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Carrot John domains in variational problems

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Abstract. In this paper, we explore carrot John domains within variational problems, dividing our examination into two distinct sections. The initial part is dedicated to establishing the lower semicontinuity of the (optimal) John constant with respect to Hausdorff convergence for bounded John domains. This result holds promising implications for both shape optimization problems and Teichmüller theory. In the subsequent section, we demonstrate that an unbounded open set satisfying the carrot John condition with a center at ∞, appearing in the Mumford–Shah problem, can be covered by a uniformly finite number of unbounded John domains (defined conventionally through cigars). These domains, in particular, support Sobolev–Poincaré inequalities.

Variaatio-ongelmiin liittyvät Johnin porkkana-alueet

Tiivistelmä. Tässä kahteen osaan jaetussa tutkimuksessa tarkastellaan variaatio-ongelmiin liittyviä Johnin porkkana-alueita. Alkuosassa osoitetaan (optimaalisen) Johnin vakion alapuolijatkuvuus rajallisten Johnin alueiden Hausdorffin suppenemisen suhteen. Tällä tuloksella on sekä muodon optimointiin että Teichmüllerin teoriaan liittyviä lupauksia. Toisessa osassa todistetaan, että Mumfordin–Shahin ongelmassa esiintyvä rajaton avoin joukko, joka toteuttaa ∞-keskisen Johnin porkkanaehdon, voidaan peittää tasaisesti äärellisellä määrällä (tavalliseen tapaan sikareiden avulla määriteltyjä) rajattomia Johnin alueita. Erityisesti näissä alueissa on voimassa Sobolevin–Poincarén epäyhtälöitä.

1. Introduction

In the realm of shape optimization problems, instances frequently arise wherein the objective is to identify the optimal class of sets, denoted as U, based on the ratio of functionals that incorporate the norm of a specific class of Sobolev functions u, the norm of its gradient Du, and the norm of its trace $u|_{\partial U}$ on ∂U. A prototypical illustration of such a scenario is the pursuit of the optimal sets $U \subset \mathbb{R}^n$ for the first p-Dirichlet eigenvalue, where for $1 < p < \infty$ and $a > 0$,

$$\min_{|U|=a} \left\{ \int_U |Du|^p \, dx \colon u \in W_0^{1,p}(U),\ \|u\|_{L^p(U)} = 1 \right\}.$$

According to the Rayleigh–Faber–Krahn inequality, it can be deduced that this quantity is not inferior to the corresponding Dirichlet eigenvalue of a Euclidean ball with a volume of a. Subsequent research, particularly through transportation techniques as explored in [18, 19], has revealed that *balls have the worst best Sobolev inequalities*. To be more specific, for any locally Lipschitz open domain Ω in $\mathbb{R}^n$ and $1 \le p < n$,

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we define

$$\Phi_\Omega^{(p)}(T) := \inf\Big\{\|\nabla f\|_{L^p(\Omega)} \colon \|f\|_{L^{p^*}(\Omega)} = 1,\ \|f\|_{L^{p^\#}(\partial\Omega)} = T,$$
$$f \in L^1_{\mathrm{loc}}(\Omega) \text{ with } \lim_{x\to\infty} f = 0\Big\},$$

where $p^* = \frac{np}{n-p}$ and $p^\# = \frac{(n-1)p}{np}$. Then the unit ball B has the lowest Φ-curve in the following sense:

$$\Phi_\Omega^{(p)}(T) \geq \Phi_B^{(p)}(T) \quad \text{on } [0, T_n(p)],$$

where $T_n(p) := \big(n|B|^{1/n}\big)^{1/p^\#}$ and $|B|$ is the Lebesgue measure of B. Additional insights and recent advancements in this domain can be found in [17] and its associated references.

Conversely, a distinctive category of domains, termed as *John domains*, supports for Sobolev–Poincaré inequalities. A (bounded) domain $\Omega \subset \mathbb{R}^n$ is J-John for some $J \geq 1$ if there exists a distinguished point $x_0 \in \Omega$ so that, for any $x \in \Omega$, one can find a curve $\gamma \subset \Omega$ joining x to x_0 satisfying

$$\ell(\gamma[x, y]) \leq Jd(y, \partial\Omega) \quad \text{for each } y \in \gamma,$$

where $\gamma[x, y]$ is the subcurve of γ joining x and y. The constant J is usually called the *John constant*. Heuristically speaking, Ω contains a uniformly linearly opened twisted cone at every $x \in \Omega$; see Figure 1. Standard examples of John domains encompass Lipschitz domains in any $\mathbb{R}^n$, and quasidisks in the plane, which include von Koch's snowflakes, see e.g. [11, Chapter 6]. Stemming from the definition of a John domain and the Lebesgue differentiation theorem, it can be deduced that the boundary of a John domain possesses a Lebesgue measure of 0.

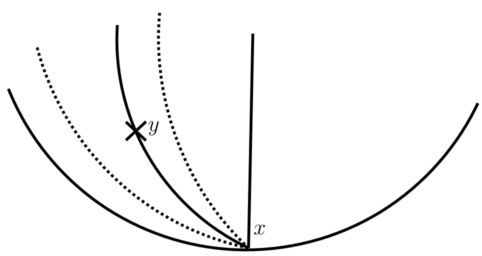

Figure 1. A domain Ω is John if, heuristically speaking, it contains a uniformly linearly opened twisted cone at every $x \in \Omega$.

For a domain $\Omega \subset \mathbb{R}^n$ supporting a (p, p^*)-Sobolev–Poincaré inequality for $1 \leq p < n$, it implies that, for every $u \in W^{1,p}(\Omega)$, one has

$$\inf_c \left(\int_\Omega |u - c|^{p^*}\, dx\right)^{\frac{1}{p^*}} \leq C(n, p, \Omega) \left(\int_\Omega |Du|^p\, dx\right)^{\frac{1}{p}},$$

where $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent. For comprehensive studies on the Sobolev–Poincaré inequality, we recommend consulting [4] and [12]. Furthermore, for an exploration of this inequality in the context of general metric measure spaces, encompassing Carnot groups, [13] serves as a valuable resource. Moreover, in the specific context of John domains, which inherently support a Poincaré inequality, one can also establish trace inequalities for Sobolev functions with additional assumptions. For example, a type of Poincaré inequality [3, Theorem 4.4] holds when the domain is inner uniform (note that an inner uniform domain is John). Then the results in [20], [15] and [21] yield the desired trace inequalities in this domain.

In contrast, Buckley and Koskela, as shown in [7], revealed that a domain $\Omega \subset \mathbb{R}^n$, possessing finite volume and adhering to a ball-separation property, supports Sobolev–Poincaré inequalities. This characteristic is particularly evident in scenarios involving conformal deformations, as discussed in [5, 2], such as bounded and simply connected domains in the plane. The implications of this discovery underscore a deep connection between shape optimization problems involving Sobolev–Poincaré inequalities and the concept of John domains.

This correlation prompts the need for a refined definition of the John constant, one that can be extended to arbitrary Euclidean domains. The ensuing definition is motivated by this imperative, and it is formulated to accommodate general Minkowski norms (defined at the beginning of Section 2) in $\mathbb{R}^n$ for potential applications in some other forthcoming research endeavors.

1.1. General Minkowski norm. In a recent manuscript [23], the authors presented an alternative proof of the seminal result obtained by Figalli, Maggi, and Pratelli [10], on the stability of isoperimetric inequality with respect to a general Minkowski norm, utilizing the John property of (almost) minimal surfaces. Partially motivated by this work, we consider John domains within the context of a general Minkowski norm in the first part of our manuscript.

Some basic notations need to be clarified here. A function

$$\|\cdot\|\colon \mathbb{R}^n \to \mathbb{R}_+$$

is a general Minkowski norm if it satisfies

$$\begin{aligned} \|x+y\| &\le \|x\| + \|y\|, \quad \forall x, y \in \mathbb{R}^n, \\ \|\lambda x\| &= \lambda \|x\|, \quad \forall x \in \mathbb{R}^n,\ \lambda > 0, \end{aligned}$$

and

$$\|x\| = 0 \quad \text{if and only if} \quad x = 0;$$

see e.g. [1, Section 2.1]. Specifically, the standard Euclidean norm is denoted by $|\cdot|$. Naturally, there exists a convex body

$$\mathcal{K}_{\|\cdot\|} := \{x \in \mathbb{R}^n : \|x\| < 1\}$$

associated to $\|\cdot\|$.

For a non-empty open set $\Omega \subsetneq \mathbb{R}^n$ and $x \in \Omega$, we denote by $\partial\Omega$ the topological boundary of Ω. We write

$$d_{\|\cdot\|}(x, \partial\Omega) := \inf_{y \in \partial\Omega} \|x - y\|,$$

and when the norm is the standard Euclidean one, we simply write

$$d(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y|.$$

For $x \in \mathbb{R}^n$ and $r > 0$, we use the notation

$$B_{\|\cdot\|}(x, r) := \{y \in \mathbb{R}^n \colon \|x - y\| < r\}$$

and by $\overline{B}_{\|\cdot\|}(x, r)$ its closure. We drop the subindices and write $B(x, r)$ when the norm is the standard Euclidean norm. Especially, we denote the ball $B_{\|\cdot\|}(0, r)$ centered at 0 by $B_{\|\cdot\|,r}$ for brevity.

Suppose that $(X, \|\cdot\|)$ is a general Minkowski space and $\gamma \subset X$ is a rectifiable curve. Using reparametrization, γ can be seen as a homeomorphism

$$\gamma\colon [0,1] \to X, \quad t \mapsto \gamma(t).$$

For every two distinct points $a_1, a_2 \in \gamma$, there exists $t_1, t_2 \in [0,1]$, such that $a_i = \gamma(t_i)$ for $i \in \{1,2\}$. We may assume $t_1 < t_2$. Then we denote the subcurve $\gamma([t_1,t_2])$ joining a_1 to a_2 by $\gamma[a_1,a_2]$. Under the assumption above, the length of the rectifiable curve $\gamma \subset X$ is written as

$$\ell_{\|\cdot\|}(\gamma) = \sup\left\{\sum_{i=0}^{N-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| : 0 = a_0 < a_1 < a_2 < \cdots < a_N = 1, \quad N \in \mathbb{N}^+\right\}.$$

If γ is the union of curves, then $\ell_{\|\cdot\|}(\gamma)$ denotes the sum of the length of these curves under the same parametrization.

Definition 1.1. For a general Minkowski norm $\|\cdot\|$ and $J \geq 1$, a (bounded) domain $\Omega \subset \mathbb{R}^n$ is J-John if there exists a distinguished point $x_0 \in \Omega$ so that, for any $x \in \Omega$, one can find a curve $\gamma \subset \Omega$ joining x to x_0 satisfying

$$\ell_{\|\cdot\|}(\gamma[x,\, y]) \leq J d_{\|\cdot\|}(y,\, \partial\Omega) \quad \text{for each } y \in \gamma.$$

Set

$$C_{\|\cdot\|} := \max_{\|x\|=1} \|-x\|. \tag{1.1}$$

We emphasize here that the value of $C_{\|\cdot\|}$ plays a crucial role in determining whether $\|\cdot\|$ constitutes a norm, as well as in influencing the length of the curve.

Remark 1.2. When $C_{\|\cdot\|} = 1$, $\|\cdot\|$ satisfies the properties of a norm. Conversely, if $C_{\|\cdot\|} \neq 1$, the convex body $\mathcal{K}_{\|\cdot\|}$ associated to $\|\cdot\|$ loses its symmetry relative to the origin. In such instances, the length of curves becomes dependent on their parametrized direction.

A straightforward illustration is the case where, for some point $x_0 \in \mathcal{K}_{\|\cdot\|}$ with $-x_0 \notin \mathcal{K}_{\|\cdot\|}$. Then the length of the line segment parametrized from 0 to x_0 is smaller than 1, while the one in the reverse direction is larger than 1.

Definition 1.3. Consider the Euclidean space $(\mathbb{R}^n,\, \|\cdot\|)$ endowed with a general Minkowski norm $\|\cdot\|$, and let $\Omega \subset \mathbb{R}^n$ be a (bounded) domain. Then for any $x \in \Omega$ and a curve $\gamma \subset \Omega$ containing x and parametrized[1] as $\gamma\colon [0,\, 1] \to \Omega$, we define a function $j(\cdot; x, \gamma, \Omega)\colon [0,1] \to \mathbb{R}$ as

$$j(t; x, \gamma, \Omega) := \frac{\ell_{\|\cdot\|}(\gamma([0,\, t]))}{d_{\|\cdot\|}(\gamma(t), \partial\Omega)} \quad \text{for any } t \in [0,1].$$

Subsequently, we set

$$J(x, \Omega; x_0) := \inf_{\beta \subset \Omega}\left\{\sup_{t\in[0,1]} j(t\,; x, \beta, \Omega)\colon \beta \subset \Omega \text{ is a curve joining } x \text{ to } x_0 \in \Omega\right\},$$

and

$$J(\Omega; x_0) := \sup_{x\in\Omega} J(x, \Omega; x_0).$$

We say that Ω satisfies the J-carrot John condition with center $x_0 \in \Omega$ if

$$J = J(\Omega; x_0) < \infty.$$

We define John$(\cdot)$ on the collection of bounded domains of $\mathbb{R}^n$ as

$$\text{John}(\Omega) := \inf_{x_0\in\Omega}\{J(\Omega; x_0)\},$$

[1] We employ the standard abuse of notation here, using the same symbol for both the map and its image of a curve.

and designate John(Ω) as the (optimal) John constant of Ω.

By the definition of John($\cdot$) and the definition of John domain, it follows that $\Omega \subset \mathbb{R}^n$ is a John domain with center $x_0 \in \Omega$ if and only if John(Ω) $< +\infty$.

In the pursuit of broader applications, we extend the definition of the J-carrot to a pair of suitable points x, $x_0 \in \dot{\mathbb{R}}^n$, where $\dot{\mathbb{R}}^n$ represents the one-point compactification of $\mathbb{R}^n$.

Definition 1.4. Let $x \in \mathbb{R}^n$ and $x_0 \in \dot{\mathbb{R}}^n$ be distinct points and $\gamma \subset \mathbb{R}^n$ be a curve joining x toward x_0. Assume that $J \geq 1$. When $x_0 \neq \infty$, we define

$$\operatorname{car}(\gamma, J) := \bigcup \big\{ B_{\|\cdot\|}(y, \ell_{\|\cdot\|}(\gamma[x,y])/J) \colon y \in \gamma \setminus \{x\} \big\},$$

while when $x_0 = \infty$, we define

$$\operatorname{car}(\gamma, J) := \bigcup \big\{ B_{\|\cdot\|}(y, \ell_{\|\cdot\|}(\gamma[x,y])/J) \colon y \in \gamma \setminus \{x, \infty\} \big\}.$$

Then the (open) set $\operatorname{car}(\gamma, J)$ is called the *J-carrot*, with core γ and vertex x, joining x to x_0.

We say an open set $\Omega \subset \mathbb{R}^n$ satisfies J-carrot John condition with center $x_0 \in \Omega \cup \{\infty\}$, if for each point $x \in \Omega$, there exists a curve $\beta \subset \Omega$ joining x toward x_0 so that $\operatorname{car}(\beta, J) \subset \Omega$. Furthermore, if Ω also satisfies connectivity, we say that Ω is a J-carrot John domain.

Remark 1.5. It is noteworthy that in the definition of $\operatorname{car}(\gamma,\, J)$, one has the flexibility to substitute $\ell_{\|\cdot\|}(\gamma[x,\, y])$ by either $\operatorname{diam}_{\|\cdot\|}(\gamma[x,\, y])$ or simply $\|y - x\|$. Importantly, these alternative formulations are equivalent in both bounded and unbounded scenarios, as elucidated in, for instance, [22, Theorem 2.14].

Remark 1.6. In the literature, an alternative definition of the John domain is sometimes employed, where the term "J-carrot" is replaced by the so-called "J-cigar". To elucidate, when considering a pair of distinct points x, $y \in \mathbb{R}^n$ and a curve $\beta \subset \mathbb{R}^n$ containing x and y, the "J-cigar" is defined as:

$$\operatorname{cig}(\beta, J) := \bigcup \big\{ B_{\|\cdot\|}(\eta, \rho(\eta)/J) \colon \eta \in \beta \setminus \{x, y\} \big\},$$

where

$$\rho(\eta) = \min \big\{ \ell_{\|\cdot\|}(\beta[x,\eta]), \ell_{\|\cdot\|}(\beta[y,\eta]) \big\}.$$

The set $\operatorname{cig}(\beta, J)$ is called the J-cigar with core β joining x and y, and Ω is J-cigar John if each pair of points x, $y \in \Omega$ can be joined by a curve $\beta \subset \Omega$ satisfying $\operatorname{cig}(\beta, J) \subset \Omega$. Heuristically speaking, in the bounded case, one can interpret a J-cigar as the union of two J-carrots. Indeed, it has been rigorously established that when $\Omega \subset \mathbb{R}^n$ is bounded, these two definitions, employing either the J-carrot or the J-cigar, are equivalent; refer to, for example, [22, Theorem 2.16], and also Lemma 3.4 in the manuscript. In addition, for a discussion of the unbounded case, see Remark 1.8.

1.2. Bounded John domains. Now we are prepared to articulate our first theorem.

Theorem 1.7. (Lower-semicontinuity of (optimal) John constants) *Let $J_0 \geq 2$ and assume that $\{\Omega_j\}_{j \in \mathbb{N}^+}$ is a sequence of uniformly bounded John domains satisfying*

$$\operatorname{John}(\Omega_j) \leq J_0 \quad \textit{and} \quad |\Omega_j| \geq c_0 |B_{\|\cdot\|}(0,1)|,$$

for some $c_0 > 0$. Then up to passing to a subsequence, $\overline{\Omega}_j$ converges to some compact set $A \subset \mathbb{R}^n$ in the Hausdorff distance d_H so that the interior Ω of A satisfies

(i) $\max_{x\in\Omega} d_{\|\cdot\|}(x, \partial\Omega) \geq c = c(n, C_{\|\cdot\|}, J_0, c_0) > 0$, *where $C_{\|\cdot\|}$ is defined in* (1.1).

(ii) *Ω is a John domain with*

$$\mathrm{John}(\Omega) \leq \liminf_{j\to\infty} \mathrm{John}(\Omega_j).$$

In Theorem 1.7, one can only anticipate lower semicontinuity, not continuity. To illustrate, consider the sequence of sets

$$\Omega_k := \mathbb{D} \setminus [0, 1] \times [-2^{-k}, 2^{-k}], \quad k \geq 1,$$

where $\mathbb{D}$ denotes the unit disk in the plane. Then $\mathrm{John}(\Omega_k)$ is uniformly bounded below, away from 1, while the limit of $\overline{\Omega}_k$ is $\overline{\mathbb{D}}$ as $k \to \infty$, whose interior has an (optimal) John constant of 1.

Corollary 1.8. *For $R \gg |D|$ satisfying that $D \subset B_{\|\cdot\|, R}$,*

$$\min\left\{\mathrm{John}(\Omega)\colon |\Omega| = |D|,\ \Omega \subset B_{\|\cdot\|, R}\right\}$$

has a solution, where $D = -\mathcal{K}_{\|\cdot\|}$. Moreover, the set of minimizers precisely consists of translations of D.

Proof. Let Ω_k be a minimizing sequence. As $\partial\Omega_k$ has Lebesgue measure 0,

$$|\Omega_k| = |\overline{\Omega}_k|.$$

Then as a direct consequence of Theorem 1.7, up to passing to a subsequence, $\overline{\Omega}_k \to \overline{\Omega}$ for some open set $\Omega \subset B_{\|\cdot\|, R}$, together with

$$\mathrm{John}(\Omega) \leq \liminf_{k\to\infty} \mathrm{John}(\Omega_k) \leq J$$

for some $J \geq 1$. Moreover, by [26, Theorem 2.8][2], Lebesgue measure is continuous with respect to the Hausdorff metric for J-carrot John domains. Thus Ω is a desired minimizer.

Now we show that a minimizer Ω must be a translation of D. Indeed, since $\mathrm{John}(\Omega) \geq 1$ and $\mathrm{John}(D) = 1$, then it follows that $\mathrm{John}(\Omega) = 1$. Now by the definition of $\mathrm{John}(\Omega)$ and Lemma 2.5, saying that the infimum of x_0 is taken away from the boundary, we conclude that for any $y \in \Omega$

$$\|y - x_0\| \leq \ell_{\|\cdot\|}(\gamma_{y, x_0}) \leq d_{\|\cdot\|}(x_0, \partial\Omega),$$

where γ_{y, x_0} is a John curve joining y to x_0 given by Lemma 2.2. Thus Ω is a translation of D. □

We expect that this outcome is intricately connected to the observation that "balls have the worst best Sobolev inequalities". In contrast, it was proven that a Jordan domain $\Omega \subset \mathbb{R}^2$ qualifies as a quasidisk if and only if both Ω and its complementary domain are John domains, as documented in [22] and [11, Theorem 6.12].

[2]Even though in [26, Theorem 2.8] it is only proved that for a J-carrot John domain $U \subset \mathbb{R}^n$ with $\mathrm{diam}(U) \leq 1$,

$$|\{x \in U\colon d(x, \partial U) < t\}| \leq \mu(t, J, n) \to 0 \quad \text{as } t \to 0.$$

However, it follows from a similar argument that for a bounded J-carrot John domain $U \subset \mathbb{R}^n$ with $|U| \leq M$, where M is a positive constant,

$$|\{x \in \mathbb{R}^n\colon d(x, \partial U) < t\}| \leq \mu(t, J, n, M) \to 0 \quad \text{as } t \to 0.$$

This, coupled with the fact that $\overline{\Omega}_k$ forms a Cauchy sequence in terms of the Hausdorff distance, leads us to the desired conclusion.

Consequently, considering the role of normalized quasidisks in modeling the universal Teichmüller space [16, Section III.1.5], Theorem 1.7 not only enables the exploration of extremal maps in quasiconformal mappings but also offers insights into the properties of quasidisks.

1.3. Unbounded open sets satisfying carrot John condition with center ∞. Advancing in our exploration, we turn our attention to the J-carrot John condition for unbounded domains $\Omega \subset \mathbb{R}^n$ (with unbounded $\partial\Omega$). Namely, for any $x \in \Omega$, there exists a curve $\gamma \subset \Omega$ from x toward ∞ in such a way that the infinite J-carrot

$$\operatorname{car}(\gamma,\, J) \subset \Omega.$$

Such domains find relevance in the exploration of the Mumford–Shah problem, as expounded in, for instance, [6, Section 19] and [9, Section 56, Proposition 7]. Also see [27] for the application of (a local version of) the following theorem.

Theorem 1.9. *Suppose that $K \subset \mathbb{R}^n$ is a closed set, $\mathbb{R}^n \setminus K$ is an unbounded open set satisfying the J-carrot John condition with center ∞ and $0 \in K$. Then for any $R \geq 0$, there exist at most N-many J'-carrot John subdomains (where some of them could be empty) $\{W_{j,R}\}_{j\in\{1,\cdots,N\}}$ of $\mathbb{R}^n \setminus K$ of $\mathbb{R}^n$ with $J' = J'(n,\, J)$ and $N = N(n,\, J)$, such that:*

(i) *We have*

$$B_R \setminus K \subset \bigcup_{j=1}^{N} \overline{W}_{j,\,R},$$

together with $W_{j,\,R} \subset B_{C'R}$ with $C' = C'(n,\, J)$ and (if it is non-empty)

$$C(n,\, J)^{-1} R^n \leq |W_{j,R}| \leq C(n,\, J) R^n. \tag{1.2}$$

In addition, for each $1 \leq k \leq N$ and $R > 0$, there exists a sequence $\{k_l\}_{l=0}^{+\infty}$ and $k_0 = k$ so that

$$C(n,\, J)^{-1} |W_{k_l,2^l R}| \leq |W_{k_l,2^l R} \cap W_{k_{l+1},2^{l+1}R}|, \quad l \geq 0. \tag{1.3}$$

(ii) *For $1 \leq j \leq N$ and some $x_j \in \mathbb{R}^n$, the set*

$$W_{j,\infty} := \bigcup_{R>|x_j|} W_{j,R} \subset \mathbb{R}^n \setminus K$$

is also a J'-carrot John subdomain centered at ∞, for which

$$\mathbb{R}^n \setminus K \subset \bigcup_{j=1}^{N} \overline{W}_{j,\infty}. \tag{1.4}$$

Moreover, for any $z, w \in W_{j,\infty}$, there exists a ball $B_{z,w} \subset W_{j,\infty}$ whose radius is $r_{z,w}$ so that there are two rectifiable curves γ_z, γ_w respectively joining z, w to the center $a_{z,w}$ of $B_{z,w}$ satisfying

$$B_{z,w} \subset \operatorname{car}(\gamma_z, J') \subset W_{j,\infty} \quad \text{and} \quad B_{z,w} \subset \operatorname{car}(\gamma_w, J') \subset W_{j,\infty}, \tag{1.5}$$

where the radius $r_{z,w}$ satisfies

$$\frac{\ell(\gamma_z[z, a_{z,w}])}{J'} = r_{z,w} = \frac{\ell(\gamma_w[w, a_{z,w}])}{J'}. \tag{1.6}$$

(iii) *In particular, as a consequence of* [4], [12] *(for bounded domains) together with* [14] *(for unbounded domains), we have*

$$\inf_c \left(\int_{W_{j,\,R}} |u-c|^{p^*}\,dx \right)^{\frac{1}{p^*}} \le C(n,\,p,\,J) \left(\int_{W_{j,\,R}} |Du|^p\,dx \right)^{\frac{1}{p}}$$

for any $u \in W^{1,p}(W_{j,\,R})$, *and*

$$\inf_c \left(\int_{W_{j,\,\infty}} |u-c|^{p^*}\,dx \right)^{\frac{1}{p^*}} \le C(n,\,p,\,J) \left(\int_{W_{j,\,\infty}} |Du|^p\,dx \right)^{\frac{1}{p}}$$

for any $u \in W^{1,p}(W_{j,\infty})$.

Remark 1.10. As noted in Remark 1.6, according to [22, Theorem 2.16], the J-carrot John condition and the J-cigar John condition are equivalent for any bounded domain, up to positive constants.

However, this equivalence does not necessarily hold for unbounded domains, and the Sobolev–Poincaré inequality in [14] is proven for unbounded cigar John domains. An example for the failure of the equivalence is given by the following: Consider the unbounded domain

$$U = \mathbb{R}^2 \setminus \big((-\infty,-1] \times \{0\} \cup [1,+\infty) \times \{0\}\big)$$

which satisfies the 1-carrot John condition with center ∞. However, it does not satisfy any cigar John condition. Nevertheless, observe that, U can be covered as the union of two sets $\mathbb{H}^+ \cup B(0,1)$ and $\mathbb{H}^-$, where $\mathbb{H}^\pm$ denote the upper/lower (open) half plane, and each of them individually satisfies the 2-cigar John condition.

In a similar vein, Theorem 1.9 establishes that any unbounded J-carrot John domain can be covered by a uniformly finite number of J'-cigar John domains, where the number of domains is uniformly bounded depending only on J and n. We remark that (1.5) is indeed equivalent to stating that every two distinct points z, w can be connected by a J'-cigar inside $W_{j,\infty}$. However, to streamline terminology, the theorem is presented in the context of carrots.

The manuscript is structured as follows: In Section 2, we provide the proof of Theorem 1.7, devoting careful attention to the continuity of functions as defined in Definition 1.3. A pivotal lemma, namely Lemma 2.1, examines the behavior of carrots under Hausdorff convergence. Another crucial aspect involves preventing the John centers of converging John domains from reaching the boundary, a concern addressed in Lemma 2.5.

The proof of Theorem 1.9 is detailed in Section 3, with an introductory overview of the proof presented at the outset of the section.

2. Lower-semicontinuity of John constant

In our manuscript, we employ the notation as follows: For any set $E \subset \mathbb{R}^n$, the closure of E with respect to the Euclidean topology is denoted as $\overline{E}$ or $\mathrm{Cl}(E)$, and its complement is denoted by E^c. Given that the Euclidean space is of finite dimension, the topology induced by the norms remains the same.

The space consisting of all nonempty compact sets in $\mathbb{R}^n$ equipped with the Hausdorff metric d_H is denoted as $(\mathcal{C}^n, d_H)$. The topologies of $(\mathcal{C}^n, d_H)$ induced by all norms in $\mathbb{R}^n$ are equivalent. For simplicity, one can think of d_H as the metric induced by the standard Euclidean norm in $\mathbb{R}^n$.

The Lebesgue measure of the set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and the s-dimension Hausdorff measure of E is denoted by $\mathcal{H}^s(E)$. A general constant is denoted by C, which may vary across different estimates, and we include all the constants it depends on within the parentheses, denoted as $C(\cdot)$.

We next prove a key lemma, which later helps us to deduce the lower-semicontinuity of both the function $J(\Omega;\cdot)\colon \Omega \to [1,+\infty)$ and the (optimal) John constant $\mathrm{John}(\cdot)$.

Lemma 2.1. *Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences of points with $x_i \in \mathbb{R}^n$ and $y_i \in \dot{\mathbb{R}}^n$ for $i \in \mathbb{N}$. Assume that $\{\gamma_i\}_{i\in\mathbb{N}}$ is a sequence of locally rectifiable curves in $\mathbb{R}^n$ joining pairs of distinct points x_i, y_i.*

$$\lim_{i\to+\infty} x_i =: x \neq \infty \quad \text{and} \quad y := \lim_{i\to+\infty} y_i$$

exist in $\dot{\mathbb{R}}^n$ and that either

$$\ell_{\|\cdot\|}(\gamma_i) \text{ is uniformly bounded,}$$

or

$$y = \infty \quad \text{and} \quad \ell_{\|\cdot\|}(\gamma_i \cap B_R),\ R \geq 1 \ \text{ uniformly bounded independent of } i \tag{2.1}$$

holds.

Moreover, let $\{J_i\}_{i\in\mathbb{N}}$ be a uniformly bounded sequence with $J \geq 1$ and $\mathrm{car}(\gamma_i, J_i)$ is the corresponding J_i-carrot joining x_i toward y_i, respectively. Then up to relabeling the sequence,

(i) *in $(\mathbb{R}^n, \|\cdot\|)$*

$$\gamma_i \to \gamma \quad \text{locally uniformly,} \tag{2.2}$$

and

$$\mathrm{car}(\gamma, J) \subset \bigcap_{m=1}^{+\infty} \bigcup_{i=m}^{+\infty} \mathrm{car}(\gamma_i, J_i), \tag{2.3}$$

where $J := \liminf_{i\to+\infty} J_i$,

(ii) *if $\ell_{\|\cdot\|}(\gamma_i)$ is uniformly bounded, then*

$$\ell_{\|\cdot\|}(\gamma) \leq \liminf_{i\to+\infty} \ell_{\|\cdot\|}(\gamma_i). \tag{2.4}$$

Proof. Case 1: $\ell_{\|\cdot\|}(\gamma_i)$ is uniformly bounded. Then our assumption implies that, for some $L > 0$,

$$l_i := \ell_{\|\cdot\|}\big(\gamma_i\big) \leq L.$$

As $\{J_i\}_{i\in\mathbb{N}}$ is a uniformly bounded sequence, we may assume

$$l_\infty := \lim_{i\to+\infty} l_i, \quad J := \lim_{i\to+\infty} J_i \tag{2.5}$$

with $l_\infty \leq L$.

By parameterizing γ_i via arc length on $[0,\ L]$, up to extending as a constant curve if necessary on the interval $[\ell_{\|\cdot\|}(\gamma_i),\ L]$, we obtain that $\{\gamma_i(\cdot)\}_{i\in\mathbb{N}}$ is equicontinuous and uniformly bounded as $x_i \to x \neq \infty$. Thus, up to passing to a subsequence, it follows from the Arzelá–Ascoli theorem that, up to extracting a subsequence,

$$\gamma_i \to \gamma \in C([0,L];\mathbb{R}^n) \quad \text{uniformly.} \tag{2.6}$$

As γ_i is 1-Lipschitz, γ is also 1-Lipschitz, and thus

$$\ell_{\|\cdot\|}(\gamma) \leq \liminf_{i\to\infty} \ell_{\|\cdot\|}(\gamma_i) \tag{2.7}$$

as desired.

In addition, for any point $\zeta \in \mathrm{car}(\gamma, J)$, there exists $t_z \in (0, L]$ with $z = \gamma(t_z) \in \gamma$ so that

$$\zeta \in B_{\|\cdot\|}(z, \ell_{\|\cdot\|}(\gamma([0, t_z]))/J), \tag{2.8}$$

which yields a positive constant $\delta := \ell_{\|\cdot\|}(\gamma([0, t_z]))/J - \|z - \zeta\| > 0$.

Note that (2.6) yields the existence of a sequence $\{z_i\}_{i\in\mathbb{N}}$ for which $z_i = \gamma_i(t_z) \in \gamma_i$ and $z_i \to z$ as $i \to \infty$. Therefore, by (2.7), for any positive $\epsilon < \delta/2$, when $i \geq i_0$ for some big integer i_0, we have $\|z_i - z\| < \epsilon$ and

$$\ell_{\|\cdot\|}(\gamma([0, t_z])) \leq \ell_{\|\cdot\|}(\gamma_i([0, t_z])) + \epsilon.$$

As a result, combining (2.8) and the triangle inequality, the estimate above gives

$$\begin{aligned}\|z_i - \zeta\| \leq \|z - \zeta\| + \|z_i - z\| &< \|z - \zeta\| + \epsilon \\ &\leq (\ell_{\|\cdot\|}(\gamma([0, t_z]))/J - \delta) + \epsilon \leq \ell_{\|\cdot\|}(\gamma_i([0, t_z]))/J + 2\epsilon - \delta \\ &< \ell_{\|\cdot\|}(\gamma_i([0, t_z]))/J,\end{aligned}$$

so that

$$\zeta \in \bigcup_{i=m}^{+\infty} B_{\|\cdot\|}\left(z_i, \ell_{\|\cdot\|}(\gamma_i([0, t_z]))/J_i\right) \subset \bigcup_{i=m}^{+\infty} \mathrm{car}(\gamma_i, J_i).$$

Consequently, $\zeta \in \bigcap_{m=1}^{+\infty}\left(\bigcup_{i=m}^{+\infty} \mathrm{car}(\gamma_i, J_i)\right)$, which implies (2.3). In conclusion, when $y \neq \infty$, Lemma 2.1 holds.

Case 2: $y = \infty$ *and* $\ell(\gamma_i) \to \infty$. In this case, as γ_i is locally rectifiable and satisfies (2.1), via suitable truncation, by Step 1 and applying a diagonal argument, we have γ_i converges locally uniformly to a curve γ parametrized by arc length on $[0, \infty)$. Similarly, by taking the union of carrots along γ_i, (2.3) is obtained also from Step 1. □

For a bounded domain Ω and any rectifiable curve $\gamma \subset \Omega$ joining x to x_0 with $x, x_0 \in \Omega$, recall the definition of $j(t; x, \gamma, \Omega)$ in Definition 1.3. Then $j(t; x, \gamma, \Omega)$ is continuous with respect to $t \in [0, 1]$, and then the compactness of $[0, 1]$ tells that there exists a point $t_0 \in [0, 1]$ such that

$$\frac{\ell_{\|\cdot\|}(\gamma([0, t_0]))}{d_{\|\cdot\|}(\gamma(t_0), \partial\Omega)} = \sup_{t\in[0,1]} j(t; x, E, \Omega) < +\infty.$$

Thus,

$$J(x, \Omega; x_0) := \inf \left\{ \sup_{t\in[0,1]} j(t; x, \beta, \Omega) \colon \beta \subset \Omega \text{ is a curve joining } x \text{ to } x_0 \right\}$$

is finite. We next show that Lemma 2.1 ensures the existence of the rectifiable curve who make this infimum be reached.

Lemma 2.2. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain. Let x, $x_0 \in \Omega$ be two distinct points with $J(x, \Omega; x_0) < +\infty$. Then there exists a rectifiable curve $\gamma \subset \Omega$ joining x to x_0 such that*

$$\sup_{t\in[0,1]} j(t; x, \gamma, \Omega) = J(x, \Omega; x_0).$$

Proof. Choose a minimizing sequence $\{\gamma_i\}_{i\in\mathbb{N}^+}$, $\gamma_i \subset \Omega$ of rectifiable curves joining x to x_0 so that

$$\lim_{i\to\infty} J_i := \lim_{i\to\infty} \sup_{t\in[0,1]} j(t; x, \gamma_i, \Omega) = J(x, \Omega; x_0) =: J.$$

Then by the Definition 1.3, the uniform boundedness of J_i implies that $\ell_{\|\cdot\|}(\gamma_i)$ is bounded uniformly. Thus by letting $\mathrm{car}(\gamma_i, J_i)$ be the J_i-carrot joining x to x_0 for $i \in \mathbb{N}$, Lemma 2.1 tells that there exists a rectifiable curve γ of a J-carrot joining x to x_0, such that

$$\mathrm{car}(\gamma, J) \subset \Omega,$$

which implies that $J \geq \sup_{t\in[0,1]} j(t; x, \gamma, \Omega)$. On the other hand, the convergence of J_i to J together with Definition 1.3, gives

$$J \leq \sup_{t\in[0,1]} j(t; x, \gamma, \Omega).$$

The proof is completed. □

Lemma 2.3. *Let $x_0 \in \Omega$ and $\Omega \subset \mathbb{R}^n$ be a bounded J-carrot John domain with $J := J(\Omega; x_0)$. Then for any $z \in \Omega$, $J(\Omega; z)$ is finite. Moreover, for any $y \in \Omega$ satisfying*

$$d_{\|\cdot\|}(y, \partial\Omega) = \max_{x\in\Omega} d_{\|\cdot\|}(x, \partial\Omega),$$

we get $J(\Omega; y) \leq C(n, C_{\|\cdot\|}, J)$.

This lemma directly follows from Theorem 3.6 in [25]. Despite their findings being initially formulated for the standard Euclidean norm, one can establish them for a general Minkowski norm in $\mathbb{R}^n$ by employing identical arguments, necessitating only notational adjustments, with additional dependency on $C_{\|\cdot\|}$[3].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain. Then*

$$J(\cdot, \Omega; \cdot)\colon \Omega \times \Omega \to [1, +\infty), \quad (x, y) \mapsto J(x, \Omega; y),$$

is locally Lipschitz continuous.

Proof. Given $(x, y) \in \Omega \times \Omega$ and $(\hat{x}, \hat{y}) \in \Omega \times \Omega$ close to (x, y), We first estimate

$$J(\hat{x}, \Omega; \hat{y}) - J(x, \Omega; y)$$

from above and below, respectively.

Step 1: Estimate $J(\hat{x}, \Omega; \hat{y}) - J(x, \Omega; y)$ from above. Let

$$J := J(x, \Omega; y).$$

Then Lemma 2.2 yields a rectifiable curve $\gamma \subset \Omega$ joining x to y together with the corresponding J-carrot $\mathrm{car}(\gamma, J)$, such that

$$\sup_{t\in[0,1]} j(t; x, \gamma, \Omega) = J(x, \Omega; y) = J \quad \text{and} \quad \mathrm{car}(\gamma, J) \subset \Omega. \tag{2.9}$$

As a consequence of the compactness of $[0, 1]$, the definition of $j(t; x, \gamma, \Omega)$ gives us a point $s \in [0, 1]$, such that

$$\frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} = J = \sup_{t\in[0,1]} j(t; x, \gamma, \Omega). \tag{2.10}$$

[3] In the proof of [25, Theorem 3.6], by choosing y as the center of the largest ball contained in Ω, for any $x \in \Omega$, the John curve γ, as the core of $\mathrm{cig}(\gamma, J) \subset \Omega$ joining x to y, is proved to be the core of $\mathrm{car}(\gamma, J_1)$ for some J_1. Due to the fact that $\mathcal{K}_{\|\cdot\|}$ might not be symmetric with respect to the origin, the upper bound estimate for $\ell_{\|\cdot\|}(\gamma[x, y])$ becomes $(1 + C_{\|\cdot\|})d_{\|\cdot\|}(y, \partial\Omega)$, different from the Euclidean case. Consequently, J_1 further depends on $C_{\|\cdot\|}$; see also Remark 1.2.

We claim that

$$d_{\|\cdot\|}(\gamma, \partial\Omega) \geq \frac{d_{\|\cdot\|}(x, \partial\Omega)}{2C_{\|\cdot\|}J}. \tag{2.11}$$

Indeed, for any $z \in \gamma \cap B_{\|\cdot\|}\left(x, \frac{1}{2}d_{\|\cdot\|}(x, \partial\Omega)\right)$, the triangle inequality gives

$$d_{\|\cdot\|}(z, \partial\Omega) \geq \frac{1}{2}d_{\|\cdot\|}(x, \partial\Omega);$$

while for $z \in \gamma \setminus B_{\|\cdot\|}\left(x, \frac{1}{2}d_{\|\cdot\|}(x, \partial\Omega)\right)$, it follows from (2.9) and the definition of $\mathrm{car}(\gamma, J)$ that

$$d_{\|\cdot\|}(z, \partial\Omega) \geq \frac{\ell_{\|\cdot\|}(\gamma[x, z])}{J} \geq \frac{\ell_{\|\cdot\|}(\gamma[z, x])}{C_{\|\cdot\|}J} \geq \frac{d_{\|\cdot\|}(x, \partial\Omega)}{2C_{\|\cdot\|}J}.$$

As $J \geq 1$, our claim (2.11) follows.

Let $(\hat{x}, \hat{y}) \in \Omega \times \Omega$ close to (x, y) with

$$\hat{x} \in B_{\|\cdot\|}\left(x, \frac{d_{\|\cdot\|}(x, \partial\Omega)}{2}\right) \quad \text{and} \quad \hat{y} \in B_{\|\cdot\|}\left(y, \frac{d_{\|\cdot\|}(y, \partial\Omega)}{2}\right).$$

Set

$$\hat{\gamma} \subset L_{\hat{x},x} \cup \gamma \cup L_{y,\hat{y}}, \tag{2.12}$$

be a rectifiable curve joining $\hat{x}$ to $\hat{y}$. where $L_{\hat{x},x}$ is the line segment joining $\hat{x}$ to x and $L_{y,\hat{y}}$ is the one joining y to $\hat{y}$. As $y \in \gamma$, we conclude from (2.10) that

$$\frac{\ell_{\|\cdot\|}(\gamma[x, y])}{d_{\|\cdot\|}(y, \partial\Omega)} \leq \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)},$$

and hence

$$\begin{aligned} &\frac{\ell_{\|\cdot\|}(\gamma[x, y])}{d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} \\ &= \left(\frac{\ell_{\|\cdot\|}(\gamma[x, y])}{d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|} - \frac{\ell_{\|\cdot\|}(\gamma[x, y])}{d_{\|\cdot\|}(y, \partial\Omega)}\right) + \left(\frac{\ell_{\|\cdot\|}(\gamma[x, y])}{d_{\|\cdot\|}(y, \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)}\right) \\ &\leq \frac{\|y - \hat{y}\|}{d_{\|\cdot\|}(y, \partial\Omega)(d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|)}\ell_{\|\cdot\|}(\gamma[x, y]). \qquad (2.13) \end{aligned}$$

For each $z \in \hat{\gamma}$, we now estimate

$$\frac{\ell_{\|\cdot\|}(\hat{\gamma}[\hat{x}, z])}{d_{\|\cdot\|}(z, \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)}$$

in three cases.

First of all, when $z \in \gamma$, as

$$\frac{\ell_{\|\cdot\|}(\gamma[x, z])}{d_{\|\cdot\|}(z, \partial\Omega)} \leq \sup_{t \in [0,1]} j(t; x, \gamma, \Omega) = \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)}$$

by (2.10) and $d_{\|\cdot\|}(\gamma, \partial\Omega) = \inf_{w \in \gamma} d_{\|\cdot\|}(w, \partial\Omega)$, then we have

$$\begin{aligned} \frac{\ell_{\|\cdot\|}(\hat{\gamma}[\hat{x}, z])}{d_{\|\cdot\|}(z, \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} &\leq \frac{\ell_{\|\cdot\|}(\gamma[x, z]) + \|x - \hat{x}\|}{d_{\|\cdot\|}(z, \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} \\ &\leq \frac{\|x - \hat{x}\|}{d_{\|\cdot\|}(\gamma, \partial\Omega)}. \qquad (2.14) \end{aligned}$$

Secondly, suppose that $z \in L_{y,\hat{y}}$. Then as (2.12) yields

$$\ell_{\|\cdot\|}(\gamma[\hat{x}, z]) \leq \|z - y\| + \ell_{\|\cdot\|}(\gamma[x, y]) + \|x - \hat{x}\| \leq \|\hat{y} - y\| + \ell_{\|\cdot\|}(\gamma[x, y]) + \|x - \hat{x}\|,$$

and the triangle inequality yields

$$d_{\|\cdot\|}(z, \partial\Omega) \geq d_{\|\cdot\|}(y, \partial\Omega) - \|y - z\| \geq d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|,$$

it follows from (2.13) that

$$\begin{aligned}
&\frac{\ell_{\|\cdot\|}(\hat{\gamma}[\hat{x}, z])}{d_{\|\cdot\|}(z, \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} \\
&\leq \frac{\|x - \hat{x}\| + \|\hat{y} - y\|}{d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|} + \frac{\ell_{\|\cdot\|}(\gamma[x, y])}{d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} \\
&\leq \frac{\|x - \hat{x}\| + C_{\|\cdot\|}\|y - \hat{y}\|}{d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|} + \frac{\|y - \hat{y}\|}{d_{\|\cdot\|}(y, \partial\Omega)(d_{\|\cdot\|}(y, \partial\Omega) - \|\hat{y} - y\|)}\ell_{\|\cdot\|}(\gamma[x, y]) \\
&\leq \frac{\|x - \hat{x}\|d_{\|\cdot\|}(y, \partial\Omega) + C_{\|\cdot\|}\|y - \hat{y}\|d_{\|\cdot\|}(y, \partial\Omega) + \|y - \hat{y}\|\ell_{\|\cdot\|}(\gamma[x, y])}{d_{\|\cdot\|}(y, \partial\Omega)\left(d_{\|\cdot\|}(y, \partial\Omega) - \|y - \hat{y}\|\right)} \\
&\leq \frac{2C_{\|\cdot\|}(d_{\|\cdot\|}(y, \partial\Omega) + \ell_{\|\cdot\|}(\gamma[x, y]))}{\left(d_{\|\cdot\|}(y, \partial\Omega)\right)^2}\left(\|x - \hat{x}\| + \|y - \hat{y}\|\right) \\
&\leq \frac{C(n, C_{\|\cdot\|},\ J)}{d_{\|\cdot\|}(y, \partial\Omega)}\left(\|x - \hat{x}\| + \|y - \hat{y}\|\right). \qquad (2.15)
\end{aligned}$$

The last case is when $z \in L_{\hat{x},x}$, $\|x - z\| \leq \|x - \hat{x}\|$ and then

$$d_{\|\cdot\|}(z, \partial\Omega) \geq d_{\|\cdot\|}(x, \partial\Omega) - \|x - z\| \geq d_{\|\cdot\|}(x, \partial\Omega) - \|x - \hat{x}\|.$$

Thus we obtain that

$$\begin{aligned}
\frac{\ell_{\|\cdot\|}(\hat{\gamma}[\hat{x}, z])}{d_{\|\cdot\|}(z, \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} &\leq \frac{\|x - \hat{x}\|}{d_{\|\cdot\|}(x, \partial\Omega) - \|x - \hat{x}\|} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} \\
&\leq \frac{2\|x - \hat{x}\|}{d_{\|\cdot\|}(x, \partial\Omega)}. \qquad (2.16)
\end{aligned}$$

All in all, we conclude from (2.14),(2.15) and (2.16) that, for any $t \in [0, 1]$,

$$\begin{aligned}
&\frac{\ell_{\|\cdot\|}(\hat{\gamma}([0, t]))}{d_{\|\cdot\|}(\hat{\gamma}(t), \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} \\
&\leq \max\left\{\frac{\|x - \hat{x}\|}{d_{\|\cdot\|}(\gamma, \partial\Omega)}, \frac{C(n, C_{\|\cdot\|},\ J)}{d_{\|\cdot\|}(y, \partial\Omega)}\left(\|x - \hat{x}\| + \|y - \hat{y}\|\right), \frac{2\|x - \hat{x}\|}{d_{\|\cdot\|}(x, \partial\Omega)}\right\}. \qquad (2.17)
\end{aligned}$$

As a result, we conclude from (2.11) that

$$\begin{aligned}
&J(\hat{x}, \Omega; \hat{y}) - J(x, \Omega; y) \\
&\leq \sup_{t\in[0,1]} \frac{\ell_{\|\cdot\|}(\hat{\gamma}([0, t]))}{d_{\|\cdot\|}(\hat{\gamma}(t), \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\gamma([0, s]))}{d_{\|\cdot\|}(\gamma(s), \partial\Omega)} \\
&\leq \max\left\{\frac{\|x - \hat{x}\|}{d_{\|\cdot\|}(\gamma, \partial\Omega)}, \frac{C(n, C_{\|\cdot\|},\ J)}{d_{\|\cdot\|}(y, \partial\Omega)}\left(\|x - \hat{x}\| + \|y - \hat{y}\|\right), \frac{2\|x - \hat{x}\|}{d_{\|\cdot\|}(x, \partial\Omega)}\right\} \\
&\leq \frac{C(n, C_{\|\cdot\|},\ J)}{d_{\|\cdot\|}(\gamma, \partial\Omega)}\left(\|x - \hat{x}\| + \|y - \hat{y}\|\right) \leq \frac{C(n, C_{\|\cdot\|},\ J)}{d_{\|\cdot\|}(x, \partial\Omega)}\left(\|x - \hat{x}\| + \|y - \hat{y}\|\right). \qquad (2.18)
\end{aligned}$$

Step 2: Estimate $J(x,\Omega;y)-J(\hat{x},\Omega;\hat{y})$ from above. Similarily, we repeat the argument and gain the following estimate:

$$J(x,\Omega;y)-J(\hat{x},\Omega;\hat{y}) \le \frac{C(n,C_{\|\cdot\|},\ J)}{d_{\|\cdot\|}(x,\partial\Omega)}\left(\|x-\hat{x}\|+\|y-\hat{y}\|\right), \tag{2.19}$$

when $\|x-\hat{x}\|+\|y-\hat{y}\|<\delta$ for a constant $\delta=\delta(x,y,C_{\|\cdot\|})$ satisfying

$$0<\delta\le\frac{1}{2}\min\left\{d_{\|\cdot\|}(x,\partial\Omega),d_{\|\cdot\|}(y,\partial\Omega)\right\}.$$

Detailed proof of (2.19) is included in the Appendix A.

Step 3: Conclusion. Combining (2.18) and (2.19), we get that $J(\cdot,\Omega,\cdot)$ is continuous and

$$|J(x,\Omega;y)-J(\hat{x},\Omega;\hat{y})| \le \frac{C(n,C_{\|\cdot\|},\ J)}{d_{\|\cdot\|}(x,\partial\Omega)}\left(\|x-\hat{x}\|+\|y-\hat{y}\|\right), \tag{2.20}$$

when $\|x-\hat{x}\|+\|y-\hat{y}\|<\delta$. Thus for any $(x,y)\in\Omega\times\Omega$, by letting

$$U_{x,y}:=\left\{(a,b)\in\Omega\times\Omega\colon \|a-x\|+\|b-y\|<\frac{1}{16C_{\|\cdot\|}}\delta\right\},$$

the estimate (2.20) yields that whenever $(x_1,y_1),(x_2,y_2)\in U_{x,y}$,

$$|J(x_1,\Omega;y_1)-J(x_2,\Omega;y_2)| \le \frac{C_{x,y}}{d_{x,y}}\left(\|x_1-x_2\|+\|y_1-y_2\|\right), \tag{2.21}$$

where

$$C_{x,y}=\max_{(a,b)\in\overline{U}_{x,y}} C(n,C_{\|\cdot\|},\ J(a,\Omega;b))<\infty$$

by the John assumption on Ω, and

$$d_{x,y}=\min_{(a,b)\in\overline{U}_{x,y}} d_{\|\cdot\|}(a,\partial\Omega).$$

From (2.21) we finally conclude that $J(\cdot,\Omega;\cdot)$ is locally Lipschitz continuous. □

Recall that

$$J(\Omega;x_0):=\sup_{x\in\Omega}J(x,\Omega;x_0).$$

Lemma 2.5. *Let $\Omega\subset\mathbb{R}^n$ be a bounded John domain. Then $J(\Omega;\cdot)\colon\Omega\to[1,+\infty)$ is a lower-semicontinuous function, such that*

(i) *For $y\in\Omega$, $r_\Omega:=\max_{z\in\Omega}\{d_{\|\cdot\|}(z,\partial\Omega)\}$, we have*

$$J(\Omega;y)\ge\frac{r_\Omega-d_{\|\cdot\|}(y,\partial\Omega)}{C_{\|\cdot\|}d_{\|\cdot\|}(y,\partial\Omega)};$$

(ii) *Let $x_\Omega\in\Omega$ be a point with $d_{\|\cdot\|}(x_\Omega,\partial\Omega)=r_\Omega$. Then $J(\Omega;\cdot)$ attains its infimum in*

$$\{x\in\Omega\colon d_{\|\cdot\|}(x,\partial\Omega)\ge r_0\},$$

where $r_0:=\frac{r_\Omega}{1+2C_{\|\cdot\|}J(\Omega;x_\Omega)}>0$.

Proof. Observe that for any $x\in\Omega$, $J(x,\Omega;\cdot)$ is a continuous function in Ω by Lemma 2.4. Then we find that $J(\Omega;\cdot)$ is a lower-semicontinuous function in Ω since

$$J(\Omega;\cdot)=\sup_{x\in\Omega}J(x,\Omega;\cdot).$$

Now we prove (i). Let $x_\Omega \in \Omega$ be a point satisfying

$$d_{\|\cdot\|}(x_\Omega, \partial\Omega) = \max_{y\in\Omega} \{d_{\|\cdot\|}(y, \partial\Omega)\} =: r_\Omega.$$

For any $y \in \Omega$, combining the definition of $J(x_\Omega, \Omega; y)$, Lemma 2.2 and the triangle inequality, there exists a rectifiable curve $\gamma \subset \Omega$ joining x_Ω to y, such that

$$J(x_\Omega, \Omega; y) = \sup_{a\in\gamma} \frac{\ell_{\|\cdot\|}(\gamma[x_\Omega, a])}{d_{\|\cdot\|}(a, \partial\Omega)} \geq \frac{\ell_{\|\cdot\|}(\gamma[x_\Omega, y])}{d_{\|\cdot\|}(y, \partial\Omega)} \geq \frac{\ell_{\|\cdot\|}(\gamma[y, x_\Omega])}{C_{\|\cdot\|} d_{\|\cdot\|}(y, \partial\Omega)} \geq \frac{r_\Omega - d_{\|\cdot\|}(y, \partial\Omega)}{C_{\|\cdot\|} d_{\|\cdot\|}(y, \partial\Omega)}.$$

Then we have

$$J(\Omega; y) \geq J(x_\Omega, \Omega; y) \geq \frac{r_\Omega - d_{\|\cdot\|}(y, \partial\Omega)}{C_{\|\cdot\|} d_{\|\cdot\|}(y, \partial\Omega)}. \tag{2.22}$$

Now we proceed to (ii). Recall that $J(\Omega; x_\Omega) < +\infty$ by Lemma 2.3. We define

$$r_0 := \frac{r_\Omega}{1 + 2C_{\|\cdot\|} J(\Omega; x_\Omega)} \quad \text{and} \quad \Omega_{r_0} := \{x \in \Omega \colon d_{\|\cdot\|}(x, \partial\Omega) > r_0\}$$

so that for any $0 < r \leq r_0$

$$\frac{r_\Omega - r}{C_{\|\cdot\|} r} \geq \frac{r_\Omega - r_0}{C_{\|\cdot\|} r_0} = 2J(\Omega; x_\Omega).$$

Then since $x_\Omega \in \Omega_{r_0}$, for any $z \in \Omega \setminus \Omega_{r_0}$, we conclude from (2.22) that

$$J(\Omega; z) \geq 2J(\Omega; x_\Omega) > J(\Omega; x_\Omega) \geq \inf_{x\in\overline{\Omega}_{r_0}} J(\Omega; x).$$

Then the above estimate yields that $\inf_{x\in\Omega} J(\Omega; x) = \inf_{x\in\overline{\Omega}_{r_0}} J(\Omega; x)$.

Notice that $\overline{\Omega}_{r_0}$ is a compact set and $J(\Omega; \cdot)$ is a lower-semicontinuous function in Ω. As a consequence, there exist a point $b \in \overline{\Omega}_{r_0}$ such that

$$J(\Omega; b) = \inf_{x\in\overline{\Omega}_{r_0}} J(\Omega; x) = \inf_{x\in\Omega} J(\Omega; x). \qquad \square$$

We further need an auxiliary lemma regarding Hausdorff convergence.

Lemma 2.6. *Suppose that $\{K_j\}_{j\in\mathbb{N}}$ is a sequence of compact sets converging to a compact set K in the Hausdorff metric and the interior of K is denoted as Ω. Assume further that*

$$\inf_{j\in\mathbb{N}} \max_{x\in K_j} d_{\|\cdot\|}(x, \partial K_j) \geq r_0.$$

Then for any $r \in (0, r_0]$ and any converging sequence $\{x_j\}_{j\in\mathbb{N}}$ satisfying $x_j \in K_j$ and

$$d_{\|\cdot\|}(x_j, \partial K_j) \geq r,$$

the limit $x := \lim_{j\to\infty} x_j$ satisfies

$$x \in \Omega \quad \text{and} \quad d_{\|\cdot\|}(x, \partial\Omega) \geq r. \tag{2.23}$$

Proof. As K_j converge to K in the Hausdorff metric by our assumption, we claim that K can be explicitly represented as

$$K = \bigcap_{m=1}^{+\infty} \mathrm{Cl}\left(\bigcup_{j=m}^{+\infty} K_j\right). \tag{2.24}$$

This conclusion can be found in [8, Exercise 7.3.4].

Now by (2.24) and the convergence of x_j, we have

$$\{x\} = \bigcap_{m=1}^{+\infty} \mathrm{Cl}\left(\bigcup_{j=m}^{+\infty} \{x_j\}\right) \subset \bigcap_{m=1}^{+\infty} \mathrm{Cl}\left(\bigcup_{j=m}^{+\infty} K_j\right) = K. \tag{2.25}$$

Choosing $\epsilon > 0$ sufficiently small and for any $r \in (0, r_0]$, there is $j_0 \in \mathbb{N}$, such that

$$\|x_j - x\| < \epsilon$$

for any $j \geq j_0$. Thus, we get

$$d_{\|\cdot\|}(x, K_j^c) \geq d_{\|\cdot\|}(x_j, K_j^c) - \|x_j - x\| \geq r - \epsilon \quad \forall j \geq j_0.$$

The estimate above yields that

$$d_{\|\cdot\|}\left(x, \left(\mathrm{Cl}\left(\bigcup_{j=m}^{+\infty} K_j\right)\right)^c\right) \geq d_{\|\cdot\|}\left(x, \left(\bigcup_{j=m}^{+\infty} K_j\right)^c\right) \geq r - \epsilon \quad \forall m \geq j_0.$$

Let $m \to +\infty$, then we have

$$d_{\|\cdot\|}(x, K^c) \geq r - \epsilon.$$

Further let $\epsilon \to 0$, from the above estimate and (2.25) we get (2.23). □

Now we are ready to show Theorem 1.7.

Proof of Theorem 1.7. Assume that

$$J := \liminf_{j \to +\infty} \mathrm{John}(\Omega_j) \leq J_0 < \infty.$$

Let

$$\Omega_{j,r} := \left\{x \in \Omega_j \colon d_{\|\cdot\|}(x, \partial\Omega_j) \geq r\right\} \quad \text{and} \quad \Omega_r := \left\{x \in \Omega \colon d_{\|\cdot\|}(x, \partial\Omega) \geq r\right\}$$

for some $r > 0$ to be determined. Further let $\{\Omega_j\}_{j \in \mathbb{N}^+}$ be a minimizing sequence and $x_{\Omega_j} \in \Omega_j$ be a point satisfying

$$d_{\|\cdot\|}(x_{\Omega_j}, \partial\Omega_j) = \max_{x \in \Omega_j} d_{\|\cdot\|}(x, \partial\Omega_j) =: r_{\Omega_j}.$$

On the other hand, by Lemma 2.5, for each $i \in \mathbb{N}$ there exists a (center) point $x_j \in \Omega_{j,r}$, such that

$$J(\Omega_j; x_j) = \mathrm{John}(\Omega_j).$$

We remark that x_j might not be x_{Ω_j}. Nevertheless, by Lemma 2.3 we have

$$J(\Omega_j; x_{\Omega_j}) \leq C(n, C_{\|\cdot\|}, J_0). \tag{2.26}$$

In addition, since $(\mathcal{C}^n, d_H)$ is complete and bounded subsets are precompact, up to passing to a subsequence, $\{\overline{\Omega}_i\}_{i \in \mathbb{N}}$ converges in the Hausdorff metric to a compact set A. We set the interior of A as Ω.

Step 1: r_{Ω_j} is uniformly bounded away from 0. To this end, for each $j \in \mathbb{N}^+$, the definition of $J(\Omega_j; x_{\Omega_j})$ and Lemma 2.2 tell that, for any $x \in \Omega_j \setminus \{x_{\Omega_j}\}$, there exists a rectifiable curve β_j joining x to x_{Ω_j}, such that

$$\sup_{a \in [0,1]} \frac{\ell_{\|\cdot\|}(\beta_j([0,a]))}{d_{\|\cdot\|}(\beta_j(a), \partial\Omega_j)} = J(x, \Omega_j; x_{\Omega_j}) \leq J(\Omega_j; x_{\Omega_j})$$

and thus by (2.26)

$$\begin{aligned}\ell_{\|\cdot\|}(\beta_j[x, x_{\Omega_j}]) &\leq \sup_{a \in [0,1]} \frac{\ell_{\|\cdot\|}(\beta_j([0,a]))}{d_{\|\cdot\|}(\beta_j(a), \partial\Omega_j)} d_{\|\cdot\|}(x_{\Omega_j}, \partial\Omega_j) \\ &\leq J(\Omega_j; x_{\Omega_j}) d_{\|\cdot\|}(x_{\Omega_j}, \partial\Omega_j) \leq C(n, C_{\|\cdot\|}, J_0) r_{\Omega_j}.\end{aligned}$$

This yields that $\Omega_j \subset B_{\|\cdot\|}(x_{\Omega_j}, C(n, C_{\|\cdot\|}, J_0) r_{\Omega_j})$, from which we conclude

$$c_0 |B_{\|\cdot\|}(0,1)| \leq |\Omega_j| \leq |B_{\|\cdot\|}(0,1)| (C(n, C_{\|\cdot\|}, J_0) r_{\Omega_j})^n.$$

As a result, we conclude that

$$r_{\Omega_j} \geq c \tag{2.27}$$

for some $c = c(n,\, C_{\|\cdot\|},\, J_0,\, c_0) > 0$.

Step 2: x_j is uniformly away from the boundary. Up to further extracting a subsequence, we may assume

$$x_\Omega := \lim_{j\to+\infty} x_{\Omega_j} \in \mathbb{R}^n.$$

Recalling Lemma 2.6 and (2.27), it follows that

$$\max_{x\in\Omega} d_{\|\cdot\|}(x, \partial\Omega) \geq d_{\|\cdot\|}(x_\Omega, \partial\Omega) \geq c > 0.$$

In addition, we can choose $r > 0$ so that

$$r \leq \inf_{j\in\mathbb{N}} \frac{r_{\Omega_j}}{1 + 2C_{\|\cdot\|} J(\Omega; x_{\Omega_j})},$$

where its existence is ensured by (2.27), (2.26) and Lemma 2.5.

Recall that $x_j \in \Omega_{j,r}$. Then up to further passing to a subsequence, we may assume that the limit x of $\{x_j\}_{j\in\mathbb{N}}$ exists, and Lemma 2.6 implies that $x \in \Omega_r$.

Step 3: Lower semicontinuity of $\mathrm{John}(\Omega_j)$. Let $J_i := \mathrm{John}(\Omega_j)$. Note that for any $y \in \Omega$, Hausdorff convergence yields the existence of a sequence $y_j \in \overline{\Omega}_j$ such that $\lim_{j\to+\infty} y_j = y$. Combining the definition of $\mathrm{John}(\Omega_j)$, Lemma 2.2 and Lemma 2.5, for each $j \in \mathbb{N}$ we obtain a rectifiable curve $\gamma_j \subset \Omega_j$ joining y_j to x_j such that the corresponding J_i-carrot $\mathrm{car}(\gamma_j, J_j)$ satisfies

$$\mathrm{car}(\gamma_j, J_j) \subset \Omega_j,$$

which, due to the fact that Ω_j is uniformly bounded and $J_j \leq J_0$ for each $j \in \mathbb{N}^+$, yields that $\ell_{\|\cdot\|}(\gamma_j[y_j, x_j]) \leq J_0 \max_{j\in\mathbb{N}^+} r_{\Omega_j}$. Then by Lemma 2.1, there exists a rectifiable curve $\gamma \subset \Omega$ joining y to x so that the J-carrot $\mathrm{car}(\gamma, J)$ (as a Euclidean open set) satisfies

$$\mathrm{car}(\gamma, J) \subset \bigcap_{m=1}^{+\infty} \left(\bigcup_{j=m}^{+\infty} \mathrm{car}(\gamma_j, J_j) \right) \subset \bigcap_{m=1}^{+\infty} \mathrm{Cl} \left(\bigcup_{j=m}^{+\infty} \overline{\Omega}_j \right) = A. \tag{2.28}$$

As Ω is the interior of A, from (2.28) we have $\mathrm{car}(\gamma, J) \subset \Omega$, which implies that

$$J(y, \Omega; x) \leq J. \tag{2.29}$$

To conclude, each $y \in \Omega$ can be joint from x by a rectifiable curve inside Ω, which implies that Ω is connected. Furthermore, the arbitrariness of y in (2.29) yields

$$\mathrm{John}(\Omega) \leq J(\Omega; x) \leq J = \liminf_{j\to+\infty} \mathrm{John}(\Omega_j).$$

We complete the second part of the proof. □

3. John component of unbounded carrot John domain

In this section, we consider the John domain defined via the standard Euclidean norm $|\cdot|$.

The proof of Theorem 1.9 is rather technical since most of the sets defined in question are open. To obtain the sets $W_{j,\infty}$ as mentioned in Theorem 1.9, we initially decompose $B_R \setminus K$ into at most $C(n,\, J)$-many sets $\{V_{j,\,R}\}$. Subsequently, for each $y \in V_{j,R}$, we create a bounded J'-carrot John domain $\Omega_{j,R,y}$ (refer to Proposition 3.2).

In the proof of Theorem 1.9, we choose sequences of points $x_{j,r} \in V_{j,R}$ along with the corresponding John curves $\gamma_{x_{j,r}}$ extending from $x_{j,r}$ towards ∞, where r is a positive number with $r \leq R$. This selection ensures that we can obtain sets

$$W_{j,R} = \Omega_{j,\,R,x_{j,r}} \subset B_{C'R}, \quad C' = C'(n,\,J).$$

In particular, by eventually choosing $x_j \in \mathbb{R}^n$ suitably,

$$W_{j,\infty} := \bigcup_{R>|x_j|} W_{j,R} \subset \mathbb{R}^n \setminus K$$

fulfills the condition that for every pair of distinct points $z, w \in W_{j,\infty}$, there exists a point $a \in \gamma_{x_j}$ to which both z and w can be connected by $\hat{\gamma}_z$ and $\hat{\gamma}_w$, respectively. Moreover, these connecting curves satisfy the properties (1.5) and (1.6).

Prior to identifying the desired bounded John domain $\Omega_{j,R,y}$, we rely on the following proposition. While this technique has been commonly employed in previous manuscripts, such as [22], it has not been explicitly formulated, to the best of our knowledge, in the context of our present work.

Proposition 3.1. *Let $J \geq 1$. Assume that $\gamma \subset \mathbb{R}^n$ is a locally rectifiable curve joining x to y, where $x \in \mathbb{R}^n$ and $y \in \dot{\mathbb{R}}^n$ (y may be ∞). Then $\mathrm{car}(\gamma, J)$ is a J-carrot John domain.*

To be more specific, for any $z \in \mathrm{car}(\gamma, J)$, we can find a rectifiable curve γ_z joining z to y, such that for some $\eta \in \gamma$, we have

$$\gamma[\eta, y] = \gamma_z[\eta, y]$$

and for each $a \in \gamma[\eta, y] \setminus \{\infty\}$,

$$\ell(\gamma_z[z,a]) \leq \ell(\gamma[x,a]), \quad \mathrm{car}(\gamma_z, J) \subset \mathrm{car}(\gamma, J). \tag{3.1}$$

Proof. For any $z \in \mathrm{car}(\gamma, J)$, the definition of $\mathrm{car}(\gamma, J)$ yields a ball

$$B(\eta, \ell(\gamma[x,\eta])/J) \subset \mathrm{car}(\gamma, J)$$

for some points $\eta \in \gamma \setminus \{x\}$ so that $z \in B(\eta, \ell(\gamma[x,\eta])/J)$.

Let $L_{z,\eta}$ be the line segment joining z to η and then $\gamma_z := L_{z,\eta} \cup \gamma[\eta, y]$ is a locally rectifiable curve joining z to y. When $a \in L_{z,\eta}$,

$$\ell(\gamma_z[z,a]) \leq d\left(a, \partial B(\eta, \ell(\gamma[x,\eta])/J)\right) \leq \ell(\gamma[x,\eta])/J. \tag{3.2}$$

When $a \in \gamma[\eta, y]$, by applying (3.2) with $a = \eta$ there, we have

$$\begin{aligned}\ell(\gamma_z[z,a]) &\leq \ell(\gamma_z[z,\eta]) + \ell(\gamma_z[\eta,a]) \leq \frac{\ell(\gamma[x,\eta])}{J} + \ell(\gamma[\eta,a]) \\ &\leq \ell(\gamma[x,\eta]) + \ell(\gamma[\eta,a]) = \ell(\gamma[x,a]).\end{aligned}$$

To conclude, we obtain that

$$\ell(\gamma_z[z,a]) \leq \ell(\gamma[x,a]),$$

which is the first formula of (3.1). The second one follows directly from our construction of $\mathrm{car}(\gamma_z, J)$ and $\mathrm{car}(\gamma, J)$, and we conclude the lemma. □

3.1. A decomposition $V_{j,R}$ of $B_R \setminus K$. Now for any $x \in \mathbb{R}^n \setminus K$, we choose a John curve $\gamma_x \subset \mathbb{R}^n \setminus K$ joining x towards ∞ with $\mathrm{car}(\gamma, J) \subset \mathbb{R}^n \setminus K$. Although there could be many choices of curves for $x \in \mathbb{R}^n \setminus K$, we just choose one of them. Let $\Gamma = \{\gamma_x\}_{x \in \mathbb{R}^n \setminus K}$ be the collection of these chosen curves. In what follows, for any points $x \in \mathbb{R}^n \setminus K$, γ_x always refers to this particular choice of John curve.

Note that for any $R > 0$, we have $B_R \cap K \neq \emptyset$ as $0 \in K$. Our first step is to decompose $B_R \setminus K$ into finitely many subsets $V_{j,R}$ so that, there exists a collection $\mathcal{B}_{j,R}$ of at most $C(n, J)$-many balls, whose centers are on ∂B_{3R} and whose radii at least $J^{-1}R$, satisfying that, for any $x \in V_{j,R}$, we can find a ball $B \in \mathcal{B}_{j,R}$ with

$$\gamma_x \cap B \neq \emptyset.$$

To this end, observe that for each $x \in B_R \setminus K$ and $\gamma_x \in \Gamma$, there exists a point

$$x_R \in \gamma_x \cap \partial B_{3R} \tag{3.3}$$

so that

$$2R \le \ell(\gamma[x, x_R]) \le J d(x_R, K). \tag{3.4}$$

Consider the collection of closed balls

$$\{\overline{B}_x\}_{x \in B_R \setminus K} := \left\{ \overline{B}\left(x_R, \frac{d(x_R, K)}{2}\right) \right\}_{x \in B_R \setminus K}. \tag{3.5}$$

Then thanks to (3.4) and $0 \in K$, we obtain that

$$\frac{R}{J} \le \frac{d(x_R, K)}{2} \le 2R, \tag{3.6}$$

and hence $B_x \cap B_R = \emptyset$.

We next let

$$A_R := \bigcup_{x \in \overline{B_R} \setminus K} \{x_R\}$$

be the collection of the centers of B_x's. By Bescovitch's covering theorem, there exists a subcollection $\{\overline{B}_i\}_{i \in \mathbb{N}}$ of $\{\overline{B}_x\}_{x \in B_R \setminus K}$ consisting of at most countably many balls, such that

$$\chi_{A_R}(z) \le \sum_{B_i} \chi_{\overline{B}_i}(z) \le C(n) \quad \forall z \in \mathbb{R}^n \setminus K; \tag{3.7}$$

see Figure 2.

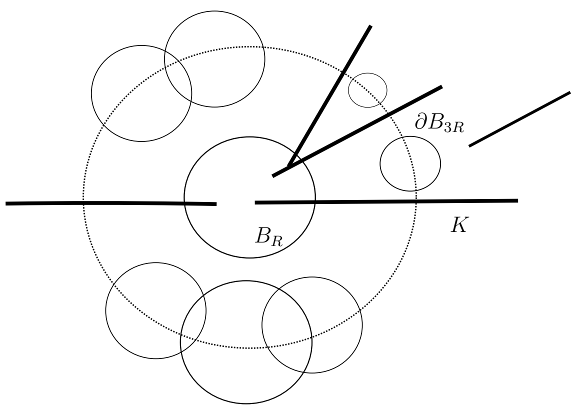

Figure 2. The set K is the union of black lines. We apply Bescovitch's covering theorem to cover the set A_R with balls centered at ∂B_{3R}.

Recall that by (3.6)

$$B_i \subset B_{5R} \setminus \overline{B}_R$$

and $|B_i| \geq c(n, J)R^n$. Thus we have at most $C(n, J)$-many elements in $\{\overline{B_i}\}$ by (3.7). As a result, the union of balls

$$\bigcup_i \overline{B}_i$$

has at most $\hat{N} = \hat{N}(n, J)$ components $U_{j,R}$ for $j \in \{1, \cdots, N_R\}$ and

$$N_R \leq \hat{N} = \hat{N}(n, J);$$

By defining $U_{j,R}$ to be empty for $j > N_R$, we may assume that there exist exactly $\hat{N}$ components $U_{j,R}$, and each $U_{j,R}$ contains at most $\hat{N}$ balls. We write

$$\mathcal{B}_{j,R} = \{B_i \colon B_i \subset U_{j,R}\} \tag{3.8}$$

Now it follows from our construction, for any $x \in B_R \setminus K$, there exists some $1 \leq j \leq N_R$ so that, $x_R \in \gamma_x$ is covered by a ball in $\mathcal{B}_{j,R}$. Thus, by defining

$$V_{j,R} := \{x \in B_R \setminus K \colon x_R \in D \text{ for some } D \in \mathcal{B}_{j,R}\}, \tag{3.9}$$

we obtain the desired decomposition of $B_R \setminus K$. The set $V_{j,R}$ is defined to be empty if $U_{j,R}$ is empty.

3.2. Construction of $\Omega_{j,R,y}$. Given $R > 0$ and $j \in \{1, \cdots, N_R\}$, recall the construction of $\mathcal{B}_{j,R}$ and $V_{j,R}$ in the last subsection. Then for each point $y \in V_{j,R}$ we set up a bounded J'-carrot John domain $\Omega_{j,R,y}$ with John center y_R, where $J' = J'(n, J)$, such that

$$V_{j,R} \subset \overline{\Omega}_{j,R,y} \quad \text{and} \quad \Omega_{j,R,y} \subset (\mathbb{R}^n \setminus K) \cap B_{C'R}$$

where $C' = C'(n, J)$; see Figure 3. We formulate it as the following proposition.

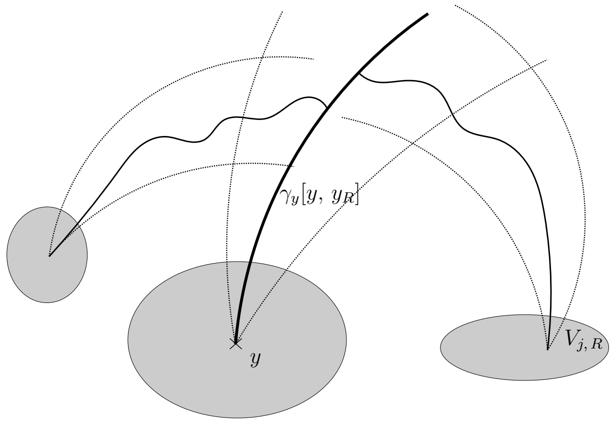

Figure 3. The set $V_{j,R}$ may not necessarily be connected. We connect each point in $V_{j,R}$ to the curve $\gamma_y[y, y_R]$ using appropriate curves. Subsequently, we take the union of the carrots surrounding these curves to form $\Omega_{j,R,y}$.

Proposition 3.2. *For fixed $y \in V_{j,R}$ and $1 \leq j \leq \hat{N}$ with $\hat{N} = \hat{N}(n, J)$ defined above, the set*

$$\Omega_{j,R,y} := \mathrm{car}(\gamma_y[y, y_R], J) \cup \bigcup_{z \in V_{j,R}} \mathrm{car}(\beta_z, J'). \tag{3.10}$$

is a J'-carrot John domain with John center y_R, where $J' = J'(n, J)$ and β_z is a rectifiable curve joining z to y_R satisfying $\gamma_z[z, z_R] \subset \beta_z$; recall that γ_x is a chosen curve joining x toward ∞.

Moreover, there exists $C_1 = C_1(n,\, J) \geq 4$ so that, the curve β_z joining $z \in V_{j,R}$ to y_R that is the core of a J'-carrot satisfying

$$\ell(\beta_z) \leq C_1 R$$

and

$$V_{j,R} \subset \overline{\Omega}_{j,R,y}, \quad \operatorname{car}(\beta_z, J') \subset \Omega_{j,R,y} \subset (\mathbb{R}^n \setminus K) \cap B_{2C_1R}. \tag{3.11}$$

Proof. Suppose that $V_{j,R}$ is non-empty, and fix $y \in V_{j,R}$. Then the corresponding point $y_R \in \gamma_y \cap \partial B_{3R}$ is covered by some ball $D_1 \in \mathcal{B}_{j,R}$ according to (3.8) and (3.9). Then we join the center $\hat{x}_1$ of D_1 to y_R by a line segment $L_{\hat{x}_1,y_R} \subset D_1$.

Now for any $z \in V_{j,R}$, we claim that there exists a rectifiable curve $\beta_z \subset \mathbb{R}^n \setminus K$ as the core of a J'-carrot joining z to y_R, such that $\gamma_z[z, z_R] \subset \beta_z$ and

$$\operatorname{car}(\beta_z, J') \subset \mathbb{R}^n \setminus K. \tag{3.12}$$

Indeed, the point $z_R \in \gamma_z \cap \partial B_{3R}$ is also covered by another ball $D_2 \in \mathcal{B}_{j,R}$ as $z \in V_{j,R}$. Likewise, we join z_R to the center $\hat{x}_2$ of D_2 by the line segment $L_{z_R,\hat{x}_2} \subset D_2$.

Recall that $U_{j,R}$ is connected and consists of at most $\hat{N}$-many balls from $\mathcal{B}_{j,R}$, where $\hat{N} = \hat{N}(n,\, J)$. This implies that $\hat{x}_1$ and $\hat{x}_2$ can be joined by a union of at most $\hat{N}$-many line segments with the endpoints being the centers of balls in $\mathcal{B}_{j,R}$. Therefore, combining with $L_{z_R,\hat{x}_2}$ and $L_{\hat{x}_1,y_R}$, we can join z_R to y_R by a polyline γ_{z_R,y_R}.

We show that

$$\beta_z := \gamma_z[z, z_R] \cup \gamma_{z_R,y_R}$$

is the desired John curve. To this end, we estimate the length of β_z and the distance $d(\eta, K)$ for any $\eta \in \beta_z$, respectively.

We start with the estimate on the length of β_z. Thanks to (3.6) and (3.8), for any pair of intersecting balls $D, D' \in \mathcal{B}_{j,R}$, the line segments L joining the center of D with radius r to the center of D' with radius r' satisfies

$$L \subset D \cup D' \quad \text{and} \quad \ell(L) \leq r + r' \leq 4R. \tag{3.13}$$

In particular, (3.6) together with the facts that $L_{z_R,\hat{x}_2} \subset D_2$ and that $L_{\hat{x}_1,y_R} \subset D_1$ also yields $\ell(L_{z_R,\hat{x}_2}) \leq 2R$, $\ell(L_{\hat{x}_1,y_R}) \leq 2R$. Therefore employing (3.13) and (3.4), the construction of β_z tells

$$\begin{aligned} \ell(\beta_z) &\leq \ell(\gamma_z[z, z_R]) + \ell(\gamma_{z_R,y_R}) \\ &\leq J d(z_R,\, K) + \ell(L_{z_R,\hat{x}_2}) + \ell(L_{\hat{x}_1,y_R}) + 4\hat{N}(n,\, J)R \\ &\leq C(n,\, J)R =: C_1 R; \end{aligned} \tag{3.14}$$

we may assume that $C_1 \geq 4$. This gives the first part of the proposition.

Towards (3.11), for any $\eta \in \beta_z$, we need to estimate the distance $d(\eta, K)$ from above. First of all, note that when $\eta \in \gamma_{z_R,y_R}$, there exists some ball $D_\eta \in \mathcal{B}_{j,R}$ containing η. Then combining (3.4),(3.5) and (3.6), we get

$$d(\eta, K) \geq d(D_\eta,\, K) \geq \frac{R}{J}. \tag{3.15}$$

Let

$$J' := C_1 J. \tag{3.16}$$

Then combining (3.4), (3.14) and (3.15), we conclude

$$\ell(\beta_z[z, \eta]) \leq \ell(\beta_z) \leq C_1 R \leq J' d(\eta, K) \quad \text{when } \eta \in \gamma_{z_R,y_R}.$$

On the other hand, when $\eta \in \gamma_x[x, x_R]$, since our construction yields $\beta_z[z, \eta] = \gamma_z[z, \eta]$, which is particularly contained in a John curve, it follows that

$$\ell(\beta_z[z, \eta]) \leq Jd(\eta, K) \leq J'd(\eta, K) \quad \text{when } \eta \in \gamma_z[z, z_R].$$

This implies (3.12). Moreover by Proposition 3.1, every point $w \in \mathrm{car}(\beta_z, J')$ also can be joined to y_R by a rectifiable curve $\hat{\gamma}_w$ satisfying

$$\ell(\hat{\gamma}_w) \leq \ell(\beta_z) \quad \text{and} \quad \mathrm{car}(\hat{\gamma}_w, J') \subset \mathrm{car}(\beta_z, J').$$

Hence, by employing (3.12), the arbitrariness of z gives the second formula in (3.11). The first formula in (3.11) holds due to $z \in \mathrm{Cl}(\mathrm{car}(\beta_z, J'))$, the closure of the carrot, for any $z \in V_{j,R}$. □

We need two more technical lemmas. The first one states how to choose a smaller carrot in the union of two carrots.

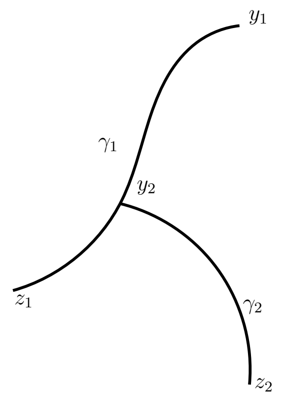

Figure 4. The two curves γ_1 and γ_2 are presented, respectively, with their end points and the intersection point y_2.

Lemma 3.3. *Let $1 \leq J_1 \leq J_2$. Assume that $z_1, z_2 \in \mathbb{R}^n$ and $y_1 \in \dot{\mathbb{R}}^n$. Let γ_1 be a rectifiable curve joining z_1 to y_1. If there exists a curve γ_2 joining z_2 to some point $y_2 \in \gamma_1$, so that*

$$\frac{\ell(\gamma_2[z_2, y_2])}{J_2} \leq \frac{\ell(\gamma_1[z_1, y_2])}{J_1}, \tag{3.17}$$

then, for any point $w \in \gamma_1[y_2, y_1)$, the curve $\hat{\gamma} := \gamma_2 \cup \gamma_1[y_2, w]$ joining z_2 to w satisfies

$$\mathrm{car}(\hat{\gamma}, J_2) \subset \mathrm{car}(\gamma_2, J_2) \cup \mathrm{car}(\gamma_1, J_1). \tag{3.18}$$

See Figure 4 for a illustration.

Proof. We first note that

$$\mathrm{car}(\hat{\gamma}[z_2, y_2], J_2) \subset \mathrm{car}(\gamma_2, J_2). \tag{3.19}$$

In addition, for any $a \in \gamma_1[y_2, w]$, the assumption $J_1 \leq J_2$ together with (3.17) yields

$$\begin{aligned} \frac{\ell(\hat{\gamma}[z_2, a])}{J_2} &= \frac{\ell(\gamma_2[z_2, y_2])}{J_2} + \frac{\ell(\gamma_1[y_2, a])}{J_2} \\ &\leq \frac{\ell(\gamma_1[z_1, y_2])}{J_1} + \frac{\ell(\gamma_1[y_2, a])}{J_1} \leq \frac{\ell(\gamma_1[z_1, a])}{J_1}. \end{aligned}$$

As a result, for any $a \in \gamma_1[y_2, w]$, the definition of $\mathrm{car}(\gamma_1, J_1)$ tells that

$$B\left(a, \frac{\ell(\hat{\gamma}[z_2, a])}{J_2}\right) \subset B\left(a, \frac{\ell(\gamma_1[z_1, a])}{J_1}\right) \subset \mathrm{car}(\gamma_1, J_1).$$

Thus, by recalling the definition of $\operatorname{car}(\hat{\gamma}, J_2)$ and (3.19), we finally get (3.18). □

Lemma 3.4. *Let $x, y, z \in \mathbb{R}^n$ and $J \geq 1$. Assume that there exist two curves $\gamma_{x,z}, \gamma_{y,z}$ respectively joining x, y to z. We denote the parametrization of*

$$\gamma := \gamma_{x,z} \cup \gamma_{y,z}$$

starting from x and ending at y as γ_1, and the one in the reversed direction, starting from y and ending at x, as γ_2. Then there exists a ball B with center $a \in \gamma$ satisfying

$$\operatorname{car}(\gamma_1[x,a], J) \cup \operatorname{car}(\gamma_2[y,a], J) \subset \operatorname{car}(\gamma_{x,z}, J) \cup \operatorname{car}(\gamma_{y,z}, J)$$

and radius r satisfying

$$r = \frac{\ell(\gamma_1[x,a])}{J} = \frac{\ell(\gamma_2[y,a])}{J}.$$

Remark 3.5. Lemma 3.4 is a corollary following from [25, Theorem 3.6] and [25, Lemma 4.3]. Since [25, Lemma 4.3] has used the concept of cigar in the statement, for the sake of completeness, we provide a proof avoiding the concept of "cigar" in the Appendix B.

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. We construct a sequence $\{W_{j,\infty}\}_{j\in\{1,\cdots,N\}}$ inductively.

Step 1: Construct $W_{1,\infty}$. We start from a point $x_1 \in \mathbb{R}^n \setminus K$ close to the origin. Then for any $R \geq 1$, the corresponding point $(x_1)_R \in \gamma_x \cap \partial B_{3R}$ is covered by some ball, say $D_R \in \mathcal{B}_{1,R}$. Thus, by (3.9), we know that

$$x_1 \in V_{1,R}. \tag{3.20}$$

Recall the definition (3.10). Let

$$W_{1,R} := \Omega_{1,R,x_1}, \quad \text{and} \quad W_{1,\infty} := \bigcup_{R\geq 1} W_{1,R}.$$

Then from (3.20) and (3.10), it follows that

$$\operatorname{car}(\gamma_{x_1}[x_1,\,(x_1)_R],\, J) \subset W_{1,\,R}, \quad \text{and} \quad \operatorname{car}(\gamma_{x_1}, J) \subset W_{1,\infty},$$

and from Proposition 3.2 that $W_{1,\,R}$ is J'-carrot John domain with $W_{1,R} \subset B_{2C_1 R}$.

Step 2: Proceeding inductively to construct $\{W_{j,\infty}\}$. We run the induction based on the two subindices j and r for $W_{j,\,r}$.

For any $r > 0$, define $x_{1,\,r} := x_1$. Suppose that for some $m \geq 1$, via the induction process, we have obtained points $\{x_j\}_{j=1}^m$ and the corresponding sets $\{W_{j,R}\}_{j=1}^m$ so that for any $1 \leq j \leq m$, $R > |x_j|$ and some $r = r(R,\,j) > 0$,

$$W_{j,R} := \Omega_{j,R,x_{j,r}}, \quad \text{and} \quad W_{j,\infty} := \bigcup_{R>|x_j|} W_{j,R} \quad \text{for some } r < R.$$

Suppose that, for some $s > 0$

$$B_s \setminus \left(K \cup \bigcup_{j=1}^{m} \overline{W}_{j,\,s} \right) \neq \emptyset. \tag{3.21}$$

This yields the existence of another point in $B_s \setminus \left(K \cup \bigcup_{j=1}^m \overline{W}_{j,\,s}\right)$. Take $r > 0$ to be (almost) the smallest $s > 0$ for which (3.21) holds. Next, we consider two cases.

Case 1: Suppose that

$$B_r \setminus K \subset \bigcup_{R \geq r} \bigcup_{j=1}^{m} V_{j,R}. \tag{3.22}$$

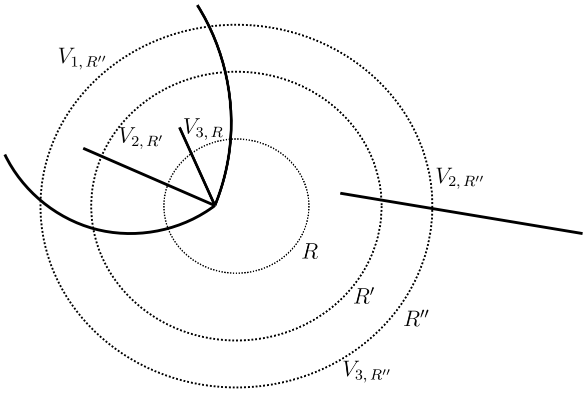

Figure 5. The set $W_{3,R}$ is contained in $W_{2,R'}$, and $W_{2,R'}$ is contained in $W_{1,R''}$, where $R \leq R' \leq R''$. Eventually they are all contained in $W_{1,\infty}$. However, we note that $W_{3,R}$ and $W_{3,R''}$ could have no intersection.

Since $V_{j,r}$ is a decomposition of $B_r \setminus K$ for any $r > 0$, (3.21) and (3.11) imply that there exists some point $x_{m+1,r} \in V_{m+1,r}$, and

$$\operatorname{car}\left(\gamma_{x_{m+1,r}}[x_{m+1,r}, (x_{m+1,r})_r], J\right) \subset \Omega_{m+1,r,x_{m+1,r}} \tag{3.23}$$

according to Proposition 3.2. Now from (3.22) it follows that for some $R > r$, we have

$$x_{m+1,r} \in V_{k,R} \neq \emptyset, \quad \text{for some } 1 \leq k \leq m. \tag{3.24}$$

Let R' be the infimum among all positive number for which (3.24) happens. Then if $R' > r$, we define

$$W_{m+1,s} := \Omega_{m+1,s,x_{m+1,r}}.$$

for $r \leq s < R'$. If $R' = r$, then we only define $W_{m+1,s} := \Omega_{m+1,s,x_{m+1,r}}$ for $s = r$.

In addition, it follows from (3.23) that

$$car\left(\gamma_{x_{m+1,r}}[x_{m+1,r}, (x_{m+1,r})_s], J\right) \subset W_{m+1,s}$$

and from Proposition 3.2 that $W_{m+1,s}$ is J'-carrot John domain with $W_{m+1,s} \subset B_{2C_1 s}$.

Next we check if

$$B_r \setminus \left(K \cup \bigcup_{j=1}^{m+1} \overline{W}_{j,r}\right) \neq \emptyset. \tag{3.25}$$

If it is non-empty, we continue to define $W_{m+2,r}$ and iterate our process. Otherwise, we define $W_{j,r} := \emptyset$ for $j \in \{m+1, \cdots, \hat{N}\}$. Then increase r until (3.25) holds for some $r' \geq r$, and consider the set $W_{m+1,r'}$.

Case 2: If (3.22) fails, then there exists a point $x_{m+1,r} \notin V_{j,R}$ for any $k \in \{1, \cdots, m\}$ and $R \geq r$. Since $\{V_{j,R}\}$ decomposes $B_R \setminus K$, then for every $R > r$, there exists $k_R \in \{m+1, \cdots, \hat{N}\}$ such that $x_{m+1,r} \in V_{k_R,R}$. Then up to relabeling the first subindex of $\{V_{j,R}\}_{j=m+1}^{\hat{N}}$, we may assume that $x_{m+1,r} \in V_{m+1,R}$.

As we need to define $W_{m+1,\infty}$ later, in order to distinguish from the first case, we write $x_{m+1} := x_{m+1,r}$ (also recall that $x_1 := x_{1,r}$ at the beginning of Step 2). Then define

$$W_{m+1,s} := \Omega_{m+1,s,x_{m+1}} \quad \text{for all } s > |x_{m+1}|,$$

and let

$$W_{m+1,\infty} := \bigcup_{R>|x_{m+1}|} W_{m+1,R}. \tag{3.26}$$

Likewise, (3.11) gives

$$\mathrm{car}(\gamma_{x_{m+1}}[x_{m+1}, (x_{m+1})_R], J) \subset W_{m+1,R} \quad \forall R > |x_{m+1}|, \quad \mathrm{car}(\gamma_{x_{m+1}}, J) \subset W_{m+1,\infty},$$

and from Proposition 3.2 that $W_{m+1,R}$ is J'-carrot John domain with $W_{m+1,R} \subset B_{2C_1R}$.

Step 3: Uniformly finitely many $W_{j,\infty}$. Our process is stopped when, for any $R > 0$,

$$B_R \setminus \left(K \cup \bigcup_{j=1}^{\hat{N}} \overline{W}_{j,R} \right) = \emptyset$$

and in particular, all $W_{j,\infty}$ have been founded so that

$$\mathbb{R}^n = \left(K \cup \bigcup_{j=1}^{N} \overline{W}_{j,\infty} \right) \tag{3.27}$$

for some $N \leq \hat{N}$. Suppose that (3.27) is not true. Then there exists a point

$$z \in \mathbb{R}^n \setminus \left(K \cup \bigcup_{j=1}^{N} \overline{W}_{j,\infty} \right).$$

Further observe that (3.26) and (3.11) give $V_{j,R} \subset \overline{W}_{j,\infty}$. Then our induction process tells that we can obtain a new set $W_{N+1,\infty}$ according to $z \notin V_{j,R}$ for any $j \in \{1, \cdots, N\}$ and those sufficiently large R, which is impossible.

Moreover, for any $1 \leq j \leq \hat{N}$, $W_{j,R}$ is a J'-carrot John domain with $W_{j,R} \subset B_{2C_1R}$, and for each $R \geq 1$, there exists $0 < r \leq R$ so that

$$\mathrm{car}(\gamma_{x_{j,r}}[x_{j,r}, (x_{j,r})_R], J) \subset W_{j,R} \tag{3.28}$$

and

$$\mathrm{car}(\gamma_{x_j}, J) \subset W_{j,\infty} \quad \text{for any } 1 \leq j \leq N. \tag{3.29}$$

Step 4: $W_{j,\infty}$ is J'-carrot John with John center ∞. Fix $j \in \{1, \cdots, N\}$. For every $z \in W_{j,\infty}$, it follows that $z \in W_{j,R}$ for some $R > |x_j|$. Hence, thanks to Proposition 3.2, z can be joined to $(x_j)_R$ by a rectifiable curve β_z as the core of a J'-carrot satisfying

$$\ell(\beta_z) \leq C_1 R \quad \text{and} \quad \mathrm{car}(\beta_z, J') \subset W_{j,\infty}, \tag{3.30}$$

In addition, by employing the definition of J' (3.16) and (3.4), Proposition 3.2 tells

$$\frac{\ell(\beta_z)}{J'} \leq \frac{R}{J} \leq \frac{\ell(\gamma_{x_j}[x_j, (x_j)_R])}{J}. \tag{3.31}$$

Further note that $(x_j)_R \in \gamma_{x_j}$. Then by employing (3.31), Lemma 3.3 tells that the curve $\zeta_z := \beta_z \cup \gamma_{x_j}[(x_j)_R, \infty)$ joining z toward ∞ satisfies

$$car(\zeta_z, J') \subset \mathrm{car}(\beta_z, J') \cup \mathrm{car}(\gamma_{x_j}, J).$$

Moreover, (3.29) and (3.30) yield

$$\mathrm{car}(\beta_z, J') \cup \mathrm{car}(\gamma_{x_j}, J) \subset W_{j,\infty}.$$

Thus the arbitrariness of z implies that $W_{j,\infty}$ is J'-carrot John with John center ∞.

Step 5: Proof of (1.5) and (1.6). In addition, for each pair of points $z, w \in W_{j,\infty}$, we can find R_z, $R_w > |x_j|$, such that $z \in W_{j,R_z}$ and $w \in W_{j,R_w}$. We may assume $R_z \le R_w$. Then Step 4 gives us two curves β_z, β_w joining z, w to $(x_j)_{R_z}, (x_j)_{R_w}$, respectively, such that

$$\mathrm{car}(\beta_w, J') \subset W_{j,\infty}, \quad \mathrm{car}(\beta_z, J') \subset W_{j,\infty} \quad \text{and} \quad \frac{\ell(\beta_z)}{J'} \le \frac{\ell(\gamma_{x_j}[x_j, (x_j)_{R_z}])}{J};$$

see (3.30) and (3.31). Therefore, applying (3.29) and Lemma 3.3 with $\gamma_1 = \gamma_{x_j}[x_j, (x_j)_{R_w}]$, $\gamma_2 = \beta_z$, and

$$J = J_1 \le J_2 = J',$$

there is a curve $\hat{\gamma} := \beta_z \cup \gamma_{x_j}[(x_j)_{R_z}, (x_j)_{R_w}]$ joining z to $(x_j)_{R_w}$, such that

$$\mathrm{car}(\hat{\gamma}, J') \subset \mathrm{car}(\beta_z, J') \cup \mathrm{car}(\gamma_{x_j}, J) \subset W_{j,\infty}.$$

Then, by Lemma 3.4, we finally arrive at (1.5) and (1.6).

Step 6: Proof of (1.2) and (1.3). The remaining task is to prove (1.2) and (1.3). As $W_{j,R}$ is a J'-carrot John domain with center $(x_{j,r})_R$ with some $r < R$, it follows from the definition of John domain that

$$\begin{aligned} B\left(x_{j,r}, \frac{\ell(\gamma_{x_{j,r}}[x_{j,r}, (x_{j,r})_R])}{J}\right) &\subset \mathrm{car}(\gamma_{x_{j,r}}[x_{j,r}, (x_{j,r})_R], J) \subset W_{j,R} \\ &\subset B\left((x_{j,r})_R, J' d((x_{j,r})_R, K)\right). \end{aligned}$$

As a result, by (3.4) and (3.6), the above estimate yields that

$$C(n,\, J)^{-1} R^n \le |W_{j,R}| \le C(n,\, J) R^n$$

and then the inequality (1.2) follows.

Furthermore, given $k \in \{1, \cdots, \hat{N}\}$, we consider the set $W_{k,\,R}$ which contains the carrot

$$\mathrm{car}(\gamma_{x_{k,\,r}}[x_{k,\,r},\, (x_{k,\,r})_R],\, J)$$

by (3.28). Then we choose $1 \le k_l \le \hat{N}$ so that $x_{k,\,r} \in V_{k_l,\,2^l R}$; such a k_l exists since $\{V_{j,\,2^l R}\}_j$ covers $B_{2^l R} \setminus K$.

Toward the inequality (1.3), recall that $W_{k_l,\,2^l R}$ is constructed via Proposition 3.2, which, in particular by the definition of $\beta_{x_{k,\,r}}$, contains the carrot

$$\mathrm{car}(\gamma_{x_{k,\,r}}[x_{k,\,r},\, (x_{k,\,r})_{2^l R}],\, J');$$

recall that

$$\gamma_{x_{k,\,r}}[x_{k,\,r},\, (x_{k,\,r})_{2^l R}] \subset \beta_{x_{k,\,r}}.$$

Especially,

$$\mathrm{car}(\gamma_{x_{k,\,r}}[x_{k,\,r},\, (x_{k,\,r})_{2^l R}],\, J') \subset W_{k_l,\,2^l R} \cap W_{k_{l+1},\,2^{l+1} R},$$

and (1.3) follows from (1.2) as

$$\left|\mathrm{car}(\gamma_{x_{k,\,r}}[x_{k,\,r},\, (x_{k,\,r})_{2^l R}],\, J')\right| \ge C(n,\, J)(2^l R)^n. \qquad \square$$

Appendix A. Proof of the estimate (2.19)

Proof. Let

$$\hat{J} := J(\hat{x}, \Omega; \hat{y}).$$

Then using Lemma 2.2, there exists a rectifiable curve $\beta \subset \Omega$ joining $\hat{x}$ to $\hat{y}$ together with the corresponding $\hat{J}$-carrot $\mathrm{car}(\beta, \hat{J})$, such that

$$\sup_{t\in[0,1]} j(t; \hat{x}, \beta, \Omega) = J(\hat{x}, \Omega; \hat{y}) = \hat{J} \quad \text{and} \quad \mathrm{car}(\beta, \hat{J}) \subset \Omega. \tag{A.1}$$

Analogously, thanks to the compactness of $[0, 1]$, the definition of $j(t; \hat{x}, \beta, \Omega)$ tells that we can find a point $\hat{s} \in [0, 1]$, such that

$$\frac{\ell_{\|\cdot\|}(\beta([0, \hat{s}]))}{d_{\|\cdot\|}(\beta(\hat{s}), \partial\Omega)} = \hat{J} = \sup_{t\in[0,1]} j(t; \hat{x}, \beta, \Omega).$$

We repeat the argument by replacing γ, x and J respectively by $\beta, \hat{x}$ and $\hat{J}$ in the proof of (2.11). Then we have

$$d_{\|\cdot\|}(\beta, \partial\Omega) \geq \frac{d_{\|\cdot\|}(\hat{x}, \partial\Omega)}{2C_{\|\cdot\|}\hat{J}}. \tag{A.2}$$

Further assume that $L_{x,\hat{x}} \subset \Omega$ is the line segment joining x to $\hat{x}$ and $L_{\hat{y},y} \subset \Omega$ is the one joining $\hat{y}$ to y. Then

$$\hat{\beta} := L_{x,\hat{x}} \cup \beta \cup L_{\hat{y},y}$$

is a rectifiable curve within Ω joining x to y. Now we also repeat the argument by replacing s, J, γ and $\hat{\gamma}$ with $\hat{s}, \hat{J}, \beta$ and $\hat{\beta}$, respectively, and swapping x, y respectively with $\hat{x}, \hat{y}$, respectively. By letting $(\hat{x}, \hat{y})$ close enough to (x, y), (2.17) changes into

$$\begin{aligned} &\frac{\ell_{\|\cdot\|}(\hat{\beta}([0, t]))}{d_{\|\cdot\|}(\hat{\beta}(t), \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\beta([0, \hat{s}]))}{d_{\|\cdot\|}(\beta(\hat{s}), \partial\Omega)} \\ &\leq \max\left\{\frac{\|\hat{x} - x\|}{d_{\|\cdot\|}(\beta, \partial\Omega)}, \frac{C(n, C_{\|\cdot\|}, \hat{J})}{d_{\|\cdot\|}(\hat{y}, \partial\Omega)} \left(\|\hat{x} - x\| + \|\hat{y} - y\|\right), \frac{2\|\hat{x} - x\|}{d_{\|\cdot\|}(\hat{x}, \partial\Omega)}\right\} \end{aligned} \tag{A.3}$$

for any $z \in \hat{\beta}$. Further note that when $\|x - \hat{x}\| + \|y - \hat{y}\| < \delta$ for a sufficiently small and positive constant $\delta = \delta(x, y, C_{\|\cdot\|})$ satisfying $\delta \leq \frac{1}{2}\min\{d_{\|\cdot\|}(x, \partial\Omega), d_{\|\cdot\|}(y, \partial\Omega)\}$ at least, by (2.9) and (A.1), the estimate (2.18) gives

$$\hat{J} \leq C(n, C_{\|\cdot\|}, J). \tag{A.4}$$

Consequently, combining the construction of $\hat{\beta}$, (A.3), (A.2) and (A.4), it follows that when $\|x - \hat{x}\| + \|y - \hat{y}\| < \delta$,

$$\begin{aligned} J(x, \Omega; y) - J(\hat{x}, \Omega; \hat{y}) &\leq \sup_{t\in[0,1]} \frac{\ell_{\|\cdot\|}(\hat{\beta}([0, t]))}{d_{\|\cdot\|}(\hat{\beta}(t), \partial\Omega)} - \frac{\ell_{\|\cdot\|}(\beta([0, \hat{s}]))}{d_{\|\cdot\|}(\beta(\hat{s}), \partial\Omega)} \\ &\leq \max\left\{\frac{\|\hat{x} - x\|}{d_{\|\cdot\|}(\beta, \partial\Omega)}, \frac{C(n, C_{\|\cdot\|}, \hat{J})}{d_{\|\cdot\|}(\hat{y}, \partial\Omega)} \left(\|\hat{x} - x\| + \|\hat{y} - y\|\right), \frac{2\|\hat{x} - x\|}{d_{\|\cdot\|}(\hat{x}, \partial\Omega)}\right\} \\ &\leq \frac{C(n, C_{\|\cdot\|}, \hat{J})}{d_{\|\cdot\|}(\beta, \partial\Omega)} \left(\|\hat{x} - x\| + \|\hat{y} - y\|\right) \leq \frac{C(n, C_{\|\cdot\|}, \hat{J})}{d_{\|\cdot\|}(\hat{x}, \partial\Omega)} \left(\|\hat{x} - x\| + \|\hat{y} - y\|\right) \\ &\leq \frac{C(n, C_{\|\cdot\|}, \hat{J})}{d_{\|\cdot\|}(x, \partial\Omega)} \left(\|\hat{x} - x\| + \|\hat{y} - y\|\right) \leq \frac{C(n, C_{\|\cdot\|}, J)}{d_{\|\cdot\|}(x, \partial\Omega)} \left(\|x - \hat{x}\| + \|y - \hat{y}\|\right), \end{aligned} \tag{A.5}$$

which yields (2.19). □

Appendix B. Proof of Lemma 3.4

Proof. We may assume that $\ell(\gamma_{x,z}) \geq \ell(\gamma_{y,z})$. Then there exists a point $a \in \gamma_{x,z}$, such that

$$\ell(\gamma_{x,z}[x,a]) = \ell(\gamma_{y,z}) + \ell(\gamma_{x,z}[z,a]). \tag{B.1}$$

Note that the construction of γ tells that

$$\gamma_1[x,a] = \gamma_{x,z}[x,a], \quad \gamma_2[y,a] = \gamma_{y,z} \cup \gamma_1[z,a]. \tag{B.2}$$

Then, due to (B.1) and (B.2), it follows that $\ell(\gamma_1[x,a]) = \ell(\gamma_2[y,a])$. This implies that for each $\eta \in \gamma_2[z,a] = \gamma_{x,z}[z,a]$,

$$\ell(\gamma_2[y,\eta]) \leq \ell(\gamma_2[y,a]) = \ell(\gamma_1[x,a]) \leq \ell(\gamma_1[x,\eta]). \tag{B.3}$$

Besides, (B.2) directly yields that

$$\operatorname{car}(\gamma_1[x,a],J) \subset \operatorname{car}(\gamma_{x,z},J), \quad \operatorname{car}(\gamma_{y,z},J) = \operatorname{car}(\gamma_2[y,z],J), \tag{B.4}$$

which, together with the definition of $\operatorname{car}(\gamma_{y,z},J)$ and (B.3), implies that

$$\operatorname{car}(\gamma_1[x,a],J) \cup \operatorname{car}(\gamma_2[y,a],J) \subset \operatorname{car}(\gamma_{x,z},J) \cup \operatorname{car}(\gamma_{y,z},J).$$

As a result, the desired ball is

$$B = B\left(a, \frac{\ell(\gamma_1[x,a])}{J}\right).$$

The proof is completed. □

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