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A clustering theorem in fractional Sobolev spaces

Fatma Gamze Düzgün, Antonio Iannizzotto and Vincenzo Vespri

Abstract. We prove a general clustering result for the fractional Sobolev space $W^{s,p}$: whenever the positivity set of a function u in a cube has measure bounded from below by a multiple of the cube's volume, and the $W^{s,p}$-seminorm of u is bounded from above by a convenient power of the cube's side, then u is positive in a universally reduced cube. Our result aims at applications in regularity theory for fractional elliptic and parabolic equations. Also, by means of suitable interpolation inequalities, we show that clustering results in $W^{1,p}$ and BV, respectively, can be deduced as special cases.

Murtoasteisten Sobolevin avaruuksien ryvästymislause

Tiivistelmä. Tässä työssä todistetaan yleinen ryvästymistulos murtoasteisissa Sobolevin avaruuksissa $W^{s,p}$: jos annetussa kuutiossa funktion u positiivisuusjoukon mitta on vähintään kuution tilavuuden monikerta, ja saman funktion $W^{s,p}$-puolinormi on korkeintaan kuution sivun sopiva potenssi, niin u on positiivinen vakiokertoimella kutistetussa kuutiossa. Tulos tähtää murtoasteisten elliptisten ja parabolisten yhtälöiden säännöllisyysteoriaa koskeviin sovelluksiin. Sopivien väliarvoepäyhtälöiden avulla osoitetaan lisäksi, että avaruuksien $W^{1,p}$ ja BV ryvästymistulokset voidaan johtaa erikoistapauksina.

1. Introduction and main result

Clustering (or local clustering) is a general property shared by the weak solutions of several types of elliptic and parabolic equations, as well as functions in De Giorgi classes. It can basically described as follows. Let u be a function defined in a cube Q_r (with side $r > 0$) and $c > 0$ be a given level, satisfying the following conditions:

(a) the measure of the level set $Q_r \cap \{u > c\}$ is bounded from below by a multiple of the measure of the cube;
(b) the seminorm of u in a certain function space with domain Q_r is bounded from above by a multiple of c times a convenient power of r.

Then, for any $\lambda \in (0, 1)$ the region $\{u > \lambda c\}$ 'clusters' at some point $x_1 \in Q_r$, occupying a cube around x_1, whose side is proportional to r by a constant independent of u.

While condition (a) and the conclusion are basically measure-theoretical properties, not affected by the regularity of the function u, condition (b) may take different forms according to the space where we pick u (which in turn depends on the differential or variational problem considered). Clustering results have been proved for $W^{1,p}(Q_r)$ $(p > 1)$ [13], $W^{1,1}(Q_r)$ [12], and $BV(Q_r)$ [27], each one with a different (b)-type condition, mainly by means of one-dimensional Poincaré inequalities. Some clustering theorems are stated on balls, rather than cubes, but essentially equivalent.

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In this note we consider the fractional Sobolev space $W^{s,p}(Q_r)$ with $s \in (0,1)$, $p \geqslant 1$, thus specializing the (b)-condition by means of the Gagliardo seminorm (see [1, 16, 23] for an introduction to this class of function spaces). Precisely, for any open set $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) we define for all measurable $u\colon \Omega \to \mathbb{R}$ the Gagliardo seminorm

$$[u]_{s,p,\Omega} = \left[\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right]^{\frac{1}{p}}.$$

We say that $u \in W^{s,p}(\Omega)$, if $u \in L^p(\Omega)$ and $[u]_{s,p,\Omega} < \infty$. In order to state our result, for all $x \in \mathbb{R}^N$ and all $r > 0$ let

$$Q_r(x) = \prod_{i=1}^{N} \Big(x^i - \frac{r}{2},\, x^i + \frac{r}{2}\Big)$$

be the N-dimensional open cube centered at x with side r, so $|Q_r(x)| = r^N$ and $\mathrm{diag}(Q_r(x)) = \sqrt{N}r$ (we denote $Q_r = Q_r(0)$, while $|A|$ always stands for the N-dimensional Lebesgue measure of any set $A \subset \mathbb{R}^N$). Our result is the following:

Theorem 1.1. *Let $s \in (0,1)$, $p \geqslant 1$, $x_0 \in \mathbb{R}^N$, $r > 0$ be given, and let $u \in W^{s,p}(Q_r(x_0))$, $c > 0$ satisfy*

(a) $|\{x \in Q_r(x_0)\colon u(x) > c\}| > \alpha r^N$ ($\alpha \in (0,1)$);

(b) $[u]_{s,p,Q_r(x_0)} \leqslant \gamma c r^{\frac{N-ps}{p}}$ ($\gamma > 0$).

Then, for all $\delta, \lambda \in (0,1)$ there exist $x_1 \in Q_r(x_0)$, $\eta \in (0,1)$ (with η independent of u) s.t.

$$|\{x \in Q_{\eta r}(x_1)\colon u(x) > \lambda c\}| > (1-\delta)(\eta r)^N.$$

We display the proof in Section 2.

Clustering results find applications in regularity theory for both elliptic and parabolic PDE's. To be more specific, in regularity theory one of the essential tools is the so-called *critical mass lemma* which states that, under suitable conditions, there exists an absolute constant $\nu \in (0,1)$ s.t. whenever u is a weak solution of an elliptic or parabolic equation, μ is the infimum of u in a ball $B_r \subset \mathbb{R}^N$ with radius $r > 0$, and the measure of the set $\{u > \mu + \varepsilon\}$ is larger than $\nu\omega_N r^N$ (where $\varepsilon > 0$ and $\omega_N > 0$ denotes the volume of the N-dimensional unit ball), then $u \geqslant \mu + \varepsilon/2$ a.e. in $B_{r/2}$. Such information is extremely important, as it marks out a region of positivity, which in turn plays a fundamental role in the proofs of Hölder continuity and Harnack estimates for u. Now the clustering theorem ensures that, provided u has enough regularity, there is some part of the domain where the hypotheses of the critical mass lemma are satisfied. This method was introduced in [13] and then applied to nonlinear PDE's, both in the elliptic [18] and the parabolic case [19]. Also, clustering is an essential tool for proving Harnack's inequality without continuity in anisotropic problems, see [11, p. 370].

The main motivation for our result is related to a reinterpretation of regularity theory for nonlinear, nonlocal equations driven by the s-fractional p-Laplacian, properly defined as the gradient of the functional

$$W^{s,p}(\mathbb{R}^N) \ni u \mapsto \frac{[u]^p_{s,p,\mathbb{R}^N}}{p}.$$

Such operator was introduced in [21] as an approximation of the p-Laplacian (for $s \to 1$) and in [25] as an approximation of the fractional infinity Laplacian (for $p \to \infty$), see also [3] for nonlinear fractional operators arising from game theory.

When $p \geqslant 2$ and u is very smooth, the fractional p-Laplacian admits an alternative, pointwise representation as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{B_\varepsilon^c(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy.$$

Hölder continuity and Harnack's inequality have been established for the fractional p-Laplacian in the elliptic case in [14, 15] through an adapted De Giorgi-Moser-Nash approach (see also [10] for an alternative approach based on fractional De Giorgi classes), and higher Hölder regularity is proved, for instance, in the recent paper [4]. On the parabolic side, regularity results for the evolutive fractional p-Laplace equation have been established in [24, 17, 28, 29].

An application of Theorem 1.1 has already appeared in [8], where local clustering is employed, along with positivity expansion and a fractional critical mass lemma, to give an alternative proof of Hölder regularity for the solutions of the fractional p-Laplace equation. Also, we aim at proving classical Harnack inequalities for the corresponding evolutive equation in the singular case ($1 < p < 2$). A secondary, but hopefully interesting, motivation of the present note is that, in fact, previous clustering theorems in classical Sobolev spaces $W^{1,p}$ (see [13] for $p > 1$, [12] for $p = 1$) and the bounded variation space BV (see [27]) can be deduced from Theorem 1.1, through convenient interpolation and scaling inequalities. So, we design here a unified approach to the problem of clustering. In this connection, we present a very simple proof of known interpolation inequalities between $W^{1,p}$ (resp., BV) and $W^{s,p}$ with a pure seminorm control (see Remark 3.3). Section 3 is devoted to this subject. We mention that a similar clustering result for the linear fractional Laplacian, with a more general kernel, has been proved independently from us in [9].

2. Proof of Theorem 1.1

First we assume that $u \in C(\overline{Q_r(x_0)})$. For any $k \in \mathbb{N}$, we partition $Q_r(x_0)$ into a family $\mathcal{F}_k$ of cubes with side r/k, so that for all $Q \in \mathcal{F}_k$ we have

$$|Q| = \left(\frac{r}{k}\right)^N, \quad \operatorname{diag}(Q) = \frac{\sqrt{N}r}{k} = \sqrt{N}|Q|^{\frac{1}{N}}.$$

Clearly $\#\mathcal{F}_k = k^N$. Let $\alpha \in (0,1)$ be as in (a) and set for all $Q \in \mathcal{F}_k$

$$\begin{cases} Q \in \mathcal{F}_k^+ & \text{if } |Q \cap \{u > c\}| \geqslant \dfrac{\alpha}{2}|Q|, \\ Q \in \mathcal{F}_k^- & \text{otherwise.} \end{cases}$$

By hypothesis (a), for all $Q \in \mathcal{F}_k^+$ we have

$$\begin{aligned} \alpha k^N |Q| &= \alpha |Q_r(x_0)| < |Q_r(x_0) \cap \{u > c\}| \\ &= \sum_{Q\in\mathcal{F}_k^+} |Q \cap \{u > c\}| + \sum_{Q\in\mathcal{F}_k^-} |Q \cap \{u > c\}| \,. \end{aligned}$$

By definition of $\mathcal{F}_k^\pm$, then,

$$\begin{aligned} \alpha k^N &< \sum_{Q\in\mathcal{F}_k^+} \frac{|Q \cap \{u > c\}|}{|Q|} + \sum_{Q\in\mathcal{F}_k^-} \frac{|Q \cap \{u > c\}|}{|Q|} \\ &\leqslant \#\mathcal{F}_k^+ + \frac{\alpha}{2}\#\mathcal{F}_k^- = \frac{\alpha}{2}k^N + \left(1 - \frac{\alpha}{2}\right)\#\mathcal{F}_k^+, \end{aligned}$$

which rephrases as

$$\#\mathcal{F}_k^+ > \frac{\alpha}{2-\alpha}k^N. \tag{2.1}$$

Now fix $\delta, \lambda \in (0,1)$. Two cases may occur:

(i) There exist $k \geqslant 2$, $Q \in \mathcal{F}_k^+$ s.t.

$$|Q \cap \{u > \lambda c\}| > (1-\delta)|Q|.$$

Then, let $x_1 \in Q_r(x_0)$ be the center of Q, $\eta = 1/k \in (0,1)$, and the conclusion follows.

(ii) For all $k \geqslant 2$ and all $Q \in \mathcal{F}_k^+$ we have

$$|Q \cap \{u > \lambda c\}| \leqslant (1-\delta)|Q|,$$

which is equivalent to

$$|Q \cap \{u \leqslant \lambda c\}| \geqslant \delta|Q|. \tag{2.2}$$

Also, since $\lambda \in (0,1)$ and $Q \in \mathcal{F}_k^+$, we have

$$\left|Q \cap \left\{u > \frac{\lambda+1}{2}\,c\right\}\right| \geqslant \frac{\alpha}{2}\,|Q|. \tag{2.3}$$

Fix now $x, y \in Q$ s.t.

$$u(x) \leqslant \lambda c, \quad u(y) > \frac{\lambda+1}{2}\,c,$$

hence

$$u(y) - u(x) > \frac{1-\lambda}{2}\,c.$$

By continuity of u, we have $|x-y| \geqslant \mu$ for some $\mu > 0$ independent of x, y (which makes all of the following integrals nonsingular). Now we start from (2.3) and integrate with respect to y, then we use Hölder's inequality:

$$\begin{aligned}
\frac{\alpha(1-\lambda)}{4}\,c|Q| &\leqslant \frac{1-\lambda}{2}\,c\left|Q \cap \left\{u > \frac{\lambda+1}{2}\,c\right\}\right| \\
&= \int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} \frac{1-\lambda}{2}\,c\,dy \\
&\leqslant \int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} |u(x)-u(y)|\,dy \\
&\leqslant \operatorname{diag}(Q)^{\frac{N+ps}{p}} \int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+ps}{p}}}\,dy \\
&\leqslant N^{\frac{N+ps}{2p}}|Q|^{\frac{N+ps}{Np}}\left[\int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy\right]^{\frac{1}{p}} \\
&\quad \cdot \left|Q \cap \left\{u > \frac{\lambda+1}{2}\,c\right\}\right|^{\frac{p-1}{p}} \\
&\leqslant N^{\frac{N+ps}{2p}}|Q|^{\frac{N+s}{N}}\left[\int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy\right]^{\frac{1}{p}},
\end{aligned}$$

hence for all $x \in Q \cap \{u \leqslant \lambda c\}$ we have

$$\left[\frac{\alpha(1-\lambda)c}{4N^{\frac{N+ps}{2p}}}\right]^p |Q|^{-\frac{ps}{N}} \leqslant \int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy. \tag{2.4}$$

Next we begin with (2.2), use (2.4) and integrate with respect to x:

$$\begin{aligned}\left[\frac{\alpha(1-\lambda)c}{4N^{\frac{N+ps}{2p}}}\right]^p \delta|Q|^{\frac{N-ps}{N}} &\leqslant |Q\cap\{u\leqslant\lambda c\}| \int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy \\ &\leqslant \int_{Q\cap\{u\leqslant\lambda c\}}\int_{Q\cap\{u>\frac{\lambda+1}{2}c\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy\,dx \\ &\leqslant \iint_{Q\times Q} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.\end{aligned}$$

Under our current assumptions, the inequality above holds for all $Q \in \mathcal{F}_k^+$. Further, we sum over $Q \in \mathcal{F}_k^+$ and use (2.1) along with hypothesis (b):

$$\begin{aligned}\frac{\alpha}{2-\alpha}k^N\left[\frac{\alpha(1-\lambda)c}{4N^{\frac{N+ps}{2p}}}\right]^p \delta|Q|^{\frac{N-ps}{N}} &\leqslant \sum_{Q\in\mathcal{F}_k^+}\iint_{Q\times Q} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\leqslant \iint_{Q_r(x_0)\times Q_r(x_0)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\leqslant \gamma^p c^p r^{N-ps}.\end{aligned}$$

Recalling that $|Q| = (r/k)^N$, we get for all $k \geqslant 2$

$$\frac{\alpha^{p+1}(1-\lambda)^p\delta}{4^p(2-\alpha)N^{\frac{N+ps}{2}}}\, r^{N-ps}\, k^{ps} \leqslant \gamma^p r^{N-ps},$$

hence

$$k^{ps} \leqslant \frac{4^p(2-\alpha)N^{\frac{N+ps}{2}}\gamma^p}{\alpha^{p+1}(1-\lambda)^p\delta}.$$

Letting $k \to \infty$ we find a contradiction.

Thus, case (i) must occur for some $k \geqslant 2$, which proves the assertion for continuous functions.

Now let $u \in W^{s,p}(Q_r(x_0))$ be arbitrary. By [16, Theorems 2.4, 5.4] we can find a sequence (u_n) in $C^\infty(\overline{Q_r(x_0)})$ s.t. $u_n \to u$ in $W^{s,p}(Q_r(x_0))$. In particular we have $u_n(x) \to u(x)$ for a.e. $x \in Q_r(x_0)$ and

$$\lim_n [u_n]_{s,p,Q_r(x_0)} = [u]_{s,p,Q_r(x_0)}.$$

Fix an arbitrary $\tilde{\gamma} > \gamma$, then by (b) and the previous convergence we have for all $n \in \mathbb{N}$ big enough

$$[u_n]_{s,p,Q_r(x_0)} < \tilde{\gamma} c r^{\frac{N-ps}{p}}. \tag{2.5}$$

For a.e. $x \in Q_r(x_0) \cap \{u > c\}$ we have

$$\chi_{Q_r(x_0)\cap\{u_n>c\}}(x) \to 1$$

(henceforth, χ_A denotes the characteristic function of any set $A \subset \mathbb{R}^N$). By Fatou's lemma, then,

$$|Q_r(x_0) \cap \{u > c\}| = \int_{Q_r(x_0)\cap\{u>c\}} 1\,dx \leqslant \liminf_n \int_{Q_r(x_0)\cap\{u>c\}} \chi_{Q_r(x_0)\cap\{u_n>c\}}(x)\,dx$$
$$\leqslant \liminf_n \Big|Q_r(x_0) \cap \{u_n > c\}\Big|.$$

By (a), for all $n \in \mathbb{N}$ big enough we have

$$|Q_r(x_0) \cap \{u_n > c\}| > \alpha r^N. \tag{2.6}$$

Fix now $\delta, \lambda \in (0, 1)$, and pick any $\tilde{\delta} \in (0, \delta)$, $\tilde{\lambda} \in (\lambda, 1)$. As in the previous case, for all $n \in \mathbb{N}$ big enough by (2.5) and (2.6) there exist $x_n \in Q_r(x_0)$ and a number $\eta \in (0, 1)$ (independent of n) s.t.

$$\Big|Q_{\eta r}(x_n) \cap \{u_n > \tilde{\lambda}c\}\Big| > (1 - \tilde{\delta})(\eta r)^N.$$

In fact, as seen before $\eta = 1/k$, and $Q_{\eta r}(x_n)$ is one of the fixed k^N cubes of the family $\mathcal{F}_k$. Passing if necessary to a subsequence, we may assume that x_n is the same for all $n \in \mathbb{N}$. Let us denote it x_1, and set $Q = Q_{\eta r}(x_1)$, so for all $n \in \mathbb{N}$

$$\Big|Q \cap \{u_n < \tilde{\lambda}c\}\Big| < \tilde{\delta}(\eta r)^N. \tag{2.7}$$

As above we have $\chi_{Q\cap\{u_n<\tilde{\lambda}c\}} \to 1$ a.e. in $Q \cap \{u < \tilde{\lambda}c\}$, hence by Fatou's lemma and (2.7) we have

$$\Big|Q \cap \{u < \tilde{\lambda}c\}\Big| \leqslant \liminf_n \Big|Q \cap \{u_n < \tilde{\lambda}c\}\Big| \leqslant \tilde{\delta}(\eta r)^N.$$

Reversing the inequality we get

$$\Big|Q \cap \{u \geqslant \tilde{\lambda}c\}\Big| \geqslant (1 - \tilde{\delta})(\eta r)^N,$$

so recalling that $\tilde{\lambda} > \lambda$ and $\tilde{\delta} < \delta$, we have

$$|Q \cap \{u > \lambda c\}| > (1 - \delta)(\eta r)^N,$$

which concludes the proof. □

3. Special cases

In this section we consider two special cases, respectively, $u \in W^{1,p}(Q_r(x_0))$ ($p \geqslant 1$) and $u \in BV(Q_r(x_0))$, previously studied in the literature. Such cases can be reduced to our framework by means of convenient interpolation inequalities (see [5, 26]), for which we present direct proofs.

We begin with the case $W^{1,p}$, setting for all open Ω and all $u \in W^{1,p}(\Omega)$

$$\|\nabla u\|_{p,\Omega} = \left[\int_\Omega |\nabla u(x)|^p\,dx\right]^{\frac{1}{p}}.$$

The following result is equivalent to [13, Proposition A.1] ($p > 1$) and [12] ($p = 1$):

Corollary 3.1. *Let $p \geqslant 1$, $x_0 \in \mathbb{R}^N$, $r > 0$ be given, and let $u \in W^{1,p}(Q_r(x_0))$, $c > 0$ satisfy (a) and*

(b') $\|\nabla u\|_{p,Q_r(x_0)} \leqslant \gamma' c r^{\frac{N-p}{p}}$ *($\gamma' > 0$).*

Then, the conclusion of Theorem 1.1 holds.

Proof. In view of Theorem 1.1, we only need to show that $u \in W^{s,p}(Q_r(x_0))$ satisfies (b), with a possibly different $\gamma > 0$ independent of u. Without loss of generality we may assume $x_0 = 0$.

First we prove that for all $s \in (0,1)$ there exists $C = C(N,s,p) > 0$ s.t. for all $v \in W^{1,p}(Q_1)$

$$[v]_{s,p,Q_1} \leqslant C\|\nabla v\|_{p,Q_1}. \tag{3.1}$$

First assume $v \in C^1(\overline{Q_1})$ and set $\sigma = N + ps - p < N$. Integrating on segments we have

$$\begin{aligned}
&\iint_{Q_1\times Q_1} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\
&= \iint_{Q_1\times Q_1} \left|\int_0^1 \nabla v(x+t(y-x))\cdot(y-x)\,dt\right|^p \frac{dx\,dy}{|x-y|^{N+ps}} \\
&\leqslant \iint_{Q_1\times Q_1} \int_0^1 |\nabla v(x+t(y-x))|^p\ dt \left[\int_0^1 |y-x|^{p'}\,dt\right]^{p-1} \frac{dx\,dy}{|x-y|^{N+ps}} \\
&= \iint_{Q_1\times Q_1} \int_0^1 |\nabla v(x+t(y-x))|^p\ dt\,\frac{dx\,dy}{|x-y|^{\sigma}} \\
&= \int_0^1 \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\nabla v(x+t(y-x))|^p}{|x-y|^{\sigma}}\chi_{Q_1\times Q_1}(x,y)\,dx\,dy\,dt.
\end{aligned}$$

Fix $t \in [0,1]$ and set $z = y - x$, $w = x + t(y-x)$. By convexity of Q_1 we have

$$\begin{cases} x = w - tz \in Q_1 \\ y = w + z - tz \in Q_1 \end{cases} \implies \begin{cases} w = (1-t)(w-tz) + t(w+z-tz) \in Q_1 \\ z = (w+z-tz) - (w-tz) \in Q_2. \end{cases}$$

So for all $t \in [0,1]$ we have

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\nabla v(x+t(y-x))|^p}{|x-y|^{\sigma}}\chi_{Q_1\times Q_1}(x,y)\,dx\,dy \\
&= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\nabla v(w)|^p}{|z|^{\sigma}}\chi_{Q_1\times Q_1}(w-tz,\ w+z-tz)\,dw\,dz \\
&\leqslant \int_{Q_2}\frac{dz}{|z|^{\sigma}} \int_{Q_1} |\nabla v(w)|^p\,dw = C\|\nabla v\|_{p,Q_1}^p,
\end{aligned}$$

with $C > 0$ only depending on N, s, p. We next integrate with respect to t and find

$$\iint_{Q_1\times Q_1} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \leqslant C\|\nabla v\|_{p,Q_1}^p,$$

which proves (3.1). If $v \in W^{1,p}(Q_1)$ is arbitrary, then we can find a sequence (v_n) in $C^\infty(\overline{Q_1})$ s.t. $v_n \to v$ in both $W^{1,p}(Q_1)$ and $W^{s,p}(Q_1)$. For all $n \in \mathbb{N}$ we have

$$[v_n]_{s,p,Q_1} \leqslant C\|\nabla v_n\|_{p,Q_1},$$

so passing to the limit we get (3.1).

Now we recall some useful scaling formulas. Let $u \in W^{1,p}(Q_r)$ be as in the assumption. Setting $v(x) = u(rx)$ for all $x \in Q_1$, we have $v \in W^{1,p}(Q_1)$ and

$$\|\nabla u\|_{p,Q_r} = r^{\frac{N-p}{p}}\|\nabla v\|_{p,Q_1}, \tag{3.2}$$

$$[u]_{s,p,Q_r} = r^{\frac{N-ps}{p}}[v]_{s,p,Q_1}. \tag{3.3}$$

Concatenating (3.3), (3.1), (3.2), and hypothesis (b') we get

$$[u]_{s,p,Q_r} = r^{\frac{N-ps}{p}}[v]_{s,p,Q_1} \leqslant Cr^{\frac{N-ps}{p}}\|\nabla v\|_{p,Q_1} = Cr^{1-s}\|\nabla u\|_{p,Q_r} \leqslant (C\gamma')cr^{\frac{N-ps}{p}}.$$

So u satisfies (b) with $\gamma = C\gamma' > 0$ (independent of u). The conclusion now follows from Theorem 1.1. □

Finally we consider the case BV. We recall that $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and the quantity

$$[u]_{BV(\Omega)} = \sup\left\{\int_\Omega u \operatorname{div}\phi\, dx \colon \phi \in C_c^\infty(\Omega, \mathbb{R}^N),\ |\phi(x)| \leqslant 1 \text{ for all } x \in \Omega\right\}$$

is finite. The following result is equivalent to [27, Lemma 1.1]:

Corollary 3.2. *Let $x_0 \in \mathbb{R}^N$, $r > 0$ be given, and let $u \in BV(Q_r(x_0))$, $c > 0$ satisfy (a) and*

(b") $[u]_{BV(Q_r(x_0))} \leqslant \gamma'' cr^{N-1}$ ($\gamma'' > 0$).

Then, the conclusion of Theorem 1.1 holds.

Proof. As in the previous case, we assume $x_0 = 0$ and fix $s \in (0,1)$. First we see that there exists $C = C(N,s) > 0$ s.t. for all $v \in BV(Q_1)$

$$[v]_{s,1,Q_1} \leqslant C[v]_{BV(Q_1)}. \tag{3.4}$$

Indeed, by classical density results (see for instance [20, Theorem 1.17]), there exists a sequence (v_n) in $C^\infty(\overline{Q_1})$ s.t. $v_n \to v$ in $L^1(Q_1)$ and

$$\lim_n \|\nabla v_n\|_{1,Q_1} = [v]_{BV(Q_1)}.$$

By (3.1) we have for all $n \in \mathbb{N}$ and some $C > 0$ independent of n

$$[v_n]_{s,1,Q_1} \leqslant C\|\nabla v_n\|_{1,Q_1}.$$

Also, up to a subsequence $v_n \to v$ in $W^{s,1}(Q_1)$. So we can pass to the limit as $n \to \infty$ and find (3.4). In particular, then, for all $s \in (0,1)$ we see that $BV(Q_1) \subseteq W^{s,1}(Q_1)$ with continuous embedding.

Let $u \in BV(Q_r)$ be as in the assumption. Setting $v(x) = u(rx)$ for all $x \in Q_1$, we have $v \in L^1(Q_1)$ and the scaling formula

$$[u]_{BV(Q_r)} = r^{N-1}[v]_{BV(Q_1)}. \tag{3.5}$$

Indeed, fix $\phi \in C_c^\infty(Q_r, \mathbb{R}^N)$ s.t. $|\phi| \leqslant 1$ in $\in Q_r$ and set $\psi(y) = \phi(ry)$ for all $y \in Q_1$, then $\psi \in C_c^\infty(Q_1, \mathbb{R}^N)$ and $|\psi| \leqslant 1$ in Q_1. Moreover,

$$\int_{Q_r} u(x) \operatorname{div}\phi(x)\, dx = r^N \int_{Q_1} u(ry) \operatorname{div}\phi(ry)\, dy = r^{N-1} \int_{Q_1} v(y) \operatorname{div}\psi(y)\, dy.$$

Taking the suprema over ϕ, ψ, respectively, we have (3.5). Now, using (3.3) (with $p = 1$), (3.4), (3.5), and hypothesis (b") we have

$$[u]_{s,1,Q_r} = r^{N-s}[v]_{s,1,Q_1} \leqslant Cr^{N-s}[v]_{BV(Q_1)} = Cr^{1-s}[u]_{BV(Q_r)} \leqslant (C\gamma'')cr^{N-s}.$$

Therefore, u satisfies (b) with $p = 1$ and $\gamma = C\gamma'' > 0$ (independent of u). The conclusion now follows from Theorem 1.1. □

Remark 3.3. A brief discussion about inequalities (3.1), (3.4) is in order. Note that in both inequalities we control a seminorm by means of another seminorm, which is a sharper result than usual embedding theorems involving full norms (which incorporate a L^p-norm as well). Inequality (3.1) is essentially contained in the proof of [5, Theorem 1], where it is obtained, for a 'smooth' domain, via a seminorm-preserving extension operator $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$. Extension operators of this type have been detected for connected, Lipschitz domains in [6, 7] (see also [22] for a more general class of Lipschitz domains). Besides, an extension operator for our cubic domain Q_1 can also be obtained by reflection. Nevertheless, we included our proof of (3.1) because it is very simple and *does not* involve any extension procedure, in addition it is easily adapted to any bounded, convex domain. A similar discussion applies to (3.4) (see also [2] for the relation between bounded variation and fractional Sobolev functions in dimension one).

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Fatma Gamze Düzgün
Università degli Studi di Cagliari
Dipartimento di Matematica e Informatica
Via Ospedale 72, 09124 Cagliari, Italy
fatmagamze.duzgun@unica.it

Antonio Iannizzotto
Università degli Studi di Cagliari
Dipartimento di Matematica e Informatica
Via Ospedale 72, 09124 Cagliari, Italy
antonio.iannizzotto@unica.it

Vincenzo Vespri
Università degli Studi di Firenze
Dipartimento di Matematica e Informatica ”U. Dini”
Viale Morgagni 67/A, 50134 Firenze, Italy
vincenzo.vespri@unifi.it