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Unique extremality of affine maps on plane domains

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Abstract. We prove that affine maps are uniquely extremal quasiconformal maps on the complement of a well distributed set in the complex plane answering a conjecture from Marković (2002). We construct the required Reich sequence using Bergman projections, and meromorphic partitions of unity.

Tasoalueiden affiinien kuvausten yksikäsitteinen ääriominaisuus

Tiivistelmä. Vastauksena Markovićin (2002) esittämään konjektuuriin todistetaan, että affiinit kuvaukset ovat kvasikonformisen homotopialuokkansa yksikäsitteisiä Beltramin kertoimen minimoijia. Tarvittava Reichin jono rakennetaan Bergmanin projektioiden ja meromorfisten yksikön ositusten avulla.

1. Introduction

Let $f\colon X \to Y$ be a quasiconformal map between Riemann surfaces X and Y. By $\mathrm{Belt}(f) = \frac{\bar\partial f}{\partial f}$ we denote the Beltrami coefficient of f. Then $\mathrm{Belt}(f)$ is a $(-1,1)$ complex form on X, while $|\mathrm{Belt}(f)|$ is a measurable function on X such that $\|\mathrm{Belt}(f)\|_\infty < 1$. We say that f is extremal if f has the smallest Beltrami coefficient in its homotopy class, that is, if $\|\mathrm{Belt}(f)\|_\infty \leq \|\mathrm{Belt}(g)\|_\infty$ for every quasiconformal map g homotopic to f. We say that f is uniquely extremal if it is the only extremal map in its homotopy class.

Every quasiconformal map is homotopic to an extremal quasiconformal map (see [1]). The classical extremal problem, first studied by Grötzsch and Teichmüller, is to describe extremal quasiconformal maps, and to decide when they are uniquely extremal. Recall that a quasiconformal map $f\colon X \to Y$ is of Teichmüller type if $\mathrm{Belt}(f) = k\frac{\overline{\phi}}{\phi}$, where ϕ is an integrable holomorphic quadratic differential on X. When X and Y are finite type Riemann surfaces then every extremal map is of Teichmüller type, and it is uniquely extremal.

In general, extremal quasiconformal maps may not be of Teichmüller type, nor they are necessarily uniquely extremal. In [9] Reich and Strebel proposed the following special case of the extremal problem.

Definition 1. Let $A\colon \mathbb{C} \to \mathbb{C}$ denote an affine map, and let $E \subset \mathbb{C}$ be a discrete set. We say that A is extremal, or uniquely extremal, on $\mathbb{C} \setminus E$, if $A\colon X \to Y$ is extremal, or uniquely extremal, as a quasiconformal map where $X = \mathbb{C} \setminus E$, and $Y = \mathbb{C} \setminus A(E)$.

Problem 1. Characterise discrete subsets $E \subset \mathbb{C}$ such that affine maps are extremal, or uniquely extremal, as quasiconformal maps.

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Remark 1. It follows from [4], [8], and [2], that if μ is the Beltrami coefficient of an extremal (uniquely extremal) quasiconformal map, then $c\mu$ is also the Beltrami coefficient of an extremal (uniquely extremal) quasiconformal map for any constant $c \in \mathbb{C}$ such that $\|c\mu\|_\infty < 1$. Let $A_1, A_2 \colon \mathbb{C} \to \mathbb{C}$ be two affine maps neither of which is the identity map. Then $\mathrm{Belt}(A_1)$ and $\mathrm{Belt}(A_2)$ are non-zero constant functions. Therefore, A_1 is extremal (uniquely extremal) on $\mathbb{C} \setminus E$ if and only A_2 is extremal (uniquely extremal) on $\mathbb{C} \setminus E$. Thus, the above problem is well posed.

The methods used to study this problem are partly geometric, and partly analytic, and it is this interplay which makes them interesting. However, even in this generality the problem is very hard. In [9] Reich and Strebel provided a somewhat complicated characterisation of discrete sets E such that affine maps are extremal on $\mathbb{C} \setminus E$. On the other hand, they were not aware of a single example such that affine maps are uniquely extremal on $\mathbb{C} \setminus E$. The first such example was provided in [5] where it was shown that affine maps are uniquely extremal on $\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})$ (here $\mathbb{Z} + i\mathbb{Z}$ denotes the integer lattice in the complex plane). This answered a conjecture by Kra–Reich–Strebel.

Definition 2. Let $c > 0$. A discrete set $E \subset \mathbb{C}$ is called c-well distributed if E intersects any disc of Euclidean radius c in the complex plane. We say that E is well distributed if it is c-well distributed for some $c > 0$.

The following conjecture was posed in [5].

Conjecture 1. If E is a well distributed set then affine maps are uniquely extremal on $\mathbb{C} \setminus E$.

Well distributed sets represent a geometric generalization of the integer lattice $\mathbb{Z}+i\mathbb{Z}$, and it is to be expected that the proof that affine maps are uniquely extremal on $\mathbb{C}\setminus(\mathbb{Z}+i\mathbb{Z})$ should be utilised to prove Conjecture 1. However, the method of proof in [5] strongly exploits the fact that $\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})$ is an amenable cover of the square once punctured torus (compare with McMullen's proof of the Kra conjecture [6]), and it can not be extended to the case of well distributed sets. The purpose of this paper is to develop a new approach and prove Conjecture 1. In particular, we provide a new and much more conceptual proof of the the main result from [5] that affine maps are uniquely extremal on the complement of the integer lattice (the old proof relies heavily on the translation invariance of the integer lattice).

Theorem 1. *If E is a well distributed set then affine maps are uniquely extremal on $\mathbb{C}\setminus E$.*

Remark 2. Every c-well distributed set can be mapped to a $\frac{1}{8}$-well distributed set by a linear isomorphism of $\mathbb{C}$. Since linear maps conjugate affine maps onto affine maps, it suffices to prove Theorem 1 for $\frac{1}{8}$-well distributed sets.

1.1. Reich sequences. The following theorem due to Reich (see Theorem B in [7]) gives a sufficient condition for an affine map on a plane domain $M \subset \mathbb{C}$ to be uniquely extremal. Let $QD^1(M)$ be the Banach space of all integrable holomorphic quadratic differentials with respect to the L^1-norm.

Theorem 2. *Let M be a plane domain. An affine map on M is uniquely extremal if there exists a sequence $\Phi_n = \phi_n\, dz^2 \in QD^1(M)$ satisfying the following three conditions:*

- $\displaystyle\lim_{n\to\infty} \phi_n(z) = 1, \quad \forall z \in M.$

- $\limsup_{n\to\infty} \int_M \big(|\phi_n| - Re(\phi_n)\big)|dz|^2 < +\infty$.
- *If* $S_{n,K} = \{z \in M : |\phi_n(z)| \geq K\}$, *then*

$$\liminf_{K\to\infty} \int_{S_{K,n}} |\phi_n(z)||dz|^2 = 0,$$

uniformly in $n \in \mathbb{N}$.

Such a sequence $\Phi_n \subset QD^1(M)$ is called a Reich sequence (we also refer to the corresponding sequence of functions ϕ_n as the Reich sequence). It is known to be hard to construct such sequences because it is difficult to estimate the L^1-norms of meromorphic functions in terms of their coefficients. We construct Reich sequences on the complements of well distributed sets.

Remark 3. It follows from [2] that the existence of a Reich sequence is equivalent to the statement that affine maps are uniquely extremal on M.

1.2. Quasilattices. By L_0 we denote the integer lattice in the complex plane $\mathbb{C}$. Points in L_0 are enumerated by $z_{k,l} = k + li$ where $k, l \in \mathbb{Z}$.

Definition 3. A countable subset L in $\mathbb{C}$ is called a quasilattice if its points can be enumerated by $w_{k,l}$, where $k, l \in \mathbb{Z}$, and $w_{k,l}$ satisfies

$$\left|z_{k,l} - w_{k,l}\right| \leq \frac{1}{8}.$$

It's easy to observe that a quasilattice is a well distributed set. On the other hand, it will be proved in the following lemma that any $\frac{1}{8}$-well distributed set E will contains a quasilattice L as subset. Then a Reich sequence for $\mathbb{C} \setminus L$ is also a Reich sequence for $\mathbb{C} \setminus E$, since $QD^1(\mathbb{C} \setminus L) \subset QD^1(\mathbb{C} \setminus E)$. Thus, it suffices to construct a Reich sequence ϕ_n for every plane domain $\mathbb{C} \setminus L$ which is a complement of a quasilattice.

Lemma 1. *Every* $\frac{1}{8}$*-well distributed set* E *contains a quasilattice* $L \subset E$.

Proof. By $\mathbb{D}_{\frac{1}{8}}(z)$ we denote the Euclidean disk of radius $\frac{1}{8}$, centered at z. Each disk $\mathbb{D}_{\frac{1}{8}}(z)$ contains at least one point from E. Therefore, we can choose points

$$w_{k,l} \in \mathbb{D}_{\frac{1}{8}}(z_{k,l}) \cap E.$$

Points $w_{k,l}$ are mutually different since they live in disjoint disks. Moreover, by construction $|w_{k,l} - z_{k,l}| < \frac{1}{8}$. Therefore $L = \{w_{k,l} : k, l \in \mathbb{Z}\}$ is a quasilattice. □

1.3. Meromorphic partition of unity. We let

$$\Omega = \left\{ z \in \mathbb{C} : d(z, L_0) > \frac{1}{4} \right\}. \tag{1}$$

Note that if L is a quasilattice then $\Omega \subset \mathbb{C} \setminus L$.

Definition 4. Let L be a quasilattice. Let $\{P_{p,q}\}$, $p, q \in \mathbb{Z}$, be a double sequence of meromorphic functions such that each $P_{p,q}\, dz^2 \in QD^1(\mathbb{C} \setminus L)$. We say that $P_{p,q}$ is a meromorphic partition of unity if there exists a constant C such that for any $z \in \Omega$, and any $p, q \in \mathbb{Z}$, we have

$$|P_{p,q}(z)| \leq \frac{C}{\exp\left(\frac{|z - z_{p,q}|}{C}\right)}, \tag{2}$$

and if

$$\sum_{k,l} P_{k,l}(z) = 1, \quad z \in \mathbb{C} \setminus L. \tag{3}$$

Remark 4. It follows from the the estimate (2) that the sequence in (3) absolutely converges for each $z \in \mathbb{C} \setminus L$.

Constructing such a meromorphic partition of unity is a key idea behind constructing a Reich sequence as illustrated by the following theorem.

Theorem 3. *Assume L is a quasilattice equipped with a meromorphic partition of unity $P_{p,q}$, $p, q \in \mathbb{Z}$. Set*

$$\phi_n(z) = \sum_{k,l} \frac{P_{k,l}(z)}{\left(|\frac{z_{k,l}}{n}| + 1\right)^4}$$

for $z \in \mathbb{C} \setminus L$. Then ϕ_n is a Reich sequence for $\mathbb{C} \setminus L$.

1.4. Organization. In the next two section we construct a meromorphic partition of unity $P_{p,q}$ on $\mathbb{C} \setminus L$. In Section 2 we recall the notion of the Bergman projection. The main result of this section is the estimate in Lemma 2. In Section 3 we define the double sequence $P_{p,q}$ using the Bergman projections of characteristic functions of squares centred at the lattice points $z_{p,q}$. Theorem 3 is proved in Section 4 by elementary computations.

2. The Bergman kernel

We recall the definition of the Bergman kernel for any hyperbolic Riemann surface and state some of its well known properties (see Section 12 in [3] for example). The Bergman kernel function on the unit disc $\mathbb{D}$ is given by

$$K(z, w) = \frac{1}{(1 - z\overline{w})^4}.$$

The Bergman kernel $B_{\mathbb{D}}$ is the differential form given by

$$B_{\mathbb{D}}(z, w) = K(z, w)\, dz^2\, d\overline{w}^2.$$

By direct computation one finds that the Bergman kernel $B_{\mathbb{D}}$ is Möbius transformation invariant, that is, for $A \in \operatorname{Aut}(\mathbb{D})$ we have

$$K(Az, Aw) A'(z)^2 \overline{A'(w)}^2 = K(z, w). \tag{4}$$

Another key property of this kernel is the following integral identity (also proved by elementary computation). Namely, let $W \subset \mathbb{D}$, and let w be the point such that $A(w) = 0$. Then

$$\rho^2_{\mathbb{D}}(w) \int_{A(W)} |dz|^2 = \int_W |K(z, w)||dz|^2, \tag{5}$$

where

$$\rho_{\mathbb{D}}(w) = \frac{1}{(1 - |w|^2)}$$

is the density of the hyperbolic metric on $\mathbb{D}$.

2.1. The Bergman projection. A word on notation first. Suppose ω is a volume form on $X \times X$ obtained as a product of two volume forms on X. Let (z, w)

denote local coordinates in $X \times X$. If $Y \subset X$, then the integral

$$\int_{Y_z} \omega$$

is a volume form on X (with local coordinate w), that is the notation Y_z means that we are integrating with respect to the z variable (and likewise for Y_w). Also, if we fix a point $p \in X$ then $\omega(p,w)$ is a volume form with respect to $w \in X$ (this is the evaluation of ω at (p,w)) (and likewise for $\omega(z,p)$).

To each measurable quadratic differential f on $\mathbb{D}$, we associate the holomorphic quadratic differential $f * B_{\mathbb{D}}$ on $\mathbb{D}$ given by

$$f * B_{\mathbb{D}} = \frac{3}{\pi} \int_{\mathbb{D}_w} f(w)\, dV_{\mathbb{D}}^{-1}(w) B_{\mathbb{D}}(z,w), \tag{6}$$

where the tensor $dV_{\mathbb{D}}^{-1}$ is given by

$$dV_{\mathbb{D}}^{-1}(w) = \frac{1}{\rho_{\mathbb{D}}^2(w)|dw|^2}.$$

This convolution formula is well defined (see Section 11.1 in [3]) when f has a finite L^1 norm in which case $f * B_{\mathbb{D}} \in QD^1(\mathbb{D})$, or if f has a finite Bers norm in which case $f * B_{\mathbb{D}} \in QD^\infty(\mathbb{D})$, where $\Phi = \phi\, dw^2 \in QD^\infty(\mathbb{D})$ if

$$\sup_{w \in \mathbb{D}} \rho^{-2}(w)|\phi(w)| < \infty.$$

If $f \in QD^1(\mathbb{D})$, or $f \in QD^\infty(\mathbb{D})$, then $f * B_{\mathbb{D}} \equiv f$ (this requirement explains the normalisation factor $\frac{3}{\pi}$ in (6)).

Let S be a hyperbolic Riemann surface given as the quotient $S = \mathbb{D}/\Gamma$, where Γ is a Fuchsian group. We use z and w to denote the local coordinates on both $\mathbb{D}$ and S. The differential form B_S, called the Bergman kernel on the surface S, is defined as follows. Denote by $\widetilde{B}_S$ the lift of B_S to the unit disc. Then $\widetilde{B}_S$ is defined by the Poincare series

$$\widetilde{B}_S(z,w) = \sum_{A \in \Gamma} K(Az,w) A'(z)^2\, dz^2\, d\overline{w}^2.$$

The Poincare series is absolutely convergent since for a fixed $w \in \mathbb{D}$ the integral of $|K|$ over $z \in \mathbb{D}$ is bounded. For any measurable quadratic differential f on S we define the holomorphic quadratic differential called the Bergman projection by

$$f * B_S = \frac{3}{\pi} \int_{S_w} f(w)\, dV_S^{-1}(w) B_S(z,w), \tag{7}$$

with the tensor

$$dV_S^{-1}(w) = \frac{1}{dV_S(w)},$$

where $dV_S(w) = \rho_S^2(w)|dw|^2$ is the volume form of the hyperbolic metric on S (here ρ_S is the density of the hyperbolic metric on S). This convolution formula is well defined when f has a finite L^1 norm in which case $f * B_S \in QD^1(S)$, or if f has a finite Bers norm in which case $f * B_S \in QD^\infty(S)$. If $f \in QD^1(S)$, or $f \in QD^\infty(S)$, then $f * B_S \equiv f$. The following lemma is our main estimate about the Bergman kernel.

Lemma 2. *Let S denote a hyperbolic Riemann surface. Given any subset $U \subset S$, and $p \in S$, we have*

$$\int_{U_z} |B_S(z,p)| \le \frac{4\pi}{\exp\big(d_S(U,p)\big)}\, dV_S(p),$$

where $d_S(U,p)$ denotes the hyperbolic distance between U and p.

Remark 5. The left hand side in the above inequality is a volume form on S (in the coordinate p). Thus, the inequality represents a pointwise comparison between two volume forms on S.

Proof. Let $V \subset \mathbb{D}$ be a fundamental domain for $U \subset S$. Let $w_p \in \mathbb{D}$ denote a lift of p. Then

$$\begin{aligned}\int_{U_z} |B_S(z,p)| &= \left(\int_V \left|\sum_{A\in\Gamma} K(Az,w_p)A'(z)^2\right||dz|^2\right)|dw|^2\\ &\le \left(\sum_{A\in\Gamma}\int_{A(V)} |K(z,w_p)|\,|dz|^2\right)|dw|^2 \qquad (8)\\ &= \left(\int_{\Gamma(V)} |K(z,w_p)||dz|^2\right)|dw|^2.\end{aligned}$$

Let $T \in \operatorname{Aut}(\mathbb{D})$ be a Möbius transformation which maps w_p to 0. By replacing $W = \Gamma(V)$ in the invariance formula (5), we obtain

$$\int_{\Gamma(V)} |K(z,w_p)||dz|^2 = \rho^2_{\mathbb{D}}(w_p)\int_{T(\Gamma(V))} |dz|^2. \tag{9}$$

Now, we have $d_S(U,p) = d_{\mathbb{D}}(\Gamma(V),w) = d_{\mathbb{D}}(T\big(\Gamma(V)),0\big)$. Thus, $T\big(\Gamma(V)\big)$ is contained in the set

$$\left\{\xi \in \mathbb{D}\colon |\xi| \ge \tanh\left(\frac{d_S(U,p)}{2}\right)\right\}.$$

We find

$$\int_{T(\Gamma(V))} |dz|^2 \le \int_{|\xi|\ge\tanh\left(\frac{d_S(U,p)}{2}\right)} |d\xi|^2 = \frac{4\pi}{\cosh^2\left(\frac{d_S(U,p)}{2}\right)} \le \frac{4\pi}{\exp\big(d_S(U,p)\big)}.$$

Combining this with the equations (8) and (9) completes the proof. ☐

3. The Bergman projection and the meromorphic partition of unity

In this section we construct an explicit meromorphic partition of unity for any quasilattice L. By B_L we denote the Bergman kernel for the Riemann surface $\mathbb{C}\setminus L$. To each measurable quadratic differential f on $\mathbb{C}\setminus L$, we associate the Bergman projection defined by

$$f * B_L = \frac{3}{\pi}\int_{(\mathbb{C}\setminus L)_w} f(w)\, dV_L^{-1}(w) B_L(z,w).$$

In the next subsection we prove:

Lemma 3. *There exists a constant C with the following properties. Set $W = [-\frac{1}{2},\frac{1}{2})\times[-\frac{1}{2},\frac{1}{2})$, and consider the measurable quadratic differential $f(z) = \chi(z)\, dz^2$,* *where χ is the characteristic function of W. Then for any quasilattice L, and any $z_0 \in \Omega = \{z \in \mathbb{C}\colon d(z, L_0) > \frac{1}{4}\}$, we have*

$$\left|\frac{(f * B_L)}{|dz|^2}\right|(z_0) \leq \frac{C}{\exp\left(\frac{|z_0|}{C}\right)}.$$

Remark 6. Note that $|(f * B_L)|$ is a volume form on $\mathbb{C} \setminus L$ since $f * B_L$ is a quadratic differential. The right hand side in the above inequality is a function on $\mathbb{C} \setminus L$. Thus, the lemma gives a pointwise comparison between two functions on $\mathbb{C} \setminus L$.

The construction of a meromorphic partition of unity is as follows. Let $W_{k,l} = z_{k,l} + W$ where $z_{k,l} = l + il$ for $k, l \in \mathbb{Z}$. The complex plane $\mathbb{C}$ is a disjoint union of the sets $W_{k,l}$. By $\chi_{k,l}(z)$ we denote the characteristic function of $W_{k,l}$, and $f_{k,l} = \chi_{k,l}\, dz^2$. For any quasilattice L, we can project the integrable quadratic differentials $f_{k,l}$ to the integrable holomorphic quadratic differentials $f_{k,l} * B_L \in QD^1(\mathbb{C} \setminus L)$. Set $P_{k,l}\, dz^2 = f_{k,l} * B_L$. Note that each $P_{k,l}$ is a meromorphic function on $\mathbb{C}$ with at most first order poles at the points $w_{k,l}$. We claim that the double sequence $P_{k,l}$ is a meromorphic partition of unity.

Since

$$\sum_{k,l} \chi_{k,l} \equiv 1,$$

by the reproducing property of the Bergman projection (see Section 11.1 in [3]) we have

$$\sum_{k,l} P_{k,l} \equiv 1.$$

Thus $P_{k,l}$ satisfies the second condition from Definition 4. It follows from Lemma 3 that $P_{k,l}$ satisfies the first condition. To complete the construction of the meromorphic partition of unity, it remains to prove Lemma 3. This is done in the next section.

3.1. Proof of Lemma 3. In the remainder of the proof we let

$$h = \frac{f * B_L}{dz^2}.$$

Then h is a holomorphic function on $\mathbb{C} \setminus L$. To prove the lemma we need to show

$$|h(z_0)| \leq \frac{C}{\exp\left(\frac{|z_0|}{C}\right)}, \tag{10}$$

for every $z_0 \in \Omega$.

By the construction of Ω, there exists a constant $t_0 > 0$ such that the injectivity radius (with respect to the hyperbolic metric on $\mathbb{C} \setminus L$) is greater than t_0 at all points in Ω. Let $\xi \in \Omega$, and choose a uniformizing map $\pi\colon \mathbb{D} \to \mathbb{C} \setminus L$ such that $\pi(0) = \xi$. Then the restriction of π to the disc $\mathbb{D}_{r_0}(0)$ is univalent where r_0 is a function of t_0.

Applying the Koebe 1/4 Theorem to the restriction of π to $\mathbb{D}_{r_0}(0)$ implies that the disc $\mathbb{D}_{r_1}(\xi) \subset \mathbb{C} \setminus L$, where

$$r_1 = \frac{r_0}{4}|\pi'(0)|.$$

By construction we know that $r_1 < 2$. This yields the estimate

$$|\pi'(0)| < \frac{8}{r_0},$$

which in turn implies

$$s_0 < \frac{ds_L}{|dz|}(\xi),$$

for every $\xi \in \Omega$, where $s_0 = \frac{r_0}{8}$. By ds_L we denote the hyperbolic length element on $\mathbb{C} \setminus L$, and by d_L the hyperbolic distance on $\mathbb{C} \setminus L$. This yields the distance estimate:

Proposition 1. *Let $z, w \in \mathbb{C} \setminus L$. Then*

$$d_L(z, w) \geq \frac{s_0}{2}|z - w| - 2s_0. \tag{11}$$

Proof. Let $\alpha \subset \mathbb{C} \setminus L$ be a smooth arc connecting z and w, and set $\beta = \alpha \cap \Omega$. Then the Euclidean length of β is greater than $(1/2)|z - w| - 2$. This implies

$$d_L(z, w) \geq s_0(\frac{1}{2}|z - w| - 2) > \frac{s_0}{2}|z - w| - 2s_0. \qquad \square$$

The proof of Lemma 3 is based on Proposition 1, and Lemma 2. Let $z_0 \in \Omega$, and let D denote the disc of radius $\frac{1}{8}$ centred at z_0. Then $D \subset \mathbb{C} \setminus L$. From the Mean Value Theorem we derive

$$h(z_0) = \frac{32}{\pi} \int_D h(\xi) |d\xi|^2.$$

Then

$$\begin{aligned} |h(z_0)| &\leq \frac{32}{\pi} \int_{D_z} \int_{\mathbb{C}_w} \left| f(w)\, dV_L^{-1}(w) B_L(z, w) \right| \\ &= \frac{32}{\pi} \int_{W_w} \left(|dw|^2\, dV_L^{-1}(w) \int_{D_z} |B_L(z, w)| \right). \end{aligned} \tag{12}$$

On the other hand, applying Lemma 2 yields

$$\int_{D_z} |B_L(z, w)| \leq \frac{4\pi}{\exp(d_L(D, w))}\, dV_L(w). \tag{13}$$

Replacing this in (12) we get

$$\begin{aligned} |h(z_0)| &\leq \frac{32}{\pi} \int_{W_w} \left(\frac{4\pi}{\exp(d_L(D, w))} \right) |dw|^2 \\ &\leq \frac{128}{\exp(d_L(D, W))} \leq \frac{128}{\exp\left(\frac{s_0}{2}|z_0| - 2s_0 - 2\right)}, \end{aligned}$$

where in the last inequality we used (11). Letting

$$C = \max\left\{ 128 \exp(2s_0 + 2), \frac{2}{s_0} \right\}$$

proves (10) and the lemma.

4. The Reich sequence ϕ_n

Let $\alpha(n) = \{\alpha_{p,q}(n)\}$ be the double sequence of complex numbers given by

$$\alpha_{p,q}(n) = \frac{1}{\left(\frac{|z_{p,q}|}{n} + 1\right)^4}.$$

Consider a quasilattice L equipped with a fixed meromorphic partition of unity $P_{p,q}$, $p, q \in \mathbb{Z}$. Set

$$\phi_n = \sum_{k,l} \alpha_{k,l}(n) P_{k,l}.$$

It follows from the first condition in Definition 4 that the L^1-norms of the quadratic differentials $P_{k,l}$ are uniformly bounded (uniform in k,l). Since $\sum_{k,l}\alpha_{k,l}(n)<\infty$ for every n, it is clear that $\phi_n(z)\subset QD^1(\mathbb{C}\setminus L)$. In this section, we prove the Theorem 3 which states that ϕ_n is a Reich sequence.

4.1. Properties of ϕ_n. In this subsection we prove two preliminary lemmas providing preliminary information about ϕ_n. For any p,q,k,l,n, we have

$$|\alpha_{p,q}(n)-\alpha_{k,l}(n)|=\alpha_{p,q}(n)\alpha_{k,l}(n)\left|\left(\frac{|z_{p,q}|}{n}+1\right)^4-\left(\frac{|z_{k,l}|}{n}+1\right)^4\right|.$$

We apply the Lagrange Theorem to obtain

$$\left|\left(\frac{|z_{p,q}|}{n}+1\right)^4-\left(\frac{|z_{k,l}|}{n}+1\right)^4\right|\le\frac{4|z_{p,q}-z_{k,l}|}{n}\left|\frac{\max\{|z_{k,l}|,|z_{p,q}|\}}{n}+1\right|^3.$$

Since

$$\begin{aligned}\alpha_{k,l}(n)\left|\frac{\max\{|z_{k,l}|,|z_{p,q}|\}}{n}+1\right|^3&=\frac{1}{\frac{|z_{k,l}|}{n}+1}\left(\frac{\max\{|z_{k,l}|,|z_{p,q}|\}+n}{|z_{k,l}|+n}\right)^3\\&\le\left(\frac{|z_{k,l}|+|z_{p,q}|+1}{|z_{k,l}|+1}\right)^3=\left(2+\frac{|z_{p,q}|-|z_{k,l}|-1}{|z_{k,l}|+1}\right)^3\\&\le\left(2+|z_{k,l}-z_{p,q}|\right)^3,\end{aligned}$$

we derive the estimate

$$|\alpha_{p,q}(n)-\alpha_{k,l}(n)|\le\frac{4\alpha_{p,q}(n)}{n}(2+|z_{k,l}-z_{p,q}|)^4. \tag{14}$$

Lemma 4. *There exists a constant C such that for any n,p,q, and any $z\in W_{p,q}\cap\Omega$, we have*

$$|\phi_n(z)-\alpha_{p,q}(n)|\le C\frac{\alpha_{p,q}(n)}{n}.$$

Proof. Let C_1 be the constant from the first property in Definition 4. Using the second property in Definition 4, we have

$$\begin{aligned}|\phi_n(z)-\alpha_{p,q}(n)|&=\left|\sum_{k,l}\alpha_{k,l}(n)P_{k,l}(z)-\alpha_{p,q}(n)\sum_{k,l}P_{k,l}(z)\right|\\&\le\sum_{k,l}|\alpha_{k,l}(n)-\alpha_{p,q}(n)|\ |P_{k,l}(z)|\\&\le C_1\sum_{k,l}\frac{|\alpha_{k,l}(n)-\alpha_{p,q}(n)|}{\exp(|z-z_{k,l}|/C_1)}\end{aligned} \tag{15}$$

for every n,p,q, and $z\in W_{p,q}\cap\Omega$. Since $z\in W_{p,q}$, we have

$$|z-z_{k,l}|\ge|z_{p,q}-z_{k,l}|-\frac{\sqrt{2}}{2},$$

so

$$\frac{1}{\exp(|z-z_{k,l}|/C_1)}\le\exp\left(\frac{\sqrt{2}}{2C_1}\right)\frac{1}{\exp(|z_{k,l}-z_{p,q}|/C_1)}$$

Combining this with the equations (14) and (15), we obtain

$$|\phi_n(z) - \alpha_{p,q}(n)| \le \left(C_1 \sum_{k,l} \frac{4(2+|z_{k,l}-z_{p,q}|)^4 \exp\left(\frac{\sqrt{2}}{2C_1}\right)}{\exp\left(\frac{|z_{k,l}-z_{p,q}|}{C_1}\right)} \right) \frac{\alpha_{p,q}(n)}{n}.$$

This proves the lemma by letting

$$C = C_1 \sum_{k,l} \frac{4(2+|z_{k,l}-z_{p,q}|)^4 \exp\left(\frac{\sqrt{2}}{2C_1}\right)}{\exp\left(\frac{|z_{k,l}-z_{p,q}|}{C_1}\right)}$$

(note that C does not depend on p and q). □

Lemma 5. *For every $z \in W_{p,q} \cap \Omega$, the inequality*

$$|\phi_n(z)| - \mathrm{Re}(\phi_n(z)) \le \frac{C^2}{n^2}\alpha_{p,q}(n),$$

holds for n large enough, where C is the constant from Lemma 4.

Proof. Note that if a complex number λ satisfies $|\lambda - 1| \le \epsilon \le \frac{1}{2}$ then

$$|\lambda| - \mathrm{Re}(\lambda) = \frac{\mathrm{Im}(\lambda)^2}{|\lambda| + \mathrm{Re}(\lambda)} \le \epsilon^2. \tag{16}$$

On the other hand, from Lemma 4 we derive the inequality

$$\left| \frac{\phi_n(z)}{\alpha_{p,q}(n)} - 1 \right| \le \frac{C}{n}.$$

Set $\lambda = \frac{\phi_n(z)}{\alpha_{p,q}(n)}$, and $\epsilon = \frac{C}{n}$. Then for n large enough we have $\epsilon < \frac{1}{2}$. The lemma now follows from (16). □

4.2. Proof of the Theorem 3. In this subsection we prove that ϕ_n is a Reich sequence. We show that ϕ_n has the three properties from Theorem 2.

The complex plane $\mathbb{C}$ is decomposed into the domain Ω, and the radius $\frac{1}{4}$ disks centred at the points $z_{k,l}$. Lemma 4 and Lemma 5 hold for points in Ω which enables us to prove the above three properties when ϕ_n is restricted to Ω. That this is sufficient follows from the following lemma proved in Proposition 3.1 and Lemma 4.1 in [5]. By $\mathbb{D}_{\frac{1}{2}}$ we denote the disk of radius $\frac{1}{2}$ centred at 0.

Lemma 6. *There exist a constant C with the following properties. Suppose f is a meromorphic function on $\overline{\mathbb{D}_{\frac{1}{2}}}$, which is holomorphic on $\mathbb{D}_{\frac{1}{2}} \setminus \{0\}$, and has a first order pole at 0. Assume that for every $z \in \partial\mathbb{D}_{\frac{1}{2}}$, we have*

$$|f(z) - 1| \le \epsilon \le \frac{1}{2}.$$

Then

$$\int_{\mathbb{D}_{\frac{1}{2}}} \big(|f| - \mathrm{Re}(f)\big)|dz|^2 \le C\epsilon^2. \tag{17}$$

Let $S_K = \{z \in \mathbb{D}_{\frac{1}{2}} : |f(z)| \ge K\}$. Then for any $K \ge 100$ we have

$$\int_{S_K} |f||dz|^2 \le \frac{C\epsilon^2}{K}. \tag{18}$$

Now, consider $z \in \Omega$. Then Lemma 4 enables us to compute the limit

$$\lim_{n\to\infty} \phi_n(z) = 1.$$

From (17) in Lemma 6, and the Mean Value Theorem for holomorphic functions, one easily concludes that the same holds for every $z \in \mathbb{C} \setminus L$. Thus, we have verified the first property of the sequence ϕ_n. It remains to do the same with the second and the third property.

Let C_1 be the constant from Lemma 4, and C_2 the constant from Lemma 6. We apply Lemma 4 to the function

$$f_{n,k,l}(z) = \frac{\phi_n(z + w_{k,l})}{\alpha_{k,l}(n)},$$

and conclude that for any n, k, l, and any $z \in \partial\mathbb{D}_{\frac{1}{2}}$, we have

$$|f_{n,k,l}(z) - 1| \leq \frac{C_1}{n}.$$

Thus, $f_{n,k,l}$ satisfies assumption from Lemma 6 for sufficiently large n. This implies

$$\int_{\mathbb{D}_{\frac{1}{2}}(w_{k,l})} \big(|\phi_n| - Re(\phi_n)\big)|dz|^2 \leq \frac{C_2C_1^2}{n^2}\alpha_{k,l}(n), \tag{19}$$

and

$$\int_{S_{n,k,l,K}} |\phi_n||dz|^2 \leq \frac{C_2C_1^2}{n^2K}\alpha_{k,l}^2(n), \tag{20}$$

where $S_{n,k,l,K} = \{z \in \mathbb{D}_{\frac{1}{2}}(w_{k,l})\colon |\phi_n(z)| \geq K\}$.

Now we prove the second property of ϕ_n. Combining the pointwise estimate on $W_{p,q} \cap \Omega$ from Lemma 5, with (19), we obtain

$$\begin{aligned}\int_{W_{k,l}} \big(|\phi_n| - \mathrm{Re}(\phi_n)\big)|dz|^2 &\leq \int_{W_{k,l}\cap\Omega} \big(|\phi_n| - \mathrm{Re}(\phi_n)\big)|dz|^2 \\ &\quad + \int_{\mathbb{D}_{\frac{1}{2}}(w_{k,l})} \big(|\phi_n| - \mathrm{Re}(\phi_n)\big)|dz|^2 \\ &\leq \frac{(1+C_2)C_1^2}{n^2}\alpha_{k,l}(n),\end{aligned} \tag{21}$$

and therefore applying (21) on each $W_{k,l}$, we get

$$\begin{aligned}\overline{\lim}_{n\to\infty} \int_{\mathbb{C}} \big(|\phi_n| - \mathrm{Re}(\phi_n)\big)|dz|^2 &= \overline{\lim}_{n\to\infty} \sum_{k,l} \int_{W_{k,l}} \big(|\phi_n| - \mathrm{Re}(\phi_n)\big)|dz|^2 \\ &\leq (1+C_2)C_1^2 \,\overline{\lim}_{n\to\infty} \sum_{k,l} \frac{1}{n^2}\frac{1}{\big(|\frac{z_{k,l}}{n}| + 1\big)^4} \\ &= (1+C_2)C_1^2 \int_{\mathbb{C}} \frac{1}{(|z|+1)^4}|dz|^2 < \infty\end{aligned}$$

confirming that ϕ_n has the second property. Here we used the following lemma.

Lemma 7. *Let $r \geq 3$. Then*

$$\lim_{n\to\infty} \sum_{k,l} \frac{1}{n^2}\frac{1}{\big(|\frac{z_{k,l}}{n}| + 1\big)^r} = \int_{\mathbb{C}} \frac{1}{(|z|+1)^r}|dz|^2 < +\infty.$$

Proof. Note that for some constant $C = C(r)$, the following holds

$$\left| \frac{1}{n^2} \frac{1}{\left(\left|\frac{z_{k,l}}{n}\right| + 1\right)^r} - \int_{W_{k,l}} \frac{1}{(|\frac{z}{n}| + 1)^r} \frac{|dz|^2}{n^2} \right| \leq \frac{C}{n^3} \frac{1}{\left(\left|\frac{z_{k,l}}{n}\right| + 1\right)^r}.$$

This yields

$$\lim_{n\to\infty} \left| \sum_{k,l} \frac{1}{n^2} \frac{1}{\left(\left|\frac{z_{k,l}}{n}\right| + 1\right)^r} - \int_{\mathbb{C}} \frac{1}{(|\frac{z}{n}| + 1)^r} \frac{|dz|^2}{n^2} \right| = 0.$$

Combining this with

$$\int_{\mathbb{C}} \frac{1}{(|\frac{z}{n}| + 1)^r} \frac{|dz|^2}{n^2} = \int_{\mathbb{C}} \frac{1}{(|z| + 1)^r} |dz|^2,$$

proves the lemma. □

Finally we prove that ϕ_n satisfies the third condition. Let $S_{n,K} = \{z \in \mathbb{C} \colon |\phi_n(z)| \geq K\}$. If $K \geq 1 + C_1$ then from Lemma 4 we conclude that $S_{n,K}$ is disjoint from Ω, and thus

$$S_{n,K} = \bigcup_{k,l} S_{n,k,l,K}.$$

We apply (18) from Lemma 6 to obtain

$$\begin{aligned} \overline{\lim}_{n\to\infty} \int_{S_{n,K}} |\phi_n||dz|^2 &\leq \overline{\lim}_{n\to\infty} \sum_{k,l} \int_{S_{n,k,l,K}} |\phi_n||dz|^2 \\ &\leq \frac{C_2 C_1^2}{K} \overline{\lim}_{n\to\infty} \sum_{k,l} \frac{1}{n^2} \frac{1}{\left(\left|\frac{z_{k,l}}{n}\right| + 1\right)^8} \\ &= \frac{C_2 C_1^2}{K} \int_{\mathbb{C}} \frac{1}{(|z| + 1)^8} |dz|^2 = O\left(\frac{1}{K}\right). \end{aligned}$$

Here we used Lemma 7 again. This completes the proof of Theorem 3.

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