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Algebraic curves and meromorphic functions sharing pairs of values

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Abstract. The 4IM+1CM-problem is to determine all pairs (f,g) of meromorphic functions in the complex plane that are not Möbius transformations of each other and share five pairs of values, one of them CM (counting multiplicities). In the present paper it is shown that each such pair parameterises some algebraic curve $K(x,y)=0$ of genus zero and low degree. Thus the search may be restricted to the pairs of meromorphic functions $(Q(e^z),\widetilde{Q}(e^z))$, where Q and $\widetilde{Q}$ are non-constant rational functions of low degree. This leads to the paradoxical situation that the 4IM+1CM-problem could be solved by a computer algebra virtuoso rather than a complex analyst.

Arvopareja keskenään jakavat meromorfiset kuvaukset ja algebralliset käyrät

Tiivistelmä. Ns. 4IM+1CM-ongelma on määrittää kaikki kompleksitason meromorfiset kuvausparit (f,g), jotka eivät ole toistensa Möbiuksen muunnoksia, ja jotka jakavat keskenään viisi arvoparia, yhden niistä kertalukuineen. Tässä työssä osoitetaan, että kukin tällainen kuvauspari parametrisoi jonkin kahvattoman, matala-asteisen algebrallisen käyrän $K(x,y)=0$. Täten etsintä voidaan rajoittaa meromorfisiin kuvauspareihin $(Q(e^z),\widetilde{Q}(e^z))$, missä Q and $\widetilde{Q}$ ovat ei-vakioarvoisia matala-asteisia rationaalikuvauksia. Tämä johtaa siihen yllättävään tilanteeseen, että 4IM+1CM-ongelman saattaisi kompleksianalyytikon sijaan ratkaista tietokonealgebravirtuoosi.

1. Introduction

Transcendental meromorphic functions f and g are said to share the *pair* (a,b) of complex numbers if $f(z)=a$ implies $g(z)=b$, and *vice versa*. Sharing CM (counting multiplicities) means that $f-a$ ($1/f$ if $a=\infty$) and $g-b$ ($1/g$ if $b=\infty$) even have the same divisor, while IM (ignoring multiplicities) means that nothing is assumed concerning multiplicities (the term 'shared IM' is established in the literature but just means 'shared'). If $a=b$ we will just say that f and g share the *value* a. The first result on functions sharing pairs of values is due Czubiak and Gundersen, who proved

Theorem A. [2] *Meromorphic functions f and g sharing more than five pairs are Möbius transformations of each other.*

In what follows, functions that are Möbius transformations of each other are *explicitly excluded* from consideration. At the centre of the 4IM+1CM-problem is the question of whether or not the functions

$$\widehat{f}(z)=\frac{e^z+1}{(e^z-1)^2} \quad \text{and} \quad \widehat{g}(z)=\frac{(e^z+1)^2}{8(e^z-1)} \tag{1}$$

are essentially the only ones that share four pairs of values IM and one pair CM. They were constructed by Gundersen [3] as an example of functions that share four

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values, namely $0, \infty, 1, -1/8$, none of them CM. Actually, the values 0 and 1 are assumed simply by $\hat{f}$ and doubly by $\hat{g}$, while for $-1/8$ and ∞ the reverse is true. Moreover, $\widehat{f}$ and $\widehat{g}$ share the pair $(-1/2, 1/4)$ CM. This was noticed by Reinders [9], who characterised the functions (1) in several ways.

Theorem B. [7, 9] *Suppose meromorphic functions f and g share four values a_ν $(1 \leq \nu \leq 4)$. Then*

$$f = M \circ \widehat{f} \circ h \quad \text{and} \quad g = M \circ \widehat{g} \circ h \tag{2}$$

holds for some Möbius transformation M and some non-constant entire function h, provided one of the following conditions is fulfilled:

(i) *f and g share some extra pair (a, b);*
(ii) *for every $\nu \in \{1, 2, 3, 4\}$, either $f - a_\nu$ or $g - a_\nu$ has only multiple zeros;*
(iii) *for every $\nu \in \{1, 2, 3, 4\}$, $(f - a_\nu)(g - a_\nu)$ has only triple zeros.*

The functions (1) also prove that the number five in Theorem A is best possible. In the context of functions sharing pairs of values, condition (2) has to be modified as follows to express the fact that $\widehat{f}$ and $\widehat{g}$ are *essentially unique:*

$$f = M_1 \circ \widehat{f} \circ h \quad \text{and} \quad g = M_2 \circ \widehat{g} \circ h. \tag{3}$$

The next step after Theorem A was made by Gundersen, Toghe, and the author. They showed that at most one of five pairs of values can be shared CM.

Theorem C. [5] *Meromorphic functions f and g that are not Möbius transformations of each other cannot share five pairs of values, two of them CM.*

Interestingly, the next Theorem was proved before Theorem C. On one hand it is stronger than Theorem C since only one CM-pair is involved, and weaker on the other by the additional hypothesis (4) imposed on the proximity functions of the CM-pair.

Theorem D. [12] *Suppose that meromorphic functions f and g share five pairs of values. Under the additional hypothesis that one of these pairs, (a, b), say, is shared CM and satisfies*

$$m(r, 1/(f - a)) + m(r, 1/(g - b)) = S(r), \tag{4}$$

either f and g are Möbius transformations of each other or else are given by (3).

The following best approximation to the corresponding 3IM+1CM-theorem for functions sharing four values is cited because it is closely related to Theorem 2 in section 3.

Theorem E. [4] *Suppose that transcendental meromorphic functions f and g share three values IM and one fourth value CM with counting function $N(r)$. Then either $N(r) \leq \frac{4}{5}T(r) + S(r)$ holds or else f and g share all four values CM.*

Condition (4) was used in [12] to derive a quadratic relation between f and g, that is, f and g parameterise some quadratic algebraic curve (necessarily of genus zero). In this paper it will be shown that also in the contrary case (when (4) is violated) f and g parameterise some algebraic curve of genus zero of low degree (at most 9) and depending on few (at most 5) parameters. In this way the 4IM+1CM-problem becomes a computer algebra problem.

Our main tool will be the theory of meromorphic functions and its interaction with uniformisation and parameterisation of algebraic curves. We will frequently make use of the Nevanlinna functions $m(r,f), N(r,f), \overline{N}(r,f), T(r,f)$ and the First and Second Main Theorem of Nevanlinna Theory. By $S(r,f)$ we denote any function satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside some exceptional set of finite measure. For details the reader is referred to Hayman's book [6]. In the next section the most important properties of algebraic curves and some examples that are related to pair-sharing meromorphic functions will be discussed.

2. Algebraic curves

Besides the functions e^z and e^{-z}, essentially three pairs of meromorphic functions f and g are known to share either four values or five pairs (Example 2 below due to Gundersen [3], Example 3 due to Reinders [8], and Example 4 due to the author [10])). In any case, f and g are algebraically dependent, that is, they satisfy some non-trivial polynomial equation $K(f(z), g(z)) = 0$. The polynomial $K(x,y)$ of two complex variables may assumed irreducible. Then the set

$$\mathcal{K} = \{(x,y) \in \mathbb{C} \times \mathbb{C} \colon K(x,y) = 0\}$$

is called an *algebraic curve*. If it is necessary or desirable to add points like (a,∞) or (∞,b) or even (∞,∞), one has to consider in addition equations $K(a+x,1/y)y^m = 0$ or $K(1/x, b+y)x^n = 0$ or $K(1/x,1/y)x^n y^m = 0$ at $(x,y) = (0,0)$; m and n denote the degree of K w.r.t. y and x, respectively. The functions f and g are said to *parameterise* $\mathcal{K}$.

The following facts are taken from [1]. If $\mathcal{K}$ is parameterised by meromorphic functions in the plane, the curve has *genus* zero or one. In case of genus *one*, any parameterisation has the form

$$f = Q \circ h \quad \text{and} \quad g = \widetilde{Q} \circ h, \tag{5}$$

where Q and $\widetilde{Q}$ are elliptic functions and h is non-constant entire. Much more important for us is the case of genus *zero*. By Theorem 1 and 4 in [1], any parameterisation again has the form (5), where now Q and $\widetilde{Q}$ are rational functions and h is any non-constant meromorphic function on the plane. Moreover,

$$\mathcal{K} = \{(f(z), g(z)) \colon z \in \mathbb{C}\} \cup \mathcal{E}$$

holds; the *exceptional set* $\mathcal{E}$ is finite and consists of *asymptotic values* (a,b) of (f,g), that is, $(f(z), g(z))$ tends to (a,b) as $z \to \infty$ on some plane curve. The rational functions Q and $\widetilde{Q}$ may be chosen such that the map $\Phi = (Q, \widetilde{Q})$ is injective on $\widehat{\mathbb{C}} \setminus E$, where E is a finite set (not to be confused with $\mathcal{E}$). With this choice, (5) holds for every parametrisation with suitably chosen meromorphic function h.

Example 1. The algebraic curve $\mathcal{C}$ defined by $x^2 + y^2 = 1$ $(x,y \in \mathbb{C})$ has genus zero and the well-known rational and entire parameterisation

$$Q(t) = \frac{1-t^2}{1+t^2}, \; \widetilde{Q}(t) = \frac{2t}{1+t^2} \quad (t \neq \pm i) \quad \text{and}$$

$$f(z) = \cos z = Q(\tan \tfrac{z}{2}), \quad g(z) = \sin z = \widetilde{Q}(\tan \tfrac{z}{2}),$$

respectively. On the other hand, $\overline{\mathcal{C}} = \mathcal{C} \cup \{(\infty,\infty)\}$ is parameterised over $\mathbb{C}$ by Jacobi's elliptic functions *sinus* and *cosinus amplitudinis*. This, however, does not mean that this curve has genus one but just says that the functions cn and sn may be written as

$$\frac{1-h^2}{1+h^2} \quad \text{and} \quad \frac{2h}{1+h^2},$$

respectively, where h is some elliptic function (with different lattice) with simple zeros and poles at the zeros of sn.

Remark 1. In case of genus one it is obvious that it suffices to consider elliptic functions $f = Q$, $g = \widetilde{Q}$ in place of $f = Q \circ h$, $g = \widetilde{Q} \circ h$. In case of genus zero we will often take the opportunity to switch from (5) to functions

$$f = Q \circ M \circ \exp \quad \text{and} \quad g = \widetilde{Q} \circ M \circ \exp \tag{6}$$

with some Möbius transformation M. This is possible by the following reason: if f and g are given by (5) and share five pairs of values, then Q and $\widetilde{Q}$ also share these pairs on $\widehat{\mathbb{C}} \setminus \{\text{Picard values of } h\}$. In any case, if h has either two Picard values ($a = M(\infty)$ and $b = M(0)$) or just one ($a = M(\infty)$) or none ($M = \mathrm{id}$), also $Q \circ M \circ \exp \circ \phi$ and $\widetilde{Q} \circ M \circ \exp \circ \phi$ for any non-constant entire function ϕ share the very same pairs IM/CM on $\mathbb{C}$. The only thing that matters is the determination of Q and $\widetilde{Q}$.

Example 2. The algebraic curve defined by the polynomial

$$4x^2 + 2cxy + y^2 - 8x \tag{7}$$

is strongly related to the 4IM+1CM-problem. Like every irreducible quadratic curve it has genus zero. For $y = tx$ we obtain $x((4 + 2ct + t^2)x - 8) = 0$, hence the curve has the rational parameterisation

$$x = Q(t) = \frac{8}{4 + 2ct + t^2}, \quad y = \widetilde{Q}(t) = \frac{8t}{4 + 2ct + t^2}. \tag{8}$$

Obviously, Q and $\widetilde{Q}$ have poles at $t = -c \pm \sqrt{c^2 - 4}$; they are simple if $c \neq \pm 2$, which together with $c \neq 0$ will henceforth be assumed. Hence the functions f and g given by (5) share the value ∞ CM for every non-constant meromorphic function h. Suppose f and g also share the finite pairs (a_ν, b_ν) $(1 \leq \nu \leq 4)$ IM and *not* CM. Then a_ν resp. b_ν is a critical value of Q resp. $\widetilde{Q}$ and the following holds:

critical values	non-critical values
$Q(\infty) = 0$	$\widetilde{Q}(\infty) = 0 = \widetilde{Q}(0)$
$Q(-c) = 8/(4 - c^2)$	$\widetilde{Q}(-c) = -8c/(4 - c^2) = \widetilde{Q}(-4/c)$
$\widetilde{Q}(2) = 4/(2 + c)$	$Q(2) = 2/(2 + c) = Q(-2 - 2c)$
$\widetilde{Q}(-2) = -4/(2 - c)$	$Q(-2) = 2/(2 - c) = Q(2 - 2c)$

If $c \neq 0, \pm 2$, Q and $\widetilde{Q}$ share the value ∞ CM and the pairs

$$(0,0), \quad \left(\frac{8}{4 - c^2}, \frac{-8c}{4 - c^2}\right), \quad \left(\frac{2}{2 + c}, \frac{4}{2 + c}\right) \quad \text{and} \quad \left(\frac{2}{2 - c}, \frac{-4}{2 - c}\right) \tag{9}$$

IM on $\widehat{\mathbb{C}} \setminus \{0, -4/c, -2-2c, 2-2c\}$. Thus there is no non-constant meromorphic function h in the plane such that the functions (5) share the value ∞ and the pairs (9)—*except when* $c = \pm 1$. Then the points $-4/c$ and $\mp 2-2c$ and also 0 and $\pm 2-2c$ coincide and the sphere is twice punctured at $-4, 0$ and $4, 0$, respectively. For any non-constant meromorphic function h with Picard values -4 and 0 resp. 4 and 0, the functions (5) share the value ∞ CM and the pairs $(0,0), (8/3, -8/3), (2/3, 4/3), (2, -4)$ resp. $(0,0), (8/3, 8/3), (2, 4), (2/3, -4/3)$. We note that then also (3) holds.

Example 3. [8] The algebraic curve defined by

$$(y-x)^4 - 16xy(x^2-1)(y^2-1) = 0$$

has genus one. For each parameterisation (f, g), the functions f and g share the values $0, 1, -1$ and ∞, with alternating multiplicities $(1{:}3)$ and $(3{:}1)$. For example, this means that f has simple and triple zeros where g has triple and simple zeros, respectively. In the most simple case, f and g are elliptic functions of elliptic order four.

Example 4. [10, 13] The algebraic curve $\mathcal{H}$ defined by

$$H(u,y) = y^3 - 3((\bar{a}-1)u^2 - 2u)y^2 - 3(2u^2 - (a-1)u)y - u^3 = 0 \tag{10}$$

with $a = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$ has genus zero and the rational parameterisation

$$u = \widetilde{R}(t) = 3(a-1)\frac{t(1+t)^2}{(1+3t)^2}, \quad y = R(t) = 3(a-1)\frac{t^2(1+t)}{1+3t}.$$

It is also parameterised by elliptic functions U and f, where U is a modified $\wp$-function satisfying $U'^2 = U(U+1)(U-a)$. The functions U and f share the values 0 and ∞ and the pairs $(a, 1)$ and $(-1, -a)$, each with alternating multiplicities $(2{:}4)$ and $(2:1)$. On its primitive lattice $\omega\mathbb{Z} + \omega'\mathbb{Z}$, U has elliptic order two, while on their common lattice $(2\omega - \omega')\mathbb{Z} + (\omega + \omega')\mathbb{Z}$ both functions have elliptic order six. We note that at $(u, y) = (0, 0)$, say, the equation $H(u, y) = 0$ has the solutions $y \sim u^2$ (this meaning $y(u) = C_2u^2 + C_3u^3 + \cdots$, $C_2 \neq 0$) and $u \sim y^2$, whose graphs parameterise $\mathcal{H}$ in a neighbourhood of $(0, 0)$. Actually there are three functions $f_1 = f$, $f_2(z) = f(z + \omega)$, and $f_3(z) = f(z + \omega')$ of this kind. They share the values $0, \infty, 1, -a$, each with alternating multiplicities $(4{:}1{:}1)$, $(1{:}4{:}1)$, and $(1{:}1{:}4)$. Up to transformations $f_\nu \mapsto M \circ f_\nu \circ h$ (M any Möbius transformation and h any non-constant entire function) this triple is unambiguously determined by the requirement that three mutually distinct meromorphic functions share four values IM. Any two of these functions parameterise some algebraic curve $\mathcal{K} : K(x, y) = 0$ of genus one. The polynomial K is an appropriate factor of the resultant of the polynomials $K(u, y)$ and $K(u, x)$ with respect to the variable u; $\mathcal{K}$ is parameterised by elliptic functions $f_j = R \circ h_j$ and $f_k = R \circ h_k$ $(1 \le j < k \le 3)$ of elliptic order six; h_j and h_k are elliptic functions of elliptic order two w.r.t. the lattice $(2\omega - \omega')\mathbb{Z} + (\omega + \omega')\mathbb{Z}$ such that $\widetilde{R} \circ h_j = \widetilde{R} \circ h_k = U$ holds. Note that $K(0, y) = y^6$, $K(1, y) = (y-1)^6$, and $K(-a, y) = (y+a)^6$. At $(x, y) = (0, 0)$, say, the equation $K(x, y) = 0$ has solutions $y = -x + O(x^2)$, $y = cx^4 + O(x^5)$, and $x = cy^4 + O(y^5)$ with $c = i\sqrt{3}/243$.

3. Results

From now on it is assumed that the following hypothesis holds:

(H) f and g are transcendental meromorphic functions that are not Möbius transformations of each other and share four finite pairs (a_ν, b_ν) IM and the value ∞ CM, with counting functions $\overline{N}(r; a_\nu, b_\nu)$ and $\overline{N}(r)$, respectively.

We note that the definitions of sharing IM and CM may be relaxed insofar as

- zeros of $f - a_\nu$ that are not zeros of $g - b_\nu$, and *vice versa*,
- poles of f that are not poles of g, and *vice versa*, and
- poles of both functions with different multiplicities

are admitted, provided these points form sequences with counting function $S(r)$. These 'generalisations' are so self-evident that any further words are superfluous.

Theorem 1. *Under hypothesis* (H)*, f and g parameterise some algebraic curve $K(x,y)=0$ of genus zero.*

The rather long and technical proof will be given in sections 4 and 5. The polynomial K is found in semi-explicit form as divisor of a rather complicated polynomial of degree 9 w.r.t. x and y, and of degree 13 w.r.t. (x,y). It turns out that this polynomial has to fulfill so many extra conditions that it is hard to believe that the 4IM+1CM-problem has solutions other that (3). In any case the following addendum to Theorem 1 is valid and will certainly prove useful in future considerations.

Addendum to Theorem 1. *For $1\le\nu\le 4$,*

I. *there exist integers $p_\nu>1$ and $q_\nu>1$ $(1\le\nu\le 4)$ with the following property: up to a sequence with counting function $S(r)$, every solution of the equation $(f(z),g(z))=(a_\nu,b_\nu)$ has multiplicity $(p_\nu{:}1)$ or $(1{:}\,q_\nu)$ or $(1{:}1)$;*
II. *the polynomials $K(x,b_\nu)$ and $K(a_\nu,y)$ vanish at $x=a_\nu$ and $y=b_\nu$, respectively, and nowhere else in the plane.*

The proof will be given in section 6.

Our next result is similar to a combination of Theorem B (iii) and Theorem E. It only formally contains Theorem D since this theorem is part of the proof.

Theorem 2. *Suppose that in addition to hypothesis* (H) *the zeros of the product $(f-a_\nu)(g-b_\nu)$ $(1\le\nu\le 4)$ have multiplicities at least three. Then either*

$$\overline{N}(r)\le\frac{5}{7}T(r)+S(r)\tag{11}$$

holds or else the conclusion of Theorem D remains valid.

The multiplicities $(p_\nu{:}1)$ and $(1{:}q_\nu)$ may alternate, that is, the equation

$$(f(z),g(z))=(a_\nu,b_\nu)$$

may have $(p_\nu{:}1)$-fold and also $(1{:}\,q_\nu)$-fold solutions, not to mention solutions with multiplicity $(1:1)$. If, however, for each pair (a_ν,b_ν) the multiplicity is always $(p_\nu{:}1)$ or always $(1{:}q_\nu)$, then the desired 4IM+1CM-theorem can be deduced from Theorem D. This is the contents of the following theorem, which may be viewed as the analogue to Theorem B (ii) for functions sharing pairs of values rather than values.

Theorem 3. *Suppose* (H) *holds and that for each ν either $f-a_\nu$ or else $g-b_\nu$ has only multiple zeros. Then the conclusion of Theorem D is valid.*

Both theorems will be proved in section 7.

Remark 2. We note that alternating multiplicities and also common simple zeros of $f-a_\nu$ and $g-b_\nu$ cannot be excluded *a priori* if only an algebraic curve serves as definition of the functions in question. This is shown by Examples 3 and 4 in section 2. We also note that *local* investigations of algebraic equations cannot utilise any information about the genus of the curve in question.

4. Proof of Theorem 1: Preliminary results

From various sources [2, 4, 5, 12] one can deduce that up to the remainder term $S(r)$ of Nevanlinna theory, meromorphic functions f and g that share four pairs IM and the value ∞ CM have the same characteristic $T(r)$, proximity function of infinity $m(r)$, and counting function of poles $N(r)=\overline{N}(r)+S(r)$; the latter relation says that up to a sequence with counting function $S(r)$, the poles of f and g are simple. Choose $(c_1, c_2, \ldots, c_5) \in \mathbb{C}^5$ non-trivially such that

$$P(x,y) = c_1x^2 + c_2xy + c_3x + c_4y + c_5$$

vanishes at $(x,y) = (a_\nu, b_\nu)$, $1 \le \nu \le 4$. Then

$$F(z) = P(f(z), g(z)) \tag{12}$$

does not vanish identically, since otherwise

$$g = -\frac{c_1f^2 + c_3f + c_5}{c_2f + c_4}$$

would be a Möbius transformation of f (which is excluded) if $c_1 = 0$ or else would satisfy $T(r,g) = 2T(r,f) + O(1)$ in contrast to $T(r,f) \sim T(r,g)$. Then the inequalities

$$\begin{aligned} m(r,F) &\le 2m(r,f) + m(r,g) + S(r) = 3m(r) + S(r),\\ N(r,F) &\le 2\overline{N}(r) + S(r),\\ T(r,F) &\le 2T(r) + m(r) + S(r) \end{aligned} \tag{13}$$

hold, and from

$$\sum_{\nu=1}^{4} \overline{N}(r; a_\nu, b_\nu) \le \overline{N}(r, 1/F) = T(r,F) - m(r,1/F) - N_1(r,1/F) + O(1) \tag{14}$$

and the Second Main Theorem it follows that

$$\begin{aligned} 3T(r) &\le \overline{N}(r) + \textstyle\sum_{\nu=1}^{4} \overline{N}(r; a_\nu, b_\nu) + S(r)\\ &\le \overline{N}(r) + T(r,F) - m(r,1/F) - N_1(r,1/F) + S(r)\\ &\le 3T(r) - m(r,1/F) - N_1(r,1/F) + S(r)\\ &\le 3T(r) + S(r). \end{aligned}$$

Thus not only in this chain of inequalities the equality sign must hold everywhere but also in (13) and (14). In particular, this means that the poles and zeros of F essentially arise from the poles and the finite shared pairs of values of (f,g) and up to sequences with counting function $S(r)$ the zeros and poles of F have multiplicity one and two, respectively. We have thus reached our first milestone.

Lemma 1. *Under hypothesis* (H) *the following holds:*

$$\begin{array}{rrcl} \text{(i)} & m(r,1/F) + N_1(r,1/F) &=& S(r),\\ \text{(ii)} & m(r,F) &=& 3m(r) + S(r),\\ \text{(iii)} & N(r,F) &=& 2\overline{N}(r) + S(r)\\ & &=& 2\overline{N}(r,F) + S(r),\\ \text{(iv)} & \sum_{\nu=1}^{4} \overline{N}(r; a_\nu, b_\nu) &=& 2T(r) + m(r) + S(r)\\ & &=& \overline{N}(r,1/F) + S(r). \end{array} \tag{15}$$

The same holds if F is replaced by $\widetilde{F}(z) = \widetilde{P}(f(z), g(z))$, where

$$\widetilde{P}(x,y) = \widetilde{c}_1y^2 + \widetilde{c}_2xy + \widetilde{c}_3x + \widetilde{c}_4y + \widetilde{c}_5$$

non-trivially satisfies $\widetilde{P}(a_\nu, b_\nu) = 0$ $(1 \le \nu \le 4)$.

For $c_1 = 0$ the better estimate $T(r,F) \le 2T(r) + S(r)$ is obtained, which implies $m(r) = S(r)$ and immediately gives (3) by Theorem D. The same is true if $\widetilde{c}_1 = 0$.

For this reason we henceforth assume $c_1 = \widetilde{c}_1 = 1$, hence

$$(16)\qquad \begin{aligned} P(x,y) &= x^2 + c_2xy + c_3x + c_4y + c_5 \quad \text{and} \\ \widetilde{P}(x,y) &= y^2 + \widetilde{c}_2xy + \widetilde{c}_3x + \widetilde{c}_4y + \widetilde{c}_5. \end{aligned}$$

Remark 3. Various polynomials of smallest possible degree vanishing at the shared pairs of values were successfully used in [2, 4, 5, 12]. The quadratic polynomials in (x, y) that vanish at the points (a_ν, b_ν) form a linear space. The polynomials (16) form a basis except when some nontrivial polynomial

$$P_0(x,y) = \widehat{c}_2xy + \widehat{c}_3x + \widehat{c}_4y + \widehat{c}_5$$

also vanishes at each (a_ν, b_ν). Then

$$b_\nu = -\frac{\hat{c}_3x + \hat{c}_5}{\hat{c}_2x + \hat{c}_4} = M(a_\nu)$$

holds, the functions f and $M^{-1} \circ g$ share four values a_ν and the pair $(\infty, M^{-1}(\infty))$, and Theorem B applies. This is the case for the pairs $(0,0)$, $(8/3,-8/3)$, $(2,-4)$, $(2/3,4/3)$; $P_0(x,y) = 3xy + 4x - 4y$ and $\widetilde{P}_0(x,y) = 12x^2 + 3y^2 - 32x + 8y$ form a basis, and the polynomials (16) are not available, that is, $c_1 = \widetilde{c}_1 = 0$ and P and $\widetilde{P}$ are constant multiples of $P_0(x,y) = 3xy + 4x - 4y$.

The proof of Theorem 1 requires a sequence of rather technical results which are based on Lemma 1 and the hypothesis

$$(17)\qquad m(r) \neq S(r).$$

Note that by Theorem D we are done if $m(r) = S(r)$. From now on this will be assumed in addition to hypothesis (H). Since the author believes that (17) is fictional, various auxiliary results will be called Claim rather than Lemma. First of all we will show that $|f|$ and $|g|$ are 'large' on disjoint sets. To be more precise, set

$$\begin{aligned} E_r &= \{\theta \in (-\pi,\pi] \colon |f(re^{i\theta})| \geq |g(re^{i\theta})|\} \quad \text{and} \\ E_r^* &= \{\theta \in (-\pi,\pi] \colon |f(re^{i\theta})| \geq K(1 + |g(re^{i\theta})|^2)\} \subset E_r; \end{aligned}$$

$K > 1$ will be fixed later. The sets $\widetilde{E}_r$ and $\widetilde{E}_r^*$ are defined in the same way with f and g interchanged.

Claim 1. $m(r) = \dfrac{1}{2\pi}\displaystyle\int_{E_r^*} \log^+|f|\,d\theta + S(r) = \dfrac{1}{2\pi}\displaystyle\int_{\widetilde{E}_r^*} \log^+|g|\,d\theta + S(r).$

Proof. The inequalities (for $F(z) = P(f(z), g(z))$)

$$\log^+|F| \leq 2\log^+|f| + O(1) \quad \text{on } E_r$$

(following from $|g| \leq |f|$) and

$$\log^+|F| \leq \log^+|f| + \log^+|g| + O(1) \quad \text{on } \widetilde{E}_r$$

(use $|f^2| \leq |f||g|$) imply

$$\begin{aligned} m(r,F) &\leq \frac{2}{2\pi}\int_{E_r} \log^+|f|\,d\theta + \frac{1}{2\pi}\int_{\widetilde{E}_r} \log^+|f|\,d\theta + \frac{1}{2\pi}\int_{\widetilde{E}_r} \log^+|g|\,d\theta + O(1) \\ &\leq 3m(r) - \frac{1}{2\pi}\int_{\widetilde{E}_r} \log^+|f|\,d\theta - \frac{1}{2\pi}\int_{E_r} \log^+|g|\,d\theta + O(1). \end{aligned}$$

Since, however, $m(r,F) = 3m(r) + S(r)$ holds by Lemma 1 (ii), this implies

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_{E_r} \log^+ |f|\, d\theta + S(r), \\ m(r,g) &= \frac{1}{2\pi}\int_{\widetilde{E}_r} \log^+ |g|\, d\theta + S(r), \quad \text{and} \end{aligned}$$

$$\frac{1}{2\pi}\int_{\widetilde{E}_r} \log^+ |f|\, d\theta + \frac{1}{2\pi}\int_{E_r} \log^+ |g|\, d\theta = S(r).$$

The assertion then follows from

$$\frac{1}{2\pi}\int_{E_r\setminus E_r^*} \log^+ |f|\, d\theta \le \log K + \frac{1}{2\pi}\int_{E_r} \log(1+|g|^2)\, d\theta = S(r). \qquad \square$$

Claim 2. *The functions* $\phi = \dfrac{f'F^2/\widetilde{F}}{\prod_{\nu=1}^4 (f-a_\nu)}$ *and* $\widetilde{\phi} = \dfrac{g'\widetilde{F}^2/F}{\prod_{\nu=1}^4 (g-b_\nu)}$ *have Nevanlinna characteristic* $S(r)$.

Proof. Up to a sequence of points with counting function $S(r)$, poles of ϕ may only occur at multiple zeros of $\widetilde{F}$ and at simple poles of $\widetilde{F}$. Since, however, again up to sequences with counting function $S(r)$, the zeros of $\widetilde{F}$ are simple by (15)(i) (for $\widetilde{F}$ in place of F) and the poles of F and $\widetilde{F}$ have order two, again by (15)(iii) (for $\widetilde{F}$ and F) we have $N(r,\phi) \le N(r, F/\widetilde{F}) + N(r, \widetilde{F}/F) = S(r)$. To prove $m(r,\phi) = S(r)$ we set

$$L = \frac{f'}{\prod_{\nu=1}^4 (f-a_\nu)} = \sum_{\nu=1}^4 A_\nu \frac{f'}{f-a_\nu}$$

and note that $m(r, 1/\widetilde{F}) = S(r)$ holds by (15)(i) (for $\widetilde{F}$ in place of F), and $m(r,L) = S(r)$ by the Lemma on the logarithmic derivative. To estimate $m(r,\phi)$ we consider the contributions of the mutually disjoint sets E_r^*, $\widetilde{E}_r^*$, and $\widehat{E}_r = (-\pi,\pi] \setminus (E_r^* \cup \widetilde{E}_r^*)$, noting that

$$\text{(18)} \qquad \frac{1}{2\pi}\int_{E_r^*\cup\widehat{E}_r} \log^+ |g|\, d\theta + \frac{1}{2\pi}\int_{\widetilde{E}_r^*\cup\widehat{E}_r} \log^+ |f|\, d\theta = S(r)$$

follows from Claim 1. On $\widetilde{E}_r^*$ (where $|g| \ge K(1+|f|^2)$ is 'large') we have $|F|^2 = O((1+|f|^2)^2|g|^2)$ and $|\widetilde{F}| > \frac{1}{2}|g|^2$ if K is chosen sufficiently large, hence $|F|^2/|\widetilde{F}| = O((1+|f|^2)^2)$ and the contribution of $\widetilde{E}_r^*$ to $m(r,\phi)$ is at most

$$m(r,L) + \frac{1}{2\pi}\int_{\widetilde{E}_r^*} \log\,(1+|f|^2)^2\, d\theta + O(1) = S(r).$$

On $\widehat{E}_r$ we have $|F| = O(|f|^2 + |g|^2 + 1)$, hence by (18) the contribution of $\widehat{E}_r$ is at most

$$\frac{1}{2\pi}\int_{\widehat{E}_r} \log(1+|f|^2+|g|^2)\, d\theta + m(r,L) + m(r, 1/\widetilde{F}) + O(1) = S(r).$$

On E_r^* write

$$\text{(19)} \qquad \phi = \frac{f'(f^2 + c_2 fg + c_3 f + c_4 g + c_5)^2}{(f-a)\prod_{\nu=1}^4 (f-a_\nu)} \times \frac{(f-a)}{\widetilde{F}},$$

where $a \neq a_\nu$ $(1 \le \nu \le 4)$ is any complex number. Since $|f| \ge |g|$ holds on E_r^* we may write

$$f^2 + c_2 fg + c_3 f + c_4 g + c_5 = c_2^* f^2 + c_3^* f + c_5,$$

where $c_2^* = 1 + c_2(g/f)$ and $c_3^* = c_3 + c_4(g/f)$ are considered as coefficients satisfying $|c_2^*| \le 1 + |c_2|$ and $|c_3^*| \le |c_3| + |c_4|$. Thus the first factor in (19) is

$$O\left(\frac{|f'|}{|f-a|} + \sum_{\nu=1}^{4} \frac{|f'|}{|f-a_\nu|}\right)$$

(partial fraction decomposition) and has proximity function $S(r)$. The First Main Theorem yields

$$m\left(r, \frac{f-a}{\widetilde{F}}\right) = m\left(r, \frac{\widetilde{F}}{f-a}\right) + N\left(r, \frac{\widetilde{F}}{f-a}\right) - N\left(r, \frac{f-a}{\widetilde{F}}\right) + O(1).$$

To estimate the first term to the right we note that on E_r^* we have

$$\left|\frac{\widetilde{F}}{f-a}\right| = O(1+|g|),$$

while $|\widetilde{F}| = O(|g|^2)$ and $|\widetilde{F}| = O(1+|f|^2+|g|^2)$ holds on $\widetilde{E}_r^*$ and $\widehat{E}_r$, respectively. Again by (18) this yields the estimate

$$\begin{aligned} m\left(r, \frac{\widetilde{F}}{f-a}\right) &\le m\left(r, \frac{1}{f-a}\right) + \frac{1}{2\pi}\int_{E_r^*} \log\,(1+|g|)\,d\theta + \frac{2}{2\pi}\int_{\widetilde{E}_r^*} \log^+|g|\,d\theta \\ &\quad + \frac{1}{2\pi}\int_{\widehat{E}_r} \log\,(1+|f|^2+|g|^2)\,d\theta + O(1) \\ &= m\left(r, \frac{1}{f-a}\right) + 2m(r) + S(r). \end{aligned}$$

Combining this with

$$N\left(r, \frac{f-a}{\widetilde{F}}\right) \ge N(r, 1/\widetilde{F}) + S(r) = 2T(r) + m(r) + S(r).$$

and

$$N\left(r, \frac{\widetilde{F}}{f-a}\right) \le N\left(r, \frac{1}{f-a}\right) + \overline{N}(r) + S(r)$$

(note that up to a sequence with counting function $S(r)$, $\tilde{F}/(f-a)$ has simple poles at the poles of f, and no others) yields

$$\begin{aligned} m\left(r, \frac{f-a}{\widetilde{F}}\right) &\le 2m(r) + m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) + \overline{N}(r) \\ &\quad - (2T(r) + m(r)) + S(r) = S(r) \end{aligned}$$

and eventually $T(r, \phi) = S(r)$. □

Claim 3. $m(r, F/\widetilde{F}) = m(r) + S(r)$.

Proof. Set $\Psi = F/\widetilde{F}$. To estimate $m(r) = m(r, f) + S(r)$ write

$$\frac{f}{\Psi} = \frac{f\widetilde{F}}{F} = \frac{fg^2 + \widetilde{c}_2 f^2 g + \widetilde{c}_3 f^2 + \widetilde{c}_4 fg + \widetilde{c}_5 f}{f^2 + c_2 fg + c_3 f + c_4 g + c_5}.$$

On E_r^*, where $|f| \ge K(1+|g|^2)$ is 'large' and thus

$$|f^2 + c_2 fg + c_3 f + c_4 g + c_5| \ge \frac{1}{2}|f|^2 \quad \text{and}$$
$$|fg^2 + \tilde c_2 f^2 g + \tilde c_3 f^2 + \tilde c_4 fg + \tilde c_5 f| = O(|f|^2(1+|g|))$$

holds we have $|f| = O((1+|g|)|\Psi|)$, hence

$$\begin{aligned} m(r) &= \frac{1}{2\pi}\int_{E_r^*} \log^+ |f|\,d\theta + S(r) \\ &\le \frac{1}{2\pi}\int_{E_r^*} \log^+ |\Psi|\,d\theta + \frac{1}{2\pi}\int_{E_r^*} \log(1+|g|)\,d\theta + S(r) \\ &\le m(r,\Psi) + S(r) \end{aligned}$$

holds by (18). Besides L we will also consider

$$\widetilde L = \frac{g'}{\prod_{\nu=1}^4 (g-b_\nu)}.$$

Since $m(r,L) + m(r,\widetilde L) = S(r)$ and

$$\frac{L}{\widetilde L}\Psi^3 = \frac{\phi}{\widetilde\phi}$$

even has characteristic $S(r)$ by Claim 2, we obtain

$$\begin{aligned} 3m(r,\Psi) &\le m(r,1/L) + m(r,\widetilde L) + T(r,\phi/\widetilde\phi) + S(r) \\ &= N(r,L) - N(r,1/L) + S(r) \\ &\le \textstyle\sum_{\nu=1}^4 \overline N(r; a_\nu, b_\nu) - 2\overline N(r) + S(r) \\ &= 2T(r) + m(r) - 2\overline N(r) + S(r) \\ &= 3m(r) + S(r). \end{aligned}$$

Altogether this is nothing but $m(r) = m(r,\Psi) + S(r) = m(r, F/\widetilde F) + S(r)$ and

$$m(r) = \frac{1}{2\pi}\int_{E_r^*} \log^+ |\Psi|\,d\theta + S(r) = \frac{1}{2\pi}\int_{\widetilde E_r^*} \log^+ |1/\Psi|\,d\theta + S(r). \qquad \square$$

Lemma 1 (i) and (iv) (also applied to $\widetilde F$) says that up to a sequence with counting function $S(r)$, the functions F and $\widetilde F$ have common simple zeros and double poles, hence $N(r,\Psi'/\Psi) = \overline N(r,\Psi'/\Psi) = S(r)$. In combination with the Lemma on the logarithmic derivative we thus obtain

Claim 4. *$\psi = \Psi'/\Psi = F'/F - \widetilde F'/\widetilde F$ has Nevanlinna characteristic $S(r)$.*

We note that ψ does not vanish identically since otherwise $\Psi = F/\widetilde F$ would be a constant in contrast to Claim 3, which says that $m(r,\Psi) = m(r) + S(r) \ne S(r)$.

5. Proof of Theorem 1: Construction of an algebraic curve

In

(20) $$\psi = \big(P_x/P - \widetilde P_x/\widetilde P\big)f' + \big(P_y/P - \widetilde P_y/\widetilde P\big)g',$$

wherein P, $\widetilde{P}$, P_x, P_y, $\widetilde{P}_x$, $\widetilde{P}_y$ have to be evaluated at $(f(z), g(z))$, the derivatives f' and g' may be replaced by

$$f' = \phi\frac{\widetilde{P}}{P^2}\prod_{\nu=1}^{4}(f-a_\nu) \quad \text{and} \quad g' = \widetilde{\phi}\frac{P}{\widetilde{P}^2}\prod_{\nu=1}^{4}(g-b_\nu),$$

respectively, thus $(x, y) = (f(z), g(z))$ satisfies some algebraic equation

$$H(z, x, y) = 0$$

over the field generated by the functions ψ, ϕ, and $\widetilde{\phi}$. Actually, H is given by

$$\psi P^3\widetilde{P}^3 - \phi(\widetilde{P}P_x - P\widetilde{P}_x)\widetilde{P}^3\prod_{\nu=1}^{4}(f-a_\nu) + \widetilde{\phi}(\widetilde{P}P_y - P\widetilde{P}_y)P^3\prod_{\nu=1}^{4}(g-b_\nu). \tag{21}$$

Our next task is to prove

Claim 5. *H is non-trivial.*

Proof. Suppose H is trivial. Since $\psi(z) \not\equiv 0$, the product $P(x, y)\widetilde{P}(x, y)$ vanishes at every point (a_μ, b_ν). We may assume $P(a_\mu, b_\nu) = 0$ for $\mu = 1, 2$, say, and two values of ν depending on μ (not necessarily the same in both cases). Then

$$P(a_\mu, b_\nu) = a_\mu^2 + c_3a_\mu + c_5 + (c_2a_\mu + c_4)b_\nu = 0$$

holds for two different values of ν, and this implies $a_\mu^2 + c_3a_\mu + c_5 = c_2a_\mu + c_4 = 0$ for $\mu = 1, 2$, hence $c_2 = c_4 = 0$ since $a_1 \neq a_2$. Of course, $P(x, y) = x^2 + c_3x + c_5$ cannot vanish at (a_ν, b_ν) for $1 \leq \nu \leq 4$, this showing that H is non-trivial. □

To get rid of the functions $\psi, \phi, \widetilde{\phi}$ we now have to attack the most involved part of the proof:

Claim 6. *The ratios ϕ/ψ and $\widetilde{\phi}/\psi$ are constant.*

Proof. The polynomial H has degree at most nine with respect to the single variables x and y, thus

$$H(z, x, y) = \sum_{j=0}^{9} h_j(z, y)x^j$$

holds. The assertion is trivially true if $h_9(z, x)$ vanishes identically: for $a_1 = b_1 = 0$, say, h_9 is given (thanks to `maple`) by

$$\begin{aligned}h_9(z, y) = {}& \widetilde{c}_3^3(\psi(z) - \widetilde{c}_3\phi(z)) + \widetilde{c}_2y\big\{3\widetilde{c}_3^2\psi(z) - 4\widetilde{c}_3^3\phi(z) + b_2b_3b_4\widetilde{\phi}(z)) \\ & + (3\widetilde{c}_2\widetilde{c}_3\psi(z) - 6\widetilde{c}_2\widetilde{c}_3^2\phi(z) - (b_2b_3 + b_3b_4 + b_4b_2)\widetilde{\phi}(z))y \\ & + (\widetilde{c}_2^2\psi(z) - 4\widetilde{c}_2^2\widetilde{c}_3\phi(z) + (b_2 + b_3 + b_4)\widetilde{\phi}(z))y^2 - (\widetilde{c}_2^3\phi(z) + \widetilde{\phi}(z))y^3\big\}.\end{aligned}$$

Assuming $h_9(z, y) \equiv 0$, either

$$\psi = \widetilde{c}_3\phi \quad (\text{and } \widetilde{c}_3 \neq 0) \tag{22}$$

holds or else $\widetilde{c}_3 = 0$. In this case, $\widetilde{\phi}(z) \not\equiv 0$ implies $\widetilde{c}_2 = 0$ and $\widetilde{P}(x, y) = y^2 + \widetilde{c}_4y$, which, of course, is impossible and proves (22).

We will now show that exactly the same is true if $h_9(z, y) \not\equiv 0$. From $H(z, x, y) = 0$ and $|x| > 1$ it follows that

$$|h_9(z, y)||x| \leq \sum_{j=0}^{8}|h_j(z, y)| = O(\max\{|\psi(z)|, |\phi(z)|, |\widetilde{\phi}(z)|\}(1 + |y|)^9),$$

hence (set $x = f(z)$, $y = g(z)$ and again use (18))

$$m(r) = \frac{1}{2\pi}\int_{E_r^*} \log|f(re^{i\theta})|\,d\theta + S(r)$$
$$\le m(r, 1/h_9(z, g(z))) + m(r,\psi) + m(r,\phi) + m(r,\widetilde{\phi}) + O(1)$$
$$+ \frac{9}{2\pi}\int_{E_r^*} \log(1 + |g(re^{i\theta})|)\,d\theta + S(r) = m(r, 1/h_9(z, g(z))) + S(r).$$

To proceed we need the following

Lemma 2. *For at least one index j, $h_9(z, b_j)$ vanishes identically.*

Proof. Suppose to the contrary that $h_9(z, b_j) \not\equiv 0$ for $1 \le j \le 4$. Then also $h(z, b_j) \not\equiv 0$ holds for every prime factor $h(z, y)$ of $h_9(z, y)$ and

$$G(z, y) = h(z, y)\prod_{\nu=1}^{4}(y - b_\nu)$$

is a square-free polynomial over the algebraic closure of the field of meromorphic functions with Nevanlinna characteristic $O(T(r,\phi) + T(r,\widetilde{\phi}) + T(r,\psi)) = S(r)$. Then from Corollary 4 in Yamanoi's paper [14] it immediately follows that

$$(\deg_y G - 1)T(r) \le \overline{N}(r) + \overline{N}(r, 1/G(z, g(z))) + \epsilon T(r) \;\|$$

holds for every $\epsilon > 0$; the symbol $\|$ means *outside some exceptional set that depends on ϵ and has finite measure with respect to $d\log\log r$*. Since, however, already

$$\overline{N}(r) + \sum_{\nu=1}^{4}\overline{N}(r, 1/(g - b_\nu)) = 3T(r) + S(r)$$

holds (without $\|$), this yields

$$\overline{N}(r, 1/h(z, g(z)) \ge (\deg_y h - \epsilon)T(r) + S(r) \;\|$$

Taking into account all prime factors according to their multiplicities we eventually obtain

$$N(r, 1/h_9(z, g(z)) \ge (\deg_y h_9 - 4\epsilon)T(r) + S(r) \;\|,$$

hence

$$m(r) \le m(r, 1/h_9(z, g(z))) + S(r) \le 4\epsilon T(r) + S(r) \;\|$$

by the first main theorem for $h_9(z, g(z))$. Although this is weaker than the condition $m(r) = S(r)$, the proof in [12] goes through in the very same way, that is, Theorem D again holds. This eventually proves $h_9(z, b_j) \equiv 0$ for some j. □

To simplify notation we may assume $j = 1$ and $a_1 = b_1 = 0$, hence $\widetilde{c}_5 = 0$ and

$$h_9(z, 0) = \widetilde{c}_3^3(\psi(z) - \widetilde{c}_3\phi(z))$$

vanishes identically. In case of $\widetilde{c}_3 = 0$ (and $\psi - \widetilde{c}_3\phi \not\equiv 0$) we obtain

$$\widetilde{P}(x, y) = y(y + \widetilde{c}_2 x + \widetilde{c}_4),$$

hence $b_\nu = -\widetilde{c}_2 a_\nu - \widetilde{c}_4 = M(a_\nu)$ holds for $2 \le \nu \le 4$ and also $\nu = 5$ with $a_5 = b_5 = \infty$. In other words, the functions f and $M^{-1} \circ g$ share four values a_ν ($2 \le \nu \le 5$) and the pair $(0, M^{-1}(0))$. By Theorem B we are done: the goal, the representation (3) has been arrived head of time. This eventually proves (22). The corresponding assertion $\psi = c_3\widetilde{\phi}$ is proved in exactly the same manner. □

From (21) and Claim 6 it then follows that $H(z,x,y) = \psi(z)H_0(x,y)$, hence also $H_0(f(z),g(z)) = 0$ holds. Of course, H_0 does not have to be irreducible, nevertheless we have

Claim 7. *(f,g) parameterises an algebraic curve $\mathcal{K} : K(x,y) = 0$ of genus zero.*

To prove Claim 7, which marks the end of proof of Theorem 1, we assume to the contrary that $\mathcal{K}$ has genus one and Q and $\widetilde{Q}$ in (5) are elliptic functions. Then also Q and $\widetilde{Q}$ share the pairs (a_ν, b_ν) IM (and not CM) and the value ∞ CM. Since

$$m(r,Q) + m(r,\widetilde{Q}) = O(1),$$

Theorem D applies. It yields $Q = M \circ \widehat{f} \circ k$ and $\widetilde{Q} = \widetilde{M} \circ \widehat{g} \circ k$ with Möbius transformations M and $\widetilde{M}$ and some non-constant entire function k. It is, however, obvious that $\widehat{f} \circ k$ and $\widehat{g} \circ k$ are not elliptic functions. Thus the genus of our algebraic curve is zero as was stated in Theorem 1, and (5) holds with rational functions Q and $\widetilde{Q}$ and some meromorphic function h. □

Remark 4. The proof of Theorem 1 is much easier for Yosida functions f and g, that is, for functions with bounded spherical derivative

$$f^\sharp = \frac{|f'|}{1+|f|^2}.$$

In this case the functions $\phi, \widetilde{\phi}, \psi$ are constants and there is no need for Claim 6 and its sophisticated and elaborate proof. The first idea therefore was to apply the re-scaling technique due to Zalcman [15] simultaneously (as in [11]) to obtain Yosida functions $\widetilde{f}(z) = \lim_{n\to\infty} f(z_n + \rho_n z)$ and $\widetilde{g}(z) = \lim_{n\to\infty} g(z_n + \rho_n z)$ for suitably chosen sequences $z_n \to \infty$ and $\rho_n \to 0$. By Hurwitz' Theorem, these functions share the same values and pairs of values as do f and g. However, it cannot be excluded that $\widetilde{f}$ and $\widetilde{g}$ are Möbius transformations of each other although f and g are not. The same problem arises if the re-scaling technique is simultaneously applied to functions f and g that share four values: while it is true that $\widetilde{f}$ and $\widetilde{g}$ share the same values as f and g, $\widetilde{f} = \widetilde{g}$ cannot be excluded although $f \neq g$.

6. Proof of the Addendum to Theorem 1

I. For the sake of simplicity let us consider the case $\nu = 1$ and $a_1 = b_1 = 0$. Assuming $(f(z_0), g(z_0)) = (0,0)$ with multiplicity $(1:q)$ and $q > 1$ yields $F'(z_0) = c_3 f'(z_0)$ and $\widetilde{F}'(z_0) = \widetilde{c}_3 f'(z_0)$, hence

$$\phi(z_0) = \frac{(c_3^2/\widetilde{c}_3)f'(z_0)}{\prod_{\mu=2}^4(-a_\mu)} \quad \text{and} \quad \widetilde{\phi}(z_0) = \frac{q(\widetilde{c}_3^2/c_3)f'(z_0)}{\prod_{\mu=2}^4(-b_\mu)}.$$

Since, however, $\widetilde{\phi}/\phi$ is constant by Claim 6, we see that q is independent of z_0.

II. Since only properties of the polynomial $K(x,y)$ are affected we may assume that our algebraic curve $\mathcal{K} : K(x,y) = 0$ is parameterised by the functions

$$f(z) = Q(e^z) \quad \text{and} \quad g(z) = \widetilde{Q}(e^z), \tag{23}$$

which share the value ∞ CM and the pairs (a_ν, b_ν) IM (and not CM). Note that the rational functions Q and $\widetilde{Q}$ share ∞ CM and the pairs (a_ν, b_ν) IM (and not CM) certainly on $\mathbb{C} \setminus \{0\}$. Also, since $m(r,f) \sim m(r,g) \sim m(r) \neq o(r)$ and f and g are 'large' on disjoint sets, Q and $\widetilde{Q}$ have a pole of the same order at $t=\infty$ and $t=0$, respectively, or *vice versa*. The exceptional set $\mathcal{E}$ in

$$\mathcal{K}=\{(f(z),g(z))\colon z\in\mathbb{C}\}\cup\mathcal{E}$$

consists of the asymptotic values

$$\begin{aligned}&\lim_{\xi\to+\infty}(Q(e^\xi),\widetilde{Q}(e^\xi))=(Q(\infty),\widetilde{Q}(\infty))=(\infty,b_\kappa)\quad\text{and}\\&\lim_{\xi\to-\infty}(Q(e^\xi),\widetilde{Q}(e^\xi))=(Q(0),\widetilde{Q}(0))=(a_\lambda,\infty)\end{aligned}\tag{24}$$

for some $1\le\kappa,\lambda\le4$. Thus $x=\widetilde{Q}(t_0)=a_\nu$ for some $t_0\in\mathbb{C}\setminus\{0\}$ implies $y=Q(t_0)=b_\nu$, that is, $K(a_\nu,y)=0$ implies $y=b_\nu$ (with multiplicity 1 or q_ν). □

7. Proof of Theorems 2 and 3

To prove Theorem 2 we need the estimate

$$\overline{N}_1(r;1/(f-a_\nu))\le m(r)+S(r)\tag{25}$$

and, of course, also $\overline{N}_1(r;1/(g-b_\nu))\le m(r)+S(r)$. To simplify notation we may assume $\nu=1$ and $a_1=b_1=0$, choose

$$R(x,y)=Ax^2+By^2+Cxy+Dx$$

non-trivially such that $R(a_\nu,b_\nu)=0$ holds for $2\le\nu\le4$ and set

$$h(z)=R(f(z),g(z)).$$

Then h vanishes at the (a_ν,b_ν)-points of (f,g), at least twice at the multiple zeros of f (with counting function $\overline{N}_1(r,1/f)$). If h does not vanish identically the Second Main Theorem again yields

$$\begin{aligned}2T(r)+m(r)&\le\textstyle\sum_{\nu=1}^4\overline{N}(r;a_\nu,b_\nu)+S(r)\\&\le N(r,1/h)-\overline{N}_1(r,1/f)+S(r)\\&\le T(r,h)-\overline{N}_1(r,1/f)+S(r)\\&\le 2N(r)+4m(r)-\overline{N}_1(r,1/f)+S(r)\\&=2T(r)+2m(r)-\overline{N}_1(r,1/f)+S(r),\end{aligned}$$

hence (25) for $a_\nu=0$. If, however, $h(z)\equiv0$, the functions f and g parameterise the algebraic curve $\mathcal{R}:Ax^2+By^2+Cxy+Dx=0$. By Remark 3 we have $AB\ne0$ and also $D\ne0$ since the algebraic curve $Ax^2+By^2+Cxy=0$ is reducible. By the elementary change of variables $(x,y)\mapsto(\gamma\alpha x,\gamma\beta y)$, the equation $R(x,y)=0$ may be transformed into

$$\gamma^2[4x^2+2cxy+y^2-8x]=0$$

with $A\alpha^2=4$, $B\beta^2=1$, $C\alpha\beta=2c$, and $D\alpha/\gamma=-8$. By Example 2 it is necessary that $c=\pm1$ and f and g are given by (3). Thus if Theorem D does not hold and $f-a_\nu$ and $g-b_\nu$ have no common simple zeros,

$$\overline{N}(r;a_\nu,b_\nu)\le\overline{N}_1(r,1/(f-a_\nu))+\overline{N}_1(r,1/(g-b_\nu))+S(r)\le2m(r)+S(r)$$

holds for $1\le\nu\le4$, and so

$$2T(r)+m(r)\le\sum_{\nu=1}^4\overline{N}(r;a_\nu,b_\nu)\le8m(r)+S(r)$$

by (15)(iv). This implies (11). □

Proof of Theorem 3. For ν fixed, the hypothesis on the multiplicities implies

$$\overline{N}(r;a_\nu,b_\nu)\le\frac{1}{2}T(r)+O(1)$$

in any case, hence $\sum_{\nu=1}^4\overline{N}(r;a_\nu,b_\nu)\le 2T(r)+O(1)$. Then (15)(iv) yields $m(r)=S(r)$ and Theorem D applies. □

8. Why the 4IM+1CM-conjecture is probably true

The polynomial in Theorem 1 has the form

$$K(x,y)=(x-a_\lambda)^s y^m+A(y-b_\kappa)^t x^n+\sum_{j,k}c_{jk}x^jy^k \tag{26}$$

and satisfies $K(a_\nu,y)=0\Leftrightarrow y=b_\nu$ and $K(x,b_\nu)=0\Leftrightarrow x=a_\nu$ $(1\le\nu\le 4)$, hence $K(a_\nu,b_\nu+y)$ and $K(a_\nu+x,b_\nu)$ are monomials. This requires

$$\frac{\partial^j K}{\partial y^j}(a_\nu,b_\nu)=\frac{\partial^\ell K}{\partial x^\ell}(a_\nu,b_\nu)=0 \tag{27}$$

for all but one j and ℓ, respectively; $m\le 9$ and $n\le 9$ are the degrees of K w.r.t. y and x, respectively, K has degree at most 13 w.r.t. (x,y), and $1\le s,t\le 4$ holds. On the other hand, since two of the pairs (a_ν,b_ν) may be prescribed ($a_1=b_1=0$ and $a_2=b_2=1$, say) and ϕ/ψ and $\widetilde{\phi}/\psi$ may be expressed in terms of the coefficients $c_j,\widetilde{c}_k$, there are only *five* free parameters a_3,b_3,a_4,b_4,A at hand to satisfy the $4(n+m-1)$ constraints (27), not to mention the fact that K has to be irreducible of genus zero. Of course, counting algebraic equations and variables does not disprove the existence of K since for special pairs (a_ν,b_ν) some of the equations might be algebraically dependent. Nevertheless it might well be that the 4IM+1CM-problem will be solved by a virtuoso in computer algebra systems rather than a complex analyst.

9. Concluding remark

It is not implausible to believe that all pairs (f,g) of meromorphic functions sharing four values or five pairs are already known in essence and are presented in Examples 2–4. A proof of this very general assertion, however, seems to be beyond the present knowledge and possibilities, and far beyond the capabilities of the author.

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