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The Landau–Bloch type theorems for certain class of holomorphic and pluriharmonic mappings in $\mathbb{C}^n$

Vasudevarao Allu and Rohit Kumar

Abstract. In this paper, we define two classes of holomorphic mappings defined on the unit ball B^n of n-dimensional complex space $\mathbb{C}^n$ and obtain the lower estimates for Bloch's constant for these classes. Also, we derive the Landau–Bloch type theorem for some subclasses of pluriharmonic mappings defined on the unit ball B^n.

Landaun–Blochin-tyyppiset lauseet avaruuden $\mathbb{C}^n$ holomorfisten ja moniharmonisten kuvausten eräälle luokalle

Tiivistelmä. Tässä työssä määritellään kaksi n-ulotteisen kompleksiavaruuden $\mathbb{C}^n$ yksikkökuulan B^n holomorfisten kuvausten luokkaa ja saadaan alarajoja näiden luokkien Blochin vakiolle. Lisäksi johdetaan Landaun–Blochin-tyyppinen lause yksikkökuulan B^n moniharmonisten kuvausten eräille alaluokille.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$. In the case of one complex variable, the following theorem of Bloch is well known (see [3]).

Theorem A. [3] *Let f be a holomorphic function on the $\overline{\mathbb{D}} = \{z\colon |z| \leq 1\}$ and satisfying $|f'(0)| = 1$. Then there exists a positive constant b such that $f(\mathbb{D})$ contains a schlicht disc of radius b.*

By a schlicht disc, we mean a disc which is the univalent image of some region in the unit disc $\mathbb{D}$. Let β_f denote the least upper bound of the radii of all schlicht discs that f carries and $\mathcal{F}$ denote the set of all holomorphic functions defined on $\overline{\mathbb{D}} = \{z\colon |z| \leq 1\}$ satisfying $|f'(0)| = 1$. Then the Bloch constant is the number defined by

$$\beta(\mathcal{F}) = \inf\{\beta_f\colon f \in \mathcal{F}\}.$$

In 1929, Landau [21] proved that if we replace the holomorphicity condition on $|z| \leq 1$ to $|z| < 1$, then the corresponding constant is also the same. If one consider the function $f(z) = z$, then clearly $\beta(\mathcal{F}) \leq 1$. However, better estimates than these are known. The exact value of $\beta(\mathcal{F})$ is still unknown.

In 1937, Ahlfors and Grunsky [1] proved that

$$0.4330 \approx \frac{\sqrt{3}}{4} \leq \beta(\mathcal{F}) \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)} \approx 0.4719.$$

It is conjectured that this upper bound is the precise value of the Bloch constant, a conjecture known as the Ahlfors–Grunsky conjecture. In 1990, Bonk [6] obtained a

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slight improvement to the lower bound as $\beta(\mathcal{F}) > \sqrt{3}/4 + 10^{-14}$. In 1996, Chen and Gauthier [7] improved Bonk's result and proved that $\beta(\mathcal{F}) > \sqrt{3}/4 + 2 \times 10^{-4}$.

Let

$$\mathbb{C}^n = \{z = (z_1, z_2, \ldots, z_n) \colon z_1, z_2, \ldots, z_n \in \mathbb{C}\}$$

be the complex space of dimension n. For any point $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$,

$$|z| = \left(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2\right)^{1/2}.$$

We denote a ball in $\mathbb{C}^n$ with center at z_0 and radius r by

$$B^n(z_0, r) = \{z \in \mathbb{C}^n \colon \ |z - z_0| < r\}$$

and the unit ball in $\mathbb{C}^n$ by

$$B^n = \{z \in \mathbb{C}^n \colon |z| < 1\}.$$

For a mapping $f = (f_1, f_2, \cdots, f_n)$ of a domain in $\mathbb{C}^n$ into $\mathbb{C}^n$, $\partial f/\partial z_k$ denotes the column vector formed by $\partial f_1/\partial z_k, \partial f_2/\partial z_k, \cdots, \partial f_n/\partial z_k$, and we denote by

$$f' = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_n}\right),$$

the matrix formed by these column vectors. For an $n \times n$ matrix A, we have the matrix norm

$$\|A\| = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$$

and the operator norm

$$|A| = \sup_{z \neq 0} \frac{|Az|}{|z|}.$$

In the case of several complex variables, the classical theorem of Bloch for holomorphic mappings in the disc fails to extend to general holomorphic mappings in the ball of $\mathbb{C}^n$. In 1967, Wu [30] pointed out that the Bloch theorem fails unless some restrictive assumptions are made on holomorphic mappings. For example, for positive integer n, setting $f_n(z_1, z_2) = (nz_1, z_2/n)$, we see that f_n is an holomorphic function on the unit ball of $\mathbb{C}^2$ for each n and $|\det {f_n}'(0)| = 1$. But the image of f_n contains a schlicht ball of radius at most $1/n$. So, the infimum of the radii of the schlicht balls anywhere when all members of $\{f_n\}$ are taken into account is zero. In order that there is a positive Bloch constant, it is necessary to restrict the class of holomorphic mappings in higher dimensions. Earlier investigations for some subclass of holomorphic mappings were conducted by Bochner [4], Hahn [19], Harris [20], Sakaguchi [27], Takahashi [28], and Wu [30]. Outside holomorphicity, there is no Bloch theorem for quasiregular mappings of the ball, but Eremenko [17] has proved that a Bloch theorem holds for entire quasiregular mappings.

In 1946, Bochner [4] defined a subclass of the class of holomorphic functions from $B^n\,(\subseteq \mathbb{C}^n)$ to $\mathbb{C}^n$. For a constant $K \geq 1$, a holomorphic mapping f from B^n to $\mathbb{C}^n$ is said to be Bochner K-mapping in B^n if f satisfies the differential inequality

$$\|f'(z)\| \leq K\,|\det f'(z)|^{1/n}$$

for all $z \in B^n$. Bochner [4] proved that, for each $K \geq 1$ and $n \geq 2$, there is a constant $\beta > 0$ such that, for each normalized Bochner K-mapping f in the unit ball, $\beta_f \geq \beta$ holds. However, Bochner has not given an estimate for this Bloch constant.

In 1951, Takahashi [28] defined a new class of normalized holomorphic mappings f satisfying the weaker condition

$$\max_{|z|<r} \|f'(z)\| \leq K \max_{|z|\leq r} |\det f'(z)|^{1/n}, \quad \text{for each } 0 \leq r < 1. \tag{1.1}$$

The holomorphic functions which satisfy (1.1) are called Takahashi K-mappings. For such normalized Takahashi K-mappings, Takahashi [28] has proved that

$$\beta \geq \frac{(n-1)^{n-2}}{12K^{2n-1}}.$$

Later in 1956, Sakaguchi [27] improved Takahashi's estimate to

$$\beta \geq \frac{(n-1)^{n-2}}{8K^{2n-1}}.$$

We now define a new class of holomorphic mappings from B^n into $\mathbb{C}^n$ which contains the Bochner K-mappings. We call them Bochner (K, K')-mappings.

Definition 1.1. For constants $K \geq 1, K' \geq 0$, a holomorphic mapping f from B^n into $\mathbb{C}^n$ is said to be a Bochner (K, K')-mapping in B^n if f satisfies the differential inequality

$$\|f'(z)\|^2 \leq K^2 |\det f'(z)|^{2/n} + K'$$

for all $z \in B^n$.

We remark that the unit ball B^n in the Definition 1.1 can be replaced by any general domain in $\mathbb{C}^n$. In particular, if $K' = 0$, then Bochner (K, K')-mappings reduce to Bochner K-mappings. We see that every Bochner K-mapping is a Bochner (K, K')-mapping for $K' = 0$, but the converse need not be true. This can be seen from the following example: Let

$$f(z_1, z_2) = \left(z_1 + z_2, \left(z_1 - \frac{1}{2}\right)^2 + \left(z_2 - \frac{1}{3}\right)^2\right)$$

in B^2. This function is clearly a holomorphic mapping on B^2. One can easily see that the mapping $f(z_1, z_2)$ is not a Bochner K-mapping for any $K \geq 1$ but it is a Bochner $(1, 20)$-mapping.

The motivation for defining this type of mappings came from the paper of Nirenberg (see [24]), where Nirenberg has defined this type of mappings in the plane. For more details on this type of mappings, we refer to [2, 10, 11, 16, 18, 24].

Let $\lambda_f^2(z)$ and $\Lambda_f^2(z)$ denote the smallest and the largest eigenvalues of the Hermitian matrix A^*A, where $A = f'(z)$ and A^* is the conjugate of A. A holomorphic mapping f from the unit ball B^n of $\mathbb{C}^n$ into $\mathbb{C}^n$ is K-quasiregular if

$$\Lambda_f(z) \leq K\lambda_f(z)$$

at every point $z \in B^n$. A mapping is said to be quasiregular if it is K-quasiregular for some $K \geq 1$. From [23] we know that for $n > 1$, a quasiregular holomorphic mappings are locally biholomorphic. In fact, Poletsky [25] also proved that quasiregular holomorphic mappings (in any bounded domain) are rather rigid. In 2000, Chen and Gauthier [8] proved that if f is a K-quasiregular holomorphic mapping with the normalization $\det f'(0) = 1$, then the image $f(B^n)$ contains a schlicht ball of radius at least $1/12K^{1-1/n}$.

Now, we define a new class of holomorphic mappings from B^n into $\mathbb{C}^n$ which contains the K-quasiregular mappings. We call such mappings (K, K')-quasiregular mappings.

Definition 1.2. For constants $K \geq 1, K' \geq 0$, a holomorphic mapping f from B^n into $\mathbb{C}^n$ is said to be (K, K')-quasiregular mapping in B^n if

$$\Lambda_f(z) \leq K\lambda_f(z) + K'$$

at each $z \in \mathbb{B}^n$.

A continuous complex-valued function ϕ defined on a domain $\Omega \subset \mathbb{C}^n$ is called a pluriharmonic mapping if, for each fixed $z' \in \Omega$ and $\theta \in \partial B^n$, the function $\phi(z' + \theta\zeta)$ is harmonic in $\{\zeta \colon |\zeta| < d_\Omega(z)\}$, where $d_\Omega(z)$ denotes the distance from z to the boundary $\partial\Omega$ of Ω. A mapping f of Ω into $\mathbb{C}^n$ is called a pluriharmonic mapping if every component of f is pluriharmonic. A mapping f of B^n into $\mathbb{C}^n$ is pluriharmonic if, and only if, f has a representation $f = g + \bar{h}$, where g and h are holomorphic mappings (see [26]).

For a continuously differentiable mapping $w = f(z) = (f_1(z), \ldots f_m(z)) : B^n \to \mathbb{C}^m$, $z = (z_1, \ldots, z_n)$, by f_z and $f_{\bar{z}}$ denote the matrices $(\partial f_j/\partial z_k)_{m\times n}$ and $(\partial f_j/\partial \bar{z}_k)_{m\times n}$, respectively. Denote the maximum dilation $\widetilde{\Lambda}_f$ and minimum dilation $\widetilde{\lambda}_f$ by

$$\widetilde{\Lambda}_f(z) = \max_{\theta\in\partial B^n} \left|f_z(z)\theta + f_{\bar{z}}(z)\bar{\theta}\right| \quad \text{and} \quad \widetilde{\lambda}_f(z) = \min_{\theta\in\partial B^n} \left|f_z(z)\theta + f_{\bar{z}}(z)\bar{\theta}\right|,$$

respectively, where θ is regarded as a column vector. In particular, when $n = 1$, these definitions coincide with the corresponding definitions for planar harmonic mappings.

Wang et al. [29] generalized the notion of K-quasiregular mappings to pluriharmonic mappings and established a lower bound estimate of Bloch constant for such mappings.

A pluriharmonic mapping f of B^n into $\mathbb{C}^n$ is said to be K-quasiregular pluriharmonic if

$$\widetilde{\Lambda}_f(z) \leq K\widetilde{\lambda}_f(z)^{1/n} \quad \text{for } z \in B^n.$$

Now, we define a new class of pluriharmonic mappings of B^n into $\mathbb{C}^n$ which contains K-quasiregular pluriharmonic mappings. We call such mappings (K, K')-quasiregular pluriharmonic mappings.

Definition 1.3. Let $f \colon B^n \to \mathbb{C}^n$ be a pluriharmonic mapping and $K \geq 1$, $K' \geq 0$. We say that f is a (K, K')-quasiregular pluriharmonic mapping if

$$\widetilde{\Lambda}_f(z) \leq K\widetilde{\lambda}_f(z)^{1/n} + K'$$

for $z \in B^n$. In particular, if $K' = 0$, then (K, K')-quasiregular pluriharmonic mappings reduce to K-quasiregular pluriharmonic mappings.

In 2011, Chen and Gauthier [9] established Landau theorems and Bloch theorems for pluriharmonic mappings $f \colon B^n \to \mathbb{C}^n$. Chen, Ponnusamy and Wang have studied Landau–Bloch constants for some specified spaces such as, pluriharmonic Bergman space, α-Bloch space and hyperbolic-harmonic Bloch space (see [12, 13, 14, 15]). In 2020, Xu and Liu [31] obtained a new version of the Bloch theorem for pluriharmonic ν-Bloch-type mappings. In 2022, Liu and Ponnusamy [22] obtained three Bloch-type theorems for pluriharmonic mappings in B^n, which improve the corresponding results of Chen and Gauthier [9].

2. The Bloch theorem for Bochner (K, K')-mappings

The following result of Takahashi [28] is useful in proving our main result in this section.

Theorem B. [28] *Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be an analytic transformation defined by n functions $f_i(z)$ of n complex variables z, each analytic in a domain D in n dimensional complex space of the variables z, and let its Jacobian $J_f(z)$ does not vanish at a point a in D. Let $\partial f_i(a)/\partial z_k \equiv \alpha_{ik}$; $i, k = 1, \cdots, n$; $A = (\alpha_{ik})$, $\det A = J_f(a) \neq 0$, it follows that the characteristic values $\lambda_1, \cdots, \lambda_n$ of A^*A are real and positive, so that $\min\{\lambda_1, \cdots, \lambda_n\} = \lambda > 0$. Next, let ρ_0 be the upper limit of ρ such that the inequality*

$$\sum_{i,k=1}^{n} \left| \frac{\partial f_i}{\partial z_k}(z) - \frac{\partial f_i}{\partial z_k}(a) \right|^2 \leq \lambda$$

is satisfied for $\sum_{k=1}^{n} |z_k - a_k|^2 \leq \rho$. Then f is univalent on a ball with center a and radius ρ_0, i.e., $B^n(a, \rho_0)$. Further, $f(B^n(a, \rho_0))$ contains a ball with center $f(a)$ and radius $2^{-1}\lambda^{\frac{1}{2}}\rho_0$. Moreover, the value $2^{-1}\lambda^{\frac{1}{2}}\rho_0$ cannot be replaced by larger one for certain analytic transformation and for some a.

The following lemma gives an estimate of the lower bound of the smallest singular values of a non-singular matrix.

Lemma A. [32] *If A is a non-singular $n \times n$ matrix. Then A^*A is a positive definite hermitian matrix, the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A^*A are therefore real and positive, so that $\lambda = \min\{\lambda_2, \ldots, \lambda_n\} > 0$. Then the following inequality is satisfied*

$$\lambda > (n-1)^{n-1} |\det A|^2 \|A\|^{-2(n-1)},$$

$\|A\|$ is the Euclidean norm of the matrix A.

In this section, we show that for each $K \geq 1$, $K' \geq 0$ and $n \geq 2$, there is a constant $\beta > 0$ such that, for each normalized Bochner (K, K')-mapping f in the ball, $\beta_f \geq \beta$.

Theorem 1. *Let $n \geq 2$ be any integer and $K \geq 1$, $K' \geq 0$ be any constants. Let $f(z)$ be Bochner (K, K')-mapping from $\overline{B}^n$ to $\mathbb{C}^n$ with $|\det f'(0)| = 1$, then f maps some subdomain of unit ball univalently onto a ball of positive radius*

$$R(n, K, K') = \frac{1}{4(K^2 + K')^{n-1}} \left(\sqrt{4K^2 + K'} + \sqrt{K^2 + K'} \right)^{-1}.$$

In other words, we say that Bloch Theorem holds for Bochner (K, K')-mappings.

Proof. Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be Bochner (K, K')-mapping from $\overline{B}^n$ to $\mathbb{C}^n$ with $|\det f'(0)| = 1$. We introduce the functions

$$M(r) = \max_{|z| \leq r} |\det f'(z)|^{1/n},$$

and

$$\phi(r) = rM(1 - r),$$

for $0 \leq r \leq 1$.

We see that

$$\phi(0) = 0 \quad \text{and} \quad \phi(1) = M(0) = |\det f'(0)|^{1/n} = 1.$$

Then there exists a number r_0, $0 < r_0 \leq 1$ such that $\phi(r_0) = 1$ and $\phi(r) < 1$ for $0 \leq r < r_0$. This implies

$$M(1 - r_0) = \frac{\phi(r_0)}{r_0} = \frac{1}{r_0},$$

and for any $0 < r < r_0$,

$$M(1-r) = \frac{\phi(r)}{r} < \frac{1}{r}. \tag{2.1}$$

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be any point inside the closed ball of radius $1 - r_0$ such that

$$|\det f'(\alpha)|^{1/n} = M(1-r_0) = \frac{1}{r_0}.$$

Define $F\colon \overline{B}^n \to \mathbb{C}^n$ by

$$F(\zeta) = F(\zeta_1, \zeta_2, \dots, \zeta_n) = (F_1(\zeta), F_2(\zeta), \cdots, F_n(\zeta)) = 2\left(f\left(\alpha + \frac{r_0}{2}\zeta\right) - f(\alpha)\right)$$

for $|\zeta| \le 1$. Then clearly $F(0) = 0$.

It is easy to see that for $|\zeta| \le 1$,

$$\left|\alpha + \frac{r_0}{2}\zeta\right| \le |\alpha| + \frac{r_0}{2}|\zeta| \le 1 - r_0 + \frac{r_0}{2} = 1 - \frac{r_0}{2} < 1.$$

Therefore, F is well-defined.

A simple computation shows that

$$\frac{\partial F_i}{\partial \zeta_j}(\zeta) = 2\frac{\partial f_i}{\partial \zeta_j}\left(\alpha + \frac{r_0}{2}\zeta\right) = r_0 \frac{\partial f_i}{\partial z_j}\left(\alpha + \frac{r_0}{2}\zeta\right)$$

where $z_j = \alpha_j + \frac{r_0}{2}\zeta_j, \quad j = 1, 2, \dots, n$. Therefore,

$$\det F'(\zeta) = {r_0}^n \det f'\left(\alpha + \frac{r_0}{2}\zeta\right),$$

which implies that

$$|\det F'(0)| = {r_0}^n |\det f'(\alpha)| = {r_0}^n (M(1-r_0))^n = {r_0}^n \left(\frac{1}{r_0}\right)^n = 1.$$

A simple computation shows that

$$\begin{aligned}
\|F'(\zeta)\|^2 &= \sum_{i,j} \left|\frac{\partial F_i}{\partial \zeta_j}(\zeta)\right|^2 = \sum_{i,j} {r_0}^2 \left|\frac{\partial f_i}{\partial z_j}\left(\alpha + \frac{r_0}{2}\zeta\right)\right|^2 \\
&= {r_0}^2 \sum_{i,j} \left|\frac{\partial f_i}{\partial z_j}\left(\alpha + \frac{r_0}{2}\zeta\right)\right|^2 = {r_0}^2 \left\|f'\left(\alpha + \frac{r_0}{2}\zeta\right)\right\|^2 \\
&\le {r_0}^2 \left[K^2 \left|\det f'\left(\alpha + \frac{r_0}{2}\zeta\right)\right|^{2/n} + K'\right].
\end{aligned}$$

Since $\left|\alpha + \frac{r_0}{2}\zeta\right| \le 1 - r_0/2$. By the definition of the function M, we obtain

$$\left|\det f'\left(\alpha + \frac{r_0}{2}\zeta\right)\right|^{2/n} \le \left(M\left(1 - \frac{r_0}{2}\right)\right)^2.$$

From (2.1), it follows that

$$\begin{aligned}
\|F'(\zeta)\|^2 &\le {r_0}^2 \left(K^2 \left(M\left(1 - \frac{r_0}{2}\right)\right)^2 + K'\right) \le {r_0}^2 \left(K^2 \frac{4}{{r_0}^2} + K'\right) \\
&= 4K^2 + {r_0}^2 K' \le 4K^2 + K'.
\end{aligned} \tag{2.2}$$

Further,

$$\|F'(0)\|^2 = r_0{}^2 \sum_{i,j} \left|\frac{\partial f_i}{\partial z_j}(\alpha)\right|^2 = r_0{}^2 \|f'(\alpha)\|^2 \le r_0{}^2 \left(K^2 |\det f'(\alpha)|^{2/n} + K'\right)$$

$$\le r_0{}^2 \left(K^2 \cdot \frac{1}{r_0^2} + K'\right) \le K^2 + K'. \tag{2.3}$$

From (2.2) and (2.3), it is easy to see that

$$\|F'(\zeta) - F'(0)\| \le \|F'(\zeta)\| + \|F'(0)\| \le \sqrt{4K^2 + K'} + \sqrt{K^2 + K'} \tag{2.4}$$

for $|\zeta| \le 1$.

By applying the Schwarz lemma for functions of several complex variables (see [5]) to (2.4), we obtain

$$\|F'(\zeta) - F'(0)\| \le \left(\sqrt{4K^2 + K'} + \sqrt{K^2 + K'}\right) |\zeta| \tag{2.5}$$

for $|\zeta| \le 1$. Since $|\det F'(0)| = 1 \ne 0$, the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $(F'(0))^* F'(0)$ are real and positive, so that $\min\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \lambda > 0$, and the inequality (2.5) takes the following form

$$\|F'(\zeta) - F'(0)\|^2 \le \lambda \quad \text{for } |\zeta| \le \frac{\lambda^{1/2}}{\sqrt{4K^2 + K'} + \sqrt{K^2 + K'}}. \tag{2.6}$$

Since the product of eigenvalues is equal to determinant of the matrix, we have

$$\lambda^n \le \lambda_1 \lambda_2 \cdots \lambda_n = |\det F'(0)|^2.$$

This implies

$$\lambda^{1/2} \le |\det F'(0)|^{1/n} = 1$$

and hence

$$\frac{\lambda^{1/2}}{\sqrt{4K^2 + K'} + \sqrt{K^2 + K'}} \le \frac{1}{\sqrt{4K^2 + K'} + \sqrt{K^2 + K'}} < 1. \tag{2.7}$$

In view of (2.7) together with inequality (2.6) and applying Theorem B, we see that $F(\zeta)$ maps some subdomain of $|\zeta| \le 1$ univalently onto a ball with center 0 and radius

$$\frac{\lambda^{1/2}}{2} \frac{\lambda^{1/2}}{\sqrt{4K^2 + K'} + \sqrt{K^2 + K'}} = \frac{\lambda}{2(\sqrt{4K^2 + K'} + \sqrt{K^2 + K'})}.$$

Using Lemma A, for $n \ge 2$ we obtain the following

$$\lambda > (n-1)^{n-1} |\det F'(0)|^2 \|F'(0)\|^{-2(n-1)} \ge |\det F'(0)|^2 \|F'(0)\|^{-2(n-1)}.$$

By (2.3), we have

$$\lambda > (K^2 + K')^{-(n-1)}.$$

Therefore, $F(\zeta)$ maps some sub-domain of the ball $|\zeta| \le 1$ univalently onto a ball with center 0 and of radius at least

$$\frac{1}{2} \frac{(K^2 + K')^{-(n-1)}}{\sqrt{4K^2 + K'} + \sqrt{K^2 + K'}}.$$

Hence by the definition of $F(\zeta)$, $f(z)$ maps some sub-domain of the unit ball univalently onto a ball of radius at least

$$R(n, K, K') = \frac{1}{4(K^2 + K')^{n-1}} \left(\sqrt{4K^2 + K'} + \sqrt{K^2 + K'}\right)^{-1}.$$

This completes the proof. □

Using Theorem 1, we obtain the following corollary.

Corollary 1. *Let $n \geq 2$ be any integer and $K \geq 1$ be any positive constant. Let $f(z)$ be Bochner K-mapping from $\overline{B}^n$ to $\mathbb{C}^n$ with $|\det f'(0)| = 1$, then f maps some subdomain of unit ball B^n univalently onto a ball of positive radius*

$$R(n, K) = \frac{1}{12K^{2n-1}}.$$

Proof. Let $f(z)$ be a Bochner K-mapping. Then we can easily see that $f(z)$ is a Bochner $(K, 0)$- mapping. Therefore, the proof follows by substituting $K' = 0$ in the Theorem 1. □

3. The Bloch theorem for (K, K')-quasiregular mappings

Following lemma is useful in proving our main result in this section.

Lemma B. [19] *Let $w = f(z)$ be a holomorphic mapping defined in a neighborhood of a point $t \in \mathbb{C}^n$ into $\mathbb{C}^n$ with $J_f(t) \neq 0$. Suppose that $\lambda_f \equiv \lambda_f(t)$ is the positive square root of the smallest characteristic value of the matrix A^*A at t, where $A \equiv (df/dz)$. Then the following hold:*

(1) *The mapping $w = f(z)$ is univalent in any open convex subset $K, t \in K$, of the set*
$$\Omega_f = \{z \colon |A(z) - A(t)| < \lambda_f\},$$
where $|A| = \sup_{|x|=1} |Ax|$ and $|x|$ denotes the euclidean norm of the n-vector x.

(2) *If r_0 is the radius of the largest ball contained in Ω_f centered at t, then $f[B(t, r_0)]$ contains the ball of radius $r_0\lambda_f/2$ centered at $f(t)$, where $B(t, r_0) = [z \colon |z - t| < r_0]$.*

In this section we show that for each $K \geq 1$, $K' \geq 0$ and $n \geq 2$, there is a constant $\beta > 0$ such that, for each normalized (K, K')-quasiregular mapping f in the ball $\beta_f \geq \beta$.

Theorem 2. *Let $f \colon B^n \to \mathbb{C}^n$ be a (K, K')-quasiregular mapping of the unit ball B^n into $\mathbb{C}^n$ with $\lambda_f(0) \geq \alpha > 0$. Then*

$$\beta_f \geq \frac{\alpha^2}{4(2K\alpha + K' + \alpha)}.$$

Proof. Let $f \colon B^n \to \mathbb{C}^n$ be a (K, K')-quasiregular mapping of the unit ball B^n into $\mathbb{C}^n$ with $\lambda_f(0) \geq \alpha > 0$. Without loss of generality, we assume that f is holomorphic on $\overline{B}^n$. We introduce the following functions

$$N(r) = \max_{|z| \leq r} \lambda_f(z)$$

and

$$\psi(r) = rN(1 - r)$$

for $0 \leq r \leq 1$. We see that

$$\psi(0) = 0 \text{ and } \psi(1) = N(0) = \lambda_f(0) = \alpha > 0.$$

Then there exists a number r_0, $0 < r_0 \leq 1$ such that $\psi(r_0) = \alpha$ and $\psi(r) < \alpha$ for $0 \leq r < r_0$. This implies

$$N(1 - r_0) = \frac{\psi(r_0)}{r_0} = \frac{\alpha}{r_0},$$

and for any $0 < r < r_0$,

$$N(1-r) = \frac{\psi(r)}{r} < \frac{\alpha}{r}. \tag{3.1}$$

Let w_0 be any point inside the closed ball of radius $1 - r_0$ such that

$$\lambda_f(w_0) = N(1-r_0) = \frac{\alpha}{r_0}.$$

Define $G\colon B^n \to \mathbb{C}^n$ by

$$G(\zeta) = \frac{r_0}{2}A^{-1}\left(f(w_0 + \frac{r_0}{2}\zeta) - f(w_0)\right) \tag{3.2}$$

for $|\zeta| \le 1$ and $A = f'(w_0)$. It is easy to see that

$$|w_0 + \frac{r_0}{2}\zeta| \le |w_0| + \frac{r_0}{2}|\zeta| \le 1 - r_0 + \frac{r_0}{2} = 1 - \frac{r_0}{2} < 1.$$

Also, $[f'(w_0)]^{-1}$ exists because $|\det f'(w_0)| \ge \lambda_f^n(w_0) > 0$. Therefore, G is well defined.

We observe that

$$\frac{dG(\zeta)}{d\zeta} = \frac{r_0}{2}A^{-1}\frac{df}{d\zeta}\left(w_0 + \frac{r_0}{2}\zeta\right),$$

which implies

$$G'(\zeta) = A^{-1}f'(w_0 + \frac{r_0}{2}\zeta). \tag{3.3}$$

By using Cauchy–Schwarz inequality, we obtain

$$\Lambda_G(\zeta) = |G'(\zeta)| \le |A^{-1}|\left|f'\left(w_0 + \frac{r_0}{2}\zeta\right)\right|.$$

Since $|A^{-1}| = \frac{1}{\lambda_f(w_0)}$, we have

$$\Lambda_G(\zeta) \le \frac{\Lambda_f(w_0 + \frac{r_0}{2}\zeta)}{\lambda_f(w_0)}.$$

Since f is (K, K')-quasiregular mapping, we have

$$\Lambda_G(\zeta) \le \frac{K\lambda_f(w_0 + \frac{r_0}{2}\zeta) + K'}{\lambda_f(w_0)}.$$

Since $|w_0 + \frac{r_0}{2}\zeta| \le 1 - r_0/2$. By the definition of the function N, we obtain

$$\lambda_f(w_0 + \frac{r_0}{2}\zeta) \le N\left(1 - \frac{r_0}{2}\right).$$

Using (3.1), we obtain

$$\begin{aligned}\Lambda_G(\zeta) &\le \frac{KN\left(1 - \frac{r_0}{2}\right) + K'}{\lambda_f(w_0)} \\ &\le \frac{2K\alpha/r_0 + K'}{\alpha/r_0} \le \frac{1}{\alpha}(2K\alpha + K'). \end{aligned} \tag{3.4}$$

From (3.3), we have $G'(0) = I_n$ and hence, $\Lambda_G(0) = \lambda_G(0) = 1$. Thus,

$$|G'(\zeta) - G'(0)| \le |G'(\zeta)| + |G'(0)| \le \frac{1}{\alpha}(2K\alpha + K') + 1 = \frac{1}{\alpha}(2K\alpha + K' + \alpha) \tag{3.5}$$

for $|\zeta| \le 1$. By the Schwarz lemma (see [5]), we obtain

$$|G'(\zeta) - G'(0)| \le \frac{1}{\alpha}(2K\alpha + K' + \alpha)|\zeta| \tag{3.6}$$

for $|\zeta| \leq 1$. Clearly (3.6) shows that

$$|G'(\zeta) - I_n| \leq 1 \quad \text{for } |\zeta| \leq \frac{\alpha}{2K\alpha + K' + \alpha}.$$

By Lemma B, $w = G(\zeta)$ maps the ball $B^n(0, \alpha/(2K\alpha + K' + \alpha))$ univalently onto a domain containing the ball $B^n(0, \alpha/(4K\alpha + 2K' + 2\alpha))$. Hence by (3.2), $w = f(z)$ maps the subdomain $B^n(w_0, r_0\alpha/(4K\alpha + 2K' + 2\alpha))$ of B^n univalently onto a ball center at $f(w_0)$ and radius

$$\frac{\alpha^2}{4(2K\alpha + K' + \alpha)}.$$

This completes the proof. □

Corollary 2. *Let $f\colon B^n \to \mathbb{C}^n$ be a (K, K')-quasiregular mapping of the unit ball B^n into $\mathbb{C}^n$ with $|\det f'(0)| = \alpha > 0$. Then*

$$\beta_f \geq \frac{\alpha^{2/n}}{4(2K\alpha^{1/n} + K' + \alpha^{1/n})}.$$

Proof. Since

$$\alpha = |\det f'(0)| \leq \lambda_f^n(0),$$

we have,

$$\lambda_f(0) \geq \alpha^{1/n}.$$

Now by replacing α by $\alpha^{1/n}$ in Theorem 2, we obtain the desired result. □

4. The Landau–Bloch type theorem for pluriharmonic mappings

In 2011, Chen and Gauthier [9] proved the following Schwarz-Pick lemma for pluriharmonic mappings:

Lemma C. [9] *Let f be a pluriharmonic mapping of B^n into B^m. Then*

$$\widetilde{\Lambda}_f(z) \leq \frac{4}{\pi}\frac{1}{1 - |z|^2} \quad \text{for } z \in B^n.$$

If $f(0) = 0$, then

$$|f(z)| \leq \frac{4}{\pi}\arctan|z| \leq \frac{4}{\pi}|z| \quad \text{for } z \in B^n.$$

The following Landau-type theorem for pluriharmonic mappings of B^n into $\mathbb{C}^n$ with bounded dilation has been proved by Wang et al. [29].

Theorem C. [29] *Let f be a pluriharmonic mapping of B^n into $\mathbb{C}^n$ such that $f(0) = 0, \widetilde{\lambda}_f(0) = 1$ and $\widetilde{\Lambda}_f(z) \leq \widetilde{\Lambda}$ for $z \in B^n$. Then f is univalent on the ball $B^n(o, \rho)$ and the range $f(B^n(0, \rho))$ covers the ball $B^n(0, R)$, where*

$$\rho = \frac{\pi}{4(\widetilde{\Lambda}_f(0) + \widetilde{\Lambda})} \quad \text{and} \quad R = \frac{\pi}{8(\widetilde{\Lambda}_f(0) + \widetilde{\Lambda})}.$$

If, in addition, $\widetilde{\Lambda}_f(0) = 1$, then f is univalent on the ball $B^n(0, \rho')$ and range $f(B^n(0, \rho'))$ covers the ball $B^n(0, R')$, where

$$\rho' = \frac{\pi}{4(1 + \widetilde{\Lambda})} \quad \text{and} \quad R' = \frac{\pi}{8(1 + \widetilde{\Lambda})}.$$

For (K, K')-quasiregular pluriharmonic mappings with bounded dilation, we prove the following Landau-type theorem.

Theorem 3. *Let $f\colon B^n \to \mathbb{C}^n$ be a (K,K')-quasiregular pluriharmonic mapping, $n>1$, such that $f(0)=0, \widetilde{\lambda}_f(0)=1$ and $\widetilde{\Lambda}_f(z)\le\widetilde{\Lambda}$ for $z\in B^n$. Then f is univalent on the ball $B^n(0,\rho)$ and $f(B^n(0,\rho))$ contains the ball $B^n(0,R)$, where*

$$\rho=\frac{\pi}{4(K+K'+\widetilde{\Lambda})}\quad\text{and}\quad R=\frac{\pi}{8(K+K'+\widetilde{\Lambda})}.$$

Proof. Let $z_1,z_2\in B^n(0,\rho)$ be two fixed distinct points and $z_1-z_2=|z_1-z_2|\theta$ for some $\theta\in\partial B^n$. Define the plurihamomic mapping

$$\phi_\theta(z)=(f_z(z)-f_z(0))\,\theta+(f_{\bar z}(z)-f_{\bar z}(0))\,\bar\theta.$$

Then, the definition of $\widetilde{\Lambda}_f(z)$ gives that

$$|\phi_\theta(z)|\le\widetilde{\Lambda}_f(z)+\widetilde{\Lambda}_f(0)\le\widetilde{\Lambda}+K\widetilde{\lambda}_f(0)^{1/n}+K'=\widetilde{\Lambda}+K+K'\quad\text{for } z\in B^n.$$

Note that $\phi_\theta(0)=0$. By Lemma C, we obtain

$$|\phi_\theta(z)|\le\frac{4}{\pi}\left(\widetilde{\Lambda}+K+K'\right)|z|\quad\text{ for } z\in B^n.$$

We have

$$\begin{aligned}
&|f(z_2)-f(z_1)|=\left|\int_{[z_1,z_2]}f_z(z)\,dz+f_{\bar z}(z)\,d\bar z\right|\\
&\ge\left|\int_{[z_1,z_2]}f_z(0)\,dz+f_{\bar z}(0)\,d\bar z\right|-\left|\int_{[z_1,z_2]}(f_z(z)-f_z(0))\,dz+(f_{\bar z}(z)-f_{\bar z}(0))\,d\bar z\right|\\
&\ge|z_2-z_1|\widetilde{\lambda}_f(0)-\int_{[z_1,z_2]}|\phi_\theta(z)|\,ds\\
&>|z_2-z_1|-\frac{4\big(\widetilde{\Lambda}+K+K'\big)\rho}{\pi}|z_2-z_1|=0.
\end{aligned}$$

Thus $f(z_1)\ne f(z_2)$. This shows that f is univalent in $B^n(0,\rho)$.

Now, let $z'\in\partial B^n(0,\rho)$. As $f(0)=0$, we have

$$\begin{aligned}
|f(z')|&=\left|\int_{[0,z']}f_z(z)\,dz+f_{\bar z}(z)\,d\bar z\right|\\
&\ge\left|\int_{[0,z']}f_z(0)\,dz+f_{\bar z}(0)\,d\bar z\right|-\left|\int_{[0,z']}(f_z(z)-f_z(0))\,dz+(f_{\bar z}(z)-f_{\bar z}(0))\,d\bar z\right|\\
&\ge\widetilde{\lambda}_f(0)\rho-\int_0^\rho\frac{4(K+K'+\widetilde{\Lambda})r}{\pi}\,dr\\
&=\rho-\frac{2(K+K'+\widetilde{\Lambda})\rho^2}{\pi}=\frac{\pi}{8(K+K'+\widetilde{\Lambda})}=R.
\end{aligned}$$

This shows that $f(B^n(0,\rho))$ contains the ball $B^n(0,R)$. This completes the proof of this theorem. ☐

Next, we establish a Bloch-type theorem for (K,K')-quasuregular pluriharmonic mappings.

Theorem 4. *Let $f\colon B^n\to\mathbb{C}^n$ be a (K,K')-quasiregular pluriharmonic mapping, $n>1$, such that $\widetilde{\lambda}_f(0)=1$. Then $f(B^n)$ contains a schlicht ball of radius b_f, with*

$$b_f\ge\frac{\pi}{16(3K+2K')}.$$

Proof. Let $f\colon B^n \to \mathbb{C}^n$ be a (K,K')-quasiregular pluriharmonic mapping of the unit ball B^n into $\mathbb{C}^n$ with $\widetilde{\lambda}_f(0)=1$. Without loss of generality, we assume that f is pluriharmonic on $\overline{B}^n$. We introduce the following functions

$$M_1(r)=\max_{|z|\le r}\widetilde{\lambda}_f(z)^{1/n},$$

and

$$\phi_1(r)=(1-r)M_1(r)$$

for $0\le r\le 1$. It is easy to see that $\phi_1(0)=M_1(0)=\widetilde{\lambda}_f(0)^{1/n}=1$ and $\phi_1(1)=0$. Then there exist r_0 such that $\phi_1(r_0)=1$ and $\phi_1(r)<1$ for $r_0<r\le 1$.

Also, since the set $\{z\colon |z|\le r_0\}$ is compact, there exist z_0 such that $|z_0|\le r_0$ and $M_1(r_0)=\widetilde{\lambda}_f(z_0)^{1/n}$, which implies

$$\phi_1(r_0)=(1-r_0)M_1(r_0)=(1-r_0)\widetilde{\lambda}_f(z_0)^{1/n}.$$

Therefore,

$$(1-r_0)\,\widetilde{\lambda}_f(z_0)^{1/n}=1. \tag{4.1}$$

Let $z\in B^n$ with $|z|=r\ge r_0$, then

$$(1-|z|)\widetilde{\lambda}_f(z)^{1/n}\le(1-r)M_1(r)\le 1. \tag{4.2}$$

In particular, we have

$$\widetilde{\lambda}_f(z)\le\widetilde{\lambda}_f(z_0)\quad\text{for } |z|=r_0. \tag{4.3}$$

We consider the following two cases.

Case 1. $r_0>0$: Fix a point w_0 with $0<|w_0|\le r_0$ and assume that $\widetilde{\Lambda}_f(w_0)=|f_z(w_0)\theta+f_{\bar z}(w_0)\bar\theta|$ with $\theta\in\partial B^n$. Define the function φ by

$$\varphi(\zeta)=f_z(\zeta w_0/|w_0|)\theta+f_{\bar z}(\zeta w_0/|w_0|)\bar\theta\quad\text{for } \zeta\in\mathbb{D}.$$

Since φ is harmonic, by the maximum modulus principle, there exists a point ζ' with $|\zeta'|=r_0$, such that

$$\widetilde{\Lambda}_f(w_0)=|\varphi(|w_0|)|\le|f_z(\zeta' w_0/|w_0|)\theta+f_{\bar z}(\zeta' w_0/|w_0|)\bar\theta|.$$

Let $z_1=\zeta' w_0/|w_0|$. Since $|z_1|=r_0$, by the definition of (K,K')-quasiregular pluriharmonic mappings and (4.3), we have

$$\widetilde{\Lambda}_f(w_0)\le\left|f_z(z_1)\theta+f_{\bar z}(z_1)\bar\theta\right|\le\widetilde{\Lambda}_f(z_1)\le K\widetilde{\lambda}_f(z_1)^{1/n}+K'\le K\widetilde{\lambda}_f(z_0)^{1/n}+K'.$$

On the other hand, by the definition of (K,K')-quasiregular pluriharmonic mappings and (4.1) with $\widetilde{\lambda}_f(0)=1$, we have

$$\widetilde{\Lambda}_f(0)\le K\widetilde{\lambda}_f(0)^{1/n}+K'=K+K'=K(1-r_0)\widetilde{\lambda}_f(z_0)^{1/n}+K'.$$

This shows that

$$\widetilde{\Lambda}_f(z)\le K\widetilde{\lambda}_f(z_0)^{1/n}+K'\quad\text{for } |z|\le r_0. \tag{4.4}$$

For $\xi\in B^n$, define

$$g(\xi)=z_0+\frac{(1-r_0)^n}{2}\xi\quad\text{and}\quad F(\xi)=2\left(f(g(\xi))-f(z_0)\right). \tag{4.5}$$

Then, it is easy to see that

$$F(0)=0\quad\text{and}\quad\widetilde{\lambda}_F(0)=(1-r_0)^n\widetilde{\lambda}_f(z_0)=1.$$

If $|g(\xi)| \le r_0$, from (4.4) and (4.1) we have

$$\begin{aligned}\widetilde{\Lambda}_F(\xi) &= (1-r_0)^n \widetilde{\Lambda}_F(g(\xi)) \le K(1-r_0)^n \widetilde{\lambda}_f(z_0)^{1/n} + (1-r_0)^n K' \\ &\le K(1-r_0)\widetilde{\lambda}_f(z_0)^{1/n} + K' = K + K',\end{aligned}$$

and if $|g(\xi)| \ge r_0$, from (4.2), we obtain

$$\begin{aligned}\widetilde{\Lambda}_F(\xi) &= (1-r_0)^n \widetilde{\Lambda}_F(g(\xi)) \le K(1-r_0)^n \widetilde{\lambda}_f(g(\xi))^{1/n} + (1-r_0)^n K' \\ &\le K(1-r_0)\widetilde{\lambda}_f((g(\xi))^{1/n} + K' \\ &= K\left(\frac{1-r_0}{1-|g(\xi)|}\right)(1-|g(\xi)|)\widetilde{\lambda}_f((g(\xi))^{1/n} + K' \\ &\le K\left(\frac{1-r_0}{1-|g(\xi)|}\right) + K' \le \frac{K(1-r_0)}{1-r_0-(1-r_0)^n|\xi|/2} + K' \\ &\le \frac{K(1-r_0)}{1-r_0-(1-r_0)|\xi|/2} + K' = \frac{2K}{2-|\xi|} + K'. \qquad (4.6)\end{aligned}$$

Case 2. $r_0 = 0$: Consider the functions g and F defined by (4.5) with $r_0 = 0$. Then $|g(\xi)| \ge r_0 = 0$ and it follows from (4.6) that

$$\widetilde{\Lambda}_F(\xi) = \frac{2K}{2-|\xi|} + K' \quad \text{for } \xi \in B^n.$$

Therefore, we conclude that

$$\widetilde{\Lambda}_F(\xi) < 2K + K' \quad \text{for } \xi \in B^n.$$

In particular, $\widetilde{\Lambda}_F(0) \le K + K'$.

Now, applying Theorem C to the mapping F, we see that $F(B^n)$ contains a schlicht ball with the center 0 and radius

$$R' = \frac{\pi}{8(3K+2K')}.$$

Consequently, $f(B^n)$ contains a schlicht ball of radius

$$R = \frac{\pi}{16(3K+2K')}.$$

This completes the proof. □

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Vasudevarao Allu
Indian Institute of Technology Bhubaneswar
School of Basic Science
Department of Mathematics
Bhubaneswar-752050, Odisha, India
avrao@iitbbs.ac.in

Rohit Kumar
Indian Institute of Technology Bhubaneswar
School of Basic Science
Department of Mathematics
Bhubaneswar-752050, Odisha, India
rohitk12798@gmail.com