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Logarithmic vanishing theorems on weakly 1-complete Kähler manifolds

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Abstract. This paper aims to investigate the vanishing theorems of cohomology groups on weakly 1-complete Kähler manifolds. Firstly, we extend a logarithmic vanishing theorem originally proposed by Huang, Liu, Wan and Yang, which says that when $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle, the q-th cohomology group of sheaf $\Omega^p(\log D) \otimes L \otimes N$ vanishes for any $p+q \geq n+k+1$ over a compact Kähler manifold X. The proof employs an approximation theorem. In the local case, the result follows directly as a corollary of the original theorem. In the global case, we construct a solution for any positive real number c, ensuring that the error term with respect to the exact solution converges to zero in the sense of norm. Finally, as applications of our results, we obtain logarithmic generalizations of several classical vanishing theorems and the corresponding corollaries.

Heikosti 1-täydellisten Kählerin monistojen logaritmiset häviämislauseet

Tiivistelmä. Työn tavoitteena on tutkia heikosti 1-täydellisten Kählerin monistojen kohomologiaryhmien häviämislauseita. Ensinnäkin yleistetään Huangin, Liun, Wanin ja Yangin alunperin esittämää logaritmista häviämislausetta, jonka sisältö on seuraava: jos $N \otimes \mathcal{O}_X([\Delta])$ on k-positiivinen $\mathbb{R}$-viivakimppu, niin lyhteen $\Omega^p(\log D) \otimes L \otimes N$ kertaluvun q kohomologiaryhmä häviää kompaktilla Kählerin monistolla X aina, kun $p+q \geq n+k+1$. Todistus käyttää likiarvoistuslausetta. Paikallisessa tapauksessa tulos seuraa suoraan alkuperäisestä lauseesta. Koko moniston tapauksessa rakennetaan jokaista reaalilukua c kohti ratkaisu ja varmistetaan, että tarkkaan ratkaisuun verrattu virhe suppenee normin mielessä nollaan. Tulosten sovelluksena saadaan lopuksi logaritmisia yleistyksiä useille klassisille häviämislauseille ja niiden seurauksille.

1. Introduction

The concept of vanishing theorems plays a central role in the realm of algebraic geometry and complex analysis. These theorems, which assert the non-existence of certain cohomology groups under specific conditions, have been instrumental in the advancement of both theoretical understanding and practical applications.

Now we will focus on logarithmic vanishing theorems on weakly 1-complete Kähler manifolds, which is a generalization of logarithmic vanishing theorems on non-compact analytic spaces. Let X be a connected complex manifold of (complex) dimension n. X is called weakly 1-complete if there exists an exhaustion function Φ which is C^∞ and plurisubharmonic on X. For every real number c, we define $X_c := \{x \in X \colon \Phi(x) < c\}$, which will be called the sublevel set of X. Since Shigeo Nakano [11, 12] established a vanishing theorem for positive bundles, there has been considerable research ([4, 9, 13, 15, 16, 17, 19, 20]) concerning the analytic cohomology groups of weakly 1-complete manifolds. The purpose of these works is

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to treat the cohomology groups from differential geometric viewpoint based on the curvature conditions on vector bundles rather than the strong pseudoconvexity of the base manifold X. So they are regarded as natural generalizations of the results obtained for compact manifolds.

The basic properties of the sheaf of logarithmic differential forms and of the sheaves with logarithmic integrable connections on smooth projective manifolds were developed by Deligne [2]. Esnault and Viehweg [5] investigated the relations between logarithmic de Rham complexes and vanishing theorems on complex algebraic manifolds, and showed that many vanishing theorems follow from the degeneration of certain Hodge to de Rham type spectral sequences.

Later on, Nakano [12, 13], Kazama [9], Takegoshi [20], and Ohsawa–Takegoshi [21] continued to develop the relevant theory. In 1978, Norimatsu [14] obtained a logarithmic vanishing theorem on compact Kähler manifolds. Recently, Huang–Liu–Wan–Yang [8] obtained the corresponding results on compact Kähler manifold by the standard analytic technique like the L^2-method. In 2022, Zou [23] established a logarithmic type Akizuki–Nakano vanishing theorem for weakly pseudoconvex (i.e., weakly 1-complete) Kähler manifolds. It is worth noting that Zou's results can be regarded as corollaries of our Theorem 4.7. Specifically, his Theorem 1.1 follows as a special case of our Corollary 5.3 when E is a positive line bundle F (i.e., $r = 1$), and his Theorem 1.2 corresponds to our Corollary 5.2 when $p = n$.

In this paper, we first follow the analytic method provided in Huang–Liu–Wan–Yang to prove the local vanishing theorems for the sheaves of logarithmic differential forms over a fixed sublevel set X_c, which is a relatively compact Kähler manifold of X. Let us consider X_c and $Y_c = X_c - D$ where $D = \sum_{i=1}^{s} D_i$ is a simple normal crossing divisor in X. Suppose that E is an Hermitian vector bundle over X_c.

Next, we derive a uniform estimate to facilitate the proof of the approximation theorem. By applying the L^2-Dolbeault isomorphism theorem, we establish the approximation theorem for logarithmic forms in cohomology, which plays a crucial role in the proof of the global vanishing theorem.

The main difficulties arise from two aspects. The first is the construction of a complete Kähler metric $\widetilde{\omega}_{Y_2}$. The second challenge lies in applying the approximation theorem to transition from a fixed sublevel set X_c to the entire weakly 1-complete Kähler manifold X.

It is well-known that various vanishing theorems are very important in complex analytic geometry and algebraic geometry. For instance, the Akizuki–Kodaira–Nakano vanishing theorem asserts that if L is a positive line bundle over a compact Kähler manifold M, then

$$H^q(M, \Omega_M^p \otimes L) = 0 \quad \forall p + q \geq \dim M + 1.$$

The main objective of this paper is to establish logarithmic Akizuki–Kodaira–Nakano vanishing theorems for the local pair (X_c, D) and the global pair (X, D).

We first generalize the result of Huang–Liu–Wan–Yang to a fixed sublevel set X_c of a weakly 1-complete Kähler manifold X of dimension n. More precisely, we establish the following result.

Theorem 1.1. (Local vanishing) *Let L be any nef holomorphic Hermitian line bundle on a sublevel set X_c of an n dimensional weakly 1-complete Kähler manifold X. Let $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor in X_c. Let N be a line bundle and $\Delta = \sum_{i=1}^{s} a_i D_i$ be an $\mathbb{R}$-divisor with $a_i \in [0, 1]$ such that $N \otimes \mathcal{O}_{X_c}([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. Then for each real number c and on the corresponding* *sublevel set X_c, we have the vanishing of cohomology groups,*

$$H^q(X_c, \Omega^p(\log D) \otimes L \otimes N) = 0 \quad \text{for any} \quad p + q \geq n + k + 1.$$

Next, we prove the approximation theorem (see 4.6) and apply it to extend the local result to the global case, which forms the central theorem of this paper.

Theorem 1.2. (Global vanishing) *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor in X. Let L be any nef holomorphic Hermitian line bundle on X and N be a line bundle such that $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle, where $\Delta = \sum_{i=1}^{s} a_i D_i$ ($a_i \in [0, 1]$) is an $\mathbb{R}$-divisor. Then, we have the vanishing of cohomology groups*

$$H^q(X, \Omega^p(\log D) \otimes L \otimes N) = 0 \quad \text{for any} \quad p + q \geq n + k + 1.$$

The structure of this paper. In Section 2, we introduce key concepts in this paper and construct a complete Kähler metric of Poincaré type for each sublevel set of a weakly 1-complete Kähler manifold, solving the first problem. And then, in Section 3, we apply the method from [8] to derive the local case of our main theorem. In Section 4, we establish the global version of the logarithmic vanishing theorem by taking the limit of a sequence of differential forms. Finally, in Section 5, we obtain logarithmic generalizations of several classical vanishing theorems and the corresponding corollaries.

2. Preliminaries

This section establishes foundational concepts in complex geometry, with particular focus on positivity properties of line bundles and vector bundles on complex manifolds. The key notions to be discussed include: nef line bundles, k-positivity in the sense of Nakano and Griffiths, ampleness characterization via curvature conditions, and analytic structures characterized by plurisubharmonic functions and pseudoconvexity. Special attention will be given to $\mathbb{R}$-divisors, as these play a crucial role in modern vanishing theorem formulations. These concepts provide the theoretical framework for our subsequent proofs in Sections 3 and 4.

2.1. Fundamental concepts.

Definition 2.1. (Nef line bundle [3, (6.11)]) A line bundle L over a compact complex manifold (X, ω) is said to be numerically effective, "nef" for short, if for every $\delta>0$, there is a smooth Hermitian metric h_δ on L such that $\sqrt{-1}\Theta(L, h_\delta) \geqslant -\delta\omega$.

For holomorphic vector bundles, positivity is studied through curvature conditions of Hermitian metrics. Nakano positivity and Griffiths positivity [7] provide two widely used criteria.

Definition 2.2. (Positive (or ample) vector bundle) A holomorphic vector bundle (E, h) is said to be

(1) positive in the sense of Nakano (resp. Nakano semipositive) if for every nonzero tensor $\phi \in TX \otimes E$, we have

$$\Theta_{E,h}(\phi, \phi) > 0 \quad (\text{resp. } \geq 0),$$

(2) positive in the sense of Griffiths (resp. Griffiths semipositive) if for every nonzero decomposable tensor $\xi \otimes e \in TX \otimes E$, we have

$$\Theta_{E,h}(\xi \otimes e, \xi \otimes e) > 0 \quad (\text{resp. } \geq 0).$$

It is clear that Nakano positivity implies Griffiths positivity and that both concepts coincide if $r = 1$. In the case of line bundle, E is merely said to be positive (resp. semipositive).

Indeed, in our proof, the concept of k-positivity is frequently used.

Definition 2.3. (k-Positive [18, Definition 6.31]) Let M be a compact complex manifold and $L \to M$ be a holomorphic line bundle over M.

(1) L is called *k-positive* $(0 \leq k \leq n-1)$ if there exists a smooth Hermitian metric h^L on L such that the curvature form $\sqrt{-1}\Theta\left(L, h^L\right) = -\sqrt{-1}\partial\bar{\partial}\log h^L$ is semipositive everywhere and has at least $n-k$ positive eigenvalues at every point of M.
(2) L is called *k-ample* $(0 \leq k \leq n-1)$, if L is semi-ample and suppose that L^m is globally generated for some $m > 0$, and the maximum dimension of the fiber of the evaluation map $X \to \mathbb{P}\left(H^0\left(M, L^m\right)^*\right)$ is $\leq k$.

Plurisubharmonic (psh) functions, which are defined on complex manifolds, generalize the notion of subharmonic functions to higher dimensions. These functions are characterized by their property that their complex Hessian is semi-positive in the sense of distributions, making them a natural extension of subharmonicity in the context of complex geometry. They play a central role in defining weakly pseudoconvex and strongly pseudoconvex manifolds, depending on whether the exhaustion function is plurisubharmonic or strictly plurisubharmonic.

Definition 2.4. (Plurisubharmonic function) A function $u\colon \Omega \to [-\infty, \infty)$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is called plurisubharmonic (psh, for short) if

(1) u is upper semi-continuous;
(2) for every complex line $L \subset \mathbb{C}^n$, $u|_{\Omega\cap L}$ is subharmonic on $\Omega \cap L$.

We now turn to some basic definitions of pseudoconvex manifolds.

Definition 2.5. (Weakly pseudoconvex = Weakly 1-complete manifolds) A function $\phi\colon X \to [-\infty, +\infty)$ on a manifold X is said to be exhaustive if all sublevel sets

$$X_c := \{x \in X\colon \phi(x) < c\} \quad (c < \sup\phi),$$

are *relatively compact.* A complex manifold X is called *weakly pseudoconvex* if there exists a smooth plurisubharmonic exhaustion function $\phi : X \to \mathbb{R}$ with $\sup\phi = +\infty$. Similarly, a complex manifold X is said *strongly pseudoconvex* if the exhaustion function is smooth strictly plurisubharmonic.

From an algebraic perspective, we define $\mathbb{R}$-divisors and $\mathbb{R}$-line bundles as generalizations of classical divisors and line bundles, where real coefficients are permitted. Particularly, this generalization provides a natural setting for extending the notion of k-positivity to $\mathbb{R}$-line bundles, formulated through curvature conditions on the associated metrics.

Definition 2.6. ($\mathbb{R}$-Divisor) T is called an $\mathbb{R}$-divisor, if it is an element of $\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathrm{Div}(X)$ is the set of divisors in X. Two divisors T_1, T_2 in $\mathrm{Div}_{\mathbb{R}}(X)$ are said to be linearly equivalent, denoted by $T_1 \sim_{\mathbb{R}} T_2$, if their difference $T_1 - T_2$ can be written as a finite sum of principal divisors with real coefficients, that is,

$$T_1 - T_2 = \sum_{i=1}^{k} r_i\left(f_i\right),$$

where $r_i \in \mathbb{R}$ and (f_i) is the principal divisor associated with a meromorphic function f_i (see [6, §5.2.3]).

Definition 2.7. ($\mathbb{R}$-Line bundle) An $\mathbb{R}$-line bundle $L = \sum_i a_i L_i$ is a finite sum with some real numbers $a_1, \cdots, a_k$ and certain line bundles $L_1, \cdots, L_k$. Note that we also use "$\otimes$" for operations on line bundles. An $\mathbb{R}$-line bundle $L = \sum_i c_i L_i$ is said to be *k-positive* if there exist smooth metrics $h_1, \cdots, h_k$ on $L_1, \cdots, L_k$ such that the curvature of the induced metric on L, which is explicitly given by

$$\sqrt{-1}\Theta(L, h) = \sum_{i=1}^{k} c_i \sqrt{-1}\Theta\left(L_i, h_i\right)$$

is k-positive.

Fine sheaf also plays a significant role in this paper.

Definition 2.8. (Fine sheaf) The sheaf $\mathcal{F}$ is called a *fine sheaf* if for any locally finite open covering $\{U_i\}$, there is a family of homomorphisms $\{f_i\}$, $f_i\colon \mathcal{F} \to \mathcal{F}$, such that

(1) $\operatorname{Supp} f_i \subset U_i$,
(2) $\sum_i f_i = 1$, i.e., $\sum_i f_i(s) = s$ for any section s.

2.2. Notations and basic formulae. Let (X, Φ) be a weakly 1-complete Kähler manifold with a fixed Kähler metric ω, where Φ is a plurisubharmonic exhaustion function. Let $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor, i.e. every irreducible component D_i is smooth, and all intersections are transverse.

Without loss of generality, we may assume Φ is positive. For any positive real number c, the sublevel set $X_c = \{x \in X\colon \Phi(x) < c\}$ is relatively compact in X and plurisubharmonic exhaustion with respect to plurisubharmonic exhaustion function $\Phi_c := \frac{1}{c-\Phi}$. Recall that there is a continuous plurisubharmonic exhaustion function in a weakly 1-complete domain. Set $\omega_c := \omega|_{X_c}$, then (X_c, ω_c, Φ_c) is again a weakly 1-complete Kähler manifold, and thus we have a weakly 1-complete sublevel set (X_c, ω_c, Φ_c).

For a fixed positive real number c, we have a sublevel set (X_c, ω_c, Φ_c). We will focus on the sublevel set X_c, which is a *relatively compact Kähler manifold.* Similar to Proposition 2.11, we set

$$\omega_{c,p} := k_c \omega_c - \frac{1}{2} \sum \partial\bar{\partial} \log \log^2 \|\sigma_i\|_i^2$$

for large positive integer k_c which depends on X_c. In a special local coordinate neighborhood U, D_i is defined by the equation $z_i = 0$ and σ_i is given by $\|\sigma_i\|_i^2 = |z_i|^2 e^u$ for some smooth function u defined on U. Then

$$-\frac{1}{2}\partial\bar{\partial} \log \log^2 \|\sigma_i\|_i^2 = \frac{1}{(\log|z_i|^2 + u)^2}\left(\frac{\mathrm{d}z_i}{z_i} + \partial u\right) \wedge \left(\frac{\mathrm{d}\bar{z}_i}{z_i} + \bar{\partial} u\right) - \frac{1}{\log|z_i|^2 + u}\partial\bar{\partial} u.$$

It is clear that $\omega_{c,p}$ is positive on X_c and of Poincaré type along D when k_c is sufficiently large. But $\omega_{c,p}$ is not complete along the boundary of X_c.

In general setting, let $\pi\colon \mathscr{L} \to X$ be a holomorphic line bundle over a complex manifold X and let $\{b_{ij}\}$ be a system of transition functions with respect to a coordinate cover $\{U_i\}_{i\in I}$ with holomorphic coordinates $(z_i^1, \ldots, z_i^n)$. We fix an Hermitian metric $\{a_i\}_{i\in I}$ along the fibers of $\mathscr{L}$ with respect to $\{U_i\}_{i\in I}$ and assume that X is equipped with a Kähler metric ω, denoted by

$$\omega = \sum_{\alpha,\beta=1}^{n} g_{i,\alpha;\bar{\beta}} \, \mathrm{d}z_i^{\alpha} \wedge \mathrm{d}\bar{z}_i^{\beta},$$

where $g_{i,\alpha\bar{\beta}}$ represents the components of the Kähler metric tensor in the local coordinate system $z_i = (z_i^1, \ldots, z_i^n)$.

Let $C^{p,q}(X, \mathscr{L})$ be the space of $\mathscr{L}$-valued smooth differential (p, q)-forms on X and let $C_0^{p,q}(X, \mathscr{L})$ be the space of the forms in $C^{p,q}(X, \mathscr{L})$ with compact supports. We express $\varphi = \{\varphi_i\}_{i\in I} \in C^{p,q}(X, \mathscr{L})$ as

$$\varphi_i = \frac{1}{p!q!} \sum_{\substack{\alpha_1,\cdots,\alpha_p \\ \beta_1,\cdots,\beta_q}} \varphi_{i,\alpha_1,\cdots,\alpha_p;\bar{\beta}_1,\cdots,\bar{\beta}_q} \, \mathrm{d}z_i^{\alpha_1} \wedge \cdots \wedge \mathrm{d}z_i^{\alpha_p} \wedge \mathrm{d}\bar{z}_i^{\beta_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_i^{\beta_q}.$$

With respect to $\{a_i\}_{i\in I}$ and ω, we set

$$\langle \varphi, \psi \rangle = a_i \sum_{A_p, B_q} \varphi_{i,A_p,\bar{B}_q} \psi_i^{A_p,\bar{B}_q}, \tag{2.1}$$

where $A_p = \{\alpha_1, \cdots, \alpha_p\}$ and $B_q = \{\beta_1, \cdots, \beta_q\}$ with $1 \le \alpha_1 < \cdots < \alpha_p \le n$ and $1 \le \beta_1 < \cdots < \beta_q \le n$.

For real valued smooth function Φ on X, we put

$$\begin{cases} \text{i)} \quad \langle \varphi, \psi \rangle_\Phi = \langle \varphi, \psi \rangle e^{-\Phi}, \\ \text{ii)} \quad (\varphi, \psi)_\Phi = \int_X \langle \varphi, \psi \rangle_\Phi \, \mathrm{d}V, \ \text{ for } \varphi, \psi \in C_0^{p,q}(X, \mathscr{L}). \end{cases} \tag{2.2}$$

In particular, we let

$$(\varphi, \psi) = (\varphi, \psi)_0, \quad \|\varphi\|^2 = (\varphi, \varphi), \quad \|\varphi\|_\Phi^2 = (\varphi, \varphi)_\Phi.$$

We now define the space of smooth L^2 integrable $\mathscr{L}$-valued (p, q)-forms with respect to $\| \ \|_\Phi$ denoted by $L^{p,q}(X, \mathscr{L}, h_\Phi^{\mathscr{L}})$. We denote by $\bar{\partial}\colon L^{p,q}(X, \mathscr{L}, h_\Phi^{\mathscr{L}}) \to L^{p,q+1}(X, \mathscr{L}, h_\Phi^{\mathscr{L}})$ the maximal closed extension of the original $\bar{\partial}$. Since $\bar{\partial}$ is a closed densely defined operator, the adjoint operator $\bar{\partial}_\Phi^*$ (resp. $\bar{\partial}^*$) with respect to $(\varphi, \psi)_\Phi$ (resp. (φ, ψ)) can be defined. We denote the domain, the range, and the nullity of $\bar{\partial}$ by $D_{\bar{\partial}}^{p,q}, \operatorname{Im} \bar{\partial}^{p,q}, \ker \bar{\partial}^{p,q}$, respectively. Similarly, $D_{\bar{\partial}_\Phi^*}^{p,q}, \operatorname{Im} \bar{\partial}_\Phi^{*,p,q}, \ker \bar{\partial}_\Phi^{*,p,q}$ are defined.

However, in the definition of the inner product on $L^{p,q}(Y_2, \mathscr{L}, h_\mu^{\mathscr{L}})$, it differs from the above general setting.

Definition 2.9. (Inner product in $L^{p,q}(Y_2, \mathscr{L}, h_\mu^{\mathscr{L}})$) For any non-negative integer μ and any $\varphi, \psi \in L^{p,q}(Y_2, \mathscr{L}, h_\mu^{\mathscr{L}})$, the inner product is defined as

$$(\varphi, \psi)_\mu := \int_{Y_2} \langle \varphi, \psi \rangle_{\tilde{\omega}_{Y_2}} \, h_\mu^{\mathscr{L}} \, \mathrm{d}V = \int_{Y_2} \langle \varphi, \psi \rangle_{\tilde{\omega}_{Y_2}} \, h^{\mathscr{L}} e^{-\mu\psi} \, \mathrm{d}V.$$

The following theorem is required for the proof.

Theorem 2.10. (Sard's theorem) *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a C^k function, where $k \ge \max\{n - m + 1, 1\}$. The set of critical values of f has measure zero in $\mathbb{R}^m$.*

Here, a point $x \in \mathbb{R}^n$ is called a critical point of f if the Jacobian matrix $Df(x)$ is not of full rank. The image of a critical point under f is called a critical value. Sard's theorem states that almost all values in the target space $\mathbb{R}^m$ are regular values, i.e., they are not critical values.

2.3. The construction of a complete Kähler metric on the complement. As shown in [5], for a compact Kähler manifold (M, ω_0) with a simple normal crossing divisor D', there exists a natural inclusion map $\tau\colon U = M \setminus D' \to M$. According to [24, P429,§3], we can choose a *special local coordinate chart* $(W; z_1, \cdots, z_n)$ of M such that the locus of D' is given by $z_1 \cdots z_k = 0$ and $U \cap W = W_r^* = (\Delta_r^*)^l \times (\Delta_r)^{n-l}$ where Δ_r (resp. Δ_r^*) is the (resp. punctured) open disk of radius r in the complex plane. Then we shall define a metric on the product $(\Delta_r^*)^l \times (\Delta_r)^{n-l}$ by

$$\omega_P = i\left(\sum_{j=1}^{l} \frac{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j}{|z_j|^2 \cdot \log^2 |z_j|^2} + \sum_{j=l+1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j\right), \tag{2.3}$$

which possesses the singularity of the Poincaré metric near the punctures (and away from the outer boundaries). Thus, it is obvious that ω_P is not complete along the boundary of U.

Now we construct the complete Kähler metric from ω_0 on U, which is of Poincaré type.

Proposition 2.11. *There exists a Kähler metric on U which in special local coordinate chart is equivalent to the metric* (2.3)*, in the sense that the two norms are mutually uniformly bounded.*

Proof. Let $[D_i']$ be the line bundle on M associated to D_i', σ_i a holomorphic section of $[D_i']$ which vanishes to first order on D_i', and $\|\cdot\|_i$ the norm from a C^∞ Hermitian metric $h^{[D']}$ on $[D_i']$ normalized so that $\|\sigma_i\|_i < 1$ on M. The desired metric is

$$\eta = \left(k\omega_0 - \frac{1}{2}\sum_{i=1}^{N} \partial\bar{\partial} \log\log^2 \|\sigma_i\|_i^2\right)$$

for k sufficiently large. In special local coordinate chart $(U; z_1, \cdots, z_n)$ of M in which D_i' is defined by $z_i = 0$,

$$\|\sigma_i\|_i^2 = |z_i|^2 e^u$$

for some function $u \in C^\infty(U)$. Then

$$\begin{aligned} -\frac{1}{2}\partial\bar{\partial}\log\log^2\|\sigma_i\|_i^2 = {} & \frac{1}{(\log|z_i|^2 + u)^2}\left(\frac{dz_i}{z_i} + \partial u\right) \wedge \left(\frac{d\bar{z}_i}{z_i} + \bar{\partial} u\right) \\ & - \frac{1}{\log|z_i|^2 + u}\,\partial\bar{\partial}u. \end{aligned} \tag{2.4}$$

It is now clear that η is positive near D', with singularities like (2.3); the term $k\omega_0$ is added to make η positive on all of U. It is also evident that the two metrics are equivalent if k is taken sufficiently large. □

Remark 2.12. Two nonnegative functions or Hermitian metrics f and g defined on $(\Delta_r^*)^l \times (\Delta_r)^{n-l}$ are said to be *equivalent* along D' if for any relatively compact subdomain V of U, there is a positive constant C such that $(1/C)g \leq f \leq Cg$ on $V \setminus D'$. In this case we shall use the notation $f \sim g$.

Then by [24, Proposition 3.4], $U = M \setminus D'$, endowed with the metric η, is a *complete manifold of finite volume.*

3. The local vanishing

3.1. L^2-method on complete Kähler manifold. Let us review the classical L^2-methods for the $\bar{\partial}$-equation.

Theorem 3.1. ($\bar{\partial}$-equation on complete Kähler manifolds) [3, Theorem 5.1] *Let (X,ω) be a complete Kähler manifold. Let (E,h^E) be an Hermitian vector bundle of rank r over X, and assume that the curvature operator $B := [i\Theta(E,h^E),\Lambda_\omega]$ is positive definite everywhere on $\bigwedge^{p,q} T_X^*\otimes E$, for some $q\geq 1$. Then for any (p,q)-form $g\in L^2(X,\bigwedge^{p,q}T_X^*\otimes E)$ satisfying $\bar{\partial}g=0$ and $\int_X\langle B^{-1}g,g\rangle dV_\omega<+\infty$, there exists a $(p,q-1)$-form $f\in L^2(X,\bigwedge^{p,q-1}T_X^*\otimes E)$ such that $\bar{\partial}f=g$ and*

$$\int_X |f|^2\,\mathrm{d}V_\omega \leq \int_X \langle B^{-1}g,g\rangle\,\mathrm{d}V_\omega.$$

Now we will follow the approach in [8] to acquire the L^2 resolution. Let (X_c,ω_c) be the fixed sublevel set, we denote the metric of the restriction of line bundle L on $Y_c := X_c\setminus D$ by $h^L_{Y_c}$. The sheaf $\Omega^{p,q}_{(2)}(X_c,L,\omega_{c,p},h^L_{Y_c})$ over X_c is defined as follows. On any open subset U of X_c, the section space $\Gamma(U,\Omega^{p,q}_{(2)}(X_c,L,\omega_{c,p},h^L_{Y_c}))$ over U consists of L-valued (p,q)-forms u with measurable coefficients such that the L^2 norms of both u and $\bar{\partial}u$ are integrable on any compact subset K of U. Here the integrability means that both $|u|^2_{\omega_{c,p}\otimes h^L_{Y_c}}$ and $|\bar{\partial}u|^2_{\omega_{c,p}\otimes h^L_{Y_c}}$ are integrable on $K\setminus D$.

If the metric $\omega_{c,p}$ is of Poincaré type as in Section 2, then it is complete along the divisor D and is of finite volume (cf. [24, Proposition 3.4]). As a consequence, the sheaf $\Omega^{p,q}_{(2)}(X_c,L,\omega_{c,p},h^L_{Y_c})$ is a fine sheaf.

3.2. L^2-Dolbeault isomorphism. Firstly, we will recall the definition of the sheaf of logarithmic differential forms.

Definition 3.2. (Sheaf of logarithmic p-forms) The sheaf of germs of differential p-forms on X with at most logarithmic poles along D denoted by $\Omega^p_X(\log D)$ is a subsheaf of $\Omega^p_X(*D)$. Its space of sections on any open subset V of X are

$$\Gamma(V,\Omega^p_X(\log D)) := \{\alpha\in\Gamma(V,\Omega^p_X\otimes\mathcal{O}_X(D)) \text{ and } \mathrm{d}\alpha\in\Gamma(V,\Omega^{p+1}_X\otimes\mathcal{O}_M(D))\},$$

i.e., α and $\mathrm{d}\alpha$ have poles along D at worst of the first order. Please see [2, §3] [1, Definition 2.1] to learn more.

This is essential for proving the following theorem.

Theorem 3.3. (An L^2-type Dolbeault isomorphism) [8, Theorem 3.1] *Let (X,ω) be a weakly pseudoconvex Kähler manifold of dimension n and $D=\sum_{i=1}^s D_i$ be a simple normal crossing divisor in X. For a fixed real number c, let X_c be the sublevel set. Let $\omega_{c,p}$ be a smooth Kähler metric on Y_c which is of Poincaré type along D as in Section 2. For a line bundle L, there exists a smooth Hermitian metric $h^L_{Y_c,\alpha_c}$ on $L|_{Y_c}$ such that the sheaf $\Omega^p(\log D)\otimes\mathcal{O}(L)$ over X_c enjoys a fine resolution given by the L^2 Dolbeault complex $(\Omega^{p,*}_{(2)}(X_c,L,\omega_{c,p},h^L_{Y_c,\alpha_c}),\bar{\partial})$, here α_c is a large positive constant depends on X_c. This is to say, we have an exact sequence of sheaves over X_c*

$$0\to\Omega^p(\log D)\otimes\mathcal{O}(L)\to\Omega^{p,*}_{(2)}(X_c,L,\omega_{c,p},h^L_{Y_c,\alpha_c}) \tag{3.1}$$

such that $\Omega^{p,q}_{(2)}(X_c,L,\omega_{c,p},h^L_{Y_c,\alpha_c})$ is a fine sheaf for each $0\leq p,q\leq n$. In particular, by Dolbeault isomorphism

$$H^q(X_c,\Omega^p(\log D)\otimes\mathcal{O}(L))\cong H^{p,q}_{(2)}(Y_c,L,\omega_{c,p},h^L_{Y_c,\alpha_c}). \tag{3.2}$$

Even though $\omega_{c,p}$ is not complete, but Y_c admits a complete Kähler metric. Indeed, let $\tilde{\omega}=\hat{\omega}+\omega_{c,p}$, here $\hat{\omega}$ is complete along the boundary of X_c like in Section 2. We know that $\tilde{\omega}$ is complete on Y_c. Hence, we can still solve the certain $\bar{\partial}$-equation on

Y_c thanks to Theorem 3.1. Now we slightly modify Huang–Liu–Wan–Yang's approach in [8] to get the local vanishing.

Theorem 3.4. (Local vanishing theorem) *Let L be any nef holomorphic Hermitian line bundle on a sublevel set X_c of an n dimensional weakly 1-complete Kähler manifold X. Let N be a line bundle and $\Delta = \sum_{i=1}^{s} a_i D_i$ be an $\mathbb{R}$-divisor with $a_i \in [0,1]$ such that $N \otimes \mathcal{O}_{X_c}([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. Then for each real number c and on the corresponding sublevel set X_c, we have the vanishing of cohomology groups,*

$$H^q(X_c, \Omega^p(\log D) \otimes L \otimes N) = 0 \quad \text{for any} \quad p+q \geq n+k+1.$$

Proof. Let ω_c be a fixed Kähler metric on X_c and $F = N \otimes \mathcal{O}_{X_c}([\Delta]), \mathscr{F} = L \otimes N$. Set $\left\{\lambda_{\omega_c}^j\left(h^F\right)\right\}_{j=1}^n$ be the increasing sequence of eigenvalues of $\sqrt{-1}\Theta\left(F, h^F\right)$ with respect to ω_c. Since F is k-positive $\mathbb{R}$-line bundle, there exists a positive constant c_0 such that for any $j \geqslant k+1$, we have

$$\lambda_{\omega_c}^j\left(h^F\right) \geqslant \lambda_{\omega_c}^{k+1}\left(h^F\right) \geqslant \min_{x \in X_c}\left(\lambda_{\omega_c}^{k+1}\left(h^F\right)(x)\right) =: c_0$$

everywhere over X_c. We construct a new metric on $\mathscr{F}|_{Y_c}$ as following,

$$h_{\alpha,\epsilon,\tau}^{\mathscr{F}} := h_\delta^L \cdot h^F \cdot \left(h^\Delta\right)^{-1} \cdot \prod_{i=1}^{s} \|\sigma_i\|_i^{2\tau_i}\left(\log^2\left(\epsilon\|\sigma_i\|_i^2\right)\right)^{\frac{\alpha}{2}}.$$

Here the constant $\alpha > 0$ is chosen to be large enough to meet the condition in Theorem 3.3 and the constants $\tau_i, \epsilon \in (0,1]$ are to be determined later and they are all depend on X_c. On Y_c, a straightforward computation gives rise to

$$\begin{aligned}\sqrt{-1}\Theta\left(\mathscr{F}, h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) &= \sqrt{-1}\Theta\left(F, h^F\right) + \sqrt{-1}\Theta\left(L, h_\delta^L\right) + \sum_{i=1}^{s}(\tau_i - a_i)c_1\left(D_i\right) \\ &\quad + \sum_{i=1}^{s} \frac{\alpha c_1\left(D_i\right)}{\log\left(\epsilon\|\sigma_i\|_i^2\right)} + \sqrt{-1}\sum_{i=1}^{s} \frac{\alpha \partial \log\|\sigma_i\|_i^2 \wedge \bar{\partial}\log\|\sigma_i\|_i^2}{\left(\log\left(\epsilon\|\sigma_i\|_i^2\right)\right)^2}.\end{aligned} \tag{3.3}$$

Set $\delta = \frac{c_0}{32n^2}$, we can choose τ_i and ϵ small enough such that

$$-\frac{\delta}{2}\omega_c \leq \sum_i(\tau_i - a_i)c_1\left(D_i\right) \leq \frac{\delta}{2}\omega_c, \quad -\frac{\delta}{2}\omega_c \leq \sum_i \frac{\alpha c_1\left(D_i\right)}{\log\left(\epsilon\|\sigma_i\|_i^2\right)} \leq \frac{\delta}{2}\omega_c, \tag{3.4}$$

hold on Y_c. Note that the constants τ_i and ϵ are thus fixed. We set

$$\omega_{Y_c} := \sqrt{-1}\Theta\left(\mathscr{F}, h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) + 2(4n+1)\delta\omega_c.$$

It follows from formula (3.3) that ω_{Y_c} is of Poincaré type Kähler form on Y_c. And it is apparent from formula (3.4) that we have

$$\sqrt{-1}\Theta\left(\mathscr{F}, h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) \geq \sqrt{-1}\Theta\left(F, h^F\right) - 2\delta\omega_c \tag{3.5}$$

on Y_c. Since $\sqrt{-1}\Theta\left(F, h^F\right)$ is a semipositive $(1,1)$-form, we know on Y_c

$$\omega_{Y_c} = \sqrt{-1}\Theta\left(\mathscr{F}, h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) + 2(4n+1)\delta\omega_c \geq 8n\delta\omega_c.$$

This implies that

$$\sqrt{-1}\Theta\left(\mathscr{F}, h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) = \omega_{Y_c} - 2(4n+1)\delta\omega_c \geq -\frac{1}{4n}\omega_{Y_c}.$$

On a local chart of Y_c, we may assume that $\omega_c = \sqrt{-1}\sum_{i=1}^{n} \eta_i \wedge \bar{\eta}_i$ and

$$\begin{aligned}\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) &= \sqrt{-1}\sum_{i=1}^{n} \lambda^i_{\omega_c}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)\eta_i \wedge \bar{\eta}_i \\ &= \sqrt{-1}\sum_{i=1}^{n} \frac{\lambda^i_{\omega_c}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)}{\lambda^i_{\omega_c}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + 2(4n+1)\delta}\eta'_i \wedge \bar{\eta}'_i\end{aligned}$$

where

$$\eta'_i = \eta_i\sqrt{\lambda^i_{\omega_c}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + 2(4n+1)\delta}$$

Note that $\omega_{Y_c} = \sqrt{-1}\sum_{i=1}^{n} \eta'_i \wedge \bar{\eta}'_i$, and so the eigenvalues of $\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)$ with respect to ω_{Y_c} are

$$\gamma_i := \frac{\lambda^i_{\omega_c}\left(h_{\alpha,\epsilon,\tau}\right)}{\lambda^i_{\omega_c}\left(h_{\alpha,\epsilon,\tau}\right) + 2(4n+1)\delta} < 1.$$

On the other hand, due to formula (3.5) one has

$$\lambda^j_{\omega_c}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geq \lambda^j_{\omega_c}(h^F) - 2\delta.$$

Hence, for any $j \geq k+1$, we have

$$\lambda^j_{\omega_c}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geq \min_{x\in X_c}\left(\lambda^{k+1}_{\omega_c}\left(h^F\right)(x)\right) - 2\delta = c_0 - 2\delta = \left(1 - \frac{1}{16n^2}\right)c_0 > 0.$$

It implies for $j \geq k+1$,

$$\gamma_j = \frac{\lambda^i_{\omega_c}\left(h_{\alpha,\epsilon,\tau}\right)}{\lambda^i_{\omega_c}\left(h_{\alpha,\epsilon,\tau}\right) + 2(4n+1)\delta} \geq 1 - \frac{1}{4n}.$$

For any section $u \in \Gamma\left(Y_c, \bigwedge^{p,q} T^*Y_c \otimes \mathscr{F}\right)$, we get

$$\begin{aligned}\left\langle\left[\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right), \Lambda_{\omega_{Y_c}}\right]u, u\right\rangle &\geq \left(\sum_{i=1}^{q}\gamma_i - \sum_{j=p+1}^{n}\gamma_j\right)|u|^2 \\ &\geq \left((q+p-n-k) - \frac{q-k}{4n} - \frac{k}{4n}\right)|u|^2 \\ &\geq \frac{1}{2}|u|^2.\end{aligned} \tag{3.6}$$

The last inequality holds because of $p+q \geq n+r$. We know ω_{Y_c} is of Poincaré type metric along D on X_c, and α is large enough, by Thorem 3.3 we have

$$H^q\left(X_c, \Omega^p(\log D) \otimes \mathscr{F}\right) \simeq H^{p,q}_{(2)}\left(Y_c, \mathscr{F}, \omega_{Y_c}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right).$$

The inequality (3.6) and Theorem 3.1 show that the vanishing of $H^{p,q}_{(2)}\big(Y_c, \mathscr{F}, \omega_{Y_c}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\big)$. Hence we get the desired local vanishing theorem. □

4. The global vanishing

Now let us check the corresponding higher direct images. Let $f\colon X \to S$ be a proper surjective morphism from a Kähler manifold X to a reduced and irreducible complex space S. Let $W \subset S$ be any Stein open subset, we put $V = f^{-1}(W)$. Then V is a holomorphically convex Kähler manifold. Let $\mathcal{F}$ be a coherent sheaf on V. Then $f^*\colon H^q(V, \mathcal{F}) \to H^0\left(W, R^q f_*\mathcal{F}\right)$ is an isomorphism of topological vector space for every $q \geq 0$. As a direct corollary of Theorem 3.4, we obtain

Corollary 4.1. *Let $f\colon X \to S$ be a proper holomorphic morphism from a Kähler manifold X of dimension n onto the reduced and irreducible complex space S. Let D be a simple normal crossing divisor for which $f|_D$ is proper. And let N be a line bundle such that $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive line bundle, where $\Delta = \sum_{i=1}^{s} a_i D_i$ ($a_i \in [0,1]$) is a $\mathbb{R}$-divisor., then*

$$R^q f_* \left(\Omega_X^p(\log D) \otimes L \otimes N\right) = 0 \quad \textit{for any} \quad p+q \geq n+k+1$$

As demonstrated previously, the local version of [8, Theorem4.1] on weakly 1-complete manifolds has been established. The remaining question pertains to its validity in the global case.

4.1. Approximation theorem. Let us choose a smooth convex increasing function $\rho(t)$ such that

i) $\rho(t)\colon (-\infty,+\infty) \to (-\infty,+\infty)$,

ii)

$$\rho(t) = \begin{cases} 0, & \text{if } t \leqslant \frac{1}{c_2-c_1}, \\ > 0, & \text{if } t > \frac{1}{c_2-c_1}, \end{cases}$$

iii)

$$\begin{cases} \text{a) } \displaystyle\int_0^{+\infty} \sqrt{\rho_\mu''(t)}\,\mathrm{d}t = +\infty, \text{ for any } \mu \geqslant 1, \\ \text{b) for every } \zeta < c_1 \text{ and any non-negative integer } v, \\ \qquad \displaystyle\lim_{\mu\to+\infty} \sup_{t\in(-\infty,\zeta)} \left|\rho_\mu^{(v)}(t) - \rho^{(v)}(t)\right| = 0, \end{cases} \tag{4.1}$$

where $\rho_\mu^{(v)}$ (resp. $\rho^{(v)}$) denotes the v-th derivative of ρ_μ (resp. ρ).

We set

$$\psi = \rho\left(\frac{1}{c_2 - \Phi}\right),$$

which is a plurisubharmonic exhaustion function on X_2 and $\psi \equiv 0$ on X_1.

The following lemma is very important in our proof. From now on, we will consistently work with the pairs (X_1, X_2) and (Y_1, Y_2).

Lemma 4.2. (Uniform estimate) *Let $\mathscr{F} := L \otimes N = L \otimes F \otimes \mathcal{O}_{X_2}(-[\Delta])$, where $F = N \otimes \mathcal{O}_{X_2}([\Delta])$ is a k-positive line bundle. And let L be any nef line bundle over X_2. There exists a positive constant C which is independent to μ such that for any $\mu \geq 1$ and $p+q \geqslant n+k+1$, we have the estimate*

$$\|\varphi\|_{\rho_\mu}^2 \leq C(\|\bar\partial\varphi\|_{\rho_\mu}^2 + \|\bar\partial_{\rho_\mu}^*\varphi\|_{\rho_\mu}^2)$$

provided $\varphi \in D_{\bar\partial}^{p,q} \bigcap D_{\bar\partial_{\rho_\mu}^*}^{p,q} \subset L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$. Here $D_{\bar\partial}^{p,q}$ is the definition of domain of $\bar\partial$ in $L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$, and $D_{\bar\partial_{\rho_\mu}^*}^{p,q}$ is similar.

Proof. Note that $h_{\alpha,\varepsilon,\tau}^{\mathscr{F}} := h_{\rho;\alpha,\varepsilon,\tau}^{\mathscr{F}}$. Since F is a k-positive $\mathbb{R}$-line bundle, there exist smooth metrics h^N and $h^{[D_i]}$ on N and $[D_i]$ respectively, such that the curvature form of the induced metric h^F on F

$$\sqrt{-1}\Theta(F, h^F) = \sqrt{-1}\Theta(N, h^N) + \sqrt{-1}\sum_{i=1}^{s} a_i \Theta([D_i], h^{[D_i]}) \tag{4.2}$$

is semipositive and has at least $n-k$ positive eigenvalues at each point of X_2.

Let $\{\lambda^j_{\omega_{c_2}}(h^F)\}_{j=1}^n$ be the eigenvalues of $\sqrt{-1}\Theta(F,h^F)$ with respect to ω_{c_2} such that $\lambda^j_{\omega_{c_2}}(h^F)\le\lambda^{j+1}_{\omega_{c_2}}(h^F)$ for all j. Thus, for any $j\ge k+1$ we have

$$\lambda^j_{\omega_{c_2}}(h^F)\ge\lambda^{k+1}_{\omega_{c_2}}(h^F)\ge\min_{x\in X_2}\left(\lambda^{k+1}_{\omega_{c_2}}(h^F)(x)\right)=:c_0>0.$$

Without loss of generality, we assume $\delta\in(0,1)$. Since L is nef, there exists a smooth metric h^L_δ on L such that

$$\sqrt{-1}\Theta(L,h^L_\delta)=-\sqrt{-1}\partial\bar\partial\log h^L_\delta>-\delta\omega_{c_2}. \tag{4.3}$$

Let σ_i be the defining section of D_i. Fix smooth metrics $h_{D_i}:=\|\cdot\|^2_{D_i}$ on line bundles $[D_i]$, such that $\|\sigma_i\|_{D_i}<\frac{1}{2}$. Write the curvature form of $[D_i]$ as $c_1(D_i)=\sqrt{-1}\Theta([D_i],h_{D_i})$. We define $h^\Delta:=\prod_{i=1}^s h^{a_i}_{D_i}$, then the curvature form of (Δ,h^Δ) is

$$-\sqrt{-1}\partial\bar\partial\log h^\Delta=-\sqrt{-1}\partial\bar\partial\log\prod_{i=1}^s h^{a_i}_{D_i}, \tag{4.4}$$

where $h^{a_i}_{D_i}:=\|\sigma_i\|^{2\tau_i}_{D_i}(\log^2(\varepsilon\|\sigma_i\|^2_{D_i}))^{\alpha/2}$. Then let the induced metric of $\mathscr{F}$ be

$$h^{\mathscr{F}}_{\alpha,\varepsilon,\tau}:=h^L\cdot h^F\cdot(h^\Delta)^{-1}\cdot\prod_{i=1}^s\|\sigma_i\|^{2\tau_i}_{D_i}(\log^2(\varepsilon\|\sigma_i\|^2_{D_i}))^{\alpha/2}.$$

Here the constant $\alpha>0$ is chosen to be large enough and the constants $\tau_i,\varepsilon\in(0,1]$ are to be determined later. Note that the smooth metric $h^F\cdot(h^\Delta)^{-1}$ on $N=F\otimes\mathcal{O}_{X_2}(-[\Delta])$ is the same as h^N up to a globally defined function over X_2. Then the corresponding curvature form is

$$\begin{aligned}\sqrt{-1}\Theta(\mathscr{F},h^{\mathscr{F}}_{\alpha,\varepsilon,\tau})&=\sqrt{-1}\Theta(F,h^F)+\sqrt{-1}\Theta(L,h^L_\delta)+\sum_{i=1}^s(\tau_i-a_i)c_1(D_i)\\&\quad+\sum_{i=1}^s\frac{\alpha c_1(D_i)}{\log(\varepsilon\|\sigma_i\|^2_{D_i})}+\sqrt{-1}\sum_{i=1}^s\frac{\alpha\partial\log\|\sigma_i\|^2_{D_i}\wedge\bar\partial\log\|\sigma_i\|^2_{D_i}}{\left(\log(\varepsilon\|\sigma_i\|^2_{D_i})\right)^2}.\end{aligned} \tag{4.5}$$

Since $a_i\in[0,1]$, for a fixed large α, we can choose $\tau_1,\cdots,\tau_s\in(0,1]$ and ε such that τ_i-a_i, ε are small enough and

$$-\frac{\delta}{2}\omega_{c_2}\le\sqrt{-1}\sum_{i=1}^s(\tau_i-a_i)c_1(D_i)\le\frac{\delta}{2}\omega_{c_2},\ -\frac{\delta}{2}\omega_{c_2}\le\sum_{i=1}^s\frac{\alpha c_1(D_i)}{\log(\varepsilon\|\sigma_i\|^2_{D_i})}\le\frac{\delta}{2}\omega_{c_2}. \tag{4.6}$$

Note that the constants τ_i and ε are thus fixed, and the choice of ε depends on α.

By (4.2), (4.3), (4.5) and (4.6), one has on Y_2

$$\sqrt{-1}\Theta\left(\mathscr{F},h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)\ge\sqrt{-1}\Theta(F,h^F)-2\delta\omega_{c_2}. \tag{4.7}$$

By the construction of ψ, we can define a *complete Kähler metric* of Y_2

$$\widetilde\omega_{Y_2}=\sqrt{-1}\Theta(\mathscr{F},h^{\mathscr{F}}_{\alpha,\varepsilon,\tau})+\kappa\delta\omega_{c_2} \tag{4.8}$$

and a new curvature form

$$\sqrt{-1}\Theta_\mu=\sqrt{-1}\Theta(\mathscr{F},h^{\mathscr{F}}_{\alpha,\epsilon,\tau})+\sqrt{-1}\partial\bar\partial\mu\psi, \tag{4.9}$$

where $\kappa>2$ will be determined later.

Since $\widetilde\omega_{Y_2}$ is a complete Kähler metric on Y_2, the classical Bochner–Kodaira–Nakano identity shows

$$\Delta''=\Delta'+[i\Theta_\mu,\Lambda_{\widetilde\omega_{Y_2}}].$$

Let $\varphi \in \mathcal{C}_0^\infty(Y_2, \Lambda^{p,q}T^*Y \otimes \mathscr{F})$ be the set of smooth $\mathscr{F}$-valued (p,q)-forms with compact support over Y_2. We have

$$\|\bar\partial\varphi\|_\mu^2 + \|\bar\partial_\mu^*\varphi\|_\mu^2 \geq \int_{Y_2} \left\langle [i\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi\right\rangle_{\widetilde{\omega}_{Y_2}} h_{\rho_\mu}^{\mathscr{F}} e^{-\mu\psi}\, \mathrm{d}V. \tag{4.10}$$

Since $\sqrt{-1}\Theta(F, h^F)$ is a semipositive (1,1) form, by (4.7) we see that on Y_2

$$\widetilde{\omega}_{Y_2} = \sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) + \kappa\delta\omega_{c_2} \geqslant (\kappa-2)\delta\omega_{c_2} \tag{4.11}$$

and then

$$\delta\omega_{c_2} \leqslant \frac{1}{\kappa-2}\widetilde{\omega}_{Y_2}. \tag{4.12}$$

This implies that

$$\sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) = \widetilde{\omega}_{Y_2} - \kappa\delta\omega_{c_2} \geqslant \frac{-2}{\kappa-2}\widetilde{\omega}_{Y_2}. \tag{4.13}$$

We can diagonalize simultaneously the Hermitian forms $\omega_{c_2}, \widetilde{\omega}_{Y_2}$ and $\sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}), \sqrt{-1}\Theta_\mu$. Without loose of generality, we can choose a suitable chart such that

$$\omega_{c_2} = \sqrt{-1}\sum_{j=1}^n \eta_j \wedge \bar\eta_j \quad \& \quad \widetilde{\omega}_{Y_2} = \sqrt{-1}\sum_{j=1}^n \eta_j' \wedge \bar\eta_j',$$

$$\sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) = \sqrt{-1}\sum_{j=1}^n \lambda_{\omega_{c_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right)\eta_j \wedge \bar\eta_j = \sqrt{-1}\sum_{j=1}^n \lambda_{\widetilde{\omega}_{Y_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right)\eta_j' \wedge \bar\eta_j',$$

$$\sqrt{-1}\Theta_\mu = \sqrt{-1}\sum_{j=1}^n \gamma_j \eta_j' \wedge \bar\eta_j',$$

where

$$\eta_j' = \eta_j \cdot \sqrt{\lambda_{\omega_{c_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) + \kappa\delta}.$$

Then (4.13) will become

$$\sqrt{-1}\Theta(\mathscr{F}, h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) \geqslant \frac{-2}{\kappa-2}\widetilde{\omega}_{Y_2} = \sum_j \frac{-2}{\kappa-2}\eta_j' \wedge \bar\eta_j'.$$

Thus we have $\lambda_{\widetilde{\omega}_{Y_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) \geqslant \dfrac{-2}{\kappa-2}$.

By (4.7) one has

$$\lambda_{\omega_{c_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) \geq \lambda_{\omega_{c_2}}^j(h^F) - 2\delta.$$

Hence, for any $j \geq k+1$, we have

$$\lambda_{\omega_{c_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) \geq \min_{x\in X}\left(\lambda_{\omega_{c_2}}^{k+1}(h^F)(x)\right) - 2\delta = c_0 - 2\delta > 0.$$

Thus, we have

$$\psi_j' = \frac{\psi_j}{\lambda_{\omega_{c_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) + \kappa\delta} < 1.$$

It also implies that for $j \geq k+1$,

$$\gamma_j := \lambda_{\widetilde{\omega}_{Y_2}}^j(h_{\alpha,\varepsilon,\tau}^{\mathscr{F}}) = \frac{\lambda_{\omega_{c_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right)}{\lambda_{\omega_{c_2}}^j\left(h_{\alpha,\epsilon,\tau}^{\mathscr{F}}\right) + \kappa\delta} \geqslant \frac{c_0 - 2\delta}{c_0 - 2\delta + \kappa\delta} = \left(1 - \frac{\kappa\delta}{c_0 + (\kappa-2)\delta}\right).$$

Let $\delta = \dfrac{c_0}{2(\kappa-2)n^2}$. Since $\gamma_j \in \left[\dfrac{-2}{\kappa-2}, 1\right)$, by (4.9) and [3, P334, Prop VI-8.3], we have

$$\begin{aligned}\left\langle[\sqrt{-1}\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi\right\rangle_{\widetilde{\omega}_{Y_2}} &\geqslant \left(\sum_{i=1}^{q}\gamma_i - \sum_{j=p+1}^{n}\gamma_j\right)|\varphi|^2 \\ &\geqslant \left[(q-k)\cdot\left(1-\frac{\kappa\delta}{c_0+(\kappa-2)\delta}\right) + k\cdot\frac{-2}{\kappa-2} - (n-p)\right]|\varphi|^2 \\ &= \left[(p+q-n-k) - \frac{2k}{\kappa-2} - \frac{(q-k)(\kappa\delta)}{c_0+(\kappa-2)\delta}\right]|\varphi|^2 \\ &= \left[(p+q-n-k) - \frac{2k}{\kappa-2} - \frac{(q-k)}{(\kappa-2)}\cdot\frac{\kappa}{(2n^2+1)}\right]|\varphi|^2.\end{aligned}$$

So we can arrange κ sufficiently large such that the operator $\left\langle[\sqrt{-1}\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi\right\rangle_{\widetilde{\omega}_{Y_2}}$ with respect to $\widetilde{\omega}_{Y_2}$ are all positive on the whole Y_2.

Hence, there exists a positive constant C_0 on Y_2, being independent on μ, so that

$$\left\langle[i\Theta_\mu, \Lambda_{\widetilde{\omega}_{Y_2}}]\varphi, \varphi\right\rangle_{\widetilde{\omega}_{Y_2}} \geq C_0|\varphi|^2.$$

Substituting this into formula (4.10) above, we obtain the desired uniform estimate:

$$\|\varphi\|_\mu^2 \leq C(\|\bar\partial\varphi\|_\mu^2 + \|\bar\partial_\mu^*\varphi\|_\mu^2)$$

for any $\varphi \in \mathcal{C}_0^\infty(Y_2, \Lambda^{p,q}T_Y^*\otimes\mathscr{F})$. Since the metric $\widetilde{\omega}_{Y_2}$ is complete, the above estimate still holds when provided $\varphi \in D_{\bar\partial}^{p,q}\bigcap D_{\bar\partial_\mu^*}^{p,q} \subset L^{p,q}(Y_2, \mathscr{F}, h^{\mathscr{F}}_{\rho_\mu;\alpha,\varepsilon,\tau})$. □

Based on the preceding lemma, we now establish the important approximation theorem, whose variant 4.6 will be served as a key role in the proof of the global vanishing theorem.

Proposition 4.3. (Approximation theorem) *If $p+q \geqslant n+k+1$, then for any $f \in L^{p,q}(\overline{Y_1}, \mathscr{F}, h^{\mathscr{F}}_\rho)$ with $\bar\partial f = 0$, then for any $\varepsilon > 0$, there exists an integer μ_0 and $\widetilde{f} \in L^{p,q}(Y_2, \mathscr{F}, h^{\mathscr{F}}_{\rho_{\mu_0}})$ with $\bar\partial\widetilde{f} = 0$ such that*

$$\left\|\widetilde{f}|_{\overline{Y_1}} - f\right\|_\rho^2 < \varepsilon. \tag{4.14}$$

In order to prove this proposition, we first give a useful lemma.

Lemma 4.4. (Riesz representation theorem) *For a continuous linear functional φ on a Hilbert space V, there exists a unique $v \in V$ such that $\varphi(v) = \langle v, w\rangle$ for all $w \in V$. And*

$$\|v\| = \|\varphi\|.$$

Consider the family of functions $\{\rho_\mu\}_{\mu\geqslant 1}$ satisfying (4.1) along with the following additional condition: there exists a constant C such that for every μ and for every $\varphi \in L^{p,q}(Y_2, \mathscr{F}, h^{\mathscr{F}}_{\rho_\mu})$, we have

$$\left\|\varphi|_{\overline{Y_1}}\right\|_\rho \leqslant C\|\varphi\|_{\rho_\mu}. \tag{4.15}$$

The existence of such functions has been proved (for any p, q) in [16], where B was assumed to be positive unnecessarily. We can easily obtain $\{\rho_\mu\}_{\mu\geqq 1}$ with required properties without any significant modifications.

Proof of proposition 4.3. First we have the equivalent proposition:

Proposition 4.5. *If $g \in L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$ and $(g, \widetilde{f}|_{\overline{Y_1}})_\rho = 0$ for any*

$$\widetilde{f} \in \bigcup_{\mu=1}^{\infty} L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$$

with $\bar\partial \widetilde{f} = 0$, then $(g, f)_\rho = 0$ for any $f \in L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$ with $\bar\partial f = 0$.

Now we take the functions $\{\rho_\mu\}_{\mu \geqslant 1}$ satisfying the notions on the beginning of Subsection 4.1.

According to (4.15), for any $v \in L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$ and for every $\mu \geqslant 1$, we have

$$\|v|_{\overline{Y_1}}\|_\rho \leqslant C\|v\|_{\rho_\mu}, \tag{4.16}$$

which introduces that when $v = (g, \cdot|_{\overline{Y_1}})_\rho$, then

$$\|(g, u|_{\overline{Y_1}})_\rho\|_\rho \leqslant C\|(g, u)_{\rho_\mu}\|_{\rho_\mu} \leqslant C\|g\|_\rho \cdot \|u\|_{\rho_\mu} \tag{4.17}$$

for any $u \in L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$.

It implies that $(g, \cdot|_{\overline{Y_1}})_\rho$ is a continuous linear functional on $L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$, by Riesz representation theorem there exists a $g_\mu \in L^{p,q}\left(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}}\right)$ such that

$$(g_\mu, u)_{\rho_\mu} = \left(g, u|_{\overline{Y_1}}\right)_\rho \quad \text{for every} \quad u \in L^{p,q}\left(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}}\right)$$

and $\|g_\mu\|_{\rho_\mu} \leqq C\|g\|_\rho$. Here

$$(g, \cdot|_{\overline{Y_1}})_\rho(u) = (g, u|_{\overline{Y_1}})_\rho = (g_\mu, u)_{\rho_\mu},$$

and by (4.17)

$$\|g_\mu\|_{\rho_\mu} = \|(g, \cdot|_{\overline{Y_1}})_\rho\|_\rho \leqslant C\|g\|_\rho. \tag{4.18}$$

Since for every $\varphi \in C_0^{p,q}\left(Y_2 \setminus \overline{Y_1}, \mathscr{F}\right)$, we have $(g_\mu, \varphi)_{\rho_\mu} = \left(g, \varphi|_{\overline{Y_1}}\right)_\rho = 0 \quad (\varphi|_{\overline{Y_1}} \equiv 0)$, which implies that

$$\operatorname{Supp}(g_\mu, \cdot)_{\rho_\mu} \subseteq \overline{Y_1} \iff \operatorname{Supp}(g_\mu) \subseteq \overline{Y_1}$$

and

$$\left\| g_\mu|_{\overline{Y_1}} \right\|_\rho \leqslant C \cdot \|g_\mu\|_{\rho_\mu} \leqslant C^2\|g\|_\rho. \tag{4.19}$$

According to (4.1) and $\operatorname{Supp}(g_\mu) \subseteq \overline{Y_1}, \mu \geqslant 1$, we have

$$\left(g_\mu|_{\overline{Y_1}}, v\right)_\rho \to (g, v)_\rho, \quad \mu \to +\infty \text{ for every } v \in C_0^{p,q}\left(\overline{Y_1}, \mathscr{F}\right).$$

Therefore, $\left\{g_\mu|_{\overline{Y_1}}\right\}$ converges weakly to g in $L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$.

On the other hand, since $g_\mu \perp N_{\bar\partial}^{p,q}$ in $L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$, then by

$$L^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}}) = \overline{R_{\bar\partial_{\rho_\mu}^*}^{p,q}} \oplus N_{\bar\partial}^{p,q} \tag{4.20}$$

we have $g_\mu \in \overline{R_{\bar\partial_{\rho_\mu}^*}^{p,q}}$, which implies that there exists a $w_\mu \in \mathcal{A}^{p,q}(Y_2, \mathscr{F}, h_{\rho_\mu}^{\mathscr{F}})$ such that

$$g_\mu = \bar\partial_{\rho_\mu}^* w_\mu$$

and from uniform estimate lemma, we shall know that

$$\|w_\mu\|_{\rho_\mu}^2 \leqslant 2\left(\left\|\bar\partial w_\mu\right\|_{\rho_\mu}^2 + \left\|\bar\partial^* w_\mu\right\|_{\rho_\mu}^2\right) = 0 + 2\left\|g_\mu\right\|_{\rho_\mu}^2. \tag{4.21}$$

With (4.16), (4.18) and (4.21), then we have

$$\left\|w_\mu|_{\overline{Y_1}}\right\|_\rho \overset{(4.16)}{\leqslant} C\left\|w_\mu\right|_{\rho_\mu} \overset{(4.21)}{\leqslant} C\cdot\sqrt{2}\left\|g_\mu\right\|_{\rho_\mu} \overset{(4.18)}{\leqslant} C\cdot C\sqrt{2}\|g\|_\rho = \sqrt{2}C^2\|g\|_\rho.$$

Thus, subsequence $\{w_{\mu_k}|_{\overline{Y_1}}\}_{k\geqq 1}$ of $\{w_\mu|_{\overline{Y_1}}\}_{\mu\geqq 1}$ converges weakly to some $w \in L^{p,q+1}\left(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}}\right)$.

For any $v \in C_0^{p,q}\left(\overline{Y_1}, \mathscr{F}\right)$, we have

$$(g,v)_\rho = \lim_{k\to+\infty}(g_{\mu_k}, v)_{\rho_{\mu_k}} \overset{\text{(I)}}{=} \lim_{k\to+\infty}\left(w_{\mu_k}, \bar\partial v\right)_{\rho_{\mu_k}} \overset{\text{(II)}}{=} \lim_{k\to+\infty}\left(w_{\mu_k}, \bar\partial v\right)_\rho \overset{\text{(III)}}{=} (w, \bar\partial v)_\rho.$$

Note that

(I) As $g_{\mu_k} = \bar\partial^*_{\rho_\mu} w_{\mu_k}$, then we have $(\bar\partial^*_{\rho_\mu} w_{\mu_k}, v)_{\rho_\mu} = (w_{\mu_k}, \bar\partial_\rho v)_{\rho_\mu}$,

(II) By (4.1), we have $\lim_{k\to\infty}\rho_{\mu_k} = \lim_{\mu\to\infty}\rho_\mu = \rho$,

(III) $\lim_{k\to\infty} w_{\mu_k} = w$.

Since $\widetilde\omega_{Y_2}$ is a complete metric on $\overline{Y_1} \subset Y_2$, $C_0^{p,q}\left(\overline{Y_1}, \mathscr{F}\right)$ is dense in $D_{\bar\partial}^{p,q} \subset L^{p,q}(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}})$ with respect to the graph norm $\left(\|\varphi\|_\rho^2 + \|\bar\partial\varphi\|_\rho^2\right)^{1/2}$ (see [22, Theorem 1.1]). Thus, $(g,v)_\rho = (w, \bar\partial v)_\rho$ for any $v \in D_{\bar\partial}^{p,q}$, whence $\bar\partial^*_\rho w = g$ in $L^{p,q}\left(Y_2, \mathscr{F}, h_\rho^{\mathscr{F}}\right)$. Therefore, for every $f \in L^{p,q}\left(\overline{Y_1}, \mathscr{F}, h_\rho^{\mathscr{F}}\right)$ with

$$\bar\partial f = 0 \quad \text{and} \quad (g,f)_\rho = \left(\bar\partial^*_\rho w, f\right) = (w, \bar\partial f)_\rho = 0. \qquad \square$$

Using the Dolbeault isomorphism from Theorem 3.3, we have

$$H^q(X_c, \Omega^p(\log D)\otimes\mathscr{F}) \cong H^{p,q}_{(2)}(Y_c, \mathscr{F}, \omega_{c,p}, h^{\mathscr{F}}_{Y_c}) = L^{p,q}(Y_c, \mathscr{F}, h^{\mathscr{F}}_{\rho_c}),$$

which leads to the following approximation theorem:

Theorem 4.6. (Approximation theorem for logarithmic forms in cohomology) *Let $X_1 \subset X_2$ be a pair of sublevel sets. For any holomorphic section $\varphi \in H^q(\overline{X_1}, \Omega^p(\log D)\otimes\mathscr{F})$, there exists a holomorphic section $\widetilde\varphi \in H^q(X_2, \Omega^p(\log D)\otimes\mathscr{F})$ such that for any $\varepsilon > 0$,*

$$\|\widetilde\varphi - \varphi\|_{X_1} < \varepsilon.$$

Theorem 4.7. (Global vanishing theorem) *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divisor in X. Let L be any nef holomorphic Hermitian line bundle on X and N be a line bundle such that $N\otimes\mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle, where $\Delta = \sum_{i=1}^s a_iD_i$ ($a_i \in [0,1]$) is an $\mathbb{R}$-divisor. Then, we have the vanishing of cohomology groups,*

$$H^q(X, \Omega^p(\log D)\otimes L\otimes N) = 0 \quad \textit{for any} \quad p+q \geq n+k+1.$$

Proof. By Sard's theorem 2.10, we can choose a sequence $\{c_v\}_{v=0,1,\dots}$ of real numbers such that

1) $c_{v+1} > c_v > 0$ and $c_v \to +\infty$ as $v \to +\infty$,
2) the boundary ∂X_{c_v} of $\{x \in X; \Phi(x) \leqslant c_v\}$ is smooth for any $v \geqslant 0$.

For any pair (c_{v+2}, c_v) $(v \geqslant 0)$ and ρ^v, we can choose a sequence of C^∞-strictly convex increasing functions $\left\{\rho_\mu^{v+2}\right\}_{\mu\geqslant 1}$ on $(-\infty, c_{v+2})$ satisfying the properties (4.1) and (4.15). The Approximation theorem holds for any pair (c_{v+2}, c_v) $(v \geqslant 0)$. We denote by $L^{p,q}_{\text{loc}}(X, \mathscr{F})$ (resp. $L^{p,q}_{\text{loc}}(X_v, \mathscr{F})$) the set of the locally square integrable (p,q)

forms on X (resp. X_v) with values in $\mathscr{F}$. For $p \geqslant 1$, there is a natural isomorphism

$$(4.22)\qquad H^q(X, \Omega^p(\log D) \otimes \mathscr{F}) \cong \frac{\{f \in L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F}); \bar\partial f = 0\}}{\{f \in L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F}); f = \bar\partial g \text{ for some } g \in L^q_{\text{loc}}(X, \Omega^{p-1}(\log D) \otimes \mathscr{F})\}}.$$

Therefore, in order to prove $H^q(X, \Omega^p(\log D) \otimes \mathscr{F}) = 0$ for $p + q \geqslant n + k$, it suffices to show that for any $\varphi \in L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F})$ with $\bar\partial\varphi = 0$, there exists a $\psi \in L^q_{\text{loc}}(X, \Omega^{p-1}(\log D) \otimes \mathscr{F})$ such that $\varphi = \bar\partial\psi$. We set $\varphi_v = \varphi|_{X_v}$ for any $v \geqslant 0$. Then from Theorem 3.4 (local theorem) and (4.22), there exists a $\psi'_v \in L^{q-1}_{\text{loc}}(X_v, \Omega^p(\log D) \otimes \mathscr{F})$ with $\varphi_v = \bar\partial\psi'_v$ for every $v \geqslant 2$.

For any $v \geqslant 1$, let $L^q(X_v, \Omega^p(\log D) \otimes \mathscr{F})$ be the completion of $C^q_0(X_v, \Omega^p(\log D) \otimes \mathscr{F})$ by the norm $\| \ \|_{X_v}$ with respect to the original Kähler metric ω_v. Inductively, we choose a sequence $\{\psi_v\}_{v \geqslant 1}$ so that

$$(4.23)\qquad \begin{cases} \text{i) } \psi_v \in L^q(X_v, \Omega^p(\log D) \otimes \mathscr{F})), \\ \text{ii) } \bar\partial\psi_v = \varphi_v, \\ \text{iii) } \|\psi_{v+1} - \psi_v\|^2_{X_{v-1}} < \frac{1}{2^v}. \end{cases}$$

First we set $\psi_1 = \psi'_2|_{X_1}$. Since $\varphi_2 = \bar\partial\psi'_2$ in $L^q_{\text{loc}}(X_2, \Omega^p(\log D) \otimes \mathscr{F})$,

$$\psi'_2|_{X_1} \in D^{p,q}_{\bar\partial} \subset L^q(X_1, \Omega^{p-1}(\log D) \otimes \mathscr{F})$$

and so $\bar\partial\psi_1 = \varphi_1$ on X_1. Suppose $\psi_1, \ldots, \psi_{v-1}$ are well chosen. Then

$$(\psi'_{v+1} - \psi_{v-1})|_{X_{v-1}} \in L^q(X_{v-1}, \Omega^{p-1}(\log D) \otimes \mathscr{F}, \rho^{v-1}(\Phi))$$

and $\bar\partial(\psi'_{v+1} - \psi_{v-1})|_{X_{v-1}} = 0$.

By Approximation theorem 4.6, for any $\varepsilon > 0$, there exists a $g \in L^q(X_{v+1}, \Omega^{p-1}(\log D) \otimes \mathscr{F}, \rho^{v+1}_{\mu_0}(\Phi))$ such that

$$\|g|_{X_{v-1}} - (\psi'_{v+1} - \psi_{v-1})|_{X_{v-1}}\|^2_{\rho^{v-1}} < \varepsilon$$

and $\bar\partial g = 0$. Since $\| \ \|_{\rho^{v-1}, X_{v-2}}$ and $\| \ \|_{X_{v-2}}$ are equivalent norms on $L^q(X_{v-2}, \Omega^{p-1}(\log D) \otimes \mathscr{F})$, we may assume

$$\|g|_{X_{v-2}} - (\psi'_{v+1} - \psi_{v-1})|_{X_{v-2}}\|^2_{X_{v-2}} < \frac{1}{2^{v-1}}.$$

We set $\psi_v = (\psi'_{v+1} - g)|_{X_v}$. Then we have

$$\begin{cases} \text{i) } \psi_v \in D^{p-1,q}_{\bar\partial} \subset L^q(X_v, \Omega^{p-1}(\log D) \otimes \mathscr{F}), \\ \text{ii) } \varphi_v = \bar\partial\psi_v, \\ \text{iii) } \|\psi_v - \psi_{v-1}\|^2_{X_{v-2}} > \dfrac{1}{2^{v-1}}. \end{cases}$$

Thus $\{\psi_v\}_{v \geqslant 1}$ has been chosen. From (4.23), for any v, $\{\psi_\mu\}_{\mu \geqslant v+1}$ converges with respect to the norm $\| \cdot \|_{X_v}$ and clearly the limit is the same as the restriction of $\lim_{\mu \geqslant \eta} \psi_\mu$ for any $\eta \geqslant v+2$. Thus, we can define an element $\psi \in L^q_{\text{loc}}(X, \Omega^{p-1}(\log D) \otimes \mathscr{F})$ by $\psi = \lim_{v \to +\infty} \psi_v$. Since $\bar\partial$ is a closed operator in $L^q(X_v, \Omega^{p-1}(\log D) \otimes \mathscr{F})$ for every $v \geqslant 1$, we have

$$\varphi_v = \bar\partial\psi \quad \text{in} \quad L^q(X_v, \Omega^p(\log D) \otimes \mathscr{F}) \quad (v \geqslant 1).$$

Hence we have $\varphi = \bar\partial\psi$ in $L^q_{\text{loc}}(X, \Omega^p(\log D) \otimes \mathscr{F})$. □

5. Logarithmic generalizations of several classical vanishing theorems

In this section, we will present several straightforward applications of Theorem 4.7 over weakly 1-complete Kähler manifolds, which are also closely related to a number of classical vanishing theorems in algebraic geometry. The first application is the following Log-type Girbau's vanishing theorem.

Theorem 5.1. (Log-type Girbau's vanishing theorem) *Let X be a weakly 1-complete Kähler manifold of dimension n and D be a simple normal crossing divisor in X. If L is a nef line bundle and N is a k-positive line bundle over X, then*

$$H^q\left(X, \Omega_X^p(\log D) \otimes L \otimes N\right) = 0 \quad \text{for any} \;\; p+q \geq n+k+1.$$

In particular, we have the following simple version.

Corollary 5.2. *Let X be a weakly 1-complete Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $L \to X$ is an ample line bundle. Then*

$$H^q\left(X, \Omega_X^p(\log D) \otimes L\right) = 0 \quad \text{for any} \;\; p+q \geq n+1.$$

The similar well-known result on compact Kähler manifolds is proved by Norimatsu [14] using analytic methods (see also Deligne–Illusie's proof [2] by the characteristic p methods). As an analogue to Corollary 5.2, we obtain the following Log-type Le Potier vanishing theorem for ample vector bundles.

Theorem 5.3. (Log-type Le Potier vanishing theorem) *Let X be a weakly 1-complete Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $E \to X$ is an ample vector bundle of rank r. Then,*

$$H^q\left(X, \Omega_X^p(\log D) \otimes E\right) = 0 \quad \text{for any} \;\; p+q \geq n+r.$$

Proof. Let $\pi\colon \mathbb{P}(E^*) \to X$ be the projective bundle of E and $\mathcal{O}_E(1)$ be the tautological line bundle. One can check that π^*D is also a simple normal crossing divisor, and for $p \geq 0$, one has

$$\pi_*\left(\Omega^p_{\mathbb{P}(E^*)}(\log \pi^*D) \otimes \mathcal{O}_{\mathbb{P}(E^*)}(1)\right) = \Omega_X^p(\log D) \otimes \mathcal{O}_X(E)$$

and

$$R^q\pi_*\left(\Omega^p_{\mathbb{P}(E^*)}(\log \pi^*D) \otimes \mathcal{O}_{\mathbb{P}(E^*)}(m)\right) = 0, \quad q>0,\; m \geq 1.$$

From [18, Lemma 5.28], we have

$$H^q\left(X, \Omega_X^p(\log D) \otimes E\right) \cong H^q\left(\mathbb{P}(E^*), \Omega^p_{\mathbb{P}(E^*)}(\log \pi^*D) \otimes \mathcal{O}_E(1)\right).$$

Hence, Corollary 5.3 follows from Corollary 5.2. □

Then by using the same strategy as in the proof of Theorem 4.7, we also obtain directly several Log-type Nakano vanishing theorems for vector bundles on weakly 1-complete Kähler manifold. For instance, we have the following proposition.

Proposition 5.4. *Let E be a vector bundle of* rank r *and L be a line bundle on weakly 1-complete Kähler manifold X.*

(1) *If E is Nakano positive (resp. Nakano semi-positive) and L is nef (resp. ample), then for any $q \geq 1$*

$$H^q\left(X, \Omega_X^n(\log D) \otimes E \otimes L\right) = 0.$$

(2) *If E is dual-Nakano positive (resp. dual-Nakano semi-positive) and L is nef (resp. ample), then for any $p \geq 1$,*

$$H^n\left(X, \Omega_X^p(\log D) \otimes E \otimes L\right) = 0.$$

(3) *If E is globally generated and L is ample, then for any $p \geq 1$,*

$$H^n\left(X, \Omega_X^p(\log D) \otimes E \otimes L\right) = 0.$$

Indeed, the vector bundle $E \otimes L$ in Proposition 5.4 is either *Nakano positive* or *dual Nakano positive* (e.g., [10]). Hence, the proof is very similar to (but simpler than) that in Theorem 4.7.

As it is well known that the Kawamata–Viehweg-type vanishing theorems have played fundamental roles in algebraic geometry and complex analytic geometry. Another corollary of Theorem 4.7 is a Log-type vanishing theorem for k-positive line bundles over weakly 1-complete Kähler manifolds, which generalizes a version of the Kawamata–Viehweg vanishing theorem over projective manifolds.

Theorem 5.5. (Generalized Log-type Kawamata–Viehweg vanishing theorem) *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divisor. Suppose F is a line bundle over X and m is a positive real number such that $mF = L + D'$, where $D' = \sum_{i=1}^s v_i D_i$ is an effective normal crossing $\mathbb{R}$-divisor and L is a k-positive $\mathbb{R}$-line bundle. Then,*

$$H^q\left(X, \Omega^p(\log D) \otimes F \otimes \mathcal{O}\left(-\sum_{i=1}^s \left(1 + \left[\frac{v_i}{m}\right]\right) D_i\right)\right) = 0 \tag{5.1}$$

for $p + q \geq n + k + 1$.

Proof. Let

$$N = F \otimes \mathcal{O}_X\left(-\sum_{i=1}^s \left(1 + \left[\frac{v_i}{m}\right]\right) D_i\right)$$

and

$$\Delta = \sum_i \left(1 + \left[\frac{v_i}{m}\right] - \frac{v_i}{m}\right) D_i.$$

We have that

$$N \otimes \mathcal{O}_X([\Delta]) = \frac{1}{m} L, \tag{5.2}$$

which is a k-positive $\mathbb{R}$-line bundle. Hence, we can apply Theorem 4.7 to complete the proof. □

There are several corollaries of this theorem.

Corollary 5.6. *Let X be a weakly 1-complete Kähler manifold $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let $[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where $D' = \sum_{i=1}^s c_i D_i$ with $c_i > 0$ and $c_i \in \mathbb{R}$. Then,*

$$H^q\left(X, \Omega^p(\log D) \otimes \mathcal{O}_X\left(-[D']\right)\right) = 0 \quad \text{for any} \;\; p + q < n - k.$$

In particular, when $[D']$ is ample,

$$H^q\left(X, \Omega^p(\log D) \otimes \mathcal{O}_X\left(-[D']\right)\right) = 0 \quad \text{for} \;\; p + q < n.$$

Proof. Let

$$N = \mathcal{O}_X(-D) \otimes [D'] \quad \text{and} \quad \Delta = \sum_i \left(1 + c_i - [c_i]\right) D_i.$$

It is easy to see that

$$N \otimes \mathcal{O}_X([\Delta]) = [D'] , \tag{5.3}$$

which is a k-positive $\mathbb{R}$-line bundle. By using Theorem 4.7, one has

$$H^q\left(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-D) \otimes [D']\right) = 0 \tag{5.4}$$

for any $p+q \geq n+k+1$. By Serre duality and the isomorphism

$$(\Omega_X^p(\log D))^* \cong \Omega_X^{n-p}(\log D) \otimes \mathcal{O}_X\left(-K_X - D\right) , \tag{5.5}$$

we see that (5.4) is equivalent to

$$H^q\left(X, \Omega^p(\log D) \otimes \mathcal{O}_X\left(-[D']\right)\right) = 0$$

for any $p+q < n-k$. □

Corollary 5.7. *Let X be a weakly 1-complete Kähler manifold and $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let $[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where $D' = \sum_{i=1}^s a_i D_i$ with $a_i > 0$ and $a_i \in \mathbb{R}$. If there exists a line bundle L over X and a real number b with $0 < a_j < b$ for all j and $bL = [D']$ as $\mathbb{R}$-line bundles. Then,*

$$H^q\left(X, \Omega^p(\log D) \otimes L^{-1}\right) = 0$$

for $p+q > n+k$ and $p+q < n-k$.

Proof. Let b' be a real number such that $\max_j a_j < b' < b$, and set

$$N = L^{-1}, \quad \Delta = \frac{D'}{b'} = \sum_{j=1}^s \frac{a_j}{b'} D_j .$$

Let

$$F = L^{-1} \otimes \mathcal{O}_X([D]) = L^{-1} + \frac{D'}{b'} = \frac{b-b'}{bb'} D' .$$

It is easy to see that F is a k-positive $\mathbb{R}$-line bundle and the coefficients of Δ are in $[0,1]$. By Theorem 4.7, we obtain

$$H^q\left(X, \Omega^p(\log D) \otimes L^{-1}\right) = 0 \quad \text{for} \quad p+q > n+k .$$

On the other hand, we can set

$$N = L \otimes \mathcal{O}_X(-D), \quad \Delta = \sum_{j=1}^s \left(1 - \frac{a_j}{2b}\right) D_j \quad \text{and} \quad F = N \otimes \mathcal{O}_X([D]) = \frac{D'}{2b} .$$

It is easy to see that F is a k-positive $\mathbb{R}$-line bundle and the coefficients of Δ are in $[0,1]$. By Theorem 4.7 again, we get

$$H^q\left(X, \Omega^p(\log D) \otimes L \otimes \mathcal{O}_X(-D)\right) = 0, \quad \text{for} \quad p+q > n+k$$

By Serre duality and the isomorphism (5.5), we have

$$H^q\left(X, \Omega^p(\log D) \otimes L^{-1}\right) = 0$$

for any $p+q < n-k$. □

Corollary 5.8. *Let X be a weakly 1-complete Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divisor in X. Suppose there exist some real constants $a_i \geq 0$ such that $\sum_{i=1}^s a_i D_i$ is a k-positive $\mathbb{R}$-divisor, then for any nef line bundle L, we have*

$$H^q\left(X, \Omega_X^p(\log D) \otimes L\right) = 0 \quad \textit{for any} \quad p+q \geq n+k+1 .$$

Proof. We can set $N = \mathcal{O}_X$ and $\Delta = \frac{1}{1+\sum_{i=1}^{s} a_i} \sum_{i=1}^{s} a_i D_i$. Then $N \otimes \mathcal{O}([\Delta]) = \mathcal{O}([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. □

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