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# A growth estimate for the planar Mumford–Shah minimizers at a tip point: An alternative proof of David–Léger

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**Abstract.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $u \in SBV(\Omega)$ be a local minimizer of the Mumford–Shah problem in the plane, with $0 \in \overline{S}_u$ being a tip point and $B_1 \subset \Omega$. Then there exist absolute constants $C > 0$ and $0 < r_0 < 1$ such that

$$|u(x) - u(0)| \le Cr^{\frac{1}{2}} \quad \text{for any} \quad x \in B_r \quad \text{and} \quad 0 < r < r_0.$$

This estimate is a local version of the original one in David–Léger (2002, Proposition 10.17). Our result is based on a dichotomy and the John structure of $\Omega \setminus \overline{S}_u$, different from the one by David–Léger (2002) or Bonnet–David (2001, Lemma 21.3).

**Kasvuarvio tason Mumfordin–Shahin minimoijille keikahduspisteessä: vaihtoehtoinen todistus Davidin ja Légerin tulokselle**

**Tiivistelmä.** Olkoon $\Omega \subset \mathbb{R}^2$ rajallinen alue ja $u \in SBV(\Omega)$ Mumfordin–Shahin taso-ongelman paikallinen minimoija, jolle $0 \in \overline{S}_u$ on keikahduspiste ja $B_1 \subset \Omega$. Tällöin on olemassa absoluuttiset vakiot $C > 0$ ja $0 < r_0 < 1$, joilla pätee epäyhtälö

$$|u(x) - u(0)| \le Cr^{\frac{1}{2}} \quad \text{kaikilla} \quad x \in B_r \quad \text{ja} \quad 0 < r < r_0.$$

Tämä arvio on Davidin ja Légerin alkuperäisen tuloksen (2002, propositio 10.17) paikallinen versio. Tulos perustuu tiettyyn kahtiajakoon sekä joukon $\Omega \setminus \overline{S}_u$ Johnin rakenteeseen, toisin kuin Davidin–Légerin (2002) ja Bonnet'n–Davidin (2001, lemma 21.3) vastaavat.

## 1. Introduction

The Mumford–Shah functional, introduced by Mumford and Shah in [21], is a well–known model in image processing. In their seminal paper [14], De Giorgi, Carriero, and Leaci established the existence of minimizers for a weaker formulation of the Mumford–Shah problem through direct methods, drawing on a lower semicontinuity result by De Giorgi and Ambrosio [13].

To be more specific, for any bounded domain $\Omega \subset \mathbb{R}^n$, it was introduced in [13] a subspace of $BV(\Omega)$, denoted by $SBV(\Omega)$, in which the functions only has jump discontinuities (see Section 2 for more details). Then for a function $u \in SBV(\Omega)$, its $\lambda$-Mumford–Shah energy on an open set $\Omega \subseteq \mathbb{R}^n$ is defined by

$$MS_\lambda(u, \Omega) := \int_\Omega |Du|^2 \, dx + \lambda \mathcal{H}^{n-1} (S_u \cap \Omega),$$

where $\lambda > 0$ and $S_u \subset \Omega$ is the set of discontinuity points of $u$. A function $u \in SBV_{\rm loc}(\Omega)$ is a local $\lambda$-minimizer if for any $x \in \Omega$ with $B_r(x) \subset \Omega$, $r > 0$ and every

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open set $U \subset\subset \Omega \cap B_r(x)$, we have $MS_\lambda(u, U) < \infty$ together with

$$MS_\lambda(u, U) \le MS_\lambda(v, U),$$

whenever $\{u \neq v\} \subset\subset U$.

The Euler–Lagrange equation [3, Theorem 7.35] for a local minimizer is that, for any $\eta \in C_0^1(\Omega; \mathbb{R}^n)$,

$$\int_{\Omega\setminus\overline{S}_u} |Du|^2 \operatorname{div} \eta - 2\langle \nabla u, \nabla u \cdot \nabla \eta \rangle \, dx + \lambda \int_{S_u} \operatorname{div}_\tau \eta \, d\mathcal{H}^{n-1} = 0, \tag{1.1}$$

where $\operatorname{div}_\tau$ denotes the tangential divergence. This in particular implies that $u$ is harmonic in $\Omega \setminus \overline{S}_u$ and satisfies the zero Neumann boundary value condition on both sides of $\overline{S}_u$; see [3, (7.42)]. Moreover the (weak) mean curvature of $\overline{S}_u$ equals to the jump of the gradient $[|Du|^2]^{\pm}$, according to [3, Theorem 7.38].

Ambrosio, Fusco and Pallara proved in [4, 2] that, when $\Omega \subset \mathbb{R}^n$ and $u \in SBV(\Omega)$ is a local Mumford–Shah minimizer, there exists a subset $\Sigma \subset \overline{S}_u$, which is relatively closed and $\mathcal{H}^{n-1}(\Sigma) = 0$, such that the set $\overline{S}_u \setminus \Sigma$ is the union of $C^{1,\frac{1}{4}}$-hypersurfaces. Moreover, both $u$ and $Du$ have a Hölder continuous extension to $\overline{S}_u \setminus \Sigma$. For more details, see also [3, Theorem 8.1], along with the survey [18], and [1, 17, 20] for more recent results.

Nevertheless, much more is known about the planar case. In particular, in an earlier result [7], Bonnet proved that an isolated connected component of $\overline{S}_u$ is a finite union of $C^{1,1}$-arcs. His result is based on the so–called Bonnet's monotonicity formula, which is applicable when the discontinuity set in the plane is connected. Later, David [10] demonstrated a version of $\epsilon$-regularity for the minimizers, and many additional results regarding Mumford–Shah local minimizers in the plane were established by Bonnet and David in [8].

In particular, in the monograph [11, Theorem 69.29], David proved that if $\overline{S}_u$ at a tip point $x \in \overline{S}_u \setminus S_u$ with $B(x, 2) =: B_2(x) \subset \Omega$, is sufficiently close to a single radius in the Hausdorff distance within (which implies $\overline{S}_u$ is connected in $B_1(x)$ [11, Lemma 69.8]), then $\overline{S}_u$ is locally $C^{1,1}_{\text{loc}}$ in $B_1(x)$. Recently, this result was independently improved by [5] and [16], showing that one actually obtains $C^{1,1}$ (and even $C^{2,\alpha}$)-regularity up to the end point of $\overline{S}_u$, again under the assumption that $\overline{S}_u \cap B_2(x)$ is sufficiently close to a single radius in the Hausdorff distance.

The planar results mentioned above generally rely on the a priori assumption that $\overline{S}_u$ is connected. In this paper, we examine the growth of $u$ near a tip point *without* assuming the connectedness of the discontinuity set. This estimate serves as a localized version of the original estimate for global minimizers presented in [12, Proposition 10.17], via a completely different argument; readers may also refer to [15, Proposition 4.6.1].

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $u \in SBV(\Omega)$ be a local minimizer for the Mumford–Shah problem, and $0 \in \Omega$ be a tip point, i.e. $0 \in \overline{S}_u \setminus S_u$. Suppose that $B_1 \subset \Omega$. Then there exist absolute constants $C > 0$ and $0 < r_0 < 1$ so that, for any $0 < r < r_0$, one has*

$$|u(x) - u(0)| \le C r^{\frac{1}{2}} \quad \text{for any } x \in B_r.$$

Our estimate is not straightforward because $u \in SBV(\Omega)$ might not satisfy a Poincaré-type inequality, even if we can estimate the growth of the $L^2$-norm of $Du$ within every disk (see Lemma 2.1). The discontinuity of $u$ poses a significant challenge.

It is worth noting that the harmonic conjugate $v$ of $u$ can be defined with Hölder continuity exponent $\frac{1}{2}$. This modulus of continuity follows directly from Lemma 2.1 as $v \in W^{1,2}(\Omega)$; see e.g. [15, Proposition 4.5.1].

**Remark 1.2.** Some readers might compare our proof with the proof of Lemma 21.3 in [8]. However, it is important to note that they are not the same.

Firstly, our estimate is directly based on an Ahlfors regularity lemma (see Lemma 2.1 below), which provides an $L^2$-estimate of $|Du|$. In contrast, the proof of Lemma 21.3 in [8] relies on an $L^\infty$-estimate, as seen in equation [8, (21.1)].

Secondly, the proof of Bonnet–David assumes that $x$ and 0 are in the same component of $B \setminus \overline{S}_u$ (see [8, (21.2)]). However, this assumption is addressed in our work through Corollary 1.4 below.

More precisely, our Lemma 3.1 and Lemma 3.2 may share some similarities with the proof of [8, Lemma 21.3]. To some extent, Lemma 3.1 can be seen as an $L^2$-version of [8, (21.14), Lemma 21.3]. Similarly, the finiteness of $N$ in Lemma 3.2 is related to the finiteness of the set $E$ in the proof of [8, Lemma 21.3]. To handle the $L^2$-integral directly, one needs to use carrot John subdomains (or Boman chains). Therefore, we must demonstrate that one can cover $B_r \setminus \overline{S}_u$ with a uniformly finite number of carrot John subdomains whose closures contain 0, provided that $r$ is smaller than some uniform constant $r_0 > 0$. This part serves as an application of a recent work in [23, Theorem 1.7]; it was somehow implicitly used in [8, Lemma 21.3].

The proof of the theorem relies on the dichotomy in Proposition 1.3, which offers a criterion for distinguishing between tip points and jump points. We will defer the proof of this proposition to Section 2.

**Proposition 1.3.** *Let $u \in SBV(\Omega)$ be a local minimizer in the plane, which is not locally constant, and $x \in K := \overline{S}_u$. We define for $r > 0$ with $B_r \subset \Omega$,*

$$\Phi_u(x,\, r) := \frac{r^{-1} \int_{B_r} |Du|^2 \, dx}{\inf_{c \in \mathbb{R}} \; ⨍_{B_r} |u - c|^2 \, dx}.$$

*Then $x$ is a jump point if and only if*

$$\Phi_u(x) := \Phi_u(x,\, 0^+) := \limsup_{r \to 0^+} \frac{r^{-1} \int_{B_r} |Du|^2 \, dx}{\inf_{c \in \mathbb{R}} \; ⨍_{B_r} |u - c|^2 \, dx} \quad \text{is finite.} \tag{1.2}$$

*In other words, $x$ is a tip point if and only if $\Phi_u(x,\, 0^+) = +\infty$.*

Note that $\Phi_u(\cdot)$ takes values in $\mathbb{R}^+ \cup \{+\infty\}$, and $\Phi_u(x,\, r)$ is continuous with repsect to $x$ for each fixed $r > 0$. In addition,

$$\Phi_u(x,\, 0^+) = \lim_{s \to 0} \sup_{0 < r < s} \Phi_u(x,\, r)$$

is the limit of a monotone non-increasing sequence as $r \to 0$. Thus $\Phi_u(\cdot)$ is upper semicontinuous. Thus we conclude the following corollary from Proposition 1.3.

**Corollary 1.4.** *The limit of a sequence of tip points must be a tip point. Especially, when 0 is a tip point of a local minimizer $u \in SBV(\Omega)$ and $B_1 \subset \Omega$, the compactness of Mumford–Shah problem yields that, there exists $r_2 > 0$ such that*

$$\operatorname{dist}(0,\, V) \geq r_2$$

*where $V$ is a connected component of $B_1 \setminus \overline{S}_u$ for which $0 \notin \partial V$. See Figure 1.*

We remark that, according to [15, Theorem 6.1.1] and Corollary 1.4, the only missing part towards Mumford–Shah conjecture (see e.g. [15, Conjecture 1.2.1]) is that, the set of tip points of $u$ is discrete in $\Omega$.

In the last section, we prove a Morrey-type estimate in John domains, Lemma 3.1, and specifically, apply a version of [23, Theorem 1.7] to cover a neighborhood of the origin with uniformly finitely many John subdomains. These results finally give us the desired estimate.

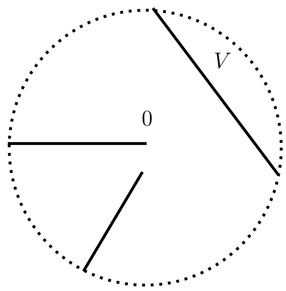


Figure 1. The black lines represents the discontinuity set $K$ of $u$. Corollary 1.4 yields that the component $V$ is uniformly away from 0.

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## 2. Preliminaries and Proof of Proposition 1.3

Let us fix some notation. We denote the $k$-dimensional Hausdorff measure by $\mathcal{H}^k$. For a given open set $\Omega \subset \mathbb{R}^n$, we denote by $BV(\Omega)$ the space of functions of bounded variation in $\Omega$, whose weak (or distributional) gradient $Du$ is a vector-valued Radon measure. We write $D^a u$ for the absolutely continuous part of $Du$ and $D^s u$ for its singular part. The set of approximately continuous points of $u$ is denoted by $\mathcal{C}_u$, each of which is a Lebesgue point of $u$. We write $S_u = \Omega \setminus \mathcal{C}_u$ the Borel set of approximate discontinuity of $u$. Furthermore, for $\mathcal{H}^{n-1}$-almost every $x \in S_u$, there exists a direction $\nu_u \in \mathbb{S}^{n-1}$ and two numbers $u_\pm \in \mathbb{R}$ so that $u_-(x) < u_+(x)$ and

$$\lim_{r\to 0} \int_{B_r^+(x,\nu_u(x))} |u(y) - u_+(x)|\, dy = 0,$$

together with

$$\lim_{r\to 0} \int_{B_r^-(x,\nu_u(x))} |u(y) - u_-(x)|\, dy = 0,$$

where

$$B_r^+(x,\, \nu_u(x)) = \{y \in B_r(x) \colon (y-x)\cdot \nu_u(x) > 0\},$$

and $B_r^-$ is defined similarly. These points are called *jump points* of $u$. When

$$D^s u = (u_+ - u_-)\nu_u \mathcal{H}^{n-1}|_{S_u},$$

then $u \in SBV(\Omega)$, the space of *special functions of bounded variation*. When $u \in SBV(\Omega)$ is a local minimizer, we usually denote by $K$ the closure of its jump set $\overline{S}_u$ unless explicitly stated otherwise. We say $x$ is a *tip point* of $u$ if $x \in \overline{S}_u \setminus S_u$, which is a Lebesgue point of $u$, particularly.

For ease of reference, we assume $\lambda = 1$ in the Mumford–Shah energy throughout the paper, unless explicitly stated otherwise. Moreover, for the ease of readablity, we suppress the subindex and write $MS := MS_1$. Suppose $u \in SBV(B_r)$ is a local 1-minimizer with $0 \in \overline{S}_u$, then it follows from [3, Remark 7.13] that

$$u_r(x) = r^{-\frac{n-1}{n}} u(rx) \in SBV(B_1)$$

is also a local 1-minimizer in $B_1$. Moreover,

$$MS(u,\, B_r) = r^{n-1} MS(u_r,\, B_1).$$

For a (rectifiable) curve $\gamma$, we denote by $\ell(\gamma)$ the Euclidean length of $\gamma$. When $\gamma$ is an arc (i.e. an injective curve), for any pair of points $x,\, y \in \gamma$, denote by $\gamma[x,\, y]$ a subarc joining them. For a measurable set $A \subset \mathbb{R}^2$, we write

$$\fint_A u\, dx := \frac{1}{|A|} \int_A u\, dx.$$

Let us begin with the following result, called the Ahlfors regularity of the local minimizer, which holds for every point in $K$. It is clear that the density

$$r^{-1} \mathcal{H}^1 \left(K \cap B_r\right)$$

is invariant under scaling. From this point forward, we focus exclusively on the planar case, specifically when $n = 2$.

**Lemma 2.1.** [18, Theorem 2.6], [9, Corollary 3.3] *Suppose that $u \in SBV(\Omega)$ is a local minimizer and $0 \in K$. There exists a constant $r_1 > 0$ so that for each $0 < r < r_1$ and $B_r \subset \Omega$, we have*

$$\frac{1}{C} \le r^{-1} \mathcal{H}^1 \left(K \cap B_r\right) \le C,$$

*and*

$$\int_{B_r} |Du|^2\, dx \le Cr.$$

We also record the following $\epsilon$-regularity of David, which provides a criterion to individuate tip points.

**Lemma 2.2.** [11, Proposition 60.1] *Suppose that $u$ is a minimizer and $0 \in K$. There exists $\epsilon_0 > 0$ and $\eta > 0$ so that, whenever $x \in K$ satisfies*

$$\int_{B_r(x)} |Du|^2\, dx \le \epsilon_0 r,$$

*then $x$ is a jump point and $K \cap B_{\eta r(x)}$ is a $C^1$-curve or a $C^1$-spider (see* [11, Section 51 & 53] *for the definitions).*

Now we are ready to show Proposition 1.3.

*Proof of Proposition 1.3.* Up to a translation, we may assume that $x$ is the origin. If (1.2) holds, then for some $c_0 > 0$ and any sequence $r_k \to 0$ we have

$$\limsup_{k\to\infty} r_k^{-1} \int_{B_{r_k}} |Du|^2\, dx \le c_0 \limsup_{k\to\infty} \left( \inf_{c\in\mathbb{R}} \fint_{B_{r_k}} |u - c|^2\, dx \right).$$

Suppose that, on the contrary, 0 is a tip point, which is particularly a Lebesgue point of $u$. Then the right-hand side of the inequality above goes to 0 as $k \to \infty$. This yields that

$$r^{-1}\int_{B_r}|Du|^2\,dx$$

is small whenever $r>0$ is sufficiently small. However, Lemma 2.2 implies that 0 cannot be a tip point. This leads to a contradiction and thus 0 must be a jump point.

Now suppose that (1.2) fails, then by Lemma 2.1, there exists a sequence $r_k\to 0$ so that for any $M>0$

$$\lim_{k\to\infty}\left(\inf_{c\in\mathbb{R}}\fint_{B_{r_k}}|u-c|^2\,dx\right)\le M^{-1}\lim_{k\to\infty}r_k^{-1}\int_{B_{r_k}}|Du|^2\,dx\le CM^{-1}.$$

Thus the origin is a Lebesgue point of $u$ as $M\to\infty$ (and $r_k\to 0$), which yields that 0 is a tip point. □

Let us also recall the definition of John domain.

**Definition 2.3.** For $J\ge 1$, a (bounded) domain $\Omega\subset\mathbb{R}^n$ is said to be $J$-John provided that, there exists a distinguished point $x_0\in\Omega$ so that, for every $x\in\Omega$, there exists an arc $\gamma\subset\Omega$ starting from $x$, ending at $x_0$ and satisfying the following condition:

$$\ell(\gamma[x,\,y])\le J\,\mathrm{dist}\,(y,\,\partial\Omega)\quad\text{for any } y\in\gamma, \tag{2.1}$$

where $\ell(\gamma[x,\,y])$ denotes the length of the subcurve of $\gamma$ joining $x$ to $y$. We usually call $x_0$ the John center of $\Omega$ and $\gamma$ the John curve joining $x_0$ and $x$.

Furthermore, let $\gamma\subset\Omega$ be a curve joining $x$ to $x_0$, and define the *J-carrot* with the vertex $x$ and the core $\gamma$ joining $x$ to $x_0$ as

$$\mathrm{car}(\gamma,J):=\bigcup\big\{B(y,\ell(\gamma[x,y])/J)\colon y\in\gamma\setminus\{x\}\big\}.$$

Then a (bounded) domain $\Omega\subset\mathbb{R}^n$ is $J$-John with the center $x_0\in\Omega\cup\{\infty\}$, if for each point $x\in\Omega$, there exists a curve $\beta\subset\Omega$ joining $x$ to $x_0$ so that $\mathrm{car}(\beta,J)\subset\Omega$.

We record the following result, which says that $\Omega\setminus K$ is locally John.

**Theorem 2.4.** [11, Proposition 68.16] *Let $u\in SBV(\Omega)$ be a local minimizer and $x\in\Omega\setminus K$. Then there exists an absolute constant $J\ge 1$ so that, for any*

$$0<r\le\frac{1}{2}\,\mathrm{dist}\,(x,\,\partial\Omega)=:r_3,$$

*one can find an arc $\gamma\subset\Omega\setminus K$ starting from $x$, escaping $B_r(x)$ (i.e. for a parametrization $\gamma\colon[0,\,1]\to\Omega$ with $\gamma(0)=0$, there exists $t\in(0,\,1)$ that $\gamma((t,\,1])\cap B_r(x)=\emptyset$), and satisfying that, for any $y\in\gamma$,*

$$\ell(\gamma[x,\,y])\le J\,\mathrm{dist}\,(y,\,K).$$

*In particular, there exists a ball with radius $J^{-1}r$ contained in $B_r(x)\cap(\Omega\setminus K)$.*

## 3. Proof of Theorem 1.1

Let $0\in\Omega$ be a tip point of $u$, and assume that $u(0)=0$ up to an additive constant. Recall that tip points are Lebesgue points of $u$, i.e.

$$\lim_{r\to 0}\fint_{B_r}|u|\,dx=0.$$

Write $K = \overline{S}_u$. Recall the definition of $J$-carrot. Then Theorem 2.4 implies that, for every $x \in \Omega \setminus K$, one can find a $J$-carrot contained in $\Omega \setminus K$ with core $\gamma$ and vertex $x$, which escapes $B_r(x)$.

The carrot condition above is equivalent to the following $M$-Boman chain condition, quantitatively: There exists $M \geq 1$ and a sequence of balls $\{U_i\}_{i=0}^{\infty}$ converging to $x$ so that,

- $U_0 = B_{\frac{\ell(\gamma)}{4J}}(x_0)$,
- $M^{-1} \operatorname{diam}(U_{i+1}) \leq \operatorname{diam}(U_i) \leq M \operatorname{diam}(U_{i+1})$,
- There exists $R_i \subset U_i \cap U_{i+1}$ so that $U_i \cup U_{i+1} \subset M R_i$,
- $\sum_i \chi_{U_i} \leq M$.

see e.g. [19, Theorem 9.3]. We record the following results for John domains.

**Lemma 3.1.** *Let $C_0 \geq 4$, $r > 0$ and $U \subset \mathbb{R}^2$ be a (bounded) $J'$-John domain with center $x_0 \in \partial B_{3r} \cap U$ such that $0 \in U \subset B_{C_0 r}$, $J' \geq 1$. Moreover, assume that there exists $C_1 > 0$ so that $u \in W^{1,2}(U)$ satisfies*

$$\int_{B_s(z) \cap U} |Du|^2 \, dx \leq C_1 s \tag{3.1}$$

*for any $z \in U$ and $0 < s < r$. Then*

$$|u(x) - u(y)| \leq C(C_0, C_1, J') r^{\frac{1}{2}}$$

*for almost every pair of points $x, y \in U \subset B_{5r}$.*

*Proof.* Fix $x, y \in U$ which are Lebegue points of $u$. Since carrot and Boman chain conditions are equivalent, we can find two sequences of balls $\{U_i\}_{i=0}^{+\infty}$ and $\{U_j\}_{j=0}^{-\infty}$ of $M$-Boman chains from $x$ and $y$ to $x_0$, respectively, where

$$M = M(J'), \quad U_0 = B_{c(C_0, J')r}(x_0), \quad U_i \to x \text{ as } i = \to +\infty, \quad U_j \to y \text{ as } j \to -\infty.$$

Then by writing in telescoping sum and applying Poincaré inequality on $u$, we have

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{k=-\infty}^{+\infty} \left| \fint_{U_k} u \, dx - \fint_{U_{k+1}} u \, dx \right| \\ &\leq \sum_{k=-\infty}^{+\infty} \fint_{U_k} \left| u - \fint_{U_{k+1}} u \right| dx \\ &\leq C(J') \sum_{k=-\infty}^{+\infty} \operatorname{diam}(U_k) \left( \fint_{U_k \cup U_{k+1}} |Du|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

As the assumption (3.1) gives

$$\fint_{U_k \cup U_{k+1}} |Du|^2 \, dx \leq C(C_1, J) \operatorname{diam}(U_k)^{-1},$$

it follows that

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} \operatorname{diam}(U_k) \left( \fint_{U_k \cup U_{k+1}} |Du|^2 \, dx \right)^{\frac{1}{2}} &\leq C(C_1, J') \sum_{k=-\infty}^{+\infty} \operatorname{diam}(U_k)^{\frac{1}{2}} \\ &\leq C(C_0, C_1, J') r^{\frac{1}{2}}, \end{aligned}$$

where we applied the fact that both $\{\operatorname{diam}(U_i)\}_{i=0}^{+\infty}$ and $\{\operatorname{diam}(U_j)\}_{j=0}^{-\infty}$ are geometric series in the last inequality. Then our lemma follows from the chain of inequalities. □

The following lemma is a local version of [23, Theorem 1.7] and we record its proof in Appendix A.

**Lemma 3.2.** *Suppose that $0 \in \overline{S}_u \subset \mathbb{R}^2$ is a tip point. Then for $r_3 > 0$ defined in Theorem 2.4 and any $0 < r < r_3/9$, there exist $C = C(J) > 0$, $N = N(J) \in \mathbb{N}$ and $J' = J'(J) > 0$ so that, we can cover $B_r \setminus \overline{S}_u$ by (the closure of) at most $N$-finitely many $J'$-John domains $W_{j,r} \subset B_{Cr} \setminus \overline{S}_u$, $1 \le j \le N$.*

*In particular, there exists a constant $C_2 = C_2(J) > 0$ such that, for every point $x \in W_{j,r}$, one can find a rectifiable curve $\beta_x \subset W_{j,r}$, which joins $x$ to a point $w_{j,r} \in \partial B_{3r} \cap W_{j,r}$, as the core of a $J'$-carrot satisfying*

$$\ell(\beta_x) \le C_2 r.$$

This lemma together with Lemma 3.1 and Lemma 3.2 implies the following theorem.

*Proof of Theorem 1.1.* Observe that Theorem 2.4 together with Lemma 3.2 implies that, for any $0 < r < r_3/9$, we can cover $B_r \setminus \overline{S}_u$ by at most $N$-finitely many (nonempty) $J'$-John domains $W_{j,r}$.

Now by recalling that $u(0) = 0$ and the $L^2$-estimate on the gradient from Lemma 2.1, we employ Lemma 3.1 to $u$ on each of the John domains $W_{j,r}$ to conclude that, for $0 < r < r_0 := \min\{r_1, r_2, r_3/9\}$,

$$|u(x) - u(0)| = |u(x)| \le C r^{\frac{1}{2}} \quad \text{for any} \quad x \in W_{j,r}.$$

Moreover, Corollary 1.4 yields that, for any $0 < r < r_0 \le r_2$, one has $0 \in \partial W_{j,r}$ for every $W_{j,r}$. This yields our desired estimate as the number of John domains is at most $N$. □

## Appendix A. Proof of Lemma 3.2

To start with, we record the following proposition. While this technique has been commonly employed in previous manuscripts, such as [22], it has not been explicitly formulated, to the best of our knowledge, in the context of our present work.

**Proposition A.1.** *Let $J \ge 1$. Assume that $\gamma \subset \mathbb{R}^2$ is a locally rectifiable curve joining $x$ to $y$, where $x, y \in \mathbb{R}^2$. Then $\operatorname{car}(\gamma, J)$ is a $J$-carrot John domain. To be more specific, for any $z \in \operatorname{car}(\gamma, J)$, we can find a rectifiable curve $\gamma_z$ joining $z$ to $y$, such that for some $\eta \in \gamma$, we have*

$$\gamma[\eta, y] = \gamma_z[\eta, y]$$

*and for each $a \in \gamma[\eta, y]$,*

$$\ell(\gamma_z[z,a]) \le \ell(\gamma[x,a]), \quad \operatorname{car}(\gamma_z, J) \subset \operatorname{car}(\gamma, J). \tag{A.1}$$

*Proof.* For any $z \in car(\gamma, J)$, the definition of $\operatorname{car}(\gamma, J)$ yields a ball

$$B(\eta, \ell(\gamma[x,\eta])/J) \subset \operatorname{car}(\gamma, J)$$

for some points $\eta \in \gamma \setminus \{x\}$ so that $z \in B(\eta, \ell(\gamma[x,\eta])/J)$.

Let $L_{z,\eta}$ be the line segment joining $z$ to $\eta$ and then $\gamma_z := L_{z,\eta} \cup \gamma[\eta, y]$ is locally rectifiable curve joining $z$ to $y$. When $a \in L_{z,\eta}$,

$$\ell(\gamma_z[z,a]) \le d\,(a, \partial B(\eta, \ell(\gamma[x,\eta])/J)) \le \ell(\gamma[x,\eta])/J. \tag{A.2}$$

When $a \in \gamma[\eta, y]$, by applying (A.2) with $a = \eta$ there, we have

$$\ell(\gamma_z[z,a]) \leq \ell(\gamma_z[z,\eta]) + \ell(\gamma_z[\eta,a]) \leq \frac{\ell(\gamma[x,\eta])}{J} + \ell(\gamma[\eta,a])$$
$$\leq \ell(\gamma[x,\eta]) + \ell(\gamma[\eta,a]) = \ell(\gamma[x,a]).$$

To conclude, we obtain that

$$\ell(\gamma_z[z,a]) \leq \ell(\gamma[x,a]),$$

which is the first formula of (A.1). The second one follows directly from out construction of $\mathrm{car}(\gamma_z, J)$ and $\mathrm{car}(\gamma, J)$, and we conclude the lemma. □

**A.1. A decomposition $V_{j,r}$ of $B_r \setminus K$.** Recall that $K = \overline{S}_u$. Now for any $x \in B_r \setminus K$ with $B_r \subset \Omega, 0 < r < r_0/9$, we choose an escaping (John) curve $\gamma_x \subset \Omega \setminus K$ from $x$ with $\mathrm{car}(\gamma, J) \subset B_r \setminus K$. Although there could be many choices of curves for $x \in \Omega \setminus K$, we just choose one of them. Let $\Gamma = \{\gamma_x\}_{x \in B_r \setminus K}$ be the collection of these chosen curves. In what follows, for any points $x \in B_r \setminus K$, $\gamma_x$ always refers to this particular choice of escaping curve.

Note that for $0 < r < r_0/9$, we have $B_r \cap K \neq \emptyset$ as $0 \in K$. Our first step is to decompose $B_r \setminus K$ into finitely many subsets $V_{j,r}$ so that, there exists a collection $\mathcal{B}_{j,r}$ of at most $C(J)$-many balls, whose center is on $\partial B_{3r}$ and whose radius is at least $J^{-1}r$, satisfying that, for any $x \in V_{j,r}$, we can find a ball $B \in \mathcal{B}_{j,r}$ with

$$\gamma_x \cap B \neq \emptyset.$$

To this end, observe that, according to Theorem 2.4, for each $x \in B_r \setminus K$ and $\gamma_x \in \Gamma$, there exists a point

$$x_r \in \gamma_x \cap \partial B_{3r} \tag{A.3}$$

so that

$$2r \leq \ell(\gamma[x, x_r]) \leq J \,\mathrm{dist}\,(x_r, K). \tag{A.4}$$

Consider the collection of closed balls

$$\{\overline{B}_x\}_{x \in B_r \setminus K} := \left\{ \overline{B}\left(x_r, \frac{\mathrm{dist}\,(x_r, K)}{2}\right)\right\}_{x \in B_r \setminus K}. \tag{A.5}$$

Then thanks to (A.4) and $0 \in K$, we obtain that

$$\frac{r}{J} \leq \frac{\mathrm{dist}\,(x_r, K)}{2} \leq \frac{3}{2} r, \tag{A.6}$$

and hence $B_x \cap B_r = \emptyset$.

We next let

$$A_r := \bigcup_{x \in \overline{B}_r \setminus K} \{x_r\}$$

be the collection of the centers of $B_x$'s. By Bescovitch's covering theorem, there exists a subcollection $\{\overline{B}_i\}_{i \in \mathbb{N}}$ of $\{\overline{B}_x\}_{x \in B_r \setminus K}$ consisting of at most countably many balls, such that

$$\chi_{A_r}(z) \leq \sum_{B_i} \chi_{\overline{B}_i}(z) \leq C, \qquad \forall z \in B_{5r} \setminus K. \tag{A.7}$$

Recall that by (A.6)

$$B_i \subset B_{5r} \setminus \overline{B}_r$$

and $|B_i| \geq c(J)r^n$. Thus we have at most $C(J)$-many elements in $\{\overline{B_i}\}$ by (A.7). As a result, the union of balls

$$\bigcup_i \overline{B}_i$$

has at most $N = N(J)$ components $U_{j,r}$ for $j \in \{1, \cdots, N_r\}$ and $N_r \leq N$. By defining $U_{j,r}$ to be empty for $j > N_r$, we may assume that there exists exactly $N$ components $U_{j,r}$, and each $U_{j,r}$ contains at most $\hat{N} = \hat{N}(J)$ balls. We write $\mathcal{B}_{j,r}$ as the collection of balls $B_i$ contained in each component $U_{j,r}$.

Now it follows from our construction, for any $x \in B_r \setminus K$, there exists some $1 \leq j \leq N_r$ so that, $x_r \in \gamma_x$ is covered by a ball in $\mathcal{B}_{j,r}$. Thus, by defining

$$V_{j,r} := \{x \in B_r \setminus K \colon x_r \in D \text{ for some } D \in \mathcal{B}_{j,r}\}, \tag{A.8}$$

we obtain the desired decomposition of $B_r \setminus K$. The set $V_{j,r}$ is defined to be empty if $U_{j,r}$ is empty.

The following lemma is a version of [23, Proposition 3.2].

**Proposition A.2.** *For $1 \leq j \leq N$ with $N = N(J)$ defined above, the set $W_{j,r}$ is either empty (if $V_{j,r}$ is empty), or for any fixed $y \in V_{j,r}$ together with the escaping point $y_r \in \partial B_{3r}$, the set*

$$W_{j,r} := \operatorname{car}(\gamma_y[y, y_r], J) \cup \bigcup_{x \in V_{j,r}} \operatorname{car}(\beta_x, J'). \tag{A.9}$$

*is a $J'$-John domain with John center $y_r$, where $J' = J'(J) > 0$ and $\beta_x$ is a rectifiable curve joining $x$ to $y_r$ satisfying $\gamma_x[x, x_r] \subset \beta_x$; recall that $\gamma_x$ is the escaping curve starting from $x$.*

*Moreover, there exists a constant $C_3 = C_3(J) \geq 4$ so that, the curve $\beta_x$ joining $x \in W_{j,r}$ to $y_r$ is the core of a $J'$-carrot satisfying*

$$\ell(\beta_x) \leq C_3 r$$

*and*

$$V_{j,r} \subset W_{j,r}, \quad \operatorname{car}(\beta_x, J') \subset W_{j,r} \subset (\Omega \setminus K) \cap B_{2C_3 r}. \tag{A.10}$$

*Proof.* Suppose that $V_{j,r}$ is non-empty, and fix a point $y \in V_{j,r}$. Then the corresponding escaping point $y_r \in \gamma_y \cap \partial B_{3r}$ is covered by some ball $D_1 \in \mathcal{B}_{j,r}$ according to (A.7). Then we can join $y_r$ to the center $\hat{x}_1$ of $D_1$ by the line segment $L_{y_r,\hat{x}_1} \subset D_1$.

Now for any $x \in V_{j,r}$, we claim that there exists a rectifiable curve $\beta_x \subset \Omega \setminus K$ as the core of a $J'$-carrot joining $x$ to $y_r$, such that $\gamma_x[x, x_r] \subset \beta_x$ and

$$\operatorname{car}(\beta_x, J') \subset \Omega \setminus K. \tag{A.11}$$

Indeed, the escaping point $x_r \in \partial B_{3r} \cap D_2$ is also covered by another ball $D_2 \in \mathcal{B}_{j,r}$ as $x \in V_{j,r}$. Moreover, we can also joint $x_r$ to the center $\hat{x}_2$ of $D_2$ by the line segment $L_{x_r,\hat{x}_2} \subset D_2$.

Recall that $U_{j,r}$ is connected and consists of at most $\hat{N}$-many balls from $\mathcal{B}_{j,r}$. This implies that $\hat{x}_1$ and $\hat{x}_2$ can be joined by a union of at most $\hat{N}$-many line segments with the endpoints being the centers of balls in $\mathcal{B}_{j,r}$. Therefore, combining with $L_{y_r,\hat{x}_1}$ and $L_{x_r,\hat{x}_2}$, we can join $x_r$ to $y_r$ by a polyline $\gamma_{x_r,y_r}$.

We show that

$$\beta_x := \gamma_x[x, x_r] \cup \gamma_{x_r, y_r}$$

is the desired John curve. To this end, we estimate the length of $\beta_x$ and the distance $\operatorname{dist}(\eta, K)$ for any $\eta \in \beta_x$, respectively.

We start with the estimate on the length of $\beta_x$. Thanks to (A.6), for any pair of intersecting balls $D, D' \in \mathcal{B}_{j,r}$, the line segments $L$ joining the center of $D$ with radius $s$ to the center of $D'$ with radius $s'$ satisfies

$$L \subset D \cup D' \quad \text{and} \quad \ell(L) \le s + s' \le 4r. \tag{A.12}$$

In particular, (A.6) together with the facts that $L_{x_r,\hat{x}_2} \subset D_2$ and that $L_{y_r,\hat{x}_1} \subset D_1$ also yields $\ell(L_{x_r,\hat{x}_2}) \le 2r$, $\ell(L_{y_r,\hat{x}_1}) \le 2r$. Therefore employing (A.12) and (A.4), the construction of $\beta_x$ tells

$$\begin{aligned}\ell(\beta_x) &\le \ell(\gamma_x[x,\, x_r]) + \ell(\gamma_{x_r,\, y_r}) \\ &\le J \operatorname{dist}(x_r,\, K) + \ell(L_{x_r,\hat{x}_2}) + \ell(L_{y_r,\hat{x}_1}) + 4\hat{N} r \\ &\le C(J) r =: C_3 r; \end{aligned} \tag{A.13}$$

we may assume that $C_3 \ge 4$. This gives the first part of the proposition.

Towards (A.10), for any $\eta \in \beta_x$ we need to estimate the distance $\operatorname{dist}(\eta, K)$ from above. First of all, note that when $\eta \in \gamma_{x_r,y_r}$, there exists some ball $D_\eta \in \mathcal{B}_{j,r}$ containing $\eta$. Then combining (A.4), (A.5) and (A.6), we get

$$\operatorname{dist}(\eta, K) \ge \operatorname{dist}(D_\eta,\, K) \ge \frac{r}{J}. \tag{A.14}$$

Let

$$J' := C_3 J. \tag{A.15}$$

Then combining (A.4), (A.13) and (A.14), we conclude

$$\ell(\beta_x[x,\eta]) \le \ell(\beta_x) \le C_3 r \le J' \operatorname{dist}(\eta, K) \quad \text{when } \eta \in \gamma_{x_r,\, y_r}.$$

On the other hand, when $\eta \in \gamma_x[x,\, x_r]$, since our construction yields $\beta_x[x,\, \eta] = \gamma_x[x,\, \eta]$, which is particularly contained in a John curve, it follows that

$$\ell(\beta_x[x,\eta]) \le J \operatorname{dist}(\eta,\, K) \le J' \operatorname{dist}(\eta, K).$$

This implies (A.11). Moreover by Proposition A.1, every point $w \in \operatorname{car}(\beta_x, J')$ also can be joined to $y_r$ by a rectifiable curve $\hat{\gamma}_w$ satisfying

$$\ell(\hat{\gamma}_w) \le \ell(\beta_x) \quad \text{and} \quad \operatorname{car}(\hat{\gamma}_w, J') \subset \operatorname{car}(\beta_x, J').$$

Hence, by employing (A.11), the arbitrariness of $x$ gives the second formula in (A.10). The first formula in (A.10) holds due to $x \in Cl(\operatorname{car}(\beta_x, J'))$, the closure of the carrot, for any $x \in V_{j,r}$. □

Now Lemma 3.2 follows immediately from Proposition A.2, where $w_{j,\,r}$ is chosen to be $y_r$ in the Proposition A.2.

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