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Continuity of solutions to complex Hessian equations via the Dinew–Kołodziej estimate

Per Åhag and Rafał Czyż

Abstract. This study extends the celebrated volume-capacity estimates of Dinew and Kołodziej, providing a foundation for examining the regularity of solutions to boundary value problems for complex Hessian equations. By integrating the techniques established by Dinew and Kołodziej and incorporating recent advances by Charabati and Zeriahi, we demonstrate the continuity of the solutions.

Kontinuitet hos lösningar till komplexa Hessianska ekvationer via ett estimat av Dinew och Kołodziej

Sammanfattning. Denna studie utvidgar det välkända volym-kapacitetsestimatet av Dinew och Kołodziej och lägger därmed grunden för att undersöka regulariteten hos lösningar till randvärdesproblem för komplexa Hessianska ekvationer. Genom att kombinera de metoder som utvecklats av Dinew och Kołodziej med nyligen gjorda framsteg av Charabati och Zeriahi, visar vi kontinuitet hos lösningarna.

1. Introduction

Let $K \subset \mathbb{C}$ be a compact subset in the complex plane with area $A(K)$ and logarithmic capacity $c(K)$. In 1928, Pólya [31] established the following inequality:

$$A(K) \leq \pi c(K)^2.$$

which has found widespread use and has been generalized in various contexts within analysis and geometry. For example, it inspired work in [1] that helped confirm a conjecture by Demailly in [14].

Of particular interest to this paper is the work by Dinew and Kołodziej [16], who proved that the volume of a relatively compact set in $\mathbb{C}^n$ can be estimated using the Hessian capacity:

$$V_{2n}(K) \leq C(\operatorname{cap}_m(K))^{\alpha}, \tag{1.1}$$

where dV_{2n} denotes the Lebesgue measure in $\mathbb{R}^{2n}$, $1 < \alpha < \frac{n}{n-m}$ and $\operatorname{cap}_m(K)$ is defined as the Hessian capacity. This result is crucial for studying the complex Hessian operator on compact Kähler manifolds.

In Theorem 3.3, we extend their volume-capacity estimate using techniques from Orlicz theory.

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Theorem 3.3. *Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. Then for any $0 < \epsilon \leq \frac{n+1}{3n}$ there exist constants $C_1, C_2 > 0$ such that for all $K \Subset \Omega$ it holds:*

$$V_{2n}(K) \leq C_1 \operatorname{cap}_m(K)^{\frac{n}{n-m}} \mathrm{W}_0 \left(C_2 \operatorname{cap}_m(K)^{\frac{-1}{m(1+\epsilon)}} \right)^{\frac{nm(1+\epsilon)}{n-m}}, \tag{1.2}$$

where W_0 *is the Lambert* W *function.*

In 1986, Vinacua [32] (see also [33]) expanded upon the foundational work of Caffarelli, Nirenberg, and Spruck [8] by introducing complex Hessian equations:

$$\mathrm{H}_m(u) = \mu, \tag{1.3}$$

bridging classical and modern theories in partial differential equations. As defined (Definition 2.1), the 1-Hessian operator, H_1, corresponds to the classical Laplace operator on subharmonic functions, herein referred to as 1-subharmonic functions. Similarly, the n-Hessian operator, H_n, corresponds to the complex Monge–Ampère operator for plurisubharmonic functions. For $k = 2, \ldots, n-1$, the m-Hessian operators form a sequence of partial differential operators, spanning from the Laplace to the complex Monge–Ampère operators.

The integrability of general m-subharmonic functions significantly differs from that of n-subharmonic functions. While all plurisubharmonic functions are locally L^p integrable for any $p > 0$, this is not necessarily true for m-subharmonic functions. Błocki [6] conjectured that m-subharmonic functions should be locally L^p integrable for $p < \frac{nm}{n-m}$, a conjecture partially verified in [2, 16].

The complex Hessian equation and m-subharmonic functions have attracted widespread attention. A significant advancement was made by Błocki in 2005, who extended these concepts to non-smooth admissible functions and introduced pluripotential methods [6]. More recently, Lu adapted Cegrell's framework [9, 10, 11] for the complex Hessian equations [22, 23, 24, 25].

In this paper, we build upon the techniques of Dinew and Kołodziej [16] and incorporate recent insights from Charabati and Zeriahi [13] to address the regularity of solutions for the complex Hessian equations.

Consider a bounded strictly m-pseudoconvex domain $\Omega \subset \mathbb{C}^n$. For a given density function f and a boundary value function $g \in \mathcal{C}(\partial\Omega)$, the problem of interest is:

$$\begin{aligned} &\mathrm{H}_m(U(f,g)) = f dV_{2n}, \\ &\lim_{z \to w} U(f,g)(z) = g(w), \quad \text{for all } w \in \partial\Omega. \end{aligned} \tag{1.4}$$

If the density function $f \in L^p$ for $p > \frac{n}{m}$, then $\mathrm{U}(f,g)$ is continuous and has been subsequently proven to be Hölder continuous, as documented in [16, 7, 12, 21, 29]. If $p < \frac{n}{m}$, then the solution to (1.4) need not to be bounded [16], which is a significant contrast to the case when $m = n$. In [3], Cegrell's energy classes played a crucial role in studying the regularity of unbounded m-subharmonic functions.

Our goal is to investigate further the regularity of solutions to (1.4) when the density function f belongs to the Orlicz space $L^{n/m}(\log L)^\alpha$ with $\alpha > 2n$. We establish the following result:

Theorem 4.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly m-pseudoconvex domain, let $f \in L^{\frac{n}{m}}(\log L)^\alpha$ for $\alpha > 2n$, and let $g \in \mathcal{C}(\partial\Omega)$. Then, the unique solution $U(f,g)$ of the Dirichlet problem for the complex Hessian operator* (4.1) *is continuous on* $\bar{\Omega}$.

Moreover, the following estimate holds:

$$\|U(f_1, g_1) - U(f_2, g_2)\|_\infty \le \|g_1 - g_2\|_\infty + C_1 \|f_1 - f_2\|_\alpha^{-\frac{1}{\gamma}} + C_2 e_{m,m}(U(|f_1 - f_2|, 0))^{\frac{1}{2m}} \exp\left(C_3 \|f_1 - f_2\|_\alpha^{-\frac{1}{\gamma}} \right),$$

where $\gamma = (1+\epsilon)m - \frac{\alpha m}{n}$ *with* $0 < \epsilon < \min\left(\frac{n+1}{3n}, \frac{\alpha}{n} - 2\right)$. *Here,* $\|f\|_\alpha$ *denotes the norm in* $L^{\frac{n}{m}}(\log L)^\alpha$, *and* C_1, C_2, C_3 *are positive universal constants. Moreover,* $e_{m,m}(u)$ *is defined as* $\int_\Omega (-u)^m \operatorname{H}_m(u)$.

This paper is organized as follows: Section 2 provides the necessary background on complex Hessian equations and the theoretical framework used throughout the paper, including discussions on the theory of Orlicz spaces. Section 3 contains the proof of Theorem 3.3, while in Section 4, we present the proof of Theorem 4.3. The final section, Section 5, studies the cases when $n < \alpha \le 2n$ and $\alpha \le n$.

We want to emphasize that the proof of Theorem 3.3 and the results in Section 5 rely heavily on the properties of plurisubharmonic functions. Notably, significant insights into m-subharmonic functions can be gained by examining the subset of plurisubharmonic functions, a phenomenon first observed by Dinew and Kołodziej [16].

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2. Preliminaries

This section is organized as follows: Section 2.1 provides the fundamental definitions of the generalized potential theory we are interested in. Section 2.2 offers basic information on Orlicz spaces, specifically $L^{\frac{n}{m}}(\log L)^\alpha$. Finally, Section 2.3 reviews some essential aspects of the Lambert W function.

2.1. Generalized potential theory. This subsection provides foundational definitions and results for m-subharmonic functions and the complex Hessian operator. Consider $\Omega \subset \mathbb{C}^n$, where $n \ge 2$, as a bounded domain, and let $1 \le m \le n$. Define $\mathbb{C}_{(1,1)}$ as the set of $(1,1)$-forms with constant coefficients. We then define

$$\Gamma_m = \left\{ \alpha \in \mathbb{C}_{(1,1)} \colon \alpha \wedge (dd^c|z|^2)^{n-1} \ge 0, \dots, \alpha^m \wedge (dd^c|z|^2)^{n-m} \ge 0 \right\}.$$

Definition 2.1. Let $n \ge 2$, and $1 \le m \le n$. Assume $\Omega \subset \mathbb{C}^n$ is a bounded domain. A function u, defined on Ω and subharmonic, is said to be *m-subharmonic* if it satisfies

$$dd^c u \wedge \alpha_1 \wedge \dots \wedge \alpha_{m-1} \wedge (dd^c|z|^2)^{n-m} \ge 0,$$

in the sense of currents for all $\alpha_1, \dots, \alpha_{m-1} \in \Gamma_m$. We denote the set of all such m-subharmonic functions on Ω by $\mathcal{SH}_m(\Omega)$.

Definition 2.2. Let $n \ge 2$, and $1 \le m \le n$. A bounded domain $\Omega \subset \mathbb{C}^n$ is termed *m-hyperconvex* if there is a non-negative, m-subharmonic exhaustion function, i.e., there exists an m-subharmonic function $\varphi\colon \Omega \to (-\infty, 0]$ such that for every $c < 0$, the closure of $\{z \in \Omega \colon \varphi(z) < c\}$ is compact within Ω.

For additional insights into m-hyperconvex domains, we refer to [5].

Definition 2.3. An open set $\Omega \subset \mathbb{C}^n$ is *strictly m-pseudoconvex* if it admits a smooth defining function ρ which is strictly m-subharmonic in a neighborhood of $\bar{\Omega}$ and satisfies $|\nabla\rho| > 0$ at each point in $\partial\Omega = \{\rho = 0\}$.

Next, we introduce function classes essential to this paper. A function φ, defined on an m-hyperconvex domain Ω and m-subharmonic, belongs to $\mathcal{E}_m^0(\Omega)$ if it is bounded,

$$\lim_{z\to\xi} \varphi(z) = 0 \text{ for every } \xi \in \partial\Omega,$$

and

$$\int_\Omega \mathrm{H}_m(\varphi) < \infty.$$

Definition 2.4. Let $n \geq 2$, $1 \leq m \leq n$, and $p \geq 0$. A function u defined on a bounded m-hyperconvex domain Ω in $\mathbb{C}^n$ belongs to $\mathcal{F}_m(\Omega)$ if there exists a decreasing sequence $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_m^0(\Omega)$, converging pointwise to u as $j \to \infty$, and $\sup_j \int_\Omega \mathrm{H}_m(\varphi_j) < \infty$.

In [22, 25], it was shown that for a function $u \in \mathcal{F}_m(\Omega)$, the complex Hessian operator $\mathrm{H}_m(u)$ is well-defined and given by

$$\mathrm{H}_m(u) = (dd^c u)^m \wedge (dd^c|z|^2)^{n-m},$$

where $d = \partial + \bar{\partial}$ and $d^c = \sqrt{-1}(\bar{\partial} - \partial)$.

Let us recall the Hessian capacity:

$$\operatorname{cap}_m(E) = \sup\left\{\int_E \mathrm{H}_m(u)\colon u \in \mathcal{SH}_m(\Omega), -1 \leq u \leq 0\right\}.$$

2.2. Orlicz spaces. This subsection introduces some notations and elementary facts concerning Orlicz spaces, which will be useful in later discussions. This section is based on [26].

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be an increasing, continuous, and convex function such that $\varphi(0) = 0$, $\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0$, and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$. We shall call such a function *admissible.*

Let X be the space of measurable functions (with respect to the Lebesgue measure) on Ω. Define a modular on X by

$$\rho(f) = \int_\Omega \varphi(|f|)\, dV_{2n},$$

and introduce the Orlicz class:

$$L_0^\varphi = \{f \in X\colon \rho(f) < \infty\}.$$

The Orlicz space L^φ is the smallest vector space containing L_0^φ. Moreover, L^φ is a Banach space equipped with the Orlicz norm

$$\|f\|_\varphi^0 = \sup\left\{\int_\Omega |fg|\, dV_{2n}\colon \int_\Omega \varphi^*(|g|)\, dV_{2n} \leq 1\right\}$$

or the equivalent Luxemburg norm

$$\|f\|_\varphi = \inf\left\{\lambda > 0\colon \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dV_{2n} \leq 1\right\},$$

where φ^*, the Legendre transform of φ is defined as

$$\varphi^*(s) = \sup_{t\geq 0}(st - \varphi(t)).$$

The Legendre transform φ^* is also called the complementary function in the sense of Young.

Let us now recall the following version of Young's inequality: for all $t, s \geq 0$,

$$st \leq \varphi(t) + \varphi^*(s),$$

and note that

$$\|f\|_\varphi \leq \|f\|_\varphi^0 \leq 2\|f\|_\varphi.$$

Hence, by the Young inequality and the definition of the Orlicz norm,

$$\begin{aligned}
\|f\|_\varphi^0 &= \sup_g \left(\int_\Omega |fg|\, dV_{2n} \colon \int_\Omega \varphi^*(|g|)\, dV_{2n} \leq 1 \right) \\
&\leq \sup_g \left(\int_\Omega \varphi(|f|) + \varphi^*(|g|)\, dV_{2n} \colon \int_\Omega \varphi^*(|g|)\, dV_{2n} \leq 1 \right) \qquad (2.1) \\
&\leq \int_\Omega \varphi(|f|)\, dV_{2n} + 1.
\end{aligned}$$

We present the following counterpart to the classical Hölder's inequality:

Theorem 2.5. *If $f \in L^\varphi$ and $g \in L^{\varphi^*}$, then*

$$\left| \int_\Omega fg\, dV_{2n} \right| \leq \|f\|_\varphi^0 \|g\|_{\varphi^*}, \quad \textit{and} \quad \left| \int_\Omega fg\, dV_{2n} \right| \leq \|f\|_\varphi \|g\|_{\varphi^*}^0.$$

Let $1 \leq m \leq n$ be integers, $n \geq 2$, and let $\alpha > 0$. A central tool in this paper is the Orlicz space generated by the function

$$\varphi(t) = (1+t)^{\frac{n}{m}} (\log(1+t))^\alpha,$$

which will be denoted by $L^{\frac{n}{m}}(\log L)^\alpha$. The corresponding Orlicz norm will be denoted for simplicity by $\|\cdot\|_\alpha$.

The following example will be used in Theorem 3.3 and Theorem 4.3

Example 2.6. Let $K \subset \Omega$ be such that $0 < V_{2n}(K) < \infty$, and let φ be an admissible function. Then,

$$\|\chi_K\|_\varphi = \frac{1}{\varphi^{-1}(V_{2n}(K)^{-1})},$$

and

$$\|\chi_K\|_\varphi^0 = V_{2n}(K)(\varphi^*)^{-1}\left(V_{2n}(K)^{-1}\right),$$

where $(\varphi^*)^{-1}$ denotes the inverse function to φ^*. If $f \in L^\varphi$, then we have:

$$\int_K f\, dV_{2n} \leq \|f\|_\varphi \|\chi_K\|_{\varphi^*}^0 = \|f\|_\varphi V_{2n}(K) \varphi^{-1}(V_{2n}(K)^{-1}), \qquad (2.2)$$

where φ^{-1} is the inverse function to φ. □

2.3. Lambert *W* function. We present in this subsection some notations and elementary facts concerning the Lambert W function, W_0, which will be helpful later on several occasions.

First recall that $\mathrm{W}_0(x)$, for $x > 0$ is defined as the unique solution to the equation

$$\mathrm{W}_0(x) \exp(\mathrm{W}_0(x)) = x.$$

It is well known that

$$\mathrm{W}_0'(x) = \frac{\mathrm{W}_0(x)}{x(1 + \mathrm{W}_0(x))} > 0.$$

By [19] we have the following estimates:

(1) for $x \geq e$

$$\frac{1}{2}\log(x) \leq \mathrm{W}_0(x) \leq \log(x);$$

(2) for $x \geq e$

$$\log(x) - \log(\log(x)) \leq \mathrm{W}_0(x) \leq \log(x) - \frac{1}{2}\log(\log(x)).$$

From the above we can deduce that for $x \geq 0$ it holds:

$$\mathrm{W}_0(x) \leq \max(1, \log x). \tag{2.3}$$

For further information on Lambert W function we refer the reader to [27].

3. Volume estimation via capacity

Dinew and Kołodziej in [16] proved that the volume of a relatively compact set can be estimated using capacity. They established that for $1 < \alpha < \frac{n}{n-m}$, the following inequality holds:

$$V_{2n}(K) \leq C(\mathrm{cap}_m(K))^{\alpha}.$$

This result has proven essential for studying the complex Hessian operator in $\mathbb{C}^n$ and on compact Kähler manifolds. However, for our aim in Section 4.1, this result is not sufficient. Consequently, we aim to refine the Dinew–Kołodziej estimate in Theorem 3.3. First, we list two known facts required for the proof.

Lemma 3.1. [20, Theorem 1] *Let h be an increasing function such that*

$$\int_1^{\infty} \frac{1}{yh^{\frac{1}{n}}(y)}\, dy < \infty,$$

and let μ be a non-negative measure on a bounded, strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Assume that there exists a constant $A > 0$ such that for any $v \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, with $v = 0$ on $\partial\Omega$ and $\int_{\Omega}(dd^c v)^n \leq 1$ it holds:

$$\int_{\Omega} (-v)^n h(-v)\, d\mu \leq A.$$

Then any bounded plurisubharmonic function u with $(dd^c u)^n = \mu$ will satisfy

$$\|u\|_{\infty} \leq B(A, h),$$

where the constant $B(A, h)$ does not depend on μ.

Lemma 3.2. [1, Theorem 4.1] *Let $v \in \mathcal{F}_n$ with $\int_{\Omega}(dd^c v)^n \leq 1$. For all $s > 0$, it follows that*

$$V_{2n}(\{v \leq -s\}) \leq C_n(1+s)^{n-1}\exp(-2ns).$$

We are now ready to present the first result of this paper.

Theorem 3.3. *Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. Then for any $0 < \epsilon \leq \frac{n+1}{3n}$ there exist constants $C_1, C_2 > 0$ such that for all $K \Subset \Omega$ it holds:*

$$V_{2n}(K) \leq C_1\, \mathrm{cap}_m(K)^{\frac{n}{n-m}}\, \mathrm{W}_0\left(C_2\, \mathrm{cap}_m(K)^{\frac{-1}{m(1+\epsilon)}}\right)^{\frac{nm(1+\epsilon)}{n-m}}, \tag{3.1}$$

where W_0 is the Lambert W function.

Proof. Without loss of generality, we may assume that Ω is a bounded strictly pseudoconvex set with $V_{2n}(\Omega) \leq 1$. Otherwise, there exists some $R > 0$ such that $\Omega \subset B(0,R)$ and $\mathrm{cap}_{K(0,R)}(K) \leq \mathrm{cap}_\Omega(K)$. Consider a compact set K with $V_{2n}(K) > 0$. If not, there is nothing to prove.

Fix $0 < \epsilon \leq \frac{n+1}{3n}$ and define the function

$$F(t) = t^{-1}(-\log t)^{-n-n\epsilon}.$$

Let φ be a plurisubharmonic solution to the Monge–Ampère equation:

$$(dd^c\varphi)^n = F(V_{2n}(K))\chi_K dV_{2n}, \quad \text{and} \quad \varphi = 0 \text{ on } \partial\Omega.$$

Then, by the inequality between mixed Monge–Ampère measures (see [15]), we obtain

$$\mathrm{H}_m(\varphi) \geq F^{\frac{m}{n}}(V_{2n}(K))\chi_K\, dV_{2n}. \tag{3.2}$$

Define

$$\Phi(t) = \exp\left(2n(1-\epsilon)(t+1)^{\frac{1}{n+n\epsilon}}\right)$$

and note that, given the choice of ϵ, the function Φ is increasing and convex.

We apply Lemma 3.1 with $h(t) = (\log(1+t))^{n+n\epsilon}$ and $\mu = F(V_{2n}(K))\chi_K dV_{2n}$. Consider $v \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, with $v = 0$ on $\partial\Omega$ and $\int_\Omega (dd^c v)^n \leq 1$. From Theorem 2.5, (2.1) and (2.2), we have

$$\begin{aligned}
&\int_\Omega (-v)^n h(-v)\, d\mu = \int_\Omega (-v)^n h(-v) F(V_{2n}(K))\chi_K\, dV_{2n} \\
&\leq \|(-v)^n h(-v)\|_\Phi F(V_{2n}(K))\|\chi_K\|^0_{\Phi^*} \leq \|(-v)^n h(-v)\|^0_\Phi F(V_{2n}(K))\|\chi_K\|^0_{\Phi^*} \\
&\leq \left(\int_\Omega \Phi((-v)^n h(-v))\, dV_{2n} + 1\right) F(V_{2n}(K))V_{2n}(K)\Phi^{-1}\left(\frac{1}{V_{2n}(K)}\right),
\end{aligned} \tag{3.3}$$

where Φ^{-1} is the inverse function,

$$\Phi^{-1}\left(\frac{1}{s}\right) = \left(-\frac{\log s}{2n(1-\epsilon)}\right)^{n+n\epsilon} - 1 \leq \left(-\frac{\log s}{2n(1-\epsilon)}\right)^{n+n\epsilon}.$$

Then

$$F(V_{2n}(K))V_{2n}(K)\Phi^{-1}\left(\frac{1}{V_{2n}(K)}\right) = \frac{1}{(2n(1-\epsilon))^{n+n\epsilon}} = \tilde{C}, \tag{3.4}$$

where $\tilde{C}$ is a constant independent of K.

On the other hand, by Lemma 3.2, we have

$$\begin{aligned}
&\int_\Omega \Phi((-v)^n h(-v))\, dV_{2n} \leq C_n + \sum_{s=0}^{\infty} \int_{\{-s-1<v<-s\}} \Phi((-v)^n h(-v))\, dV_{2n} \\
&\leq C_n \sum_{s=0}^{\infty} (1+s)^{n-1} \exp(-2ns) \exp\left(2n(1-\epsilon)((s+1)^n(\log(2+s))^{n+n\epsilon})^{\frac{1}{n+n\epsilon}}\right) \\
&\leq A,
\end{aligned} \tag{3.5}$$

where the constant A does not depend on v.

Combining (3.3), (3.4), and (3.5), we conclude

$$\int_\Omega (-v)^n h(-v)\, d\mu \leq \tilde{C}(A+1),$$

thereby, by Lemma 3.1, a constant d, independent of K, exists such that the solution φ satisfies $\|\varphi\|_\infty \leq \frac{1}{d}$.

Now, let $\psi = d\varphi$, then

$$\begin{aligned}\operatorname{cap}_m(K) &\geq \int_\Omega \mathrm{H}_m(\psi) = d^m \int_K \mathrm{H}_m(\varphi) \geq d^m \int_K F^{\frac{m}{n}}(V_{2n}(K))\, dV_{2n} \\ &= d^m (V_{2n}(K))^{1-\frac{m}{n}} (-\log(V_{2n}(K)))^{-(1+\epsilon)m}.\end{aligned} \tag{3.6}$$

Define the function $G_{p,q}(t) = t^q(-\log t)^p$, for $p < 0$, $q > 0$, which is increasing for $t \in (0,1)$, with its inverse given by

$$G^{-1}_{p,q}(s) = \frac{\left(-\frac{q}{p}\right)^{\frac{p}{q}} s^{\frac{1}{q}}}{\mathrm{W}_0\left(-\frac{q}{p} s^{\frac{1}{p}}\right)^{\frac{p}{q}}}.$$

In our scenario $p = -(1+\epsilon)m$ and $q = 1 - \frac{m}{n}$. Finally, it follows from (3.6) that

$$\begin{aligned}V_{2n}(K) &\leq G^{-1}_{p,q}\left(d^{-m} \operatorname{cap}_m(K)\right) \\ &= C_1 \operatorname{cap}_m(K)^{\frac{n}{n-m}} \mathrm{W}_0(C_2 \operatorname{cap}_m(K)^{\frac{-1}{(1+\epsilon)m}})^{\frac{nm(1+\epsilon)}{n-m}},\end{aligned}$$

for some universal constants C_1 and C_2. The proof is complete. □

Remark. We recall that an inequality similar to (3.6), applicable to any open set in $\mathbb{C}^n$, was previously established by Nguyen in [28]. Notably, our approach, which leverages Orlicz space techniques, differs from the methods employed by Nguyen. Although a statement analogous to Theorem 3.3 is not explicitly formulated in [28], it can be obtained from the inequality resembling (3.6).

Corollary 3.4. *Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. Then for any $0 < \epsilon \leq \frac{n+1}{3n}$ there exist constants $D_1, D_2 > 0$ such that for all $K \Subset \Omega$ it holds:*

$$V_{2n}(K) \leq D_1 \operatorname{cap}_m(K)^{\frac{n}{n-m}} \max(1, 1 - D_2 \log(\operatorname{cap}_m(K))^{\frac{nm(1+\epsilon)}{n-m}} \tag{3.7}$$

Proof. It follows from Theorem 3.3 and the estimate (2.3) for the Lambert W function. □

4. Continuous solution

Let $n \geq 2$ and $1 \leq m \leq n$. In this section, we confine our discussion to a bounded strictly m-pseudoconvex domain $\Omega \subset \mathbb{C}^n$, as defined in Definition 2.3. Consider the following Dirichlet problem for the complex Hessian equation given a density function f and a boundary value function $g \in \mathcal{C}(\partial\Omega)$:

$$\begin{aligned}&\mathrm{H}_m(U(f,g)) = f dV_{2n}, \\ &\lim_{z\to w} U(f,g)(z) = g(w), \quad \text{for all } w \in \partial\Omega,\end{aligned} \tag{4.1}$$

where dV_{2n} represents the Lebesgue measure in $\mathbb{R}^{2n}$. Theorem 4.3 establishes that if the density function f is in $L^{\frac{n}{m}}(\log L)^\alpha$ with $\alpha > 2n$, then the solution exists and is continuous on $\bar{\Omega}$. The proof is using the following result recently proved in [13].

Theorem 4.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly m-pseudoconvex domain and let μ be a positive finite Borel measure on Ω such that for all compact sets $K \subset \Omega$ the following holds:*

$$\mu(K) \leq A \operatorname{cap}_m(K) F(\operatorname{cap}_m(K)),$$

where $F\colon (0,\infty) \to (0,\infty)$ is a continuous increasing function that satisfies

$$\int_{0^+} \frac{F(t)^{\frac{1}{m}}}{t}\, dt < \infty.$$

Then, for any positive continuous boundary function $g \in \mathcal{C}(\partial\Omega)$, there exists a unique continuous solution $U(\mu, g)$ of the Dirichlet problem for the complex Hessian equation (4.1).

To provide an L^∞-estimate of the solution we shall use the following lemma from [13].

Lemma 4.2. *Let $h\colon [0,\infty) \to [0,\infty)$ be a decreasing, right-continuous function with $\lim_{s\to\infty} h(s) = 0$. Let $\eta\colon [0,\infty) \to [0,\infty)$ be a non-decreasing function that satisfies the integrability condition:*

$$\int_{0^+} \frac{\eta(t)}{t}\, dt < \infty.$$

Assume for any $t \in [0,1]$ and any $s > 0$, the inequality

$$th(s+t) \leq h(s)\eta(h(s))$$

holds. Then $h(s) = 0$ for all $s \geq S_\infty$, where S_∞ is defined as

$$S_\infty = s_0 + e\int_0^{eh(s_0)} \frac{\eta(t)}{t}\, dt,$$

and s_0 is determined by the condition $\eta(h(s_0)) \leq \frac{1}{e}$.

Using the above theorem and lemma, we proceed with our proof that the solution is continuous under the stated conditions.

Theorem 4.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly m-pseudoconvex domain, let $f \in L^{\frac{n}{m}}(\log L)^\alpha$ for $\alpha > 2n$, and let $g \in \mathcal{C}(\partial\Omega)$. Then, the unique solution $U(f,g)$ of the Dirichlet problem for the complex Hessian operator (4.1) is continuous on $\bar{\Omega}$. Moreover, the following estimate holds:*

$$\begin{aligned}\|U(f_1,g_1)-U(f_2,g_2)\|_\infty &\leq \|g_1-g_2\|_\infty + C_1\|f_1-f_2\|_\alpha^{-\frac{1}{\gamma}} \\ &\quad + C_2 e_{m,m}(U(|f_1-f_2|,0))^{\frac{1}{2m}} \exp\Big(C_3\|f_1-f_2\|_\alpha^{-\frac{1}{\gamma}}\Big),\end{aligned} \tag{4.2}$$

where $\gamma = (1+\epsilon)m - \frac{\alpha m}{n}$ with $0 < \epsilon < \min\left(\frac{n+1}{3n}, \frac{\alpha}{n} - 2\right)$. Here, $\|f\|_\alpha$ denotes the norm in $L^{\frac{n}{m}}(\log L)^\alpha$, and C_1, C_2, C_3 are positive universal constants. Moreover, $e_{m,m}(u)$ is defined as $\int_\Omega (-u)^m \operatorname{H}_m(u)$.

Proof. Define the function

$$G_{\alpha,\frac{n}{m}}(t) = (1+t)^{\frac{n}{m}}(\log(1+t))^\alpha,$$

which is increasing and convex. Its inverse is given by

$$G^{-1}_{\alpha,\frac{n}{m}}(s) = \frac{\left(\frac{n}{\alpha m}\right)^{\frac{\alpha m}{n}} s^{\frac{m}{n}}}{\operatorname{W}_0\left(\frac{n}{\alpha m} s^{\frac{1}{\alpha}}\right)^{\frac{\alpha m}{n}}} - 1,$$

and the function $t\Big(G^{-1}_{\frac{n}{m},\alpha}(\frac{1}{t}) + 1\Big)$ is increasing. Assuming $0 < \epsilon < \min\left(\frac{n+1}{3n}, \frac{\alpha}{n} - 2\right)$ and choosing a relatively compact subset $K \subset \Omega$, using the Corollary 3.4, (2.1) and

(2.2), we have:

$$\mu(K) = \int_K f\, dV_{2n} = \int_\Omega f\chi_K\, dV_{2n} \leq \|f\|_\alpha \|\chi_K\|^0_{G^*_{\alpha,\frac{n}{m}}}$$
$$\leq \left(\int_\Omega G_{\alpha,\frac{n}{m}}(f)\, dV_{2n} + 1\right) V_{2n}(K) G^{-1}_{\frac{n}{m},\alpha}\left(\frac{1}{V_{2n}(K)}\right) \tag{4.3}$$
$$\leq D_1 \operatorname{cap}_m(K) \max(1, 1 - D_2 \log(\operatorname{cap}_m(K)))^\gamma,$$

where $\gamma = (1+\epsilon)m - \frac{\alpha m}{n}$. Define

$$F(t) = \max(1, 1 - D_2 \log t)^\gamma.$$

Note that since by our assumption $\alpha > (2+\epsilon)n$, the function $t^{-1}F^{\frac{1}{m}}(t)$ is locally integrable near zero, therefore by Theorem 4.1 there exists a unique solution $U(f,g)$ of the Dirichlet problem for the complex Hessian operator (4.1) which is continuous on $\bar{\Omega}$.

Now we shall provide the L^∞-estimate of the solution $U(f,g)$. First, note that it follows from the comparison principle that:

$$|U(f_1,g_1) - U(f_2,g_2)| \leq -U(|f_1-f_2|, -|g_1-g_2|)$$
$$\leq -U(|f_1-f_2|, 0) + \|g_1-g_2\|_\infty. \tag{4.4}$$

Therefore, it is sufficient to prove the result for the density function belonging to the Orlicz space $L^{\frac{n}{m}}(\log L)^\alpha$ and with boundary values equal to zero. Set, $u = U(|f_1-f_2|, 0)$.

From [13], for all $s,t>0$, it follows

$$t^m \operatorname{cap}_m(\{u < -s-t\}) \leq \int_{\{u<-s\}} \mathrm{H}_m(u) \leq \frac{1}{t^m} e_{m,m}(u). \tag{4.5}$$

Define

$$h(t) = \operatorname{cap}_m(\{u<-t\})^{\frac{1}{m}},$$

then (4.3) and (4.5) yields

$$th(t+s) \leq h(s)\eta(h(s))$$

with

$$\eta(t) = D_1^{\frac{1}{m}} \left(\max\left(1, 1 - \frac{D_2}{m}\log t\right)\right)^{\frac{\gamma}{m}}.$$

By Lemma 4.2, there exists S_∞ such that

$$h(t) = 0 \quad \text{for } t \geq S_\infty,$$

provided $\int_{0^+} \frac{\eta(t)}{t} < \infty$. Note that $\frac{\gamma}{m} < -1$, since $\alpha > (2+\epsilon)n$. The result then follows from the fact that if

$$\operatorname{cap}_m(\{u<-t\}) = 0 \Rightarrow V_{2n}(\{u<-t\}) = 0, \quad \text{for} \quad t \geq S_\infty,$$

then u is bounded by S_∞.

Next, we shall estimate the norm of u. We need to examine S_∞ and s_0. Assume s_0 is such that $\eta(h(s_0)) \leq \frac{1}{e}$, then the expression for $\eta(t)$ can be simplified to, see (4.3),

$$\eta(t) = d_1^{\frac{1}{m}} \|f\|_\alpha^{\frac{1}{m}} \left(d_2 - \frac{1}{m}\log t\right)^{\frac{\gamma}{m}},$$

where $d_1, d_2 > 0$ are universal constants. The condition $\eta(h(s_0)) = \frac{1}{e}$ leads us to the equation

$$d_2 - \frac{1}{m} \log h(s_0) = e^{-\frac{m}{\gamma}} d_1^{-\frac{1}{\gamma}} \|f\|_\alpha^{-\frac{1}{\gamma}}.$$

This equality implies that

$$\begin{aligned} e\int_0^{eh(s_0)} \frac{\eta(t)}{t}\,dt &= -\frac{ed_1^{\frac{1}{m}} m^2}{\gamma+m} \|f\|_\alpha^{\frac{1}{m}} \left(d_2 - \frac{1}{m}\log(eh(s_0))\right)^{\frac{\gamma+m}{m}} \\ &= -\frac{ed_1^{\frac{1}{m}} m^2}{\gamma+m} \|f\|_\alpha^{\frac{1}{m}} (e^{-\frac{m}{\gamma}} d_1^{-\frac{1}{\gamma}} \|f\|_\alpha^{-\frac{1}{\gamma}} - m^{-1})^{\frac{\gamma+m}{m}} \\ &\le C_1 \|f\|_\alpha^{\frac{1}{m}} \|f\|_\alpha^{-\frac{\gamma+m}{\gamma m}} = C_1 \|f\|_\alpha^{-\frac{1}{\gamma}}, \end{aligned} \tag{4.6}$$

where $C_1 > 0$ is a universal constant.

The condition $\eta(h(s_0)) \le \frac{1}{e}$ is equivalent to

$$\mathrm{cap}_m(\{u < -s_0\})^{\frac{1}{m}} = h(s_0) \le \exp\left(md_2 - me^{-\frac{m}{\gamma}} d_1^{-\frac{1}{\gamma}} \|f\|_\alpha^{-\frac{1}{\gamma}}\right).$$

Therefore, applying (4.5) with $s = t$, we find s_0 that satisfies

$$\left(\frac{4^m}{s_0^{2m}} e_{m,m}(u)\right)^{\frac{1}{m}} = \exp\left(md_2 - me^{-\frac{m}{\gamma}} d_1^{-\frac{1}{\gamma}} \|f\|_\alpha^{-\frac{1}{\gamma}}\right),$$

which can be rewritten as

$$s_0 = C_2 e_{m,m}(u)^{\frac{1}{2m}} \exp\left(C_3 \|f\|_\alpha^{-\frac{1}{\gamma}}\right), \tag{4.7}$$

where $C_2, C_3 > 0$ are universal constants.

To finalize the proof, observe that the estimation (4.2) follows from (4.7) and (4.6). □

Remark. Theorem 4.3 states that for $\alpha > 2n$, the solution $U(f, g)$ is continuous up to the boundary of a bounded strictly m-pseudoconvex domain $\Omega \subset \mathbb{C}^n$. It is unclear whether our assumption on the power α is optimal. Moreover, whether the condition on the domain Ω can be relaxed to an m-hyperconvex domain remains an open question.

5. Bounded solution

In the previous section, we proved in Theorem 4.3 that under the assumption $f \in L^{\frac{n}{m}}(\log L)^\alpha$ for $\alpha > 2n$, and $g \in \mathcal{C}(\partial\Omega)$, the unique solution $U(f, g)$ of the Dirichlet problem for the complex Hessian operator (4.1) is continuous on $\bar{\Omega}$. This section focuses on the case where $\alpha > n$ and $\Omega \subset \mathbb{C}^n$ is a bounded, strictly n-pseudoconvex domain. In Theorem 5.1, we show that the solution to (4.1) is bounded, and in Example 5.2, we observe that when $\alpha \le n$, the solution may be unbounded.

Lu and Guedj proved a result similar to Theorem 5.1 in the setting of compact Kähler manifolds (see Theorem B and the remark just above the acknowledgments in [18]).

Theorem 5.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly n-pseudoconvex domain, and let $f \in L^{\frac{n}{m}}(\log L)^\alpha$ for $\alpha > n$ and let $g \in \mathcal{C}(\partial\Omega)$. Then, there exists the unique solution $U(f, g)$ of the Dirichlet problem for the complex Hessian operator* (4.1) *and it is bounded.*

Proof. Without loss of generality, we assume that $g \leq 0$. Assume that $(1+f)^{n/m}(\log(1+f))^{\alpha} \in L^1$ for some $\alpha > n$, and set $h = (1+f)^{n/m}$. Then $h(\log h)^{\alpha} \in L^1$, and by [20], there exists a bounded plurisubharmonic (and hence m-subharmonic) solution v to

$$(dd^c v)^n = h\, dV_{2n}, \quad \text{and } v = g \text{ on } \partial\Omega.$$

By the mixed Monge–Ampère inequality [15], we have

$$(dd^c v)^m \wedge \beta^{n-m} \geq h^{m/n}\, dV_{2n} \geq f\, dV_{2n}.$$

Additionally, as shown in [5], $U(0,g) \in \mathcal{SH}_m(\Omega)$ is a maximal and continuous m-subharmonic function satisfying

$$\lim_{z\to w} U(0,g)(z) = g(w) \quad \text{for all } w \in \partial\Omega.$$

It follows from [17] that there exists $U(f,g)$ in the Cegrell class $\mathcal{F}_m$ with generalized boundary value $U(0,g)$, and such that $\mathrm{H}_m(U(f,g)) = f\, dV_{2n}$. Applying the comparison principle [17] gives

$$v \leq U(f,g) \leq U(0,g),$$

and thus $U(f,g)$ is a bounded m-subharmonic function satisfying

$$\lim_{z\to w} U(f,g)(z) = g(w) \quad \text{for all } w \in \partial\Omega. \qquad \square$$

Example 5.2. Let $B(0,1) \subset \mathbb{C}^n$ be the unit ball, and consider a radially symmetric density function f with $\alpha \leq n$. Set $g = f^{\frac{m}{n}}$. Let U_m be a solution to $\mathrm{H}_m(U_m) = f\, dV_{2n}$, with $\lim_{z\to w} U_m(z) = 0$ for all $w \in \partial B(0,1)$, and U_n be a solution to $(dd^c U_n)^n = g\, dV_{2n}$, with $\lim_{z\to w} U_n(z) = 0$ for all $w \in \partial B(0,1)$.

Then by [30], we have

$$-U_m(z) = \int_{|z|}^1 t^{1-\frac{2n}{m}} F(t)^{\frac{1}{m}}\, dt, \quad F(t) = \frac{1}{2^{2n-m-1}(n-1)!}\int_0^t r^{2n-1} f(r)\, dr;$$

$$-U_n(z) = \int_{|z|}^1 t^{-1} G(t)^{\frac{1}{n}}\, dt, \quad G(t) = \frac{1}{2^{n-1}(n-1)!}\int_0^t r^{2n-1} g(r)\, dr.$$

Using Hölder's inequality, we obtain

$$\begin{aligned} G(t) &= \frac{1}{2^{n-1}(n-1)!}\int_0^t r^{2n-1} g(r)\, dr \\ &\leq \frac{1}{2^{n-1}(n-1)!}\left(\int_0^t r^{2n-1} f(r)\, dr\right)^{\frac{m}{n}} \left(\int_0^t r^{2n-1}\, dr\right)^{\frac{n-m}{n}} \\ &= C(n,m) F(t)^{\frac{m}{n}} t^{2n-2m}, \end{aligned}$$

where $C(n,m)$ is a constant depending only on n and m. Again, using Hölder's inequality, we have

$$\begin{aligned} -U_n(z) &= \int_{|z|}^1 t^{-1} G(t)^{\frac{1}{n}}\, dt \leq C(n,m)^{\frac{1}{n}} \int_{|z|}^1 F(t)^{\frac{m}{n^2}} t^{1-\frac{2m}{n}}\, dt \\ &\leq C(n,m)^{\frac{1}{n}} \left(\int_{|z|}^1 t^{1-\frac{2n}{m}} F(t)^{\frac{1}{m}}\, dt\right)^{\frac{m^2}{n^2}} \left(\int_{|z|}^1 t^{1-\frac{2m}{n}}\, dt\right)^{\frac{n^2-m^2}{n^2}} \qquad (5.1) \\ &= D(n,m)(-U_m(z))^{\frac{m^2}{n^2}} \left(1-|z|^{\frac{2n-2m}{n}}\right)^{\frac{n^2-m^2}{n^2}}, \end{aligned}$$

where $D(n,m)$ is a constant depending only on n and m.

Therefore, we have shown that if U_n is unbounded, then U_m is also unbounded. By Example 3.3 in [4], there exists an unbounded plurisubharmonic function U_n for which $(dd^cU_n)^n = g\,dV_{2n}$ with $g \in L(\log L)^n$. Therefore, by (5.1), the m-subharmonic solution U_m to $\mathrm{H}_m(U_m) = g^{\frac{n}{m}}dV_{2n}$ is also unbounded.

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Per Åhag
Umeå University
Department of Mathematics and Mathematical Statistics, SE-901 87 Umeå, Sweden
per.ahag@umu.se

Rafał Czyż
Jagiellonian University
Faculty of Mathematics and Computer Science
Łojasiewicza 6, 30-348 Kraków, Poland
rafal.czyz@im.uj.edu.pl