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On the sharp second order Caffarelli–Kohn–Nirenberg inequality

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Abstract. The purpose of this paper is twofold. By developing a new technique, we first prove some sharp Caffarelli–Kohn–Nirenberg inequalities of second order for non-radial functions. As a direct consequence, we recover the sharp second order uncertainty principle in Cazacu, Flynn and Lam (2022). For the class of radial functions, we next establish a sharp inequality which can be regarded as the second order version of Xia's inequality (2007).

Tarkasta toisen asteen Caffarellin–Kohnin–Nirenbergin epäyhtälöstä

Tiivistelmä. Tällä työllä on kaksi tavoitetta. Ensinnäkin kehitetään uusi menetelmä, jolla todistetaan eräitä tarkkoja toisen asteen Caffarellin–Kohnin–Nirenbergin epäyhtälöitä ei-säteittäisille funktioille. Tämän suorana seurauksena saadaan Cazacun, Flynnin ja Lamin (2022) tarkka toisen asteen epätarkkuusperiaate. Säteittäisille funktioille todistetaan vielä tarkka epäyhtälö, jota voidaan pitää Xian epäyhtälön (2007) toisen asteen muunnelmana.

1. Introduction

The Heisenberg uncertainty principle in quantum mechanics states that the position and the momentum of a given particle cannot both be determined exactly at the same time (see [19]). The rigorous mathematical formulation of this principle is established by Kennard [21] and Weyl [29] (who attributed it to Pauli) stating that the function itself and its Fourier transform cannot be sharply localized at the origin simultaneously.

Mathematically, the Heisenberg–Pauli–Weyl uncertainty principle is described by the following inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2\,dx \int_{\mathbb{R}^n} |x|^2|u|^2\,dx \ge \frac{n^2}{4}\left(\int_{\mathbb{R}^n} |u|^2\,dx\right)^2 \tag{1.1}$$

for any function $u \in H^1(\mathbb{R}^n)$ (the first order Sobolev space in $\mathbb{R}^n$) such that

$$\int_{\mathbb{R}^n} |x|^2|u|^2\,dx < \infty.$$

It is well known that the constant $n^2/4$ is sharp and is attained only by Gaussian functions (see [17]).

In [30], Xia extends the inequality (1.1) and obtains the following inequality

$$\frac{n-t\gamma}{t}\int_{\mathbb{R}^n} \frac{|u|^t}{|x|^{t\gamma}}\,dx \le \left(\int_{\mathbb{R}^n} \frac{|\nabla u|^p}{|x|^{\alpha p}}\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \frac{|u|^{\frac{p(t-1)}{p-1}}}{|x|^{\beta}}\,dx\right)^{\frac{p-1}{p}}, \tag{1.2}$$

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where $n \geq 2$, $1 < p < t$, and α, β, γ satisfy the following conditions

$$n - \alpha p > 0, \quad n - \beta > 0, \quad n - t\gamma > 0,$$

and the balance condition

$$\gamma = \frac{1+\alpha}{t} + \frac{\beta}{tp}.$$

Moreover, when $1 + \alpha - \frac{\beta}{r} > 0$ and

$$n - \beta < \left(1 + \alpha - \frac{\beta}{r}\right) \frac{p(t-1)}{t-p}$$

then the inequality (1.2) is sharp and the extremal functions are given by

$$u(x) = \left(\lambda + |x|^{1+\alpha-\frac{\beta}{r}}\right)^{\frac{1-p}{t-p}}, \quad \lambda > 0.$$

The inequality (1.2) is extended to the Riemannian manifolds in [26], the Finsler manifolds in [20] and the stratified Lie groups in [25].

Both the Heisenberg–Pauli–Weyl principle (1.1) and the Xia inequality (1.2) belong to a larger class of the first order interpolation inequalities which are called Caffarelli–Kohn–Nirenberg (CKN) inequalities established in [4] to study the Navier–Stokes equation and the regularity of particular solutions [3]. The class of CKN inequalities contains many well-known inequalities such as the Sobolev inequality, the Hardy inequality, the Hardy–Sobolev inequality, the Gagliardo–Nirenberg inequality, etc. They play an important role in theory of partial differential equations and have been extensively studied in many settings. Concerning to the sharp version of CKN inequalities, we refer the reader to the papers [28, 1, 22, 12, 13, 9, 6, 5, 10, 14, 2].

The higher order CKN inequalities were established by Lin [23]. In contrast to the first order inequalities, much less is known on the sharp version of the higher order CKN inequalities except the Rellich inequality [27] and the sharp higher order Sobolev inequality [22, 11]. In recent paper [7], the authors have proved the following sharp second order uncertainty principle which is a special case of the second order CKN inequalities

$$\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \int_{\mathbb{R}^n} |x|^2 |\nabla u|^2 \, dx \geq \left(\frac{n+2}{2}\right)^2 \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx\right)^2. \tag{1.3}$$

The constant $(n+2)^2/4$ above is sharp and is attained by Gaussian functions. In fact, the inequality (1.3) without the sharp constant can be derived from (1.1) and Cauchy–Schwartz inequality (see the comment in the introduction in [7]). The inequality (1.3) is motivated by an open question of Maz'ya [24] on finding the sharp constant in (1.1) when we replace u by a divergence-free vector field U. In particular, the inequality (1.3) answers affirmatively the question of Maz'ya in the case $n = 2$. In higher dimension, this open question has been solved recently by Hamamoto [18].

The aim in this paper is to extend the inequality (1.3) to a larger class of parameters in spirit of (1.2). For α, β satisfying the conditions $n - 2\alpha > 0$ and $n - \beta > 0$, we denote by $H^2_{\alpha,\beta}(\mathbb{R}^n)$ the second order Sobolev space which is completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{H^2_{\alpha,\beta}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha} \, dx + \int_{\mathbb{R}^n} |\nabla u|^2 |x|^{-\beta} \, dx\right)^{\frac{1}{2}}, \quad u \in C_0^\infty(\mathbb{R}^n).$$

The first main result in this paper reads as follows.

Theorem 1.1. *Let $n \geq 1$ and $\alpha \in \mathbb{R}$ satisfy $n - 2\alpha > 0$, $n + 2\alpha > 0$ and $n + 2 + 4\alpha > 0$. Then the following inequality*

$$\int_{\mathbb{R}^n} \frac{|\Delta u|^2}{|x|^{2\alpha}} \, dx \int_{\mathbb{R}^n} |x|^{2\alpha} |\nabla u \cdot x|^2 \, dx \geq \left(\frac{n + 4\alpha + 2}{2}\right)^2 \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx\right)^2 \tag{1.4}$$

holds true for any function $u \in H^2_{\alpha,-2-2\alpha}(\mathbb{R}^n)$. Furthermore, if $1 + \alpha > 0$, then the inequality (1.4) *is sharp and is attained by function*

$$U_0(x) = \exp\left(-\frac{|x|^{2(1+\alpha)}}{2(1+\alpha)}\right).$$

In particular, when $\alpha = 0$, Theorem 1.1 implies the following.

Corollary 1.2. *Let $n \geq 1$. Then there holds*

$$\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \int_{\mathbb{R}^n} |\nabla u \cdot x|^2 \, dx \geq \left(\frac{n+2}{2}\right)^2 \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx\right)^2 \tag{1.5}$$

for any $u \in H^2_{0,-2}(\mathbb{R}^n)$. This inequality is sharp and is attained by the Gaussian function $\exp(-|x|^2/2)$.

Evidently, we have $|\nabla u \cdot x| \leq |\nabla u||x|$. Hence, the sharp second order uncertainty principle (1.3) obtained in [7] is a direct consequence of (1.5). Note that Theorem 1.1 is one part of our manuscript [15]. The proof of Theorem 1.1 provides a preparation for a stability inequality for the second order uncertainty principle in [16]. Later, a different proof of Theorem 1.1 was given by Cazacu, Flynn and Lam [8].

For radial functions, we obtain the following inequality which can be seen as the second order version of Xia inequality.

Theorem 1.3. *Let $n \geq 1$, $t \geq 2$ and α, β, γ be such that*

$$n - 2\alpha > 0, \quad n - \beta > 0, \quad n - t\gamma > 0 \tag{1.6}$$

and

$$\gamma = \frac{1+\alpha}{t} + \frac{\beta}{2t}. \tag{1.7}$$

Then the following inequality

$$\int_{\mathbb{R}^n} \frac{|\Delta u|^2}{|x|^{2\alpha}} \, dx \int_{\mathbb{R}^n} \frac{|\nabla u|^{2(t-1)}}{|x|^{\beta}} \, dx \geq \left(\frac{n + t(1 + 2\alpha - \gamma)}{t}\right)^2 \left(\int_{\mathbb{R}^n} \frac{|\nabla u|^t}{|x|^{t\gamma}} \, dx\right)^2 \tag{1.8}$$

holds true for any radial function $u \in H^2_{\alpha,\beta}(\mathbb{R}^n)$. Moreover, under the following conditions

$$(1 + 2\alpha)(t - 2) + 1 + \alpha - \frac{\beta}{2} > 0, \quad t < 3 + \alpha - \frac{\beta}{2}, \tag{1.9}$$

$$n + 2\alpha > 0, \tag{1.10}$$

and

$$n - \beta < \frac{2(t-1)}{t-2}\left(1 + \alpha - \frac{\beta}{2}\right), \tag{1.11}$$

then the constant $(n + t(1 + 2\alpha - \gamma))^2/t^2$ is sharp and is attained only up to a dilation and a multiplicative constant by the function the form

$$U_1(x) = \int_{|x|}^{\infty} s^{1+2\alpha} \exp\left(-\frac{s^{1+\alpha-\frac{\beta}{2}}}{1 + \alpha - \frac{\beta}{2}}\right) ds,$$

if $t = 2$, *and*

$$U_2(x) = \int_{|x|}^{\infty} r^{1+2\alpha} \left(1 + (t-2)\frac{r^{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}}{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}\right)^{\frac{1}{2-t}} dr,$$

if $t > 2$.

In the special case where $\alpha = \beta = 0$, $t = 2$ and $\gamma = 1/2$, we recover the second result in [7] for radial functions in $H^2_{0,0}(\mathbb{R}^n)$ which is the sharp second order Hydrogen uncertainty principle

$$\int_{\mathbb{R}^n} |\Delta u|^2\, dx \int_{\mathbb{R}^n} |\nabla u|^2\, dx \geq \frac{(n+1)^2}{4}\left(\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|}\, dx\right)^2. \tag{1.12}$$

In fact, in that paper, the authors proved the inequality (1.12) for any function u (without radiality assumption) for any $n \geq 5$. The condition $n \geq 5$ appears due to the technical restrictions of decomposing a smooth function into spherical harmonics. They conjectured that the inequality (1.12) still holds for $n = 2, 3, 4$. The inequality (1.12) is sharp and an extremal function is given by $u(x) = c(1 + a|x|)e^{-a|x|}$ with $c \in \mathbb{R}$ and $a > 0$.

In the general case of parameters, comparing (1.8) with the first order inequality (1.2), the sharp constant changes from $((n - \gamma t)/t)^2$ to $((n + t(1 + 2\alpha - \gamma))/t)^2$. Moreover, to obtain the sharpness and the attainability of constant, we need some more conditions on the parameters (see (1.9) and (1.10)). Indeed, these conditions ensure that the function U_1 (and U_2) is well-defined, and ΔU_1 (and ΔU_2) exists in the distributional sense and belongs to $H^2_{\alpha,\beta}(\mathbb{R}^n)$.

Let us finish the introduction by giving the idea of proof. Notice that the proof of (1.3) in [7] is quite long and complicated. The authors have used the decomposition of function u into spherical harmonic and integral estimates for radial functions. Our approach to prove Theorem 1.1 is new and is completely different with the one in [7]. Indeed, the key step in our proof is to establish a new identity

$$\int_{\mathbb{R}^n} \frac{|\Delta u + \nabla u \cdot x|x|^{2\alpha}|^2}{|x|^{2\alpha}}\, dx = \int_{\mathbb{R}^n} \frac{|\Delta u|^2}{|x|^{2\alpha}}\, dx + \int_{\mathbb{R}^n} |x|^{2\alpha} |\nabla u \cdot x|^2 dx - (n-2)\int_{\mathbb{R}^n} |\nabla u|^2\, dx.$$

Then, using a factorization of u as $u = vU_0$, we are able to show that

$$\int_{\mathbb{R}^n} \frac{|\Delta u + \nabla u \cdot x|x|^{2\alpha}|^2}{|x|^{2\alpha}}\, dx \geq 2(n+2\alpha)\int_{\mathbb{R}^n} |\nabla u|^2\, dx.$$

Combining two estimates above and a simple minimizing argument, we obtain (1.4).

The proof of (1.8) in Theorem 1.3 is elementary. Nevertheless, the construction of extremal functions $U_1(x)$ and $U_2(x)$ in Theorem 1.3 is nontrivial and quite interesting.

The rest of the paper is devoted to the proof of Theorem 1.1 and Theorem 1.3.

2. Proof of main results

In this section, we provide the proof of the second order CKN inequalities in Theorems 1.1 and 1.3. We also show that under the conditions of parameters in these theorems, the obtained inequalities are sharp and we exhibit a class of extremal functions. We begin with the proof of Theorem 1.1.

2.1. Proof of Theorem 1.1. For any function $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |\Delta u + \nabla u \cdot x|x|^{2\alpha}|^2 |x|^{-2\alpha}\, dx \\ &= \int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha}\, dx + |\nabla u \cdot x|^2 |x|^{2\alpha}\, dx + 2\int_{\mathbb{R}^n} \Delta u \nabla u \cdot x\, dx.\end{aligned} \tag{2.1}$$

Using an integration by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta u \nabla u \cdot x\, dx &= -\int_{\mathbb{R}^n} \nabla^2 u(\nabla u)\cdot x\, dx - \int_{\mathbb{R}^n} |\nabla u|^2\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^n} \nabla(|\nabla u|^2)\cdot x\, dx - \int_{\mathbb{R}^n} |\nabla u|^2\, dx \\ &= \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u|^2\, dx.\end{aligned}$$

Inserting this equality in (2.1), we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta u + \nabla u \cdot x|x|^{2\alpha}|^2 |x|^{-2\alpha}\, dx &= \int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha}\, dx + \int_{\mathbb{R}^n} \left|\nabla u \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx \\ &\quad + (n-2)\int_{\mathbb{R}^n} |\nabla u|^2\, dx.\end{aligned} \tag{2.2}$$

We first consider the case $\alpha \neq -1$. Setting $u = vU_0(x)$ with

$$U_0(x) = \exp\left(-\frac{|x|^{2+2\alpha}}{2+2\alpha}\right),$$

we have

$$\nabla u = \nabla v U_0(x) + v(x)\nabla U_0(x), \quad \Delta u = \Delta v U_0 + 2\nabla v \nabla U_0 + v\Delta U_0$$

and

$$\nabla U_0(x) = -x|x|^{2\alpha} e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}, \quad \Delta U_0(x) = -(n+2\alpha)|x|^{2\alpha} e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}} + |x|^{2+4\alpha} e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}.$$

Plugging this expression into $\Delta u + \nabla u \cdot x|x|^{2\alpha}$ and using simple computations imply

$$\Delta u + \nabla u \cdot x|x|^{2\alpha} = \left(\Delta v - \nabla v \cdot x|x|^{2\alpha} - (n+2\alpha)v|x|^{2\alpha}\right) e^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}.$$

Hence, it holds

$$\begin{aligned}&\int_{\mathbb{R}^n} |\Delta u + \nabla u \cdot x|x|^{2\alpha}|^2 |x|^{-2\alpha}\, dx \\ &= \int_{\mathbb{R}^n} |\Delta v - \nabla v \cdot x|x|^{2\alpha}|^2 |x|^{-2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\, dx + (n+2\alpha)^2 \int_{\mathbb{R}^n} v^2 |x|^{2\alpha} e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx \\ &\quad - 2(n+2\alpha)\int_{\mathbb{R}^n} (\Delta v - \nabla v \cdot x|x|^{2\alpha}) v e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\, dx.\end{aligned}$$

We next compute the last integral in the right-hand side of the preceding equality. Notice that the function $v = ue^{\frac{|x|^{2+2\alpha}}{2+2\alpha}}$ is not C^2 at origin in general. So we can not use integration by parts directly. To overcome this difficulty, we first notice that under the assumption $u \in C_0^\infty(\mathbb{R}^n)$, $n + 2\alpha > 0$ and $n + 2 + 4\alpha > 0$, we have

$$v\Delta v e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} = u(\Delta u + 2\nabla u \cdot x|x|^{2\alpha} + (n+2\alpha)|x|^{2\alpha}u + |x|^{2+4\alpha}u) \in L^1(\mathbb{R}^n)$$

and

$$v\nabla v \cdot x|x|^{2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}} = u(\nabla u \cdot x|x|^{2\alpha} - u|x|^{2+4\alpha}) \in L^1(\mathbb{R}^n).$$

Therefore it holds

$$\int_{\mathbb{R}^n}(\Delta v-\nabla v\cdot x|x|^{2\alpha})ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx=\lim_{\epsilon\to 0^+}\int_{B_\epsilon^c}(\Delta v-\nabla v\cdot x|x|^{2\alpha})ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx,$$

where $B_\epsilon^c=\{x\in\mathbb{R}^n\colon |x|\ge\epsilon\}$. Using integration by parts we have

$$\begin{aligned}
&\int_{B_\epsilon^c}(\Delta v-\nabla v\cdot x|x|^{2\alpha})ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx=\int_{B_\epsilon^c}\operatorname{div}\left(\nabla ve^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}\right)ve^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}\,dx\\
&=-\int_{B_\epsilon^c}\nabla ve^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}}\nabla(ve^{-\frac{|x|^{2+2\alpha}}{2+2\alpha}})\,dx+\int_{\{|x|=\epsilon\}}\nabla v\cdot\frac{x}{|x|}ve^{-2\frac{\epsilon^{2+2\alpha}}{2+2\alpha}}\,ds\\
&=-\int_{B_\epsilon^c}|\nabla v|^2e^{-\frac{|2x|^{2+2\alpha}}{2+2\alpha}}\,dx+\frac12\int_{B_\epsilon^c}\nabla v^2\cdot x|x|^{2\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx+\int_{\{|x|=\epsilon\}}\nabla v\cdot\frac{x}{|x|}ve^{-2\frac{\epsilon^{2+2\alpha}}{2+2\alpha}}\,ds\\
&=-\int_{B_\epsilon^c}|\nabla v|^2e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx-\frac{n+2\alpha}{2}\int_{B_\epsilon^c}v^2|x|^{2\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx+\int_{B_\epsilon^c}v^2|x|^{2+4\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx\\
&\quad+\frac12\epsilon^{1+2\alpha}e^{-2\frac{\epsilon^{2+2\alpha}}{2+2\alpha}}\int_{\{|x|=\epsilon\}}v^2\,ds+\int_{\{|x|=\epsilon\}}\nabla v\cdot\frac{x}{|x|}ve^{-2\frac{\epsilon^{2+2\alpha}}{2+2\alpha}}\,ds.
\end{aligned}$$

Letting $\epsilon\to 0^+$ and using the assumptions $n+2\alpha>0$, $n+2+4\alpha>0$, we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_{B_\epsilon^c}(\Delta v-\nabla v\cdot x|x|^{2\alpha})ve^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx\\
&=-\int_{\mathbb{R}^n}|\nabla v|^2e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx-\frac{n+2\alpha}{2}\int_{\mathbb{R}^n}v^2|x|^{2\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx\\
&\quad+\int_{\mathbb{R}^n}v^2|x|^{2+4\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx.
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\Delta u+\nabla u\cdot x|x|^{2\alpha}|^2|x|^{-2\alpha}\,dx\\
&=\int_{\mathbb{R}^n}|\Delta v-\nabla v\cdot x|x|^{2\alpha}|^2|x|^{-2\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx+2(n+2\alpha)\int_{\mathbb{R}^n}|\nabla v|^2e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx\\
&\quad+2(n+2\alpha)^2\int_{\mathbb{R}^n}v^2|x|^{2\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx-2(n+2\alpha)\int_{\mathbb{R}^n}v^2|x|^{2+4\alpha}e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\,dx. \qquad (2.3)
\end{aligned}$$

Again, we have

$$\int_{\mathbb{R}^n}|\nabla u|^2\,dx=\lim_{\epsilon\to 0}\int_{B_\epsilon^c}|\nabla u|^2\,dx$$

and

$$\begin{aligned}
&\int_{B_\epsilon^c}|\nabla u|^2\,dx=\int_{B_\epsilon^c}|\nabla vU_0(x)+v(x)\nabla U_0(x)|^2\,dx\\
&=\int_{B_\epsilon^c}|\nabla v-v(x)x|x|^{2\alpha}|^2e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\,dx\\
&=\int_{B_\epsilon^c}|\nabla v|^2e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\,dx+\int_{B_\epsilon^c}v^2|x|^{4\alpha+2}e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\,dx-\int_{B_\epsilon^c}\nabla v^2\cdot x|x|^{2\alpha}e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\,dx
\end{aligned}$$

$$= \int_{B_\epsilon^c} |\nabla v|^2 e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx + \int_{B_\epsilon^c} v^2 |x|^{4\alpha+2} e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx + (n+2\alpha) \int_{B_\epsilon^c} v^2 |x|^{2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\, dx$$
$$- 2\int_{B_\epsilon^c} v^2 |x|^{4\alpha+2} e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx + \epsilon^{1+2\alpha} e^{-2\frac{\epsilon^{2+2\alpha}}{2+2\alpha}} \int_{\{|x|=\epsilon\}} v^2\, ds$$
$$= \int_{B_\epsilon^c} |\nabla v|^2 e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx - \int_{B_\epsilon^c} v^2 |x|^{4\alpha+2} e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx + (n+2\alpha) \int_{B_\epsilon^c} v^2 |x|^{2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\, dx$$
$$+ \epsilon^{1+2\alpha} e^{-2\frac{\epsilon^{2+2\alpha}}{2+2\alpha}} \int_{\{|x|=\epsilon\}} v^2\, ds.$$

As above, using again the assumptions $n+2\alpha > 0$, $n+2+4\alpha > 0$ and letting $\epsilon \to 0^+$, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2\, dx = \int_{\mathbb{R}^n} |\nabla v|^2 e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx + (n+2\alpha) \int_{\mathbb{R}^n} v^2 |x|^{2\alpha} e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx$$
$$- \int_{\mathbb{R}^n} v^2 |x|^{2+4\alpha} e^{-2\frac{|x|^{2+2\alpha}}{2+2\alpha}}\, dx. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we obtain

$$\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha}\, dx + \int_{\mathbb{R}^n} \left|\nabla u \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx + (n-2)\int_{\mathbb{R}^n} |\nabla u|^2\, dx$$
$$= \int_{\mathbb{R}^n} |\Delta v - \nabla v \cdot x|x|^{2\alpha}|^2 |x|^{-2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\, dx + 2(n+2\alpha)\int_{\mathbb{R}^n} |\nabla u|^2\, dx$$

which is equivalent to

$$\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha}\, dx + \int_{\mathbb{R}^n} \left|\nabla u \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx$$
$$= \int_{\mathbb{R}^n} |\Delta v - \nabla v \cdot x|x|^{2\alpha}|^2 |x|^{-2\alpha} e^{-\frac{2|x|^{2+2\alpha}}{2+2\alpha}}\, dx + (n+4\alpha+2)\int_{\mathbb{R}^n} |\nabla u|^2\, dx. \tag{2.5}$$

By density argument, (2.5) still holds for any functions $u \in H^2_{\alpha,-2-2\alpha}(\mathbb{R}^n)$. It follows from (2.5) that

$$\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha}\, dx + \int_{\mathbb{R}^n} \left|\nabla u \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx \geq (n+4\alpha+2)\int_{\mathbb{R}^n} |\nabla u|^2\, dx$$

for any $u \in H^2_{\alpha,-2-2\beta}(\mathbb{R}^n)$. Replacing u by function $u_\lambda(x) = \lambda^{\frac{n-2}{2}} u(\lambda x)$ with $\lambda > 0$, we get

$$\lambda^{2+2\alpha} \int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha}\, dx + \lambda^{-2-2\alpha} \int_{\mathbb{R}^n} \left|\nabla u \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx \geq (n+4\alpha+2)\int_{\mathbb{R}^n} |\nabla u|^2\, dx.$$

The left-hand side of the preceding inequality is minimized by

$$\lambda_0 = \left(\frac{\int_{\mathbb{R}^n} |\Delta u|^2 |x|^{-2\alpha}\, dx}{\int_{\mathbb{R}^n} \left|\nabla u \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx} \right)^{\frac{1}{4+4\alpha}}.$$

Hence, by taking $\lambda = \lambda_0$, we obtain (1.4) for $\alpha \neq -1$. The case $\alpha = -1$ follows by letting $\alpha \to -1$.

It remains to check the sharpness of (1.4) when $1+\alpha > 0$. Taking $u = U_0$ implies $v \equiv 1$. Hence, (2.5) becomes

$$\int_{\mathbb{R}^n} |\Delta U_0|^2 |x|^{-2\alpha}\, dx + \int_{\mathbb{R}^n} \left|\nabla U_0 \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx = (n+4\alpha+2)\int_{\mathbb{R}^n} |\nabla U_0|^2\, dx.$$

Furthermore, by the direct computations and integration by parts, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\Delta U_0|^2 |x|^{-2\alpha}\, dx &= (n+2\alpha)^2 \int_{\mathbb{R}^n} |x|^{2\alpha} U_0^2\, dx - 2(n+2\alpha)\int_{\mathbb{R}^n} |x|^{2+4\alpha} U_0^2\, dx \\
&\quad + \int_{\mathbb{R}^n} |x|^{4+6\alpha} U_0^2\, dx \\
&= (n+2\alpha)^2 \int_{\mathbb{R}^n} |x|^{2\alpha} U_0^2\, dx + (n+2\alpha)\int_{\mathbb{R}^n} \nabla U_0^2 \cdot x|x|^{2\alpha}\, dx \\
&\quad + \int_{\mathbb{R}^n} \left|\nabla U_0 \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx \\
&= \int_{\mathbb{R}^n} \left|\nabla U_0 \cdot \frac{x}{|x|}\right|^2 |x|^{2+2\alpha}\, dx.
\end{aligned}$$

This implies that the equality occurs in (1.4) with $u = U_0$. Hence, the inequality (1.4) is sharp and U_0 is an extremal function. This completes the proof of Theorem 1.1. $\square$

2.2. Proof of Theorem 1.3. By density argument, it is enough to prove (1.8) for radial functions $u \in C_0^\infty(\mathbb{R}^n)$. Let $u \in C_0^\infty(\mathbb{R}^n)$ be a radial function, by using an integration by parts, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|\nabla u|^t}{|x|^{t\gamma}}\, dx &= |S^{n-1}| \int_0^\infty |u'|^t r^{n-t\gamma-1}\, dr \\
&= \frac{1}{n-t\gamma}|S^{n-1}| \int_0^\infty |u'|^t (r^{n-t\gamma})'\, dr \\
&= -\frac{t}{n-t\gamma}|S^{n-1}| \int_0^\infty |u'|^{t-2} u' u'' r^{n-t\gamma}\, dr \\
&= -\frac{t}{n-t\gamma}|S^{n-1}| \int_0^\infty |u'|^{t-2} u' \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right) r^{n-t\gamma}\, dr \\
&\quad - \frac{t(1+2\alpha)}{n-t\gamma}|S^{n-1}| \int_0^\infty |u'|^t r^{n-t\gamma-1}\, dr.
\end{aligned}$$

This gives

$$\begin{aligned}
&\frac{n+t(1+2\alpha-\gamma)}{t}\int_{\mathbb{R}^n} \frac{|\nabla u|^t}{|x|^{t\gamma}}\, dx \\
&= -|S^{n-1}| \int_0^\infty |u'|^{t-2} u' \Big(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\Big) r^{n-2\gamma}\, dr \\
&= -|S^{n-1}| \int_0^\infty |u'|^{t-2} u' r^{-\frac{\beta}{2}} \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right) r^{-\alpha} r^{n-1}\, dr, \qquad (2.6)
\end{aligned}$$

here we have used (1.7). By density argument, (2.6) still holds for radial function $u \in H^2_{\alpha,\beta}(\mathbb{R}^n)$. Using Hölder inequality, we arrive at

$$\begin{aligned}
&\left|\frac{n+t(1+2\alpha-\gamma)}{t}\right| \int_{\mathbb{R}^n} \frac{|\nabla u|^t}{|x|^{t\gamma}}\,dx \\
&\le \left(|S^{n-1}| \int_0^\infty \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right)^2 r^{n-2\alpha-1}\,dr\right)^{\frac{1}{2}} \\
&\quad \times \left(|S^{n-1}| \int_0^\infty |u'|^{2(t-1)} r^{n-\beta-1}\,dr\right)^{\frac{1}{2}} \qquad (2.7)\\
&= \left(|S^{n-1}| \int_0^\infty \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right)^2 r^{n-2\alpha-1}\,dr\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{|\nabla u|^{2(t-1)}}{|x|^\beta}\,dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Furthermore, using an integration by parts, we have

$$\begin{aligned}
&\int_0^\infty \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right)^2 r^{n-2\alpha-1}\,dr \\
&= \int_0^\infty (\Delta u(r))^2 r^{n-2\alpha-1}\,dr + (n+2\alpha)^2 \int_0^\infty (u')^2 r^{n-2\alpha-3}\,dr \\
&\quad - 2(n+2\alpha) \int_0^\infty \left(u'' + \frac{n-1}{r}u'\right) \frac{u'}{r} r^{n-2\alpha-1}\,dr \\
&= \int_0^\infty (\Delta u(r))^2 r^{n-2\alpha-1}\,dr - (n+2\alpha)(n-2\alpha-2) \int_0^\infty (u')^2 r^{n-2\alpha-3}\,dr \\
&\quad - (n+2\alpha) \int_0^\infty ((u')^2)' r^{n-2\alpha-2}\,dr \\
&= \int_0^\infty (\Delta u(r))^2 r^{n-2\alpha-1}\,dr.
\end{aligned}$$

Consequently, it holds

$$|S^{n-1}| \int_0^\infty \left(u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u'\right)^2 r^{n-2\alpha-1}\,dr = \int_{\mathbb{R}^n} \frac{(\Delta u)^2}{|x|^{2\alpha}}\,dx.$$

Inserting this equality into (2.7), we obtain (1.8).

Suppose that a nonzero radial function $u \in H^2_{\alpha,\beta}(\mathbb{R}^n)$ is an extremal function for (1.8). Notice that under the conditions (1.9) and (1.10), we have

$$n + t(1+2\alpha-\gamma) = n + 2\alpha + (1+2\alpha)(t-2) + 1 + \alpha - \frac{\beta}{2} > 0.$$

Hence, the equation holds when applying the Hölder inequality to (2.6) if and only if

$$u'' + \frac{n-1}{r}u' - \frac{n+2\alpha}{r}u' = -\lambda |u'|^{t-2} u' r^{\alpha-\frac{\beta}{2}}$$

for some $\lambda > 0$, which is equivalent to

$$u'' - \frac{1+2\alpha}{r}u' + \lambda |u'|^{t-2} u' r^{\alpha-\frac{\beta}{2}} = 0.$$

Denote $u' = r^{1+2\alpha} w$, then w satisfies the equation

$$w' + \lambda r^{(1+2\alpha)(t-2)+\alpha-\frac{\beta}{2}} |w|^{t-2} w = 0.$$

We have following two cases:

Case 1: $t = 2$. In this case, we have $w' + \lambda r^{\alpha - \frac{\beta}{2}} w = 0$ which implies $w(r) = c \exp(-\lambda r^{1+\alpha-\frac{\beta}{2}}/(1+\alpha-\frac{\beta}{2}))$ for some $c \in \mathbb{R}$. Hence,

$$u'(r) = c r^{1+2\alpha} \exp\left(-\lambda \frac{r^{1+\alpha-\beta/2}}{1+\alpha-\beta/2}\right)$$

and

$$u(x) = c \int_{|x|}^{\infty} r^{1+2\alpha} \exp\left(-\lambda \frac{r^{1+\alpha-\beta/2}}{1+\alpha-\beta/2}\right) dr.$$

Case 2: $t > 2$. In this case, we have $(|w|^{2-t})' = \lambda(t-2)\lambda r^{(1+2\alpha)(t-2)+\alpha-\frac{\beta}{2}}$ which implies

$$|w(r)| = \left(c + \lambda(t-2) \frac{r^{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}}{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}\right)^{\frac{1}{2-t}},$$

for some $c > 0$, here we have used (1.9). From this expression, up to a multiplicative constant 1 or -1, we can assume that

$$w(r) = \left(c + \lambda(t-2) \frac{r^{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}}{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}\right)^{\frac{1}{2-t}}.$$

Therefore, the extremal function has the form

$$u(x) = \int_{|x|}^{\infty} r^{1+2\alpha} \left(c + \lambda(t-2) \frac{r^{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}}{(1+2\alpha)(t-2)+1+\alpha-\frac{\beta}{2}}\right)^{\frac{1}{2-t}} dr$$

as desired. The proof is then complete. □

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