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CMC hypersurfaces with polynomial volume growth in warped products and the nonexistence of entire solutions to the minimal hypersurface equation

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Abstract. We investigate constant mean curvature (CMC) complete two-sided hypersurfaces with polynomial volume growth in a class of warped products satisfying a suitable curvature constraint. In this setting, we establish the nonexistence of such a CMC hypersurface under mild hypotheses involving the mean curvature and the warping function. Applications to Einstein warped product, pseudo-hyperbolic, Schwarzschild and Reissner–Nordström spaces are also given. Furthermore, we present a nonparametric version of our main result which, in particular, guarantees the nonexistence of entire solutions with finite C^2 norm of the the minimal hypersurface equation on a complete Riemannian manifold with polynomial volume growth.

Poimutulojen polynomitilavuuskasvuiset vakiokeskikaarevat hyperpinnat ja minimihyperpintayhtälön kokonaisten ratkaisujen puuttuminen

Tiivistelmä. Työssä tutkitaan sopivan kaarevuusehdon toteuttavien poimutulojen luokassa täydellisiä kaksipuolisia hyperpintoja, joiden keskikaarevuus on vakio ja tilavuus kasvaa polynomivauhtia. Keskikaarevuutta ja poimufunktiota koskevilla maltillisilla lisäoletuksilla osoitetaan, että tällaisia hyperpintoja ei ole olemassa. Tälle esitellään Einsteinin poimutuloihin sekä pseudohyperbolisiin, Schwarzschildin ja Reissnerin–Nordströmin avaruuksiin liittyviä sovelluksia. Lisäksi annetaan päätuloksen parametriton muotoilu, joka erityisesti takaa, että polynomitilavuuskasvuisen Riemannin moniston mimimihyperpintayhtälöllä ei ole C^2-normiltaan äärellisiä, kokonaisia ratkaisuita.

1. Introduction

The study of constant mean curvature (CMC) hypersurfaces immersed in a Riemannian manifold constitutes a classical but still fruitful thematic in differential geometry. Into this branch, Montiel [15] investigated constant mean curvature compact hypersurfaces immersed in warped products of the type $I \times_f M^n$, whose Ricci curvature Ric_M of the fiber $(M^n, \langle\, ,\rangle_M)$ and the (positive) warping function $f\colon I \subset \mathbb{R} \to \mathbb{R}$ satisfy the following curvature constraint:

$$\mathrm{Ric}_M \geq (n-1)\sup_I (f'^2 - ff'')\langle\, ,\rangle_M. \tag{1.1}$$

In this context, he obtained the analogous to the classical Jellett–Liebmann and Alexandrov theorems for hypersurfaces in Euclidean space. It is also worth to point out that Einstein warped product, pseudo-hyperbolic, Schwarzschild and Reissner–Nordström spaces also constitute examples of warped product models satisfying curvature constraint (1.1). For more details, see Section 4.

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Later on, Alías and Dajczer [4] studies complete properly immersed surfaces contained in a slab of a warped product $\mathbb{R} \times_f M^2$, where M^2 is complete with nonnegative Gaussian curvature. Under certain restrictions on the mean curvature of the surface they showed that such an immersion does not exists or must be a *slice*, that is, a leaf of the trivial totally umbilical foliation $t \in \mathbb{R} \mapsto \{t\} \times M^2$. Afterwards, these same authors [5] reobtained Montiel's results [15] considering complete, not necessarily compact, hypersurfaces immersed in $\mathbb{R} \times_f M^n$. A few years later, the author named jointly with Aquino [6] and Caminha [10] obtained rigidity results for complete vertical graphs with constant mean curvature in $I \times_f M^n$, assuming appropriate restrictions on the values of the mean curvature and the norm of the gradient of the height function h of these graphs. Next, supposing that the gradient of h is Lebesgue integrable and that the mean curvature function takes values in the interval $(0, 1]$, the author named jointly with Camargo and Caminha [9] showed that complete hypersurfaces lying in a slab of a pseudo-hyperbolic space $\mathbb{R} \times_{e^t} M^n$ must be slices.

In [3], the named author jointly with Alías and Colares proved uniqueness results for entire graphs in a warped product space satisfying (1.1), under suitable restrictions on the higher order mean curvatures. In particular, they obtained applications to the study of the rigidity of minimal and radial graphs in the Euclidean space. Afterwards, Aledo and Rubio [1] provided uniqueness and nonexistence of entire solutions to the minimal surface equation in warped products of the type $\mathbb{R} \times_f \mathbb{R}^2$ and, as a consequence of their results, they extended the classical Bernstein's Theorem. Later, Romero, Rubio and Salamanca [17] proved several Moser–Bernstein type results when the ambient space is a warped product $I \times_f M^n$ whose fiber M^n is parabolic and such that $\log f$ is convex.

More recently, the named author jointly with Araújo and Gomes [7] studied complete two-sided hypersurfaces immersed in a warped product space $I \times_f M^n$. Under appropriate restrictions on the warping function f, on the sectional curvature of the fiber M^n and on the mean curvature of such a hypersurface Σ^n, they applied some maximum principles to show that Σ^n must be a slice of $I \times_f M^n$. They also obtained Moser–Bernstein type results concerning entire graphs constructed over M^n, as well as applications to pseudo-hyperbolic spaces $I \times_{e^t} M^n$.

Proceeding with this picture, here we investigate constant mean curvature (CMC) complete two-sided hypersurfaces with polynomial volume growth in the class of warped products satisfying the curvature constraint (1.1). In this setting, we establish the nonexistence of such a CMC hypersurface under mild hypotheses involving the mean curvature and the warping function (see Theorem 3.2). Applications to Einstein warped product, pseudo-hyperbolic, Schwarzschild and Reissner–Nordström spaces are also given (see Section 4). Furthermore, we study the nonexistence of entire solutions of the mean curvature equation (see Theorem 5.1). As a consequence of this study, we obtain the following result concerning the nonexistence of entire solutions to the minimal hypersurface equation on a complete Riemannian manifold with polynomial volume growth:

Let $(M^n, \langle\,,\rangle_M)$ be a complete Riemannian manifold with polynomial volume growth and let $f\colon I \to \mathbb{R}$ be a smooth positive function defined on a open interval $I \subset \mathbb{R}$ such that $f'(t) > 0$ and $(\log f)''(t) \geq 0$. If (1.1) is satisfied, then there is no smooth function $u\colon M^n \to I$ with finite C^2 norm which is an entire solution to the nonlinear elliptic equation

$$\operatorname{div}_M\left(\frac{Du}{f(u)\sqrt{f^2(u)+|Du|_M^2}}\right)=\frac{f'(u)}{\sqrt{f^2(u)+|Du|_M^2}}\left(n-\frac{|Du|_M^2}{f^2(u)}\right).$$

2. Basic setup

Let $(M^n,\langle\,,\rangle_M)$ be a connected, n-dimensional oriented Riemannian manifold, $I\subset\mathbb{R}$ an open interval and $f\colon I\to\mathbb{R}$ a positive smooth function. In the product differentiable manifold $I\times M^n$, let π_I and π_M denote the projections onto the I and M^n factors, respectively. A particular class of Riemannian manifolds is the one obtained by furnishing $I\times M^n$ with the metric

$$\langle v,w\rangle(p)=\langle(\pi_I)_*v,(\pi_I)_*w\rangle_{\mathbb{R}}(\pi_I(p))+(f(\pi_I(p)))^2\langle(\pi_M)_*v,(\pi_M)_*w\rangle_M(\pi_M(p)),\tag{2.1}$$

for all $p\in\overline{M}^{n+1}$ and all $v,w\in T_p\overline{M}$. Indeed, according to [15, Proposition 1], the smooth vector field

$$V=(f\circ\pi_I)\partial_t$$

is conformal and closed (in the sense that its dual 1-form is closed), with conformal factor $\phi=f'$, where the prime denotes differentiation with respect to $t\in I$. Such a space is called a *warped product*, and in what follows we will write $\overline{M}^{n+1}=I\times_f M^n$ to denote it.

Throughout this paper, we will study two-sided hypersurfaces $\psi\colon\Sigma^n\to\mathbb{R}\times_f M^n$ oriented by a globally defined unit vector field N. Let $\overline{\nabla}$ and ∇ denote the Levi–Civita connections in $\mathbb{R}\times_f M^n$ and Σ^n, respectively. Then the Gauss and Weingarten formulas for such hypersurfaces are given by

$$\overline{\nabla}_X Y=\nabla_X Y+\langle AX,Y\rangle N\tag{2.2}$$

and

$$AX=-\overline{\nabla}_X N,\tag{2.3}$$

for every tangent vector fields $X,Y\in\mathfrak{X}(\Sigma)$. Here $A\colon\mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ stands for the second fundamental form of Σ^n with respect to the Gauss map N.

Remark 2.1. In the warped product $\overline{M}^{n+1}=I\times_f M^n$ there exists a remarkable family of two-sided hypersurfaces: its slices $M_{t_0}=\{t_0\}\times M$, with $t_0\in I$. The second fundamental form and the mean curvature of M_{t_0} with respect to $N=-\partial_t$ are, respectively, $A_{t_0}=\frac{f'(t_0)}{f(t_0)}I$, where I denotes the identity operator, and $H_{t_0}=\frac{1}{n}\operatorname{tr}(A_{t_0})=\frac{f'(t_0)}{f(t_0)}$.

Now, let us consider two particular functions naturally attached to such hypersurfaces, namely, the (vertical) height function $h=(\pi_I)|_\Sigma$ and the angle function $\Theta=\langle N,\partial_t\rangle$.

Let us denote by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics of $I\times_f M^n$ and Σ^n, respectively. Then, a simple computation shows that the gradient of π_I on $I\times_f M^n$ is given by

$$\overline{\nabla}\pi_I=\langle\overline{\nabla}\pi_I,\partial_t\rangle\partial_t=\partial_t,\tag{2.4}$$

so that the gradient of h on Σ^n is

$$\nabla h=(\overline{\nabla}\pi_I)^\top=\partial_t^\top=\partial_t-\Theta N,\tag{2.5}$$

where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n. Thus, we get

$$|\nabla h|^2 = 1 - \Theta^2, \tag{2.6}$$

where $|\ |$ denotes the norm of a vector field on Σ^n.

The following lemma can be found in [10, Propositions 3.1 and 3.2]. See also [13, Proposition 1.1].

Lemma 2.2. *Let $\psi\colon \Sigma^n \to \mathbb{R}\times_f M^n$ be a two-sided hypersurface with orientation N. Then,*

$$\Delta h = (\log f)'(h)(n - |\nabla h|^2) + nH\Theta,$$

where $H = \frac{1}{n}\mathrm{tr}(A)$ is the mean curvature of Σ^n with respect to N. Moreover, if H is constant,

$$\Delta(f(h)\Theta) = -f(h)\left\{\mathrm{Ric}_M(N^*, N^*) + (n-1)(\log f)''(h)|\nabla h|^2 + |A|^2\right\}\Theta - nHf'(h),$$

where Ric_M denotes the Ricci curvature of the fibre M^n, $N^* = (\pi_M)_* N$ and $|A|$ is the Hilbert–Schmidt norm of the second fundamental form A of Σ^n.

3. Main result

We start this section quoting the analytical tool that will be used to prove our results. For this, let Σ^n be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p, r)$ the geodesic ball centered at p and with radius r. Given a polynomial function $\sigma\colon (0, +\infty) \to (0, +\infty)$, we say that Σ^n has polynomial volume growth like $\sigma(r)$ if there exists $p \in \Sigma^n$ such that

$$\mathrm{vol}(B(p, r)) = \mathcal{O}(\sigma(r)),$$

as $r \to +\infty$, where vol denotes the canonical Riemannian volume of Σ^n. As it was already observed in the beginning of [2, Section 2], if $p, q \in \Sigma^n$ are at distance d from each other, we can verify that

$$\frac{\mathrm{vol}(B(p,r))}{\sigma(r)} \geq \frac{\mathrm{vol}(B(q, r-d))}{\sigma(r-d)} \cdot \frac{\sigma(r-d)}{\sigma(r)}.$$

Consequently, the choice of p in the notion of volume growth is immaterial, and we will just say that Σ^n has polynomial volume growth.

Keeping in mind the previous digression, we have the following lemma which corresponds to a particular case of [2, Theorem 2.1] due to Alías, Caminha and do Nascimento.

Lemma 3.1. *Let Σ^n be a connected, oriented, complete noncompact Riemannian manifold, and let $\zeta \in \mathcal{C}^\infty(\Sigma)$ be nonnegative and such that $\Delta\zeta \geq \alpha\zeta$ on Σ^n, for some positive constant $\alpha \in \mathbb{R}$. If Σ^n has polynomial volume growth and $|\nabla\zeta|$ is bounded on Σ^n, then ζ vanishes identically on Σ^n.*

From now on, we will orient the two-sided hypersurfaces in such a way that $\Theta \leq 0$. In this setting, we obtain the following nonexistence result:

Theorem 3.2. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product satisfying (1.1). There is no CMC complete two-sided hypersurface $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ with polynomial volume growth, bounded second fundamental form A, lying between two slices of*

$\overline{M}^{n+1}$ *such that* $f'(h) > 0$, $(\log f)''(h) \geq 0$ *and whose mean curvature* H *satisfies*

$$H < \inf_{\Sigma} \frac{f'(h)}{f(h)}. \tag{3.1}$$

Proof. Let us assume, by contradiction, the existence of such a hypersurface $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ and let us define on it the function $\zeta\colon \Sigma^n \to \mathbb{R}$ given by

$$\zeta = f(h)(1 + \Theta).$$

By computing the Laplacian of ζ with the aid of the formulas of Lemma 2.2, we get

$$\begin{aligned}\Delta\zeta = &- \left(\mathrm{Ric}_M(N^*, N^*) - (n-1)\frac{f'(h)^2 - f(h)f''(h)}{f(h)^2}|\nabla h|^2\right) f(h)\Theta \\ &+ \left(\frac{f(h)f''(h) - f'(h)^2}{f(h)}\right)|\nabla h|^2 \\ &+ n\frac{f'(h)^2}{f(h)} - nHf'(h) + \left(nHf'(h) - f(h)|A|^2\right)\Theta.\end{aligned}$$

On the other hand, taking into account the curvature constraint (1.1), a straightforward computation gives us

$$\mathrm{Ric}_M(N^*, N^*) \geq \frac{(n-1)}{f(h)^2} \sup_{\Sigma}(f'(h)^2 - f(h)f''(h))|\nabla h|^2. \tag{3.2}$$

Thus, since $\Theta \leq 0$, from (3.2) and (3.2) we obtain that

$$\begin{aligned}\Delta\zeta \geq &\left(\frac{f(h)f''(h) - f'(h)^2}{f(h)}\right)|\nabla h|^2 + n\frac{f'(h)^2}{f(h)} \\ &- nHf'(h) + \left(nHf'(h) - f(h)|A|^2\right)\Theta.\end{aligned} \tag{3.3}$$

Consequently, since we are also supposing that $(\log f)''(h) \geq 0$, from (3.3) we get

$$\Delta\zeta \geq n\frac{f'(h)^2}{f(h)} - nHf'(h) + \left(nHf'(h) - f(h)|A|^2\right)\Theta. \tag{3.4}$$

Then, using that $|A|^2 \geq nH^2$ in (3.4), we arrive at

$$\Delta\zeta \geq n\left(\frac{f'(h)}{f(h)} - H\right)\left(f'(h) + Hf(h)\Theta\right). \tag{3.5}$$

Hence, since we are assuming that Σ^n lies between two slices and $f'(h) > 0$, from (3.1) and (3.5) we obtain

$$\Delta\zeta \geq \alpha\zeta, \tag{3.6}$$

where

$$\alpha = n \inf_{\Sigma} \frac{f'(h)}{f(h)} \left(\inf_{\Sigma} \frac{f'(h)}{f(h)} - H\right)$$

is a positive constant.

Moreover, since

$$\nabla\zeta = (f'(h)\,\mathrm{Id} - f(h)A)(\nabla h),$$

where Id stands for the identity operator on $\mathfrak{X}(\Sigma)$, we infer that

$$|\nabla\zeta| \leq f'(h) + f(h)|A|.$$

So, using the hypotheses that Σ^n lies in a slice and that $|A|$ is bounded, we have that $|\nabla\zeta|$ is also bounded.

When Σ^n is noncompact, since we are also assuming that Σ^n has polynomial volume growth and taking into account that ζ is a nonnegative function, from (3.6) we can apply Lemma 3.1 to conclude that ζ vanishes identically on Σ^n, which implies that $\Theta \equiv -1$ and, hence, Σ^n must be a slice, contradicting hypothesis (3.1). Finally, in the case that Σ^n is compact, we can use divergence theorem in (3.6) to also get that ζ is identically zero on Σ^n, arriving that the same conclusion. □

Remark 3.3. Taking into account Remark 2.1, we see that hypotheses $f'(h) > 0$ and $(\log f)''(h) \geq 0$ in Theorem 3.2 just mean that the mean curvature (with respect to $-\partial_t$) of the slices which foliate the region of $\overline{M}^{n+1}$ where Σ^n is contained constitutes a strictly positive and nondecreasing function.

From Theorem 3.2 we derive the following consequence:

Corollary 3.4. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product satisfying* (1.1). *There is no complete minimal two-sided hypersurface $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ with polynomial volume growth, bounded second fundamental form and lying between two slices of $\overline{M}^{n+1}$, such that $(\log f)''(h) \geq 0$ and*

$$\inf_{\Sigma} \frac{f'(h)}{f(h)} > 0.$$

4. Applications to some standard models

From [8, Corollary 9.107], it follows that the warped product space $(\overline{M}^{n+1}, \langle , \rangle)$ is Einstein with its Ricci curvature tensor satisfying $\overline{\mathrm{Ric}} = \overline{c}\langle\, , \rangle$, $\overline{c}$ being a constant, if and only if

i) the fiber M^n has constant Ricci curvature c, and
ii) the warping function f satisfies the following differential equations

$$\frac{f''}{f} = -\frac{\overline{c}}{n} \quad \text{and} \quad \frac{\overline{c}(n-1)}{n} = \frac{c-(n-1)f'^2}{f^2}.$$

In particular, we have that $c = (n-1)(f'^2 - ff'') =$ constant and $\overline{M}^{n+1}$ satisfies (1.1) with $\mathrm{Ric}_M = c\langle , \rangle_M = (n-1)(f'^2 - ff'')\langle\, , \rangle_M$. So, in this context Corollary 3.4 reads as follows:

Corollary 4.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be an Einstein warped product. There is no CMC complete two-sided hypersurface $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ with polynomial volume growth, bounded second fundamental form and lying between two slices of $\overline{M}^{n+1}$, such that $f'(h) > 0$, $(\log f)''(h) \geq 0$ and whose mean curvature satisfies* (3.1).

According to the terminology introduced by Tashiro in [18], when the warping function is exponential the corresponding warped product $I \times_{e^t} M^n$ is referred to as a *pseudo-hyperbolic space*. Tashiro's terminology is due to the fact that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is isometric to the warped product $\mathbb{R} \times_{e^t} \mathbb{R}^n$, where the slices constitute a family of horospheres sharing a fixed point in the asymptotic boundary $\partial_\infty \mathbb{H}^{n+1}$ and giving a complete foliation of $\mathbb{H}^{n+1}$. For more details about these spaces see, for instance, [4, 5, 14, 15].

We observe that a pseudo-hyperbolic space $I \times_{e^t} M^n$, whose fiber M^n has nonnegative Ricci curvature satisfies (1.1). So, from Corollary 3.4 we obtain the following consequence:

Corollary 4.2. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n has nonnegative Ricci curvature. There is no CMC complete two-sided hypersurface with polynomial volume growth, bounded second fundamental form, lying between two slices of $\overline{M}^{n+1}$ and whose mean curvature satisfies $H < 1$.*

Given a mass parameter $\mathfrak{m} > 0$, the *Schwarzschild space* is defined to be the product $\overline{M}^{n+1} = (r_0(\mathfrak{m}), +\infty) \times \mathbb{S}^n$ furnished with the metric $\bar{g} = V_{\mathfrak{m}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$,

$$V_{\mathfrak{m}}(r) = 1 - 2\mathfrak{m}r^{1-n}$$

stands for its potential function and

$$r_0(\mathfrak{m}) = (2\mathfrak{m})^{1/(n-1)}$$

is the unique positive root of $V_{\mathfrak{m}}(r) = 0$. Its importance lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m}}(r)dt^2 + \bar{g}$ is a solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance, [11, Section 4.7] and [16, Chapter 13] for more details concerning Schwarzschild geometry).

As it was observed in [12, Example 1.3], $\overline{M}^{n+1}$ can be reduced in the form $(0, +\infty) \times_f \mathbb{S}^n$ with metric (2.1) via the following change of variables:

$$t = \int_{r_0(\mathfrak{m})}^{r} \frac{d\sigma}{\sqrt{V_{\mathfrak{m}}(\sigma)}}, \quad f(t) = r(t), \;\; t \in (0, +\infty). \tag{4.1}$$

As it was noted in [12, Example 4.1], since $V_{\mathfrak{m}}(r)$ is strictly increasing on $(r_0(\mathfrak{m}), +\infty)$, it follows from (5.5) that the warping function f satisfies:

$$f'(t) = \frac{dr}{dt} = \sqrt{V_{\mathfrak{m}}(r(t))} > 0 \quad \text{and} \quad f''(t) = \frac{1}{2}\frac{dV_{\mathfrak{m}}}{dr}(r(t)) > 0. \tag{4.2}$$

Hence, from (5.5) and (4.2) it is not difficult to verify that $(\log f)''(t) \geq 0$ and (1.1) is satisfied if only if $f(t) \leq (\mathfrak{m}(n+1))^{1/(n-1)}$.

Taking into account this previous digression, from Theorem 3.2 we obtain the following application:

Corollary 4.3. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Schwarzschild space with mass parameter $\mathfrak{m} > 0$. There is no CMC complete two-sided hypersurface $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ with polynomial volume growth, bounded second fundamental form, lying between two slices of $\overline{M}^{n+1}$ such that $f(h) \leq (\mathfrak{m}(n+1))^{1/(n-1)}$ and whose mean curvature H satisfies* (3.1).

Given a mass parameter $\mathfrak{m} > 0$ and an electric charge $\mathfrak{q} \in \mathbb{R}$, with $|\mathfrak{q}| \leq \mathfrak{m}$, the *Reissner–Nordström space* is defined to be the product $\overline{M}^{n+1} = (r_0(\mathfrak{m}, \mathfrak{q}), +\infty) \times \mathbb{S}^n$ endowed with the metric $\bar{g} = V_{\mathfrak{m},\mathfrak{q}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$,

$$V_{\mathfrak{m},\mathfrak{q}}(r) = 1 - 2\mathfrak{m}r^{1-n} + \mathfrak{q}^2 r^{2-2n}$$

stands for its potential function and

$$r_0(\mathfrak{m}, \mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}}\right)^{1/(n-1)}$$

is the largest positive zero of $V_{\mathfrak{m},\mathfrak{q}}(r)$. The importance of this model lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m},\mathfrak{q}}(r)dt^2 + \bar{g}$

is a charged black-hole solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance, [11, Remark 4.5] and [19, Section 12.3]).

As in the Schwarzschild space, $\overline{M}^{n+1}$ can be reduced in the form $(0,+\infty)\times_f \mathbb{S}^n$ with metric (2.1) via the same change of variables as in (5.5). Furthermore, following the same previous steps, the warping function f has positive first and second derivatives. Moreover, we can verify that $(\log f)''(t)\geq 0$ and (1.1) is satisfied if only if $|\mathfrak{q}|<\frac{\mathfrak{m}(n+1)}{2\sqrt{n}}$ and $x_{**}^{1/(1-n)}\leq f(t)\leq x_{*}^{1/(1-n)}$, where $x_*<x_{**}$ are the two positive real roots of the polynomial function $\mathcal{P}(x)$ given by

$$\mathcal{P}(x)=\mathfrak{q}^2nx^2-\mathfrak{m}(n+1)x+1. \tag{4.3}$$

Keeping in mind our previous discussion, from Theorem 3.2 we obtain the following application:

Corollary 4.4. *Let $\overline{M}^{n+1}=I\times_f\mathbb{S}^n$ be the Reissner–Nordström space with mass parameter $\mathfrak{m}>0$ and an electric charge $\mathfrak{q}\in\mathbb{R}$, with $|\mathfrak{q}|<\frac{\mathfrak{m}(n+1)}{2\sqrt{n}}$. There is no CMC complete two-sided hypersurface $\psi\colon\Sigma^n\to\overline{M}^{n+1}$ with polynomial volume growth, bounded second fundamental form, lying between two slices of $\overline{M}^{n+1}$ such that $x_{**}^{1/(1-n)}\leq f(h)\leq x_{*}^{1/(1-n)}$, where $x_*<x_{**}$ are the two positive real roots of the polynomial function $\mathcal{P}(x)$ defined in* (4.3)*, and whose mean curvature H satisfies* (3.1)*.*

5. Entire graphs

Let us consider Ω a domain in M^n. A function $u\in C^\infty(\Omega)$ such that $u(\Omega)\subseteq\mathbb{R}$ defines a vertical graph in the product space $\overline{M}^{n+1}=I\times M^n$. In such a case, $\Sigma(u)$ will denote the graph over Ω determined by u, that is,

$$\Sigma(u)=\{(u(p),p)\colon p\in\Omega\}\subset\overline{M}^{n+1}.$$

The graph $\Sigma(u)$ is said to be entire if $\Omega=M^n$. Observe that $h(u(p),p)=u(p)$, $p\in\Omega$. Hence, h and u can be identified in a natural way. The metric induced on Ω from the Riemannian metric of the ambient space via $\Sigma(u)$ is

$$g_u=du^2+f(u)^2\langle\,,\rangle_M. \tag{5.1}$$

The unit vector field

$$N(p)=-\frac{1}{f(u)\sqrt{f(u)^2+|Du(p)|^2_{M^n}}}\left(f(u)^2\partial_t|_{(u(p),p)}-Du(p)\right),\quad p\in\Omega, \tag{5.2}$$

where Du stands for the gradient of u in M^n and $|Du|_{M^n}=\langle Du,Du\rangle_{M^n}^{\frac{1}{2}}$, gives an orientation of $\Sigma(u)$ with respect to which we have $\Theta\leq 0$. The corresponding second fundamental form is given by

$$\begin{aligned}AX=&\left(\frac{\langle D_XDu,Du\rangle_M}{f(u)\left(f^2(u)+|Du|^2_M\right)^{3/2}}+\frac{f'(u)\langle Du,X\rangle_M}{\left(f^2(u)+|Du|^2_M\right)^{3/2}}\right)Du\\&-\frac{1}{f(u)\sqrt{f^2(u)+|Du|^2_M}}D_XDu+\frac{f'(u)}{\sqrt{f^2(u)+|Du|^2_M}}X,\end{aligned} \tag{5.3}$$

for any vector field X tangent to Ω, where D is the Levi–Civita connection in M^n. Consequently, if $\Sigma(u)$ is a vertical graph over a domain $\Omega\subseteq M^n$ and denoting by

div_{M^n} the divergence operator computed in the metric $\langle , \rangle_{M^n}$, it is not difficult to verify from (5.3) that the mean curvature function $H(u)$ of $\Sigma(u)$ is given by:

$$(5.4)\quad H(u) = -\mathrm{div}_M\left(\frac{Du}{f(u)\sqrt{f^2(u)+|Du|_M^2}}\right) + \frac{f'(u)}{\sqrt{f^2(u)+|Du|_M^2}}\left(n - \frac{|Du|_M^2}{f^2(u)}\right).$$

We recall that $u \in C^\infty(M)$ has finite C^2 norm when

$$\|u\|_{C^2(M)} := \sup_{|\gamma|\le 2} |D^\gamma u|_{L^\infty(M)} < +\infty.$$

In this context, we close our paper establishing a nonparametric version of Theorem 3.2:

Theorem 5.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product satisfying curvature constraint* (1.1) *and whose fiber M^n is complete with polynomial volume growth. There is no CMC entire graph $\Sigma(u)$ determined by a function $u \in C^\infty(M)$ with finite C^2 norm such that $f'(u) > 0$, $(\log f)''(u) \ge 0$ and whose mean curvature $H(u)$ satisfies*

$$H(u) < \inf_M \frac{f'(u)}{f(u)}.$$

Proof. By contradiction, let us assume the existence of such a entire graph $\Sigma(u)$. Since we are assuming that u has finite C^2, it follows from (5.3) that the second fundamental form of $\Sigma(u)$ is bounded. Moreover, we note that the finiteness of the C^2 norm of u also implies, in particular, that u is bounded, which, in turn, guarantees that $\inf_M f(u) > 0$. So, since M^n is complete, from (5.1) we infer that $\Sigma(u)$ furnished with the metric g_u is also complete.

On the other hand, from (5.1) we have that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma$ stand for the Riemannian volume elements of $(M^n, \langle\, ,\rangle_M)$ and $(\Sigma(u), g_u)$, respectively, and $G = \det(g_{ij})$ with

$$g_{ij} = g_u(E_i, E_j) = E_i(u)E_j(u) + f(u)^2\delta_{ij}.$$

Here, $\{E_1, \ldots, E^n\}$ denotes a local orthonormal frame with respect to the metric $\langle\, ,\rangle_M$. Consequently, we obtain that

$$|G| = f(u)^{2(n-1)}(f(u)^2 + |Du|_M^2).$$

Thus, we arrive at the following relation

$$(5.5)\qquad d\Sigma = f(u)^{n-1}\sqrt{f(u)^2 + |Du|_M^2}\,dM.$$

Hence, since we are assuming that $(M^n, \langle\, ,\rangle_M)$ has polynomial volume growth and that $u \in C^\infty(M)$ has finite C^2 norm, from (5.5) we conclude that $(\Sigma(u), g_u)$ also has polynomial volume growth. Therefore, by applying Theorem 3.2, we reach a contradiction. □

It follows from Theorem 5.1 jointly with the mean curvature equation (5.4) the following nonexistence result:

Corollary 5.2. *Let $(M^n, \langle\, ,\rangle_M)$ be a complete Riemannian manifold with polynomial volume growth and let $f\colon I \to \mathbb{R}$ be a smooth positive function defined on a open interval $I \subset \mathbb{R}$ such that $f'(t) > 0$ and $(\log f)''(t) \ge 0$. If* (1.1) *is satisfied, then there is no smooth function $u\colon M^n \to I$ with finite C^2 norm which is an entire* *solution to the nonlinear elliptic equation*

$$\operatorname{div}_M\left(\frac{Du}{f(u)\sqrt{f^2(u)+|Du|_M^2}}\right)=\frac{f'(u)}{\sqrt{f^2(u)+|Du|_M^2}}\left(n-\frac{|Du|_M^2}{f^2(u)}\right).$$

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References

[1] Aledo, J. D., and R. M. Rubio: A Bernstein problem in warped products. - Ann. Acad. Sci. Fenn. Math. 41:2, 2016, 699–704.

[2] Alías, L. J., A. Caminha, and F. Y. do Nascimento: A maximum principle related to volume growth and applications. - Ann. Mat. Pura Appl. 200, 2021, 1637–1650.

[3] Alías, L. J., A. G. Colares, and H. F. de Lima: Uniqueness of entire graphs in warped products. - J. Math. Anal. Appl. 430, 2015, 60–75.

[4] Alías, L. J., and M. Dajczer: Uniqueness of constant mean curvature surfaces properly immersed in a slab. - Comment. Math. Helv. 81, 2006, 653–663.

[5] Alías, L. J., and M. Dajczer: Constant mean curvature hypersurfaces in warped product spaces. - Proc. Edinburg Math. Soc. 50, 2007, 511–526.

[6] Aquino, C. P., and H. F. de Lima: On the rigidity of constant mean curvature complete vertical graphs in warped products. - Diff. Geom. Appl. 29, 2011, 590–596.

[7] Araújo, J. G., H. F. de Lima, and W. F. Gomes: Rigidity of hypersurfaces and Moser–Bernstein type results in certain warped products, with applications to pseudo-hyperbolic spaces. - Aequat. Math. 96, 2022, 1159–1177.

[8] Besse, A. L.: Einstein manifolds. - Springer-Verlag, Berlin, 1987.

[9] Camargo, F., A. Caminha, and H. F. de Lima: Bernstein-type theorems in semi-Riemannian warped products. - Proc. Amer. Math. Soc. 139, 2011, 1841–1850.

[10] Caminha, A., and H. F. de Lima: Complete vertical graph with constant mean curvature in semi-Riemannian warped products. - Bull. Belg. Math. Soc. Simon Stevin 16, 2009, 91–105.

[11] Chen, B. Y.: Differential geometry of warped product manifolds and submanifolds. - World Scientific, London, 2017.

[12] Colombo, G., L. Mari, and M. Rigoli: Remarks on mean curvature flow solitons in warped products. - Discrete Contin. Dyn. Syst. 13, 2020, 1957–1991.

[13] de Lima, H. F., G. M. Bisci, and M. A. L. Velásquez: Immersions in warped product spaces. - Front. Math., Birkhäuser Cham, Springer Nature Switzerland AG, 2025.

[14] García-Martínez, S. C., D. Impera, and M. Rigoli: A sharp height estimate for compact hypersurfaces with constant k-mean curvature in warped product spaces. - Proc. Edinburg Math. Soc. 58, 2015, 403–419.

[15] Montiel, S.: Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds. - Indiana Univ. Math. J. 48, 1999, 711–748.

[16] O'Neill, B.: Semi-Riemannian geometry, with applications to relativity. - Academic Press, New York, 1983.

[17] Romero, A., R. M. Rubio, and J. J. Salamanca: New examples of Moser–Bernstein type problems for some nonlinear elliptic partial differential equations arising in geometry. - Ann. Fenn. Math. 46, 2021, 781–794.

[18] Tashiro, Y.: Complete Riemannian manifolds and some vector fields. - Trans. Amer. Math. Soc. 117, 1965, 251–275.

[19] Wald, R. M.: General relativity. - The University of Chicago Press, Chicago and London, 1984.

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