^p(\mathbb{R}_+, dm_\lambda)$ with $p \in [1, \infty]$, the Poisson semigroup $\{P_t^{[\lambda]}\}_{t>0}$ associated with the operator Δ_λ is defined by

$$P_t^{[\lambda]} f(x) := \int_0^\infty P_t^{[\lambda]}(x, y) f(y) y^{2\lambda}\, dy,$$

where

$$P_t^{[\lambda]}(x, y) := \frac{2\lambda t}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy\cos\theta)^{\lambda+1}}\, d\theta, \quad t, x, y \in \mathbb{R}_+;$$

see [3, 26].

We define $\nabla_{x,t} := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$ and denote by $\mathrm{HMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ the class of all $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ functions $u(x, t)$ which are the solutions of (1.2) and satisfy the following Carleson type condition

$$\|u\|^2_{\mathrm{HMO}_\lambda} := \sup_{I \subset \mathbb{R}_+} \frac{1}{m_\lambda(I)} \int_0^{|I|} \int_I t \left|\nabla_{y,t} u(y, t)\right|^2 dm_\lambda(y)\, dt < \infty,$$

where and in the sequel, I under the supremum always represents an interval on $\mathbb{R}_+$.

In order to state our result in this paper, we give the following definition.

Definition 1.1. A function u belongs to $\mathrm{HCMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ if $u \in \mathrm{HMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$, and satisfies the following limiting conditions:

$$\lim_{a \to 0^+} \sup_{m_\lambda(I) \le a} \left(\frac{1}{m_\lambda(I)} \int_0^{|I|} \int_I t \left|\nabla_{y,t} u(y, t)\right|^2 dm_\lambda(y)\, dt \right)^{\frac{1}{2}} = 0; \tag{1.3}$$

$$\lim_{a \to \infty} \sup_{m_\lambda(I) \ge a} \left(\frac{1}{m_\lambda(I)} \int_0^{|I|} \int_I t \left|\nabla_{y,t} u(y, t)\right|^2 dm_\lambda(y)\, dt \right)^{\frac{1}{2}} = 0; \tag{1.4}$$

and

$$\lim_{R \to \infty} \sup_{I \subset [R, \infty)} \left(\frac{1}{m_\lambda(I)} \int_0^{|I|} \int_I t \left|\nabla_{y,t} u(y, t)\right|^2 dm_\lambda(y)\, dt \right)^{\frac{1}{2}} = 0. \tag{1.5}$$

We endow $\mathrm{HCMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ with the norm of $\mathrm{HMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$.

We state our main result as follows.

Theorem 1.2. *Let $\lambda \ge \frac{1}{2}$ and u be a function on $\mathbb{R}_+ \times \mathbb{R}_+$. Then the following statements are equivalent:*

(c_i) *There exists some $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ such that $u(x, t) = P_t^{[\lambda]}(f)(x)$, $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$;*

(c_{ii}) *$u \in \mathrm{HCMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$.*

Moreover, the quantities $\|f\|_{*,\lambda}$ and $\|u\|_{\mathrm{HMO}_\lambda}$ are equivalent.

Remark 1.3. We mention that the assumption $\lambda \ge \frac{1}{2}$ was made in the proof of [18, Theorem 1.2], which is useful in the proof of Theorem 1.2. It is unknown if the conclusion of Theorem 1.2 holds for $\lambda \in (0, \frac{1}{2})$; see also Remark 4.6 in [18].

In order to show the implication $(c_{ii}) \Longrightarrow (c_i)$ of Theorem 1.2, we establish a characterization of the space $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ in terms of tent spaces in Section 2. More precisely, in Section 2, we introduce the tent spaces T_2^p with $p \in [1, \infty]$ in the Bessel setting, and provide a characterization of the space $T^\infty_{2,C}$ via the limiting conditions, where $T^\infty_{2,C}$ is a subspace of T^∞_2; see Proposition 2.1 in Subsection 2.1. In Subsection 2.2, we apply Proposition 2.1 to establish a connection between $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ and $T^\infty_{2,C}$ via $\{P^{[\lambda]}_t\}_{t>0}$; see Theorem 2.5. Section 3 is devoted to the proof of Theorem 1.2. In Section 4, we introduce the operator π_λ and show the connection between tent spaces and some classical function spaces on $\mathbb{R}_+$ via the operator π_λ.

We now make some preliminaries. In what follows, for every $x, r \in \mathbb{R}_+$, define

$$I(x,r) := (x-r, x+r) \cap \mathbb{R}_+.$$

Observe that for $x, r \in (0,\infty)$, $x < r$,

$$I(x,r) = (0, x+r) = I\left(\frac{x+r}{2}, \frac{x+r}{2}\right).$$

Thus, in the sequel, for a given interval $I(x,r)$, without any specific condition, we may always assume that $x \geq r$. For $k \in \mathbb{R}_+$ and any interval $I := I(x,r)$ for some $x, r \in \mathbb{R}_+$, $kI := I(x, kr)$. It is easy to see that for every interval $I(x,r)$, $x, r \in \mathbb{R}_+$,

$$m_\lambda(I(x,r)) \sim \begin{cases} x^{2\lambda} r, & x > r; \\ r^{2\lambda+1}, & x \leq r. \end{cases} \tag{1.6}$$

Moreover, it is known that for every $I \subset \mathbb{R}_+$,

$$\min\{2, 2^{2\lambda}\} m_\lambda(I) \leq m_\lambda(2I) \leq 2^{2\lambda+1} m_\lambda(I); \tag{1.7}$$

see [12, Proposition 2.1].

Throughout the paper, we use the notation $f \lesssim g$ and $f \sim g$ which mean that there exists $C > 0$ such that $f \leq Cg$ and $f/C \leq g \leq Cf$, respectively. The letter C denotes a positive constant that can change from one line to the next.

2. A connection of $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ and tent spaces

In this section, we introduce tent spaces to study the space $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$. In Subsection 2.1, we introduce the spaces T^p_2 $(1 \leq p \leq \infty)$, $T^\infty_{2,0}$ and $T^\infty_{2,C}$, and provide a characterization of $T^\infty_{2,C}$. By using the theory of tent spaces, we further obtain a Carleson type characterization of $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ in Subsection 2.2.

2.1. Preliminaries for tent spaces. To begin with, we denote by $\Gamma_+(x)$ the cone whose vertex is $x \in \mathbb{R}_+$, i.e.,

$$\Gamma_+(x) := \big\{(y,t) \in \mathbb{R}_+ \times \mathbb{R}_+ : |x-y| < t\big\}.$$

For any closed set $E \subset \mathbb{R}_+$, $\mathcal{R}(E)$ means the union of the cones with vertices in E, i.e., $\mathcal{R}(E) := \bigcup_{x\in E} \Gamma_+(x)$. Let O be the open set in $\mathbb{R}_+$ which is the complement of E, $O := E^c$. Then the tent over O, denoted by $\widehat{O}$, is given as $\widehat{O} := (\mathcal{R}(E))^c$. Hence, for any open interval $I \subset \mathbb{R}_+$, we see that

$$\widehat{I} = \big\{(y,t) \in \mathbb{R}_+ \times \mathbb{R}_+ : I(y,t) \subset I\big\}.$$

For a given measurable function f on $\mathbb{R}_+ \times \mathbb{R}_+$, we define $\Psi(f)$ and $\Phi(f)$ as follows: for any $x \in \mathbb{R}_+$,

$$\Psi(f)(x) := \left(\iint_{\Gamma_+(x)} |f(y,t)|^2 \frac{dm_\lambda(y)}{m_\lambda(I(y,t))} \frac{dt}{t}\right)^{\frac{1}{2}},$$

and

$$\Phi(f)(x) := \sup_{I \ni x} \left(\frac{1}{m_\lambda(I)} \iint_{\widehat{I}} |f(y,t)|^2 \frac{dm_\lambda(y)\, dt}{t}\right)^{\frac{1}{2}}.$$

By the fact that $\widehat{I} \subset I \times (0, |I|) \subset \widehat{3I}$ for any interval $I \subset \mathbb{R}_+$ and (1.7), it is obvious that for any $x \in \mathbb{R}_+$,

$$\Phi(f)(x) \sim \sup_{I \ni x} \left(\frac{1}{m_\lambda(I)} \int_0^{|I|} \int_I |f(y,t)|^2 \frac{dm_\lambda(y)\, dt}{t} \right)^{\frac{1}{2}}. \tag{2.1}$$

In the following, we introduce the space T_2^p, $1 \le p \le \infty$. Let $L^0(\mathbb{R}_+ \times \mathbb{R}_+)$ be the set of all measurable functions on $\mathbb{R}_+ \times \mathbb{R}_+$. For $1 \le p < \infty$, we define

$$T_2^p := \left\{ f \in L^0(\mathbb{R}_+ \times \mathbb{R}_+) \colon \Psi(f) \in L^p(\mathbb{R}_+, dm_\lambda) \right\}$$

and endow T_2^p with the norm $\|f\|_{T_2^p} := \|\Psi(f)\|_{L^p(\mathbb{R}_+, dm_\lambda)}$. For $p = \infty$, we define

$$T_2^\infty := \left\{ f \in L^0(\mathbb{R}_+ \times \mathbb{R}_+) \colon \Phi(f) \in L^\infty(\mathbb{R}_+, dm_\lambda) \right\}$$

with the norm $\|f\|_{T_2^\infty} := \|\Phi(f)\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}$.

Let $T_{2,c}^p$, $1 \le p < \infty$, be the subset of all $f \in T_2^p$ with compact support in $\mathbb{R}_+ \times \mathbb{R}_+$ and $T_{2,0}^\infty$ be the subset of all $f \in T_2^\infty$ such that

$$\lim_{a \to 0^+} \sup_{m_\lambda(I) \le a} \left(\frac{1}{m_\lambda(I)} \iint_{\widehat{I}} |f(y,t)|^2 \frac{dm_\lambda(y)\, dt}{t} \right)^{\frac{1}{2}} = 0. \tag{2.2}$$

And we endow $T_{2,0}^\infty$ with the norm of T_2^∞. Then we have the inclusion

$$T_{2,c}^2 \subset T_{2,0}^\infty;$$

see, for example, [25, p. 226] in the setting of spaces of homogeneous type. Based on this fact, we further denote by $T_{2,C}^\infty$, the closure of the set $T_{2,c}^2$ in $T_{2,0}^\infty$ and endow $T_{2,C}^\infty$ with the norm of T_2^∞. Then we have an equivalent characterization of $T_{2,C}^\infty$; see [25, Lemma 3.3] for the proof.

Proposition 2.1. *Let* $f \in T_2^\infty$. *Then* $f \in T_{2,C}^\infty$ *if and only if* f *satisfies* (2.2),

$$\lim_{a \to \infty} \sup_{m_\lambda(I) \ge a} \left(\frac{1}{m_\lambda(I)} \iint_{\widehat{I}} |f(y,t)|^2 \frac{dm_\lambda(y)\, dt}{t} \right)^{\frac{1}{2}} = 0, \tag{2.3}$$

and

$$\lim_{R \to \infty} \sup_{I \subset [R, \infty)} \left(\frac{1}{m_\lambda(I)} \iint_{\widehat{I}} |f(y,t)|^2 \frac{dm_\lambda(y)\, dt}{t} \right)^{\frac{1}{2}} = 0. \tag{2.4}$$

Remark 2.2. We remark that $T_{2,C}^\infty$ is a proper subspace of $T_{2,0}^\infty$. In fact, let

$$f(x,t) := \begin{cases} 1, & (x,t) \in \bigcup_{k=1}^\infty E_k; \\ 0, & \text{otherwise,} \end{cases}$$

where $E_k := [5 \cdot 2^{k-3}, 7 \cdot 2^{k-3}] \times [1,2]$. It is easy to see that $\{E_k\}_{k=1}^\infty$ are pairwise disjoint and for every $k \in \mathbb{N}$,

$$\iint_{E_k} |f(y,t)|^2 \frac{dm_\lambda(y)\, dt}{t} = \frac{(7^{2\lambda+1} - 5^{2\lambda+1}) \ln 2}{2\lambda + 1} 2^{(k-3)(2\lambda+1)}.$$

By this fact, it can be seen that the function $f \in T_{2,0}^\infty$ does not satisfy (2.3) and (2.4) of Proposition 2.1.

2.2. Carleson type characterizations of $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$. We start with the Carleson characterization of $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ established in [18]. Recall that a positive measure μ on $\mathbb{R}_+ \times \mathbb{R}_+$ is an m_λ-Carleson measure, if there exists $C > 0$ such that for every interval $I \subset \mathbb{R}_+$,

$$\mu(I \times (0, |I|)) \leq C m_\lambda(I).$$

In [18, Theorem 1.1], the authors established the following characterization of the space $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ via the m_λ-Carleson measure that a function $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ if and only if $(1 + x^{2\lambda+2})^{-1} f \in L^1(\mathbb{R}_+, dm_\lambda)$, and

$$t\frac{\partial}{\partial t} P_t^{[\lambda]}(f) \in T_2^\infty, \quad t \in \mathbb{R}_+. \tag{2.5}$$

Moreover, $\|f\|_{*,\lambda} \sim \left\| t\frac{\partial}{\partial t} P_t^{[\lambda]}(f) \right\|_{T_2^\infty}$.

We now gather some known pointwise estimates of derivatives of $P_t^{[\lambda]}(x, y)$ as follows; for the proof see, for example, [32, Proposition 2.1 (iii)] or [26, p. 86 (b)].

Lemma 2.3. *There exists a positive constant C such that for any $x, y, t \in (0, \infty)$,*

$$\left|\frac{\partial}{\partial t} P_t^{[\lambda]}(x, y)\right| \leq C \min\left\{\frac{1}{(|y - x|^2 + t^2)^{\lambda+1}}, \frac{1}{(yx)^\lambda (|y - x|^2 + t^2)}\right\}, \tag{2.6}$$

$$\left|\frac{\partial}{\partial x} P_t^{[\lambda]}(x, y)\right| \leq C \min\left\{\frac{t}{(|y - x|^2 + t^2)^{\lambda+\frac{3}{2}}}, \frac{t}{(yx)^\lambda (|y - x|^3 + t^3)}\right\}, \tag{2.7}$$

and

$$\left|\frac{\partial}{\partial y}\frac{\partial}{\partial t} P_t^{[\lambda]}(x, y)\right| \leq C \min\left\{\frac{1}{(|y - x|^2 + t^2)^{\lambda+\frac{3}{2}}}, \frac{1}{(yx)^\lambda (|y - x|^3 + t^3)}\right\}. \tag{2.8}$$

Let $p \in [1, \infty)$ and

$$M_{\lambda,p}(f, I) := \left(\frac{1}{m_\lambda(I)} \int_I |f(y) - f_{I,\lambda}|^p y^{2\lambda}\, dy\right)^{1/p}.$$

The following characterization of $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ is an extension of [12, Theorem 3.1] where the case $p = 1$ was considered. The proof for $p \in (1, \infty)$ follows immediately from the John–Nirenberg inequality for functions in $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ and [12, Theorem 3.1], and is omitted.

Lemma 2.4. *Let $p \in [1, \infty)$ and $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$. Then $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ if and only if f satisfies*

$$\lim_{a\to 0^+} \sup_{m_\lambda(I)\leq a} M_{\lambda,p}(f, I) = \lim_{a\to\infty} \sup_{m_\lambda(I)\geq a} M_{\lambda,p}(f, I) = \lim_{R\to\infty} \sup_{I\subset[R,\infty)} M_{\lambda,p}(f, I) = 0.$$

We state the main result in this subsection as follows.

Theorem 2.5. *Let $\lambda > 0$. Then the following statements are equivalent:*

(b_i) $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$;

(b_{ii}) $(1 + x^{2\lambda+2})^{-1} f \in L^1(\mathbb{R}_+, dm_\lambda)$ *and*

$$t\frac{\partial}{\partial t} P_t^{[\lambda]}(f) \in T_{2,C}^\infty, \quad t \in \mathbb{R}_+. \tag{2.9}$$

Moreover, the quantities $\|f\|_{*,\lambda}$ and $\left\| t\frac{\partial}{\partial t} P_t^{[\lambda]}(f) \right\|_{T_2^\infty}$ are equivalent.

Proof. $(b_i) \Rightarrow (b_{ii})$: Assume that $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$. Then by [18, Theorem 1.1], we obtain that $(1+x^{2\lambda+2})^{-1}f \in L^1(\mathbb{R}_+, dm_\lambda)$, (2.5) and

$$\left\| t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)\right\|_{T_2^\infty} \lesssim \|f\|_{*,\lambda}.$$

Let $I := I(x_0, r_0)$, $x_0, r_0 \in \mathbb{R}_+$. Write

$$f = (f - f_{2I,\lambda})\chi_{2I} + (f - f_{2I,\lambda})\chi_{\mathbb{R}_+\setminus 2I} + f_{2I,\lambda} =: f_1 + f_2 + f_{2I,\lambda}.$$

By

$$\int_0^\infty P_t^{[\lambda]}(x,y)\, dm_\lambda(y) = 1, \quad x, t \in \mathbb{R}_+; \tag{2.10}$$

see [3, p. 208], we deduce $t\frac{\partial}{\partial t}P_t^{[\lambda]}(f_{2I,\lambda})(x) = 0$, $x, t \in \mathbb{R}_+$. Hence, we have

$$\begin{aligned}
&\left(\frac{1}{m_\lambda(I)}\iint_{\widehat{I}}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)(y)\right|^2 \frac{dm_\lambda(y)\,dt}{t}\right)^{\frac12} \\
&\le \left(\frac{1}{m_\lambda(I)}\iint_{\widehat{I}}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f_1)(y)\right|^2 \frac{dm_\lambda(y)\,dt}{t}\right)^{\frac12} \\
&\quad + \left(\frac{1}{m_\lambda(I)}\iint_{\widehat{I}}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f_2)(y)\right|^2 \frac{dm_\lambda(y)\,dt}{t}\right)^{\frac12} \\
&=: K(f_1) + K(f_2).
\end{aligned} \tag{2.11}$$

For $K(f_1)$, by the boundedness of the Littlewood–Paley g-function on $L^2(\mathbb{R}_+, dm_\lambda)$ (see [18, Lemma 3.1] or [29, 5]), we get

$$\begin{aligned}
(K(f_1))^2 &\le \frac{1}{m_\lambda(I)}\int_I\int_0^\infty \left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f_1)(y)\right|^2 \frac{dt}{t}\, dm_\lambda(y) \\
&\lesssim \frac{1}{m_\lambda(2I)}\int_{2I} |f(y) - f_{2I,\lambda}|^2\, dm_\lambda(y) = \big(M_{\lambda,2}(f, 2I)\big)^2.
\end{aligned}$$

Then by Lemma 2.4 with $p = 2$, we have

$$\lim_{a\to 0}\sup_{m_\lambda(I)\le a} K(f_1) = \lim_{a\to\infty}\sup_{m_\lambda(I)\ge a} K(f_1) = \lim_{R\to\infty}\sup_{I\subset[R,\infty)} K(f_1) = 0. \tag{2.12}$$

As for $K(f_2)$, (2.6) together with (1.6) implies that for any $x, y, t \in \mathbb{R}_+$,

$$\left|\frac{\partial}{\partial t}P_t^{[\lambda]}(x,y)\right| \lesssim \frac{1}{m_\lambda(I(x, |y-x|+t))}\frac{1}{|y-x|+t}. \tag{2.13}$$

Moreover, for $x \in I$ and $y \in \mathbb{R}_+\setminus 2I$, we have

$$|y - x_0| \sim |y - x| \quad \text{and} \quad m_\lambda(I(x, |y-x|)) \sim m_\lambda(I(x_0, |y-x_0|)).$$

Hence, by (2.13), we see that $x \in I$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f_2)(x)\right| &\le \int_{\mathbb{R}_+\setminus 2I}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(x,y)f_2(y)\right| dm_\lambda(y)\\ &\lesssim \sum_{k=1}^\infty \int_{2^{k+1}I\setminus 2^kI}\frac{1}{m_\lambda(I(x_0,|y-x_0|))}\frac{t}{|y-x_0|+t}|f(y)-f_{2I,\lambda}|\,dm_\lambda(y)\\ &\lesssim \frac{t}{|I|}\sum_{k=1}^\infty \frac{1}{2^k}\frac{1}{m_\lambda(2^kI)}\int_{2^{k+1}I\setminus 2^kI}|f(y)-f_{2I,\lambda}|\,dm_\lambda(y) =: \frac{t}{|I|}H.\end{aligned}$$

By the above estimate, we have

$$\begin{aligned}(K(f_2))^2 &\lesssim \frac{1}{m_\lambda(I)}\int_0^{|I|}\int_I\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f_2)(x)\right|^2\frac{dm_\lambda(x)\,dt}{t}\\ &\lesssim \frac{1}{m_\lambda(I)}\int_0^{|I|}\int_I\left|\frac{t}{|I|}H\right|^2\frac{dm_\lambda(x)\,dt}{t}\lesssim H^2.\end{aligned}$$

Hence, we have for $N_0 \in \mathbb{N}$,

$$\begin{aligned}K(f_2) &\le \sum_{k=1}^\infty \frac{1}{2^k}\frac{1}{m_\lambda(2^kI)}\int_{2^{k+1}I}|f(y)-f_{2I,\lambda}|\,dm_\lambda(y)\\ &\lesssim \left(\sum_{k=1}^{N_0}+\sum_{k=N_0}^\infty\right)\frac{1}{2^k}\frac{1}{m_\lambda(2^{k+1}I)}\int_{2^{k+1}I}|f(y)-f_{2I,\lambda}|\,dm_\lambda(y)\\ &=: II_1 + II_2.\end{aligned}$$

For II_2, using the fact that for any $k \in \mathbb{N}$,

$$|f_{2^{k+1}I,\lambda} - f_{2I}| \lesssim k\|f\|_{*,\lambda},$$

we have

$$II_2 \lesssim \sum_{k=N_0}^\infty \frac{k}{2^k}\|f\|_{*,\lambda} \lesssim \frac{\|f\|_{*,\lambda}}{2^{N_0/2}}.$$

Note that if N_0 large enough, then we have that II_2 is sufficiently small.

Regarding II_1, it is obvious that for every $k \in \{1, 2, \dots N_0\}$,

$$|f_{2^{k+1}I,\lambda} - f_{2^kI,\lambda}| \le \frac{2^{2\lambda+1}}{m_\lambda(2^{k+1}I)}\int_{2^{k+1}I}|f(y)-f_{2^{k+1}I,\lambda}|\,dm_\lambda(y).$$

Then

$$\begin{aligned}&\frac{1}{m_\lambda(2^{k+1}I)}\int_{2^{k+1}I}|f(y)-f_{2I,\lambda}|dm_\lambda(y)\\ &\le \frac{1}{m_\lambda(2^{k+1}I)}\int_{2^{k+1}I}|f(y)-f_{2^{k+1}I,\lambda}|\,dm_\lambda(y)\\ &\quad + \sum_{j=0}^k \frac{2^{2\lambda+1}}{m_\lambda(2^{j+1}I)}\int_{2^{j+1}I}|f(y)-f_{2^{j+1}I,\lambda}|\,dm_\lambda(y)\\ &\lesssim \sum_{j=0}^k \frac{1}{m_\lambda(2^{j+1}I)}\int_{2^{j+1}I}|f(y)-f_{2^{j+1}I,\lambda}|\,dm_\lambda(y).\end{aligned}$$

From this fact, we see

$$II_1 \lesssim \sum_{k=0}^{N_0} \frac{N_0 - k + 2}{2^k} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y) \tag{2.14}$$

$$\lesssim N_0 \sum_{k=0}^{N_0} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y).$$

On the other hand, for every $k \in \{0, 1, 2, \dots N_0\}$, we have

$$\begin{aligned}
&\sup_{m_\lambda(I)\le a} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y)\\
&\le \sup_{m_\lambda(2^{k+1}I)\le 2^{(k+1)(2\lambda+1)}a} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y)\\
&\le \sup_{m_\lambda(2^{k+1}I)\le 2^{(N_0+1)(2\lambda+1)}a} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y),
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{m_\lambda(I)\ge a} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y)\\
&\le \sup_{m_\lambda(2^{k+1}I)\ge a} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y).
\end{aligned}$$

Then by Lemma 2.4 and (2.14), we see

$$\lim_{a\to 0} \sup_{m_\lambda(I)\le a} II_1 = \lim_{a\to\infty} \sup_{m_\lambda(I)\ge a} II_1 = 0. \tag{2.15}$$

Moreover, we claim that for every $k \in \{0, 1, 2, \dots N_0\}$,

$$\lim_{R\to\infty} \sup_{I\subset[R,\infty)} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}|\, dm_\lambda(y) = 0.$$

In fact, fix $\epsilon > 0$. From Lemma 2.4, there exists $b > 0$ such that

$$\sup_{m_\lambda(I)\ge b} \frac{1}{m_\lambda(I)} \int_I |f(y) - f_{I,\lambda}| y^{2\lambda}\, dy < \epsilon. \tag{2.16}$$

Again, by Lemma 2.4, we choose $M > 0$ such that $m_\lambda((M, 2M)) \ge b$ and

$$\sup_{I\subset[M,\infty)} \frac{1}{m_\lambda(I)} \int_I |f(y) - f_{I,\lambda}| y^{2\lambda}\, dy < \epsilon.$$

This together with (2.16) implies that

$$\sup_{I\subset[2M,\infty)} \frac{1}{m_\lambda(2^{k+1}I)} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I,\lambda}| y^{2\lambda}\, dy < \epsilon.$$

Thus, the claim holds. Hence, by (2.14), we see

$$\lim_{R\to\infty} \sup_{I\subset[R,\infty)} II_1 = 0,$$

which along with (2.15) and II_2 further implies that

$$\lim_{a\to 0} \sup_{m_\lambda(I)\le a} K(f_2) = \lim_{a\to\infty} \sup_{m_\lambda(I)\ge a} K(f_2) = \lim_{R\to\infty} \sup_{I\subset[R,\infty)} K(f_2) = 0. \tag{2.17}$$

Since $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$, we have (2.5). Hence, from (2.17), (2.11), (2.12) and Proposition 2.1, we conclude that for any $t > 0$, $t\frac{\partial}{\partial t}P_t^{[\lambda]}(f) \in T^\infty_{2,C}$.

$(b_{ii}) \Rightarrow (b_i)$: By the condition (b_{ii}) and (2.5), it is easy to deduce that $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$. For any interval $I := I(x_0, r_0)$, $x_0, r_0 \in \mathbb{R}_+$, by a duality argument, we have

$$\begin{aligned}(2.18)\qquad \|f - f_{I,\lambda}\|_{L^1(I,dm_\lambda)} &= \sup_{\|g\|_{L^\infty(I,dm_\lambda)}\le 1}\left|\int_I \big(f(y) - f_{I,\lambda}\big)g(y)\,dm_\lambda(y)\right|\\ &= \sup_{\|g\|_{L^\infty(I,dm_\lambda)}\le 1}\left|\int_I \big(g(y) - g_{I,\lambda}\big)f(y)\,dm_\lambda(y)\right|\\ &= \sup_{\|g\|_{L^\infty(I,dm_\lambda)}\le 1}\left|\int_0^\infty \big(g(y) - g_{I,\lambda}\big)\chi_I(y)f(y)\,dm_\lambda(y)\right|.\end{aligned}$$

Assume that $g \in L^\infty(I, dm_\lambda)$ with $I \subset \mathbb{R}_+$ such that $\|g\|_{L^\infty(I,dm_\lambda)} \le 1$. Let $g_0 := \big(g - g_{I,\lambda}\big)\chi_I$. Then $\operatorname{supp} g_0 \subset I$,

$$(2.19)\qquad \int_0^\infty g_0(y)\,dm_\lambda(y) = 0 \quad\text{and}\quad \|g_0\|_{L^\infty(I,dm_\lambda)} \le 2.$$

From the definition of g_0 and [18, Proposition 3.4], we have

$$(2.20)\quad \frac{1}{4}\int_0^\infty f(y)g_0(y)\,dm_\lambda(y) = \int_0^\infty\int_0^\infty t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)(y)\,t\frac{\partial}{\partial t}P_t^{[\lambda]}(g_0)(y)\frac{dm_\lambda(y)\,dt}{t}.$$

Let

$$F(y,t) := t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)(y) \quad\text{and}\quad G(y,t) := t\frac{\partial}{\partial t}P_t^{[\lambda]}(g_0)(y), \quad y, t \in \mathbb{R}_+.$$

Write

$$\begin{aligned}(2.21)\qquad &\left|\int_0^\infty\int_0^\infty F(y,t)G(y,t)\frac{dm_\lambda(y)\,dt}{t}\right|\\ &\le \left(\iint_{\widehat{2I}} + \sum_{k=1}^\infty \iint_{\widehat{2^{k+1}I}\setminus\widehat{2^kI}}\right)\big|F(y,t)G(y,t)\big|\frac{dm_\lambda(y)\,dt}{t}\\ &=: A_0 + \sum_{k=1}^\infty A_k.\end{aligned}$$

Consider A_0. By $\|g_0\|_{L^\infty(I,dm_\lambda)} \le 2$ and by the boundedness of the Littlewood–Paley g-function on $L^2(\mathbb{R}_+, dm_\lambda)$, we have

$$\iint_{\widehat{2I}} |G(y,t)|^2\frac{dm_\lambda(y)\,dt}{t} \lesssim \|g_0\|^2_{L^2(\mathbb{R}_+,dm_\lambda)} \lesssim m_\lambda(I).$$

By using Hölder's inequality, we see

$$\begin{aligned}(2.22)\qquad A_0 &\le \left(\iint_{\widehat{2I}} |F(y,t)|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}\left(\iint_{\widehat{2I}} |G(y,t)|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}\\ &\lesssim m_\lambda(I)\left(\frac{1}{m_\lambda(2I)}\iint_{\widehat{2I}} |F(y,t)|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}.\end{aligned}$$

For A_k, $k \in \mathbb{N}$, using Hölder's inequality again,

$$A_k \le \left(\iint_{\widehat{2^{k+1}I}\setminus\widehat{2^kI}} |F(y,t)|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}\left(\iint_{\widehat{2^{k+1}I}\setminus\widehat{2^kI}} |G(y,t)|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}.$$

We now estimate

$$E_k := \left(\iint_{\widehat{2^{k+1}I}\backslash\widehat{2^kI}} |G(y,t)|^2 \frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}.$$

By (2.19), (2.8) and the mean value theorem, we have that for $y \in 2^{k+1}I\backslash 2^kI$ and $t \in (0,\infty)$,

$$\begin{aligned}\left|\frac{\partial}{\partial t}P_t^{[\lambda]}(g_0)(y)\right| &= \left|\int_0^\infty \left[\frac{\partial}{\partial t}P_t^{[\lambda]}(y,x) - \frac{\partial}{\partial t}P_t^{[\lambda]}(y,x_0)\right] g_0(x)\,dm_\lambda(x)\right| \\ &\le \int_0^\infty |x-x_0| \left|\frac{\partial}{\partial x}\frac{\partial}{\partial t}P_t^{[\lambda]}(y,x)\right|_{x=\eta} |g_0(x)|\,dm_\lambda(x) \\ &\lesssim \int_I \frac{|g_0(x)|}{m_\lambda(I(x_0,|y-x_0|))}\frac{|x-x_0|}{(|y-x_0|+t)^2}\,dm_\lambda(x) \\ &\lesssim \frac{m_\lambda(I)}{2^k}\frac{1}{m_\lambda(2^kI)}\frac{1}{2^kr_0},\end{aligned}$$

where $\eta := (1-s)x_0 + sx$ for some $s \in (0,1)$. Hence, we see

$$\begin{aligned}E_k &\lesssim \frac{m_\lambda(I)}{2^k}\left(\int_0^{2^{k+1}r_0}\int_{2^{k+1}I} t\left|\frac{1}{m_\lambda(2^kI)}\frac{1}{2^kr_0}\right|^2 dm_\lambda(y)\,dt\right)^{1/2} \\ &\lesssim \frac{m_\lambda(I)}{2^k}\left(\frac{1}{m_\lambda(2^kI)}\right)^{1/2}\left(\frac{1}{2^{2k}r_0^2}\int_0^{2^{k+1}r_0} t\,dt\right)^{1/2} \sim \frac{m_\lambda(I)}{2^k}\left(\frac{1}{m_\lambda(2^kI)}\right)^{1/2},\end{aligned}$$

which further leads to

$$A_k \lesssim \frac{1}{2^k}m_\lambda(I)\left(\frac{1}{m_\lambda(2^{k+1}I)}\iint_{\widehat{2^{k+1}I}} |F(y,t)|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}.$$

This together with (2.21) and (2.22) implies that

$$\begin{aligned}&\left|\int_0^\infty\int_0^\infty F(y,t)G(y,t)\frac{dm_\lambda(y)\,dt}{t}\right| \\ &\lesssim m_\lambda(I)\sum_{k=0}^\infty \frac{1}{2^k}\left(\frac{1}{m_\lambda(2^{k+1}I)}\iint_{\widehat{2^{k+1}I}} |F(y,t)|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}.\end{aligned}$$

Combining (2.18) and (2.20), we conclude

$$\begin{aligned}&\frac{1}{m_\lambda(I)}\int_I |f(y)-f_{I,\lambda}|y^{2\lambda}\,dy \\ &\lesssim \sum_{k=0}^\infty \frac{1}{2^k}\left(\frac{1}{m_\lambda(2^{k+1}I)}\iint_{\widehat{2^{k+1}I}} \left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)(y)\right|^2 \frac{dm_\lambda(y)\,dt}{t}\right)^{1/2},\end{aligned}$$

which implies that $\|f\|_{*,\lambda} \lesssim \left\|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)\right\|_{T_2^\infty}$. Moreover, by using an argument similar to the proof of $(b_i) \Rightarrow (b_{ii})$, and Proposition 2.1, we see that for given f such that

$(1+x^{2\lambda+2})^{-1}f\in L^1(\mathbb{R}_+,dm_\lambda)$ and $t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)\in T_{2,C}^\infty$,

$$\begin{aligned}&\lim_{a\to 0^+}\sup_{m_\lambda(I)\le a}\frac{1}{m_\lambda(I)}\int_I|f(y)-f_{I,\lambda}|y^{2\lambda}\,dy\\&=\lim_{a\to\infty}\sup_{m_\lambda(I)\ge a}\frac{1}{m_\lambda(I)}\int_I|f(y)-f_{I,\lambda}|y^{2\lambda}\,dy\\&=\lim_{R\to\infty}\sup_{I\subset[R,\infty)}\frac{1}{m_\lambda(I)}\int_I|f(y)-f_{I,\lambda}|y^{2\lambda}\,dy=0.\end{aligned}$$

This via Lemma 2.4 implies that $f\in \mathrm{CMO}(\mathbb{R}_+,dm_\lambda)$. Therefore, we complete the proof of Theorem 2.5. □

Remark 2.6. (1) Let $\{W_t^{[\lambda]}\}_{t>0}$ be the heat semigroup associated with Δ_λ defined by setting for all $f\in\bigcup_{1\le p\le\infty}L^p(\mathbb{R}_+,dm_\lambda)$ and $x\in\mathbb{R}_+$,

$$W_t^{[\lambda]}f(x):=\int_0^\infty W_t^{[\lambda]}(x,y)f(y)\,dm_\lambda(y),$$

where

$$W_t^{[\lambda]}(x,y):=\frac{2^{(1-2\lambda)/2}}{\Gamma(\lambda)\sqrt{\pi}}t^{-\lambda-\frac12}\int_0^\pi\exp\left(-\frac{x^2+y^2-2xy\cos\theta}{2t}\right)(\sin\theta)^{2\lambda-1}\,d\theta.$$

We remark the conclusion of Theorem 2.5 holds if (2.9) is replaced by

$$t^2\frac{\partial}{\partial s}W_s^{[\lambda]}(f)\big|_{s=t^2}\in T_{2,C}^\infty.$$

(2) Let $\lambda>0$ and $\mathrm{VMO}(\mathbb{R}_+,dm_\lambda)$ be the subspace of functions $f\in\mathrm{BMO}(\mathbb{R}_+,dm_\lambda)$ satisfying

$$\lim_{a\to0^+}\sup_{m_\lambda(I)\le a}M_{\lambda,p}(f,I)=0.$$

Then for $f\in\mathrm{BMO}(\mathbb{R}_+,dm_\lambda)$, the following statements are equivalent:

(W_i) $f\in\mathrm{VMO}(\mathbb{R}_+,dm_\lambda)$;

(W_{ii}) $(1+x^{2\lambda+2})^{-1}f\in L^1(\mathbb{R}_+,dm_\lambda)$ and

$$t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)\in T_{2,0}^\infty,\quad t\in\mathbb{R}_+;$$

(W_{iii}) $(1+x^{2\lambda+2})^{-1}f\in L^1(\mathbb{R}_+,dm_\lambda)$ and

$$t^2\frac{\partial}{\partial s}W_s^{[\lambda]}(f)\big|_{s=t^2}\in T_{2,0}^\infty,\quad t\in\mathbb{R}_+.$$

Moreover, the quantities $\|f\|_{*,\lambda}$ and $\left\|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)\right\|_{T_2^\infty}$, $\left\|t^2\frac{\partial}{\partial s}W_s^{[\lambda]}(f)\big|_{s=t^2}\right\|_{T_2^\infty}$ are equivalent.

3. The proof of Theorem 1.2

In this section, we provide the proof of Theorem 1.2. Before that, we first recall a characterization of $\mathrm{BMO}(\mathbb{R}_+,dm_\lambda)$ via $\mathrm{HMO}_\lambda(\mathbb{R}_+\times\mathbb{R}_+)$ obtained in [18].

Lemma 3.1. *Let $\lambda\ge\frac12$ and u be a function on $\mathbb{R}_+\times\mathbb{R}_+$. Then the following statements are equivalent:*

(a_i) *There exists some $f\in\mathrm{BMO}(\mathbb{R}_+,dm_\lambda)$ such that $u(x,t)=P_t^{[\lambda]}(f)(x)$, $(x,t)\in\mathbb{R}_+\times\mathbb{R}_+$;*

(a_{ii}) $u\in\mathrm{HMO}_\lambda(\mathbb{R}_+\times\mathbb{R}_+)$.

Moreover, the quantities $\|f\|_{*,\lambda}$ *and* $\|u\|_{\mathrm{HMO}_\lambda}$ *are equivalent.*

Proof of Theorem 1.2. $(c_{ii}) \Longrightarrow (c_i)$. If $u \in \mathrm{HCMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$, then $u \in \mathrm{HMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$. By Lemma 3.1, there exists a function $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ such that $u(x,t) = P_t^{[\lambda]}(f)(x)$, $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$ and

$$\|f\|_{*,\lambda} \lesssim \|u\|_{\mathrm{HMO}_\lambda}.$$

It follows from $u(x,t) = P_t^{[\lambda]}(f)(x) \in \mathrm{HCMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ that

$$t\frac{\partial}{\partial t}P_t^{[\lambda]}(f) \in T^\infty_{2,C}, \quad t \in \mathbb{R}_+.$$

By Theorem 2.5, we have $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$.

$(c_i) \Longrightarrow (c_{ii})$. If $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$, then $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$. From Lemma 3.1, we have that $u(x,t) = P_t^{[\lambda]}(f)(x) \in \mathrm{HMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$ and

$$\|u\|_{\mathrm{HMO}_\lambda} \lesssim \|f\|_{*,\lambda}.$$

From Theorem 2.5, we have

$$t\frac{\partial}{\partial t}P_t^{[\lambda]}(f) \in T^\infty_{2,C}, \quad t \in \mathbb{R}_+.$$

Hence, in order to prove $u \in \mathrm{HCMO}_\lambda(\mathbb{R}_+ \times \mathbb{R}_+)$, it suffices to show that

$$\lim_{a\to 0^+} \sup_{m_\lambda(I)\le a} \left(\frac{1}{m_\lambda(I)}\int_0^{|I|}\int_I t\left|\frac{\partial}{\partial y}P_t^{[\lambda]}(f)(y)\right|^2 dm_\lambda(y)\,dt\right)^{\frac12} = 0;$$

$$\lim_{a\to \infty} \sup_{m_\lambda(I)\ge a} \left(\frac{1}{m_\lambda(I)}\int_0^{|I|}\int_I t\left|\frac{\partial}{\partial y}P_t^{[\lambda]}(f)(y)\right|^2 dm_\lambda(y)\,dt\right)^{\frac12} = 0;$$

and

$$\lim_{R\to \infty} \sup_{I\subset[R,\infty)} \left(\frac{1}{m_\lambda(I)}\int_0^{|I|}\int_I t\left|\frac{\partial}{\partial y}P_t^{[\lambda]}(f)(y)\right|^2 dm_\lambda(y)\,dt\right)^{\frac12} = 0.$$

By (2.7) and an argument similar to the proof of $(b_i) \Rightarrow (b_{ii})$ in Theorem 2.5, we get

$$t\frac{\partial}{\partial x}P_t^{[\lambda]}(f)(x) \in T^\infty_{2,C}, \quad t \in \mathbb{R}_+.$$

Thus, we complete the proof of Theorem 1.2. □

4. Boundedness of the operator π_λ

In this section, we study the close connection between $T_2^p(1 \le p \le \infty)$ and some classical function spaces by considering the operator π_λ, $\lambda > 0$ defined on $T^p_{2,c}\,(1 \le p < \infty)$ by

$$\pi_\lambda f(x) := \int_0^\infty\int_0^\infty \frac{\partial}{\partial t}P_t^{[\lambda]}(x,y)f(y,t)\,dm_\lambda(y)\,dt, \quad x \in \mathbb{R}_+.$$

Such operator was first introduced by Coifman, Meyer and Stein [8] in the study of tent spaces on $\mathbb{R}^{n+1}_+$; see also [13, 10, 28].

We first claim that for any $f \in T^p_{2,c}$ with compact support $K \subset \mathbb{R}_+ \times \mathbb{R}_+$,

$$\left(\iint_K |f(y,t)|^2\,dm_\lambda(y)\,dt\right)^{1/2} \le C_K\|\Psi(f)\|_{L^p(\mathbb{R}_+,dm_\lambda)}, \tag{4.1}$$

where and in the sequel, the constant C_K only depends on K. In fact, there exists $R > 0$ such that for any $(y,t) \in K$, we have $y + t \leq R$. Then by Minkowski's inequality,

$$\begin{aligned}
&\left(\iint_K |f(y,t)|^2\,dm_\lambda(y)\,dt\right)^{1/2}\\
&\leq C_K\left(\iint_K\left|\int_0^R \chi_{\{x\in\mathbb{R}_+:|x-y|<t\}}(x)f(y,t)\,dm_\lambda(x)\right|^2\frac{dm_\lambda(y)\,dt}{m_\lambda(I(y,t))t}\right)^{1/2}\\
&\leq C_K\int_0^R\left(\iint_K\left|\chi_{\{x\in\mathbb{R}_+:|x-y|<t\}}(x)f(y,t)\right|^2\frac{dm_\lambda(y)\,dt}{m_\lambda(I(y,t))t}\right)^{1/2}dm_\lambda(x)\\
&\leq C_K\int_0^R\left(\iint_{\Gamma_+(x)}\left|f(y,t)\right|^2\frac{dm_\lambda(y)\,dt}{m_\lambda(I(y,t))t}\right)^{1/2}dm_\lambda(x)\leq C_K\|\Psi(f)\|_{L^p(\mathbb{R}_+,dm_\lambda)}.
\end{aligned}$$

Thus, the integral $\pi_\lambda f$ is well defined. From (4.1), we further deduce that $\pi_\lambda f \in L^2(\mathbb{R}_+, dm_\lambda)$. In fact, by (2.6), we see

$$|\pi_\lambda f(x)| \leq C_K\left(\iint_K |f(y,t)|^2\,dm_\lambda(y)\,dt\right)^{1/2}, \quad x\in\mathbb{R}_+.$$

On the other hand, by (4.1) and the fact that for $x, y, t \in \mathbb{R}_+$,

$$\left|\frac{\partial}{\partial t}P_t^{[\lambda]}(x,y)\right| \lesssim \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(x^2+y^2+t^2-2xy\cos\theta)^{\lambda+1}}\,d\theta,$$

we have

$$\begin{aligned}
\|\pi_\lambda f\|_{L^2(\mathbb{R}_+,dm_\lambda)} &\leq \iint_K\left(\int_0^\infty\left|\frac{\partial}{\partial t}P_t^{[\lambda]}(x,y)\right|^2dm_\lambda(x)\right)^{1/2}|f(y,t)|\,dm_\lambda(y)\,dt\\
&\leq C_K\iint_K\frac{1}{t^{\lambda+3/2}}\left(\int_0^\infty\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(x,y)\right|dm_\lambda(x)\right)^{1/2}|f(y,t)|\,dm_\lambda(y)\,dt\\
&\leq C_K\iint_K|f(y,t)|\,dm_\lambda(y)\,dt\\
&\leq C_K\left(\iint_K|f(y,t)|^2\,dm_\lambda(y)\,dt\right)^{1/2}\leq C_K\|\Psi(f)\|_{L^p(\mathbb{R}_+,dm_\lambda)}.
\end{aligned}$$

The proof of the following lemma is similar to [18, Proposition 3.3] and we omit the details.

Lemma 4.1. *Let F, G be measurable functions on $\mathbb{R}_+\times\mathbb{R}_+$. Then there exists a constant $C > 0$ independent of F and G such that*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty |F(y,t)G(y,t)|\,dm_\lambda(y)\frac{dt}{t}\\
&\leq C\min\left\{\int_0^\infty\Psi(F)(x)\Psi(G)(x)\,dm_\lambda(x),\int_0^\infty\Phi(F)(x)\Psi(G)(x)\,dm_\lambda(x)\right\}.
\end{aligned}$$

To state our result, we now recall the Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ in [3, 33]. The space $H^1(\mathbb{R}_+, dm_\lambda)$ is defined by

$$H^1(\mathbb{R}_+,dm_\lambda) := \left\{f\in L^1(\mathbb{R}_+,dm_\lambda)\colon \sup_{s>0}\left|P_s^{[\lambda]}(f)\right|\in L^1(\mathbb{R}_+,dm_\lambda)\right\}$$

with norm

$$\|f\|_{H^1(\mathbb{R}_+,dm_\lambda)} := \|f\|_{L^1(\mathbb{R}_+,dm_\lambda)} + \left\|\sup_{s>0}\left|P_s^{[\lambda]}(f)\right|\right\|_{L^1(\mathbb{R}_+,dm_\lambda)}.$$

Now we are in a position to state our main result in this section. From the theorem below, the operator π_λ can be seen as the reverse direction mapping of T defined by $T(f) := t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)$, $t > 0$, where f is a suitable function on $\mathbb{R}_+$, from $L^p(\mathbb{R}_+, dm_\lambda)(1 < p < \infty)$, $H^1(\mathbb{R}_+, dm_\lambda)$, $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ and $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ to T_2^p, T_2^1, T_2^∞ and $T_{2,C}^\infty$, respectively.

Theorem 4.2. *Let $\lambda > 0$ and $1 \le p < \infty$. Then the operator π_λ initially defined $T_{2,c}^p$ extends to a bounded linear operator:*

(i) *from T_2^p to $L^p(\mathbb{R}_+, dm_\lambda)$, $1 < p < \infty$;*
(ii) *from T_2^1 to $H^1(\mathbb{R}_+, dm_\lambda)$;*
(iii) *from T_2^∞ to $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$;*
(iv) *from $T_{2,C}^\infty$ to $\mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$;*
(v) *from $T_{2,0}^\infty$ to $\mathrm{VMO}(\mathbb{R}_+, dm_\lambda)$.*

Proof. (i): Suppose that $g \in L^q(\mathbb{R}_+, dm_\lambda)$ where $\frac{1}{p} + \frac{1}{q} = 1$. By using Lemma 4.1, we have

$$\begin{aligned}\left|\int_0^\infty \pi_\lambda f(x)g(x)\,dm_\lambda(x)\right| &= \left|\int_0^\infty\int_0^\infty\int_0^\infty \frac{\partial}{\partial t}P_t^{[\lambda]}(x,y)f(y,t)\,dm_\lambda(y)g(x)\,dt\,dm_\lambda(x)\right|\\ &= \left|\int_0^\infty\int_0^\infty t\frac{\partial}{\partial t}P_t^{[\lambda]}(g)(y)f(y,t)\,dm_\lambda(y)\frac{dt}{t}\right|\\ &\lesssim \left|\int_0^\infty \Psi\left(t\frac{\partial}{\partial t}P_t^{[\lambda]}(g)\right)(y)\Psi(f)(y)\,dm_\lambda(y)\right|\\ &\lesssim \|\Psi(f)\|_{L^p(\mathbb{R}_+,dm_\lambda)}\left\|\Psi\left(t\frac{\partial}{\partial t}P_t^{[\lambda]}(g)\right)\right\|_{L^q(\mathbb{R}_+,dm_\lambda)}\\ &\lesssim \|f\|_{T_2^p}\|g\|_{L^q(\mathbb{R}_+,dm_\lambda)},\end{aligned}$$

where the last inequality follows from the $L^p(\mathbb{R}_+, dm_\lambda)$-boundednesss of the Littlewood–Paley S-function; see [19, Proposition 2.17]. This implies that

$$\|\pi_\lambda f\|_{L^p(\mathbb{R}_+,dm_\lambda)} \lesssim \|f\|_{T_2^p}.$$

We now prove (ii). Let $f \in T_2^1$ and $g \in C_c^\infty(\mathbb{R}_+)$. It follows from the Fubini theorem and Lemma 4.1 that

$$\begin{aligned}\left|\int_0^\infty \pi_\lambda f(x)g(x)\,dm_\lambda(x)\right| &= \left|\int_0^\infty\int_0^\infty t\frac{\partial}{\partial t}P_t^{[\lambda]}(g)(y)f(y,t)\frac{dm_\lambda(y)\,dt}{t}\right|\\ &\lesssim \left|\int_0^\infty \Phi\left(t\frac{\partial}{\partial t}P_t^{[\lambda]}(g)\right)(y)\Psi(f)(y)\,dm_\lambda(y)\right|\\ &\lesssim \|\Psi(f)\|_{L^1(\mathbb{R}_+,dm_\lambda)}\left\|\Phi\left(t\frac{\partial}{\partial t}P_t^{[\lambda]}(g)\right)\right\|_{L^\infty(\mathbb{R}_+,dm_\lambda)}.\end{aligned}$$

By (2.1) and Theorem 2.5, we have

$$\left\|\Phi\left(t\frac{\partial}{\partial t}P_t^{[\lambda]}(g)\right)\right\|_{L^\infty(\mathbb{R}_+,dm_\lambda)} \lesssim \|g\|_{*,\lambda}.$$

Hence, we see

$$\left|\int_0^\infty \pi_\lambda f(x)g(x)\,dm_\lambda(x)\right| \lesssim \|f\|_{T_2^1}\|g\|_{*,\lambda}.$$

According to [9, Theorem (4.1)], $H^1(\mathbb{R}_+,dm_\lambda)$ is dual space of $\mathrm{CMO}(\mathbb{R}_+,dm_\lambda)$. Then we have that $\|\pi_\lambda f\|_{H^1(\mathbb{R}_+,dm_\lambda)} \lesssim \|f\|_{T_2^1}$. Thus (ii) holds.

Moreover, from an argument analogous to the proof of (ii) and the boundedness of the Littlewood–Paley S-function from $H^1(\mathbb{R}_+,dm_\lambda)$ to $L^1(\mathbb{R}_+,dm_\lambda)$ in [19, Theorem 2.21], and the fact that $\mathrm{BMO}(\mathbb{R}_+,dm_\lambda)$ is dual space of $H^1(\mathbb{R}_+,dm_\lambda)$; see [9, Theorem B], we see that (iii) holds

We continue to prove (iv). Let $f\in T^\infty_{2,C}$. To prove that $\pi_\lambda f\in \mathrm{CMO}(\mathbb{R}_+,dm_\lambda)$, by Theorem 2.5, it suffices to prove

$$(4.2)\qquad (1+x^{2\lambda+2})^{-1}\pi_\lambda(f)\in L^1(\mathbb{R}_+,dm_\lambda)\quad\text{and}\quad t\frac{\partial}{\partial t}P_t^{[\lambda]}(\pi_\lambda f)\in T^\infty_{2,C},\quad t\in\mathbb{R}_+.$$

From (iii), we have that $\pi_\lambda f\in \mathrm{BMO}(\mathbb{R}_+,dm_\lambda)$, which in turn implies that $(1+x^{2\lambda+2})^{-1}\pi_\lambda(f)\in L^1(\mathbb{R}_+,dm_\lambda)$ and

$$t\frac{\partial}{\partial t}P_t^{[\lambda]}(\pi_\lambda f)\in T_2^\infty,\quad t\in\mathbb{R}_+.$$

Suppose $I:=I(x_0,r_0)$, $x_0,r_0\in\mathbb{R}_+$. Let

$$f_0:=f\chi_{\widehat{2I}}\quad\text{and}\quad f_k:=f\chi_{\widehat{2^{k+1}I}\setminus\widehat{2^kI}},\quad k=1,2\ldots.$$

Then $f=\sum_{k=0}^\infty f_k$. We write

$$(4.3)\qquad \left(\iint_{\widehat{I}}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(\pi_\lambda f)(y)\right|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2} \le \sum_{k=0}^\infty\left(\iint_{\widehat{I}}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(\pi_\lambda f_k)(y)\right|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2} =: \sum_{k=0}^\infty H_k.$$

For H_0, we use the boundedness of Littlewood–Paley g-function on $L^2(\mathbb{R}_+,dm_\lambda)$ (see [29, 5] or [18]) together with (i) to obtain

$$\begin{aligned}H_0&\lesssim \|\pi_\lambda f_0\|_{L^2(\mathbb{R}_+,dm_\lambda)}\lesssim\|f_0\|_{T_2^2}\\&=\left(\int_0^\infty\iint_{\Gamma_+(x)}\left|f(y,t)\chi_{\widehat{2I}}(y,t)\right|^2\frac{dm_\lambda(y)}{m_\lambda(I(y,t))}\frac{dt}{t}dm_\lambda(x)\right)^{1/2}\\&\lesssim \left(m_\lambda(I)\right)^{1/2}\left(\frac{1}{m_\lambda(2I)}\iint_{\widehat{2I}}\left|f(y,t)\right|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2}.\end{aligned}$$

Regarding H_k, $k \in \mathbb{N}$, by (iii) and the property of Poisson semigroup, for any $y \in I$,

$$\begin{aligned}\left|\frac{\partial}{\partial t}P_t^{[\lambda]}(\pi_\lambda f_k)(y)\right| &= \left|\int_0^\infty \frac{\partial}{\partial t}P_t^{[\lambda]}(y,x)\pi_\lambda f_k(x)\,dm_\lambda(x)\right| \\ &\leq \int_0^\infty\int_0^\infty \left|\int_0^\infty \frac{\partial}{\partial t}P_t^{[\lambda]}(y,x)\frac{\partial}{\partial s}P_s^{[\lambda]}(x,z)f_k(z,s)\,dm_\lambda(x)\right| dm_\lambda(z)\,ds \\ &= \int_0^\infty\int_0^\infty \left|\frac{\partial}{\partial t}\frac{\partial}{\partial s}\int_0^\infty P_t^{[\lambda]}(y,x)P_s^{[\lambda]}(x,z)\,dm_\lambda(x)\right| |f_k(z,s)|\,dm_\lambda(z)\,ds \\ &= \int_0^\infty\int_0^\infty |f_k(z,s)| \left|\frac{\partial}{\partial t}\frac{\partial}{\partial s}P_{t+s}^{[\lambda]}(y,z)\right| dm_\lambda(z)\,ds.\end{aligned}$$

By a computation, for $y \in I$ and $z \in 2^{k+1}I\backslash 2^kI$, we have

$$\left|\frac{\partial}{\partial t}\frac{\partial}{\partial s}P_{t+s}^{[\lambda]}(y,z)\right| \lesssim \frac{1}{m_\lambda(I(x_0,|y-z|))}\frac{1}{\left(|y-z|+t+s\right)^2}.$$

Here the implicit constant is independent of k. Using this estimate, we obtain for $y \in I$,

$$\left|\frac{\partial}{\partial t}P_t^{[\lambda]}(\pi_\lambda f_k)(y)\right| \lesssim \frac{1}{m_\lambda(2^kI)}\iint_{\widehat{2^{k+1}I}\backslash\widehat{2^kI}} \frac{|f(z,s)|}{\left(|y-z|+t+s\right)^2}\,dm_\lambda(z)\,ds.$$

By Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}H_k &\lesssim \frac{1}{m_\lambda(2^kI)}\left(\iint_{\widehat{I}} t\left|\iint_{\widehat{2^{k+1}I}\backslash\widehat{2^kI}} \frac{|f(z,s)|}{\left(|y-z|+t+s\right)^2}\,dm_\lambda(z)\,ds\right|^2 dm_\lambda(y)\,dt\right)^{1/2} \\ &\lesssim \frac{1}{m_\lambda(2^kI)}\iint_{\widehat{2^{k+1}I}\backslash\widehat{2^kI}}\left(\iint_{\widehat{I}} \frac{t|f(z,s)|^2}{\left(|y-z|+t+s\right)^4}\,dm_\lambda(y)\,dt\right)^{1/2} dm_\lambda(z)\,ds \\ &\lesssim \frac{1}{m_\lambda(2^kI)}\iint_{\widehat{2^{k+1}I}\backslash\widehat{2^kI}}\left(\int_0^{|I|}\int_I \frac{t|f(z,s)|^2}{\left(|y-z|+t+s\right)^4}\,dm_\lambda(y)\,dt\right)^{1/2} dm_\lambda(z)\,ds \\ &\lesssim \frac{\left(m_\lambda(I)\right)^{1/2}}{m_\lambda(2^kI)}\frac{1}{2^{2k}|I|}\iint_{\widehat{2^{k+1}I}\backslash\widehat{2^kI}} |f(z,s)|\,dm_\lambda(z)\,ds \\ &\lesssim \frac{\left(m_\lambda(I)\right)^{1/2}}{(m_\lambda(2^kI))^{1/2}}\frac{2^kr_0}{2^{2k}|I|}\left(\iint_{\widehat{2^{k+1}I}} |f(z,s)|^2\frac{dm_\lambda(z)\,ds}{s}\right)^{1/2} \\ &\lesssim \frac{1}{2^k}\left(m_\lambda(I)\right)^{1/2}\left(\frac{1}{m_\lambda(2^kI)}\iint_{\widehat{2^{k+1}I}} |f(z,s)|^2\frac{dm_\lambda(z)\,ds}{s}\right)^{1/2}.\end{aligned}$$

Combining all the estimates of H_k, $k \in \mathbb{N}\cup\{0\}$ and applying (4.3), we have

$$\begin{aligned}&\left(\frac{1}{m_\lambda(I)}\iint_{\widehat{I}}\left|t\frac{\partial}{\partial t}P_t^{[\lambda]}(\pi_\lambda f)(y)\right|^2\frac{dm_\lambda(y)\,dt}{t}\right)^{1/2} \\ &\lesssim \sum_{k=0}^\infty \frac{1}{2^k}\left(\frac{1}{m_\lambda(2^kI)}\iint_{\widehat{2^{k+1}I}} |f(z,s)|^2\frac{dm_\lambda(z)\,ds}{s}\right)^{1/2}.\end{aligned}$$

With an argument similar to the proof of $(b_i) \Rightarrow (b_{ii})$ of Theorem 2.5, we see that (4.2) holds, from which we further get $\pi_\lambda f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$. Thus (iv) holds.

Since the argument of (v) is analogous to that of (iv), hence we omit the details. Therefore, we complete the proof of Theorem 4.2. □

Remark 4.3. If we substitute the operator π_λ with the operator

$$W_\lambda f(x) := \int_0^\infty \int_0^\infty t^2 \frac{\partial}{\partial s} W_s^{[\lambda]}(x,y)\Big|_{s=t^2} f(y,t) \frac{dm_\lambda(y)\,dt}{t}, \quad x \in \mathbb{R}_+,$$

where $f \in T^p_{2,c}$, $1 \le p < \infty$, the conclusions of Theorem 4.2 also hold.

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