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Normalized solutions for logarithmic Schrödinger equation with a perturbation of power law nonlinearity

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Abstract. We study the existence of normalized solutions to the following logarithmic Schrödinger equation

$$-\Delta u + \lambda u = \alpha u \log u^2 + \mu |u|^{p-2} u, \quad x \in \mathbb{R}^N,$$

under the mass constraint

$$\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c^2,$$

where $\alpha, \mu \in \mathbb{R}$, $N \geq 2$, $p > 2$, $c > 0$ is a constant, and $\lambda \in \mathbb{R}$ appears as Lagrange multiplier. Under different assumptions on α, μ, p and c, we prove the existence of ground state solutions and excited state solutions. The asymptotic behavior of the ground state solution as $\mu \to 0$ is also investigated. Our results include the case $\alpha < 0$ or $\mu < 0$, which is less studied in the literature.

Potenssityyppisen epälineaarisen häiriön sisältävän logaritmisen Schrödingerin yhtälön normitetut ratkaisut

Tiivistelmä. Työssä tarkastellaan logaritmisen Schrödingerin yhtälön

$$-\Delta u + \lambda u = \alpha u \log u^2 + \mu |u|^{p-2} u, \quad x \in \mathbb{R}^N,$$

normitettujen ratkaisujen olemassaoloa, kun oletetaan massaehto

$$\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c^2,$$

missä $\alpha, \mu \in \mathbb{R}$, $N \geq 2$, $p > 2$, $c > 0$ on vakio, ja $\lambda \in \mathbb{R}$ on Lagrangen kertoja. Perus- ja viritystilaratkaisujen olemassaolo todistetaan suureita α, μ, p ja c koskevilla eri oletuksilla. Lisäksi tutkitaan perustilaratkaisun asymptoottista käytöstä, kun $\mu \to 0$. Tulokset kattavat kirjallisuudessa vähän tutkitun tapauksen, jossa $\alpha < 0$ tai $\mu < 0$.

1. Introduction

In this paper, for prescribed $c > 0$, we look for solutions $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ satisfying

$$-\Delta u + \lambda u = \alpha u \log u^2 + \mu |u|^{p-2} u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

and

$$\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c^2, \tag{1.2}$$

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where $\alpha, \mu \in \mathbb{R}$ and $2 < p \leq 2^* = \frac{2N}{N-2}$ ($2^* = \infty$ if $N = 1, 2$). Equation (1.1) is closely related to the following time-dependent Schrödinger equation

$$-i\partial_t \Phi - \Delta\Phi - \alpha\Phi \log |\Phi|^2 - \mu|\Phi|^{p-2}\Phi = 0 \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Equation (1.3) admits plenty of applications related to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose–Einstein condensation, see [1, 12, 14, 34, 36, 38] and the references therein.

Normalized solutions for equation (1.1) seem to be particularly meaningful from the physical viewpoint, as pointed out in [32, 33], these solutions often offer a good insight of the dynamical properties of the stationary solutions for the nonlinear Schrödinger equation (1.3), such as stability and instability [8, 15].

In the past three decades, the existence and multiplicity of normalized solutions for Schrödinger equation has been investigated by many authors, see [6, 5, 13, 16, 9, 10, 23, 24, 26, 25, 27, 28, 30, 32, 33, 39, 20, 21, 4]. In particular, if we replace the term $\alpha u \log u^2$ by $|u|^{q-2}u$, then equation (1.1) turns into

$$-\Delta u + \lambda u = |u|^{q-2}u + \mu|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $\mu \in \mathbb{R}$. If $2 < p \leq 2 + \frac{4}{N} \leq q \leq 2^*$, Soave et al. [32, 33, 39, 24, 26] studied the existence and multiplicity of normalized solutions for equation (1.4). Thereafter, normalized solutions to elliptic PDEs have attracted much attention of researchers e.g. [25, 27, 28, 30, 4].

In the case $\mu = 0$, equation (1.1) is the so-called logarithmic Schrödinger equation. Recently, there has been increasing interest in studying logarithmic Schrödinger equation, specially on the existence of positive solutions, multiple solutions, ground states and semiclassical states, see for examples, [2, 11, 22, 31, 42, 36, 35]. In particular, Cazenave [13] investigated the existence and multiplicity of normalized solutions for equation (1.1)–(1.2) with $\alpha = 1$ and $\mu = 0$.

Inspired by the above results, an interesting question is that whether one can find normalized solutions for equation (1.1)–(1.2). The goal of the present paper is to give an affirmative answer. Although equation (1.1) has, at least formally, a variational structure related to the energy functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (\lambda + \alpha)u^2 \,\mathrm{d}x - \frac{\alpha}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x,$$

this energy functional is not well defined on the natural Sobolev space $H^1(\mathbb{R}^N)$. In [13], the idea is to work on the Banach space

$$W = \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} u^2 |\log u^2| \,\mathrm{d}x < \infty \right\}, \tag{1.5}$$

which is equipped the norm given by

$$\|u\|_W = \|u\|_{H^1(\mathbb{R}^N)} + \inf\left\{ k > 0 \mid \int_{\mathbb{R}^N} A(k^{-1}|u|) \,\mathrm{d}x \leq 1 \right\},$$

where $A(s) = -s^2 \log s^2$ on $[0, e^{-3}]$ and $A(s) = 3s^2 + 4e^{-3}s - e^{-6}$ on $[e^{-3}, \infty)$. In fact, by [13, Proposition 2.7], $I_\lambda \colon W \to \mathbb{R}$ is well defined and $\mathcal{C}^1$ smooth.

Solutions to equations (1.1)-(1.2) can be obtained by finding critical points of the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \alpha u^2 \,\mathrm{d}x - \frac{\alpha}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x$$

under the constraint

$$S(c) := \left\{ u \in W \;\middle|\; \|u\|_2 = c \right\}.$$

Before introducing our main results, we give some definitions (see also [7]).

Definition 1.1. We say that u_0 is a ground state solution for equation (1.1)–(1.2) on $S(c)$ if

$$dI|_{S(c)}(u_0) = 0 \quad \text{and} \quad I(u_0) = \inf \left\{ I(u) \text{ s.t. } dI|_{S(c)}(u) = 0 \text{ and } u \in S(c) \right\}.$$

We say that v_0 is an excited state solution for (1.1)–(1.2) on $S(c)$ if

$$dI|_{S(c)}(v_0) = 0 \quad \text{and} \quad I(v_0) > \inf \left\{ I(u) \text{ s.t. } dI|_{S(c)}(u) = 0 \text{ and } u \in S(c) \right\}.$$

For $2 < p < 2^*$, the Gagliardo–Nirenberg inequality (see [40]) is

$$\|u\|_p \leq C(N,p) \|\nabla u\|_2^{\gamma_p} \|u\|_2^{1-\gamma_p}, \quad \text{for each } \; u \in H^1(\mathbb{R}^N), \tag{1.6}$$

where $C(N,p)$ is the best constant in the Gagliardo–Nirenberg inequality, $\gamma_p = \frac{N(p-2)}{2p}$.

We shall mainly concentrate here on the cases $\alpha = \pm 1$ and $\mu \in \mathbb{R}$. For $\alpha = 0$, we refer the reader to [32, 33]. Our main results of the paper can be stated as follows. We first consider the existence of global minima on $S(c)$. Define

$$m(c) := \inf_{u \in S(c)} I(u).$$

Theorem 1.1. *Let $\alpha = 1$. Suppose $N \geq 2$, $c > 0$ and that one of the following three conditions holds*

$$\text{(i) } \mu \leq 0 \text{ and } p > 2; \qquad \text{(ii) } \mu > 0 \text{ and } 2 < p < 2 + \frac{4}{N};$$

$$\text{(iii) } \mu > 0,\; p = \bar{p} := 2 + \frac{4}{N}, \text{ and } c < \left(\frac{N+2}{\mu N} \right)^{\frac{N}{4}} \left(\frac{1}{C(N,\bar{p})} \right)^{\frac{N+2}{2}};$$

where $C(N,\bar{p})$ is the best constant of (1.6) with $p = \bar{p}$, then the infimum $m(c)$ admits a minimizer $\tilde{u} \in S(c)$, which is positive, radially symmetric and decreasing in $r = |x|$. Moreover, $\tilde{u}$ is a ground state of (1.1)–(1.2).

If $2 + \frac{4}{N} < p \leq 2^*$, we can easily verify $\inf_{u \in S(c)} I(u) = -\infty$. In spirit of [5], we introduce the following Pohozaev manifold

$$\mathcal{P}_c = \left\{ u \in S(c) \mid P(u) = 0 \right\},$$

where

$$P(u) =: \|\nabla u\|_2^2 - \mu \gamma_p \|u\|_p^p - \frac{N}{2} \alpha c^2, \tag{1.7}$$

with $\gamma_p = \frac{N(p-2)}{2p}$. If $u \in W$ is a weak solution of (1.1), then the Pohozaev identity $P(u) = 0$ holds. For $u \in S(c)$ and $s \in \mathbb{R}$, we define

$$s \star u(x) := e^{\frac{N}{2}s} u\left(e^s x\right), \quad \text{for a.e. } x \in \mathbb{R}^N.$$

The Pohozaev manifold $\mathcal{P}_c$ is quite closely related to the fiber map

$$\begin{aligned} \Psi_u(s) := I(s \star u) = \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \frac{\alpha}{2} \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x - \alpha \frac{Ns}{2} \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \\ - \frac{\alpha}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \,\mathrm{d}x - \frac{e^{p\gamma_p s}}{p} \mu \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x. \end{aligned}$$

For $u \in S(c)$ and $s \in \mathbb{R}$, we have

$$\Psi_u'(s) = P\big(s \star u\big),$$

where P is defined by (1.7). We shall see that the critical points of $\Psi_u(s)$ allow us to project a function on $\mathcal{P}_c$. Thus, the monotonicity and convexity properties of $\Psi_u(s)$ strongly affect the structure of $\mathcal{P}_c$. Generally, $\mathcal{P}_c$ can be divided into the disjoint union $\mathcal{P}_c = \mathcal{P}_c^+ \cup \mathcal{P}_c^0 \cup \mathcal{P}_c^-$, where

$$\begin{aligned}
\mathcal{P}_c^+ &:= \{u \in S(c) \mid \Psi_u'(0) = 0,\ \Psi_u''(0) > 0\},\\
\mathcal{P}_c^0 &:= \{u \in S(c) \mid \Psi_u'(0) = 0,\ \Psi_u''(0) = 0\},\\
\mathcal{P}_c^- &:= \{u \in S(c) \mid \Psi_u'(0) = 0,\ \Psi_u''(0) < 0\}.
\end{aligned}$$

Define

$$m^+(c) := \inf_{u \in \mathcal{P}_c^+} I(u), \quad m^-(c) := \inf_{u \in \mathcal{P}_c^-} I(u)$$

and

$$c_0 := \left[\frac{p 2^{\frac{p\gamma_p}{2}}}{\mu (p\gamma_p)^{\frac{p\gamma_p+2}{2}}}\left(\frac{p\gamma_p - 2}{N}\right)^{\frac{p\gamma_p-2}{2}} C^{-p}(N,p)\right]^{\frac{1}{p-2}}.$$

If $\alpha = 1$, $\mu > 0$, $2 + \frac{4}{N} < p < 2^*$, then the functional $I|_{S(c)}$ present a convex-concave geometry structure for $c > 0$ small. We thus show that (1.1)–(1.2) admits two solutions u_c^+ and u_c^-, which can be characterized respectively as local minima and mountain pass critical point of I restricted to $S(c)$.

Theorem 1.2. *Let $\alpha = 1$, $\mu > 0$, $2 + \frac{4}{N} < p < 2^*$, $N \geq 2$. If $0 < c < c_0$, then*

(1) *$m^+(c)$ is achieved by some u_c^+;*

(2) *$m^-(c)$ is achieved by some u_c^-.*

Moreover, u_c^+, u_c^- are positive, radially symmetric and decreasing in $r = |x|$. In addition there exist λ_c^+, $\lambda_c^- \in \mathbb{R}$ such that (λ_c^+, u_c^+) and (λ_c^-, u_c^-) are solutions to equation (1.1)-(1.2), and u_c^+ is a ground state solution, u_c^- is a excited solution.

Next, we give the existence of normalized solutions to equation (1.1)–(1.2) with Sobolev critical exponent.

Theorem 1.3. *Let $\alpha = 1$, $\mu > 0$, $p = 2^*$, $N \geq 3$. If $0 < c < c_0$, then*

(1) *$m^+(c)$ is achieved by some u_c^+;*

(2) *$m^-(c)$ is achieved by some u_c^-.*

Moreover, u_c^+, u_c^- are positive, radially symmetric and decreasing in $r = |x|$. In addition there exist λ_c^+, $\lambda_c^- \in \mathbb{R}$ such that (λ_c^+, u_c^+) and (λ_c^-, u_c^-) are solutions to equation (1.1)–(1.2), and u_c^+ is a ground state solution, u_c^- is a excited solution.

Remark 1.1. (1) For $2+\frac{4}{N} < p \leq 2^*$ and $0 < c \leq c_0$, we first study the following local minimization problem

$$m_c := \inf_{u \in V_{k_0}} I(u), \quad \text{where } V_{k_0} = \big\{u \in S(c) \mid \|\nabla u\|_2^2 < k_0\big\},\ k_0 = \frac{p\gamma_p N c^2}{2(p\gamma_p - 2)}.$$

In order to find more than one solution, we decompose the natural constraint $\mathcal{P}_c$ into three disjoint subsets $\mathcal{P}_c^+$, $\mathcal{P}_c^0$ and $\mathcal{P}_c^-$. If $0 < c < c_0$, then u_c^+ is a global minimizer of I restricted on $\mathcal{P}_c$, which can characterized as a local minimizer of I on the set V_{k_0}. Moreover, the second solution u_c^- corresponds to a critical point of mountain-pass type for I on $S(c)$. It is worth mentioning that the existence of such two critical points on $S(c)$ for Schrödinger equation has been studied in [32, 16, 33, 25, 39].

(2) If $p = 2^*$, we have $\gamma_p = 1$ and $C(N,p) = \mathcal{S}^{-\frac{1}{2}}$, then c_0 can be written as $c_0 := \left[\frac{N^2-2N}{4\mu}\left(\frac{4\mathcal{S}}{N^2}\right)^{\frac{2^*}{2}}\right]^{\frac{1}{2^*-2}}$, where $\mathcal{S}$ is the optimal constant in the Sobolev embedding. Compared to the works [16, 25], we can even prove the existence of at least one positive solution for $c = c_0$, see Lemma 5.2.

(3) The condition $0 < c < c_0$ in Theorems 1.2 and 1.3 not only ensures that the corresponding energy functional I admits a convex-concave geometry, but also guarantees that the Pohozaev manifold $\mathcal{P}_c$ is a natural constraint, on which the critical points of I are indeed normalized solutions for (1.1)–(1.2). For $p = 2^*$, we drive a better energy estimate on the associated mountain pass level

$$m^-(c) < \frac{\mu^{-\frac{N-2}{2}}}{N}\mathcal{S}^{\frac{N}{2}} + m^+(c),$$

i.e., the mountain pass energy level is less than the usual critical threshold plus the ground state energy, and thus ensures the compactness of Palais–Smale sequence and a mountain pass type solution follows.

Now, we study the asymptotic behavior of ground states as $\mu \to 0$.

Theorem 1.4. *Assume* (λ_μ, u_μ^+) *is the normalized ground state solution of* (1.1)–(1.2) *obtained by Theorem 1.1 (Theorem 1.2 or Theorem 1.3), then, up to a subsequence, we have*

$$u_\mu^+ \to w_0 := c\pi^{-\frac{N}{4}}e^{-\frac{|x|^2}{2}} \quad \textit{strongly in } W,$$

and $\lambda_\mu \to \lambda_0 := -N - \log(c^{-2}\pi^{\frac{N}{2}})$ *as* $\mu \to 0$, *where* (λ_0, w_0) *is a normalized ground state solution for* (1.1)–(1.2) *with* $\mu = 0$ *(see Section 6).*

Next, we consider the case of $\alpha = -1$. The term $-u \log|u|^2$ makes the geometry structure of $I|_{S(c)}$ much more complex, we follow the ideas introduced by Cingolani and Jeanjean [16] and Jeanjean and Le [25] to catch the normalized solutions. The authors in [16, 25] studied the existence of normalized solutions for the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \lambda u + \gamma\phi u = a|u|^{p-2}u & \text{in } \mathbb{R}^d,\\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^d, \end{cases}$$

for $d = 2$ or $d = 3$ respectively. Under different assumptions on γ, $a \in \mathbb{R}$ and p, they proved several existence and multiplicity results.

Theorem 1.5. (i) *If* $p := \bar{p} = 2 + \frac{4}{N}$, $\mu > 0$ *and* $c < \left(\frac{N+2}{\mu N}\right)^{\frac{N}{4}}\left(C(N,\bar{p})\right)^{-\frac{N+2}{2}}$, *then* I *does not have critical points on* $S(c)$.

(ii) *If* $\mu \le 0$, *for any* $p > 2$ *and* $c > 0$, *then* I *does not have critical points on* $S(c)$.

(iii) *Let* $p > 2$, $N \ge 2$ *and* $c > 0$, *then equation* (1.1)–(1.2) *has no positive radial solution in* $W \cap H_r^1(\mathbb{R}^N)$ *for all* $\mu \in \mathbb{R}$.

Define

$$D := \left(\frac{Np\gamma_p}{2(2-p\gamma_p)}\right)^{\frac{2-p\gamma_p}{2(p-2)}}\left(\frac{\mu p\gamma_p^2}{2}C^p(N,p)\right)^{-\frac{1}{p-2}}.$$

By Lemma 7.2, we obtain

$$\inf_{u\in\mathcal{P}_c} I(u) = -\infty \quad \text{if } c \ge D.$$

However

$$\sup_{u\in\mathcal{P}_c} I(u) < +\infty \quad \text{if } c \geq D.$$

We are able to prove that the existence of a global maximal on $\mathcal{P}_c$.

Theorem 1.6. *Let $\alpha = -1$. If $p \in \left(2, 2+\frac{8}{N(N+2)}\right) \cup \left(2+\frac{8}{N(N+2)}, 2+\frac{4}{N}\right)$, $N \geq 2$, $\mu > 0$ and $c = D$, there exists $\bar{u} \in S(c)$ such that*

$$I(\bar{u}) = \sup_{u\in\mathcal{P}_c\cap H^1_r(\mathbb{R}^N)} I(u).$$

Moreover, $\bar{u}$ is a critical point of I restricted to $S(c)$, and $\bar{u}$ is non-positive and radially symmetric.

Remark 1.2. (1) If $\alpha < 0$, for fixed frequency case, the study of (1.1) is still an open field of investigation. Fortunately, with the help of prescribed L^2-norm and Pohozaev manifold, we get partial existence and non-existence results.

(2) By Lemma 7.1, $\mathcal{P}_c$ is not empty if and only if $c \geq D$. In addition, if $0 < c \leq D$, $\mathcal{P}_c$ is a smooth manifold of codimension 2 in $W \cap H^1_r(\mathbb{R}^N)$ (see Lemma 7.5). Therefore, for $c = D$, we prove the existence of one global maximizer of I on $\mathcal{P}_c$. In particular, in the mass-subcritical case $2 < p < 2+\frac{4}{N}$, it could be natural to expect that there exists a second radial critical point on $S(c)$. However, we can not ensure that the corresponding energy functional I admits a concave-convex geometry structure on $S(c)$, and can not guarantees that the Pohozaev manifold $\mathcal{P}_c$ is a natural constraint.

This paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we study the existence of the associated global minimizer and prove Theorem 1.1. In Section 4, we mainly focus on mass-supercritical and Sobolev subcritical case. In Section 5, mass-supercritical and Sobolev critical case is investigated, two pairs normalized solutions are obtained. While in Section 6, we give a precise asymptotic behavior of the normalized ground state solutions as $\mu \to 0$. Finally, in Section 7, we prove some existence and non-existence results of (1.1)–(1.2) in the case of $\alpha < 0$.

Notation. In the paper, we use the following notations. $L^p = L^p(\mathbb{R}^N)$ with norm $\|u\|_{L^p(\mathbb{R}^N)} = \|u\|_p$, $H^1(\mathbb{R}^N)$ is the usual Sobolev space with norm $\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,\mathrm{d}x\right)^{\frac{1}{2}}$. C, C_i, $i = 1, 2, \cdots$, and so on denote universal positive constants, which we need not to specify, and which may vary from line to line.

2. Preliminaries

In this section, we give some preliminary results. Similar as [13], we define

$$A(s) := \begin{cases} -s^2 \log s^2, & \text{if } 0 \leq s \leq e^{-3}, \\ 3s^2 + 4e^{-3}s - e^{-6}, & \text{if } e^{-3} \leq s, \end{cases}$$

and

$$B(s) := s^2 \log s^2 + A(s).$$

From Lemma 1.2 in [13], A is a positive convex increasing function. Moreover, for every $s > 0$, there exists $K_q > 0$ such that

$$B(s) \leq K_q s^q \quad \text{for all } q \in \left(2, 2+\frac{4}{N}\right). \tag{2.1}$$

Denote

$$V := \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \mid A(|u|) \in L^1(\mathbb{R}^N)\right\}.$$

Lemma 2.1. [13, Lemma 2.1]

(i) *V is the Orlicz space associated to A. V equipped with the norm $\|\cdot\|_V$ defined by*

$$\|u\|_V := \inf\left\{k > 0 \mid \int_{\mathbb{R}^N} A(k^{-1}|u|)\,\mathrm{d}x \le 1\right\}$$

is a reflexive Banach space.

(ii) *For any $u \in V$, $\inf\{\|u\|_V, \|u\|_V^2\} \le \int_{\mathbb{R}^N} A(|u|)\,\mathrm{d}x \le \sup\{\|u\|_V, \|u\|_V^2\}$.*

(iii) *If $u_n \to u$ a.e. in $\mathbb{R}^N$ and $\int_{\mathbb{R}^N} A(|u_n|)\,\mathrm{d}x \to \int_{\mathbb{R}^N} A(|u|)\,\mathrm{d}x < \infty$, then $\|u_n - u\|_V \to 0$ as $n \to \infty$.*

Define

$$W_r := W \cap H_r^1(\mathbb{R}^N),$$

where W is defined by (1.5). Obviously, W is a reflexive Banach space, since $H^1(\mathbb{R}^N)$ and V are both reflexive Banach spaces.

Lemma 2.2. [13, Proposition 2.7, Proposition 3.1] *The following facts hold:*

(i) *$I \in \mathcal{C}^1(W, \mathbb{R})$ and for any $u \in W$ one has $DI(u) = Lu$, where*

$$Lu = -\Delta u - \alpha u \log u^2 - \mu |u|^{p-2} u.$$

(ii) *The embedding from W_r to $L^2(\mathbb{R}^N)$ is compact.*

Lemma 2.3. [6, Lemma 3.6] *For $u \in S(c)$ and $t \in \mathbb{R}$, the map $T_u S(c) \to T_{t\star u} S(c)$ defined by $\psi \to t \star \psi$ is a linear isomorphism, where*

$$T_u S(c) = \left\{v \in H^1(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} uv\,\mathrm{d}x = 0\right\}.$$

3. The proof of Theorem 1.1

In this section, we prove Theorem 1.1. Define

$$m(c) := \inf_{u \in S(c)} I(u) \quad \text{and} \quad m_r(c) := \inf_{u \in S(c) \cap H_r^1(\mathbb{R}^N)} I(u).$$

Lemma 3.1. *Under the assumption of Theorem 1.1, then $m(c) > -\infty$ and $m(c) = m_r(c)$.*

Proof. We first prove $m(c) > -\infty$. For case (i): $\mu \le 0$ and $p > 2$. Let $2 < q < 2 + \frac{4}{N}$, for each $u \in W$, by using (2.1), we can deduce that

$$\begin{aligned} I(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} A(|u|)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} B(|u|)\,\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,\mathrm{d}x - \frac{1}{2}K_q \int_{\mathbb{R}^N} |u|^q\,\mathrm{d}x \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}C^q(N,q) K_q c^{q(1-\gamma_q)} \|\nabla u\|_2^{q\gamma_q}. \end{aligned} \tag{3.1}$$

For case (ii): $\mu > 0$ and $2 < p < 2 + \frac{4}{N}$. Similar as (3.1), for each $u \in W$, we have

$$\begin{aligned} I(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} A(|u|) - B(|u|)\,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}C^q(N,q) K_q c^{q(1-\gamma_q)} \|\nabla u\|_2^{q\gamma_q} - \frac{\mu}{p} C^p(N,p) c^{p(1-\gamma_p)} \|\nabla u\|_2^{p\gamma_p}. \end{aligned} \tag{3.2}$$

For case (iii): $\mu > 0$, $p = \bar{p}$, and $c < \left(\mu^{-1}\frac{N+2}{N}\right)^{\frac{N}{4}} \left(C(N,\bar{p})\right)^{-\frac{N+2}{2}}$. It follows from (3.2) that

$$I(u) \geq \left(\frac{1}{2} - C^{\bar{p}}(N,\bar{p})\frac{\mu c^{\frac{4}{N}}}{\bar{p}}\right)\|\nabla u\|_2^2 - C^q(N,q)K_q c^{q(1-\gamma_q)}\|\nabla u\|_2^{q\gamma_q}. \tag{3.3}$$

Since $q\gamma_q$, $p\gamma_p < 2$ for $p, q \in [2, 2+\frac{4}{N})$, we then conclude that, under the assumption of Theorem 1.1, I is coercive on $S(c)$. Therefore, $m(c) > -\infty$.

On the other hand, one can easily conclude $m(c) \leq m_r(c)$, so we only need to prove $m(c) \geq m_r(c)$.

Assume that $\{u_n\} \subset S(c)$ is a minimizing sequence for $m(c)$. Denote u_n^* be the symmetric decreasing rearrangement of u_n. Thus, by [29, (iv)–(v) of Chapter 3.3, and Lemma 7.17], one has

$$\int_{\mathbb{R}^N} |\nabla u_n^*|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla u_n|^2\,\mathrm{d}x, \quad \int_{\mathbb{R}^N} |u_n^*|^r\,\mathrm{d}x = \int_{\mathbb{R}^N} |u_n|^r\,\mathrm{d}x \quad \text{for } r \in [2, 2^*].$$

Recall that A, B are positive, convex, increasing functions on $(0, +\infty)$, by using [29, (v) of Chapter 3.3], we get

$$\int_{\mathbb{R}^N} A(u_n^*)\,\mathrm{d}x = \int_{\mathbb{R}^N} A(u_n)\,\mathrm{d}x, \quad \int_{\mathbb{R}^N} B(u_n^*)\,\mathrm{d}x = \int_{\mathbb{R}^N} B(u_n)\,\mathrm{d}x,$$

which implies

$$\int_{\mathbb{R}^N} |u_n^*|^2 \log|u_n^*|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |u_n|^2 \log|u_n|^2\,\mathrm{d}x.$$

Therefore

$$m_r(c) = \inf_{u \in S(c)\cap H_r^1(\mathbb{R}^N)} I(u) \leq \inf_{u\in S(c)} I(u) = m(c). \qquad \square$$

Lemma 3.2. *Under the assumption of Theorem 1.1, then the infimum $m(c)$ is achieved by some $u \in S(c)$, which is a positive, radially symmetric and decreasing in $r = |x|$.*

Proof. Assume that $\{u_n\} \subset S(c) \cap H_r^1(\mathbb{R}^N)$ be a minimizing sequence for $m(c)$. From (3.1)–(3.3), we deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and $\{\int_{\mathbb{R}^N} A(|u_n|)\mathrm{d}x\}$ is bounded. By Lemma 2.1, we conclude $\{u_n\}$ is bounded in W_r. Therefore, $u_n \rightharpoonup u$ weakly in W_r, it follow from Lemma 2.2 that $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. If $\mu > 0$, we deduce that

$$B(|u_n|) \to B(|u|) \text{ strongly in } L^1(\mathbb{R}^N) \quad \text{and} \quad u_n \to u \text{ strongly in } L^p(\mathbb{R}^N).$$

Therefore

$$\begin{aligned} m(c) \leq I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2\log u^2\,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 + u_n^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} A(|u_n|)\,\mathrm{d}x\right) \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N} B(|u|)\,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x \\ &\leq \liminf_{n\to\infty} I(u_n) = m(c). \end{aligned} \tag{3.4}$$

Hence, $I(u) = m(c)$, $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$ and $A(|u_n|) \to A(|u|)$ in $L^1(\mathbb{R}^N)$. It thus follows from Lemma 2.1 that $u_n \to u$ strongly in V as $n \to \infty$. Therefore, $u_n \to u$ strongly in W_r.

If $\mu \leq 0$, we apply the same argument. By the weak lower semi-continuity, we have

$$\liminf_{n\to\infty}\left(-\frac{\mu}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x\right)\leq -\frac{\mu}{p}\int_{\mathbb{R}^N}|u_n|^p\,\mathrm{d}x.$$

Similar as (3.4), we obtain $u_n \to u$ strongly in W_r. Since u is a nonnegative nontrivial weak solution of (1.1)-(1.2). Moreover, by elliptic regularity theory, we obtain $u \in \mathcal{C}^2(\mathbb{R}^N)$. For $a > 0$ small enough, we have

$$\Delta u = \lambda u - u\log u^2 - \mu u^{p-1} \leq \beta(u) \quad \text{in } \{x \in \mathbb{R}^N \mid 0 < u(x) < a\},$$

where $\beta(s) = \lambda s - s\log s^2$ for $\mu > 0$ and $\beta(s) = (\lambda-\mu)s - s\log s^2$ for $\mu \leq 0$. Since β is continuous, nondecreasing for s small, $\beta(0) = 0$ and $\beta(\sqrt{e^\lambda}) = 0$ for $\mu > 0$, $\beta(\sqrt{e^{\lambda-\mu}}) = 0$ for $\mu \leq 0$, by [37, Theorem 1], we have that $u > 0$. □

Proof of Theorem 1.1. The proof follows directly from Lemma 3.2. □

4. Mass-supercritical and Sobolev subcritical case

In this section, we deal with the mass supercritical case $2+\frac{4}{N} < p < 2^* = \frac{2N}{(N-2)^+}$, $N \geq 2$, $\alpha > 0$ and $\mu > 0$, the functional I is unbounded from below on $S(c)$, it will not be possible to find a global minimizer.

4.1. Existence of a local minima on $S(c)$.

For $\alpha = 1$, $c > 0$ and $2 + \frac{4}{N} < p < 2^*$, $N \geq 2$, define

$$k_0 = \frac{p\gamma_p N c^2}{2(p\gamma_p - 2)}, \quad c_0 = \left[\frac{p2^{\frac{p\gamma_p}{2}}}{\mu(p\gamma_p)^{\frac{p\gamma_p+2}{2}}}\left(\frac{p\gamma_p-2}{N}\right)^{\frac{p\gamma_p-2}{2}} C^{-p}(N,p)\right]^{\frac{1}{p-2}}. \tag{4.1}$$

Lemma 4.1. *Suppose that $\mu > 0, c > 0$, and $2 + \frac{4}{N} < p < 2^*$. If $P(u) \leq 0$ and $\|\nabla u\|_2^2 = k_0$, then $c \geq c_0$. Moreover, if $P(u) \leq 0$ and $c < c_0$, then $\|\nabla u\|_2^2 \neq k_0$.*

Proof. Since $P(u) \leq 0$, we have

$$\|\nabla u\|_2^2 \leq \mu\gamma_p\|u\|_p^p + \frac{N}{2}c^2.$$

By using the Gagliardo–Nirenberg inequality (1.6), it follows that

$$\|\nabla u\|_2^2 \leq \mu\gamma_p C^p(N,p)c^{p(1-\gamma_p)}\|\nabla u\|_2^{p\gamma_p} + \frac{N}{2}c^2.$$

If $\|\nabla u\|_2^2 = k_0$, then

$$\frac{N}{p\gamma_p - 2} \leq \mu\gamma_p C^p(N,p)\left(\frac{Np\gamma_p}{2(p\gamma_p-2)}\right)^{\frac{p\gamma_p}{2}} c^{p-2},$$

which follows that $c \geq c_0$. Therefore, we deduce that if $P(u) \leq 0$ and $c < c_0$, then $\|\nabla u\|_2^2 \neq k_0$. We obtain the conclusion of the lemma. □

Now, we define

$$V_{k_0} := \{u \in S(c) \mid \|\nabla u\|_2^2 < k_0\}, \tag{4.2}$$

where k_0 is defined by (4.1). Next, for each $0 < c \leq c_0$, we study the following minimization problem

$$m_c := \inf_{u \in V_{k_0}} I(u). \tag{4.3}$$

Lemma 4.2. *Let $\mu > 0$ and $2 + \frac{4}{N} < p < 2^*$. If $0 < c \le c_0$, then m_c is achieved by some u, which is a positive, radially symmetric critical point of I on $S(c)$.*

Proof. Let $\{u_n\}$ be a minimizing sequence for m_c, similar as the proof of Lemma 3.2, we can deduce $u_n \to u$ strongly in W_r. We only need to prove that $u \in V_{k_0}$. In fact, if $\|\nabla u\|_2^2 = k_0$ and $0 < c < c_0$, it directly follows from Lemma 4.1 that $P(u) > 0$. Therefore, there exist $t_0 < 0$ such that $t_0 \star u \in V_{k_0}$, and $I(t_0 \star u) < I(u) = m_c$. This is a contradiction. On the other hand, if $\|\nabla u\|_2^2 = k_0$ and $c = c_0$, the discussion is divided into three cases.

Case 1: If $P(u) < 0$, similar to the proof in Lemma 4.1, we have $c > c_0$, which is a contradiction.

Case 2: If $P(u) = 0$, we have

$$\|\nabla u\|_2^2 \le \mu\gamma_p C^p(N,p) c^{p(1-\gamma_p)} \|\nabla u\|_2^{p\gamma_p} + \frac{N}{2}c^2. \tag{4.4}$$

The equality in (4.4) holds only for the best constant in the Gagliardo–Nirenberg inequality is achieved, from [40], u satisfies

$$\|\nabla u\|_2^2 = \|u\|_2^2. \tag{4.5}$$

Combining $P(u) = 0$ and (4.5), which contradicts with $N \ge 2$. Therefore,

$$\|\nabla u\|_2^2 < \mu\gamma_p C^p(N,p) c^{p(1-\gamma_p)} \|\nabla u\|_2^{p\gamma_p} + \frac{N}{2}c^2,$$

we deduce that $c > c_0$ because $\|\nabla u\|_2^2 = k_0$. This is also a contradiction.

Case 3: If $P(u) > 0$, then there is $t_1 < 0$ such that $t_1 \star u \in V_{k_0}$, and $I(t_1 \star u) < I(u) = m_c$. This is a contradiction.

In summarization, $u \in V_{k_0}$ and $I(u) = m_c$. Therefore, by combining $\|\nabla u\|_2^2 < k_0$ with [30, Proposition A.1], we conclude that u is a positive, radially symmetric critical point of I on $S(c)$. □

4.2. Multiplicity of solutions. In this subsection, we are interested in the multiplicity of solutions. For any $0 < c < c_0$, we prove that (1.1)–(1.2) admits two solutions, which can be characterized respectively as a local minimizer or a mountain pass critical point of the energy functional I restricted to $S(c)$. We first study the structure of the Pohozaev manifold $\mathcal{P}_c$. Recalling the decomposition of $\mathcal{P}_c = \mathcal{P}_c^+ \cap \mathcal{P}_c^0 \cap \mathcal{P}_c^-$.

Lemma 4.3. *Let $\mu > 0$ and $2 + \frac{4}{N} < p < 2^*$. If $0 < c < c_0$, then $\mathcal{P}_c^0 = \emptyset$, and the set $\mathcal{P}_c$ is a $\mathcal{C}^1$-submanifold of codimension 1 in $S(c)$.*

Proof. Assume by contradiction that there exists $u \in \mathcal{P}_c^0$ such that $P(u) = 0$ and

$$\Psi_u''(0) = 2\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - p\gamma_p^2 \mu \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x = 0.$$

Let

$$\begin{aligned} f(t) :&= t\Psi_u'(0) - \Psi_u''(0) \\ &= (t-2)\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - (t - p\gamma_p)\gamma_p\mu \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x - \frac{N}{2}tc^2, \end{aligned}$$

we see that $f(t) = 0,\ \forall t \in \mathbb{R}$. Therefore, it follows from $f(p\gamma_p) = 0$ that

$$(p\gamma_p - 2)\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x = p\gamma_p \frac{N}{2}c^2. \tag{4.6}$$

From (4.6), we have $\|\nabla u\|_2^2 = k_0$. Since $f(2) = 0$ and $\|\nabla u\|_2^2 = k_0$, we get

$$N \le \mu(p\gamma_p - 2)\gamma_p C^p(N,p)\left(\frac{Np\gamma_p}{2(p\gamma_p - 2)}\right)^{\frac{p\gamma_p}{2}} c^{p-2},$$

which is a contradiction with $c < c_0$. This proves that $\mathcal{P}_c^0 = \emptyset$.

We now prove that $\mathcal{P}_c$ is a smooth manifold of codimension 1 in $S(c)$. We know that $\mathcal{P}_c$ is defined by $P(u) = 0$ and $G(u) = 0$ where

$$G(u) = \|u\|_2^2 - c^2.$$

Since P and G are of $\mathcal{C}^1$-class, the proof is complete provided we show that the differential $(dP(u), dG(u)) \colon W \to \mathbb{R}^2$ is surjective, for every

$$u \in G^{-1}(0) \cap P^{-1}(0).$$

If this is not true, $dP(u)$ has to be linearly dependent from $dG(u)$, i.e. there exists $\nu \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} 2\nabla u \nabla\varphi_1 + 2\nu u\varphi_1 \,\mathrm{d}x = \int_{\mathbb{R}^N} p\gamma_p\mu|u|^{p-2}u\varphi_1 \,\mathrm{d}x, \quad \text{for each } \varphi_1 \in W.$$

Therefore, u satisfies

$$-2\Delta u + 2\nu u = p\gamma_p\mu|u|^{p-2}u.$$

The Pohozaev identity for the above equation is

$$2\|\nabla u\|_2^2 = p\gamma_p^2\mu\|u\|_p^p,$$

which contradicts to $\mathcal{P}_c^0 = \emptyset$. □

Lemma 4.4. *If $2 + \frac{4}{N} < p < 2^*$, $N \ge 2$ and $0 < c < c_0$, for each $u \in S(c)$, then the function $h(s) := \Psi_u(s)$ has exactly two critical points s_u, t_u with $s_u < t_u$. Moreover,*

(i) *$s_u \star u \in \mathcal{P}_c^+$ and s_u is a strict local minimum point for $h(s)$; $t_u \star u \in \mathcal{P}_c^-$ and t_u is a strict local maximum point for $h(s)$;*

(ii) *The maps $u \mapsto s_u \in \mathbb{R}$ and $u \mapsto t_u \in \mathbb{R}$ are of class $\mathcal{C}^1$.*

Proof. (i) For each fixed $u \in S(c)$ and $\mu > 0$, there exists $\bar t = \frac{1}{p\gamma_p - 2}\log\left(\frac{2\|\nabla u\|_2^2}{p\gamma_p^2\mu\|u\|_p^p}\right)$ such that

$$2e^{2\bar t}\|\nabla u\|_2^2 = p\gamma_p^2 e^{p\gamma_p \bar t}\mu\|u\|_p^p,$$

i.e.

$$2\|\bar t \star (\nabla u)\|_2^2 = p\gamma_p^2\mu\|\bar t \star u\|_p^p.$$

It follows that

$$2\|t \star (\nabla u)\|_2^2 > p\gamma_p^2\mu\|t \star u\|_p^p \quad \text{for all } -\infty < t < \bar t \tag{4.7}$$

and

$$2\|t \star (\nabla u)\|_2^2 < p\gamma_p^2\mu\|t \star u\|_p^p \quad \text{for all } t > \bar t.$$

From (4.7), for $-\infty < t < \bar t$, we have

$$\begin{aligned}
\Psi_u'(t) &= \|t \star (\nabla u)\|_2^2 - \mu\gamma_p\|t \star u\|_p^p - \frac{N}{2}c^2 \\
&> \|t \star (\nabla u)\|_2^2 - \frac{2}{p\gamma_p}\|t \star (\nabla u)\|_2^2 - \frac{N}{2}c^2 \\
&= \frac{p\gamma_p - 2}{p\gamma_p}\|t \star (\nabla u)\|_2^2 - \frac{N}{2}c^2.
\end{aligned} \tag{4.8}$$

Now, we claim that if $c < c_0$, then

$$\|\bar{t} \star (\nabla u)\|_2^2 > \frac{p\gamma_p N c^2}{2(p\gamma_p - 2)} = k_0.$$

In fact, since $c < c_0$, we have

$$\begin{aligned}\frac{p\gamma_p - 2}{p\gamma_p} e^{2\bar{t}} \|\nabla u\|_2^2 - \frac{N}{2}c^2 &= \left(\frac{2\|\nabla u\|_2^2}{p\gamma_p^2 \mu \|u\|_p^p}\right)^{\frac{2}{p\gamma_p - 2}} \frac{p\gamma_p - 2}{p\gamma_p}\|\nabla u\|_2^2 - \frac{N}{2}c^2 \\ &\geq \left(\frac{2}{p\gamma_p^2\mu}\right)^{\frac{2}{p\gamma_p-2}} C^{-\frac{2p}{p\gamma_p-2}}(N,p)\frac{p\gamma_p - 2}{p\gamma_p} c^{-\frac{2p(1-\gamma_p)}{p\gamma_p-2}} - \frac{N}{2}c^2 \\ &> 0.\end{aligned}$$

This gives $\|\bar{t} \star (\nabla u)\|_2^2 > k_0$. It follows that there exists $\delta > 0$ such that for any $(\bar{t} - \delta, \bar{t})$

$$\|t \star (\nabla u)\|_2^2 > k_0.$$

From (4.8), we get that for any $(\bar{t} - \delta, \bar{t})$, $\Psi_u'(t) > 0$ and thus $\Psi_u(t)$ increasing in $(\bar{t} - \delta, \bar{t})$.

Notice that

$$\lim_{s\to-\infty} \Psi_u(s) = +\infty \quad \text{and} \quad \lim_{s\to+\infty} \Psi_u(s) = -\infty,$$

we thus conclude that $\Psi_u(s)$ has a local minimum point $s_u < \bar{t}$ and has a local maximum point $t_u > \bar{t}$. Since $s_u < \bar{t}$, by (4.7), we deduce that

$$2\|s_u \star (\nabla u)\|_2^2 > p\gamma_p^2 \mu \|s_u \star u\|_p^p. \tag{4.9}$$

In addition, combining (4.9) and $\Psi_u'(s_u) = 0$, we have

$$\Psi_u''(s_u) = 2\|s_u \star (\nabla u)\|_2^2 - p\gamma_p^2\mu\|s_u \star u\|_p^p > 0.$$

This implies that s_u is a strict minimum point for $\Psi_u'(t)$ and $s_u \star u \in \mathcal{P}_c^+$.

Next, we prove that s_u is unique. By contradiction, we assume that s_u^* is another local minimum point. We then claim that $s_u^* < \bar{t}$. If not, we assume $s_u^* \geq \bar{t}$, then

$$\Psi_u''(s_u^*) = 2\|s_u^* \star (\nabla u)\|_2^2 - p\gamma_p^2\mu\|s_u^* \star u\|_p^p < 0,$$

which is a contradiction. Therefore, $s_u^* < \bar{t}$ and $s_u^* \star u \in \mathcal{P}_c^+$. Moreover, there exists a another point, $s_u^0 < \bar{t}$ such that s_u^0 is a local maximum point for Ψ_u. From (4.7), we have $\Psi_u''(s_u^0) > 0$, which is a contradiction. Thus s_u is unique. Similar to the proof of s_u is unique, we can prove t_u is also unique.

(ii) Finally, applying the Implicit Function Theorem to the $\mathcal{C}^1$ function $g(s,u)\colon \mathbb{R}\times S(c) \to \mathbb{R}$ defined by $g_u(s) = g(s,u) = \Psi_u'(s)$. Therefore, we have that $u \mapsto s_u$ is of class $\mathcal{C}^1$ because $g_u(s_u) = 0$ and $\partial_s g_u(s_u) = \Psi_u''(s_u) > 0$. Similarly, we can prove that $u \mapsto t_u \in \mathbb{R}$ is of class $\mathcal{C}^1$. □

Lemma 4.5. *Let $\mu > 0$ and $2 + \frac{4}{N} < p < 2^*$. Suppose that $0 < c < c_0$. Then*

$$m_c := \inf_{u\in V_{k_0}} I(u) = m^+(c),$$

where V_{k_0} is defined in (4.2).

Proof. From the definition of $m^+(c)$, we can deduce that $\mathcal{P}_c^+ \subset V_{k_0}$, then $m_c \leq m^+(c)$. On the other hand, if $u \in V_{k_0}$, $s_u \star u \in \mathcal{P}_c^+ \subset V_{k_0}$, we have

$$I(s_u \star u) = \min\{I(s \star u) \mid s \in \mathbb{R} \text{ and } \|s \star (\nabla u)\|_2^2 < k_0\} \leq I(u),$$

it follows that $m^+(c) = \inf_{\mathcal{P}_c^+} I \le m_c$. $\square$

Now, we define

$$\mathcal{P}_{r,c}^+ := \mathcal{P}_c^+ \cap H_r^1(\mathbb{R}^N) \quad \text{and} \quad \mathcal{P}_{r,c}^- := \mathcal{P}_c^- \cap H_r^1(\mathbb{R}^N).$$

Lemma 4.6. *Let $\mu > 0$ and $2 + \frac{4}{N} < p < 2^*$. If $0 < c < c_0$, then*

$$m^+(c) := \inf_{u \in \mathcal{P}_c^+} I(u) = \inf_{u \in \mathcal{P}_{r,c}^+} I(u) \quad \textit{and} \quad m^-(c) := \inf_{u \in \mathcal{P}_c^-} I(u) = \inf_{u \in \mathcal{P}_{r,c}^-} I(u).$$

Proof. Since $\mathcal{P}_{r,c}^+ \subset \mathcal{P}_c^+$, $\mathcal{P}_{r,c}^- \subset \mathcal{P}_c^-$, we have

$$\inf_{u \in \mathcal{P}_{r,c}^+} I(u) \ge \inf_{u \in \mathcal{P}_c^+} I(u) \quad \text{and} \quad \inf_{u \in \mathcal{P}_{r,c}^-} I(u) \ge \inf_{u \in \mathcal{P}_c^-} I(u).$$

On the other hand, by Lemma 4.3, for each $u \in S(c)$, there exist $s_u, t_u \in \mathbb{R}$ such that $s_u \star u \in \mathcal{P}_c^+, t_u \star u \in \mathcal{P}_c^-$, and

$$\begin{aligned} \inf_{u \in \mathcal{P}_c^+} I(u) &= \inf_{u \in S(c)} \min_{-\infty < t \le s_u} I(t \star u), \\ \inf_{u \in \mathcal{P}_c^-} I(u) &= \inf_{u \in S(c)} \max_{s_u < t \le t_u} I(t \star u). \end{aligned}$$

For $u \in S(c)$, let $u^* \in S_r(c)$ be the Schwarz rearrangement of $|u|$. By [29, Chapter 3], we have for all $t > 0$, $I(t \star u^*) \le I(t \star u)$. Recall that $\Psi_u'(t) = P(t \star u)$, we have

$$\lim_{s \to -\infty} \Psi_{u^*}'(s) \le \lim_{s \to -\infty} \Psi_u'(s) = +\infty \quad \text{and} \quad \Psi_{u^*}''(t) \le \Psi_u''(t), \quad \text{for each} \quad t \in \mathbb{R}.$$

This implies that $-\infty < s_u \le s_{u^*} < t_{u^*} \le t_u$. Therefore, we get that

$$\min_{-\infty < t < s_{u^*}} I(t \star u^*) \le \min_{-\infty < t < s_u} I(t \star u)$$

and

$$\max_{s_{u^*} < t \le t_{u^*}} I(t \star u^*) \le \max_{s_u < t \le t_u} I(t \star u).$$

Therefore, $\inf_{u \in \mathcal{P}_{r,c}^+} I(u) \le \inf_{u \in \mathcal{P}_c^+} I(u)$ and $\inf_{u \in \mathcal{P}_{r,c}^-} I(u) \le \inf_{u \in \mathcal{P}_c^-} I(u)$. $\square$

Lemma 4.7. *Let $\mu > 0$ and $2 + \frac{4}{N} < p < 2^*$. If $0 < c < c_0$, then $m^+(c)$ and $m^-(c)$ can be achieved by some u_c^+ and u_c^-, respectively, which are positive, radially symmetric and decreasing in $r = |x|$.*

Proof. By Lemma 4.2 and Lemma 4.5, the proof for $m^+(c)$ is completed. We only give the proof for $m^-(c)$. Let $\{u_n\} \subset \mathcal{P}_c^-$ be a minimizing sequence of $m^-(c)$. From Lemma 4.6, then by taking $|u_n|$ and adapting the Schwarz symmetrization to $|u_n|$ if necessary, we can obtain a new minimizing sequence (up to a subsequence), such that u_n are all nonnegative, radially symmetric and decreasing in $r = |x|$. Since $u_n \in \mathcal{P}_c^-$, we have

$$\begin{aligned} I(u_n) &= \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right) \|\nabla u_n\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \log u_n^2 \, dx + \frac{p}{2(p-2)} c^2 \\ &\ge \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right) \|\nabla u_n\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} A(|u_n|) \, dx - \frac{1}{2} \int_{\mathbb{R}^N} B(|u_n|) \, dx \\ &\ge \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right) \|\nabla u_n\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} A(|u_n|) \, dx - \frac{K_q}{2} \int_{\mathbb{R}^N} |u_n|^q \, dx, \end{aligned}$$

where $2 < q < 2 + \frac{4}{N}$. Thus, by the Gagliardo–Nirenberg inequality (1.6), $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} A(|u_n|) \, dx$ is bounded. By Lemma 2.1, we get $\{u_n\}$ is bounded in W_r. By Lemma 2.2, $u_n \rightharpoonup u_0$ weakly in W_r, $u_n \to u_0$ strongly in $L^2(\mathbb{R}^N)$

and $u_n \to u_0$ a.e. in $\mathbb{R}^N$, and thus $u_0 \in S(c)$, $P(u_0) \le 0$. From Lemma 4.6, there exists $s_0 \in \mathbb{R}$ such that $s_0 \star u_0 \in \mathcal{P}_c^-$,

$$m^-(c) + o_n(1) = I(u_n) \ge I(s_0 \star u_n) \ge I(s_0 \star u_0) + o_n(1) \ge m^-(c).$$

$I(s_0 \star u_0) = m^-(c)$, $A(|u_n|) \to A(|s_0 \star u_0|)$ in $L^1(\mathbb{R}^N)$. It thus follows from Lemma 2.1 that $u_n \to s_0 \star u_0$ in V as $n \to \infty$. Therefore, $u_n \to s_0 \star u_0$ in W. Finally, it follows from the proof in Lemma 2.3 that $s_0 \star u_0 > 0$. We deduce that $m^-(c)$ is attained by $s_0 \star u_0$ which is positive radially symmetric and decreasing in $r = |x|$. □

Proof of Theorem 1.2. The proof follows directly from Lemma 4.7. □

5. Sobolev-critical case

In this section, we study the case $\alpha = 1$, $\mu > 0$ and $p = 2^*$. Denote

$$\mathcal{S} = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

Then the following Sobolev inequality hold

$$\mathcal{S}\|u\|_{2^*}^2 \le \|\nabla u\|_2^2, \quad \text{for each } \ u \in D^{1,2}(\mathbb{R}^N), \tag{5.1}$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{D^{1,2}} := \|\nabla u\|_2$. For $c > 0$, $N \ge 3$, we denote

$$k_0 = \frac{N^2 c^2}{4}, \quad c_0 = \left[\frac{N^2 - 2N}{4\mu} \left(\frac{4\mathcal{S}}{N^2} \right)^{\frac{2^*}{2}} \right]^{\frac{1}{2^*-2}}. \tag{5.2}$$

5.1. Existence of a local minima on $S(c)$.

Lemma 5.1. *Suppose that $\mu > 0$, $c > 0$ and $p = 2^*$. If $P(u) \le 0$ and $\|\nabla u\|_2^2 = k_0$, then $c \ge c_0$. Moreover, if $P(u) \le 0$ and $c < c_0$, then $\|\nabla u\|_2^2 \ne k_0$.*

Proof. Since $P(u) \le 0$, we have

$$\|\nabla u\|_2^2 \le \mu \|u\|_{2^*}^{2^*} + \frac{N}{2} c^2.$$

By the Sobolev inequality (5.1), it is easy to verify that

$$\|\nabla u\|_2^2 \le \mu \mathcal{S}^{-\frac{2^*}{2}} \|\nabla u\|_2^{2^*} + \frac{N}{2} c^2.$$

By using the fact $\|\nabla u\|_2^2 = k_0$ that

$$\frac{N^2 - 2N}{4} \le \mu \mathcal{S}^{-\frac{2^*}{2}} \left(\frac{N^2}{4} \right)^{\frac{2^*}{2}} c^{2^*-2},$$

it follows that $c \ge c_0$. Therefore, we deduce that if $P(u) \le 0$ and $c < c_0$, then $\|\nabla u\|_2^2 \ne k_0$. □

Define

$$V_{k_0} := \left\{ u \in S(c) \mid \|\nabla u\|_2^2 < k_0 \right\}. \tag{5.3}$$

where k_0 is defined by (5.2). Thus, for any $0 < c \le c_0$, we consider the following minimization problem

$$m_c := \inf_{u \in V_{k_0}} I(u). \tag{5.4}$$

Lemma 5.2. *Let $\mu > 0$ and $p = 2^*$, $N \geq 3$. Suppose that $0 < c \leq c_0$. Then m_c is achieved by some u, which is a positive, radially symmetric critical point of I on $S(c)$.*

Proof. Let $\{w_n\}$ be a minimizing sequence for m_c. Using the symmetric decreasing rearrangement, we obtain a minimizing sequence $\{w_n^*\} \subset H_r^1(\mathbb{R}^N) \cap V_{k_0}$, w_n^* is radial decreasing for each n. Thus, by the Ekeland's variational principle, we can obtain a nonnegative radial Palais–Smale sequence $\{u_n\}$ for $I|_{S(c)}$ at level m_c, i.e.

$$\lim_{n\to\infty} I(u_n) = m_c \quad \text{and} \quad dI|_{S(c)}(u_n) \to 0 \ \text{ as } \ n \to \infty.$$

Since

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \,\mathrm{d}x - \frac{\mu}{2^*}\int_{\mathbb{R}^N} |u_n|^{2^*} \,\mathrm{d}x + \frac{1}{2}c^2 \\ &\geq \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} A(|u_n|)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} B(|u_n|)\,\mathrm{d}x - \frac{\mu}{2^*}\mathcal{S}^{-\frac{2^*}{2}} k_0^{\frac{2^*}{2}}, \end{aligned}$$

by the same method as the proof of Lemma 4.7, we can show that $\{u_n\}$ is bounded in W_r, then $u_n \rightharpoonup u$ weakly in W_r, $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$. In fact, by the Lagrange multiplier's rule (see [10, Lemma 3]) there is $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \big(\nabla u_n \nabla \phi + \lambda_n u_n \phi - \mu |u_n|^{p-2} u_n \phi - u_n \phi \log u_n^2\big)\,\mathrm{d}x = o_n(1)\|\phi\|_W, \tag{5.5}$$

for each $\phi \in W$. In particular, if we take $\phi = u_n$, we can prove that $\{\lambda_n\}$ is bounded. Hence, up to a subsequence, we assume $\lambda_n \to \lambda \in \mathbb{R}$. Passing to the limit in (5.5), we get that u satisfies

$$-\Delta u + \lambda u = u \log u^2 + \mu |u|^{2^*-2} u.$$

Set $v_n := u_n - u$, we get

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + A(|v_n|)\,\mathrm{d}x - \mu \int_{\mathbb{R}^N} |v_n|^{2^*} \mathrm{d}x = o_n(1).$$

Therefore

$$\begin{aligned} m_c \leq I(u) &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \,\mathrm{d}x - \frac{\mu}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 + A(|v_n|)\,\mathrm{d}x - \frac{\mu}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + u_n^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \,\mathrm{d}x - \frac{\mu}{2^*}\int_{\mathbb{R}^N} |u_n|^{2^*}\,\mathrm{d}x + o_n(1) \\ &= \liminf_{n\to\infty} I(u_n) = m_c. \end{aligned}$$

Next, we only need to prove $u \in V_{k_0}$. In fact, if $\|\nabla u\|_2^2 = k_0$ and $0 < c < c_0$, it follows from Lemma 5.1 that $P(u) > 0$, then there exists $t_0 < 0$ such that $t_0 \star u \in V_{k_0}$ and $I(t_0 \star u) < I(u) = m_c$. This is a contradiction. On the other hand, if $\|\nabla u\|_2^2 = k_0$ and $c = c_0$, the proof is similar to Lemma 4.2. Therefore, $u \in V_{k_0}$ and $I(u) = m_c$. According to the proof in Lemma 3.1, we conclude that $u > 0$. □

5.2. Multiplicity of solutions. In this subsection, we are interested in the multiplicity of solutions. The following two Lemmas are similar to Lemma 4.3 and Lemma 4.4, so we omit the proof.

Lemma 5.3. *Let $\mu > 0$ and $p = 2^*$. If $0 < c < c_0$, then $\mathcal{P}_c^0 = \emptyset$, and the set $\mathcal{P}_c$ is a $\mathcal{C}^1$-submanifold of codimension 1 in $S(c)$.*

Lemma 5.4. *If $p = 2^*$ and $0 < c < c_0$, for $u \in S(c)$, then the function $\Psi_u(s)$ has exactly two critical points $s_u < t_u \in \mathbb{R}$. Moreover,*

(i) *$s_u \star u \in \mathcal{P}_c^+$ and s_u is a strict local minimum point for Ψ_u. $t_u \star u \in \mathcal{P}_c^-$ and t_u is a strict local maximum point for Ψ_u.*
(ii) *The maps $u \mapsto s_u \in \mathbb{R}$ and $u \mapsto t_u \in \mathbb{R}$ are of class $\mathcal{C}^1$.*

In view of Lemma 5.4, we can define

$$m^+(c) := \inf_{u \in \mathcal{P}_c^+} I(u) \quad \text{and} \quad m^-(c) := \inf_{u \in \mathcal{P}_c^-} I(u).$$

Based on σ-homotopy stable family of compact subsets of $\mathcal{P}_c$, we aim to construct bounded Palais–Smale sequences on a manifold by the Ghoussoub minimax principle [19, Theorem 3.2], (see also [23, Section2.2], [25, Lemma 3.7] and [33, Section 5]).

Define the functions

$$J^+ \colon S(c) \to \mathbb{R}, \quad J^+ := I(s_u \star u),$$
$$J^- \colon S(c) \to \mathbb{R}, \quad J^- := I(t_u \star u).$$

Similar as Lemma 5.4 (ii), we can prove the maps $u \mapsto s_u$ and $u \mapsto t_u$ are of class $\mathcal{C}^1$, thus the functionals J^+, J^- are of class $\mathcal{C}^1$.

Lemma 5.5. *For any $0 < c < c_0$, we have that*

$$dJ^+[\psi] = dI(s_u \star u)[s_u \star \psi] \quad \text{and} \quad dJ^-[\psi] = dI(t_u \star u)[t_u \star \psi],$$

for any $u \in S(c)$ and $\psi \in T_u S(c)$.

Proof. We first give the proof for J^+. Let $\psi \in S(c)$, and $\psi = h'(0)$ where $h \colon (-\epsilon, \epsilon) \mapsto S(c)$ is a $\mathcal{C}^1$-curve with $h(0) = u$. We consider the incremental quotient

$$\frac{J^+(h(t)) - J^+(h(0))}{t} = \frac{I(s_t \star h(t)) - I(s_0 \star h(0))}{t},$$

where $s_t := s_{h(t)}$ and $s_0 := s_{h(0)} = s_u$. It follows from Lemma 5.4 that s_0 is a strict local minimum of $s \mapsto I(s \star u)$, we have

$$\begin{aligned}
&I(s_t \star h(t)) - I(s_0 \star h(0)) \geq I(s_t \star h(t)) - I(s_t \star h(0)) \\
&= \frac{e^{2s_t}}{2}\left(\|\nabla h(t)\|_2^2 - \|\nabla h(0)\|_2^2\right) + \left(\frac{1}{2} - \frac{N s_t}{2}\right)\left(\|h(t)\|_2^2 - \|h(0)\|_2^2\right) \\
&\quad - \frac{1}{2}\left(\int_{\mathbb{R}^N} |h(t)|^2 \log|h(t)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} |h(0)|^2 \log|h(0)|^2 \,\mathrm{d}x\right) \\
&\quad - \frac{\mu e^{2^* s_t}}{2^*}\left(\|h(t)\|_{2^*}^{2^*} - \|h(0)\|_{2^*}^{2^*}\right) \\
&= e^{2s_t} \int_{\mathbb{R}^N} \nabla h(\tau_1 t) \nabla h'(\tau_1 t) t \,\mathrm{d}x + (1 - N s_t) \int_{\mathbb{R}^N} h(\tau_2 t) h'(\tau_2 t) t \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} h(\tau_3 t) h'(\tau_3 t) \log|h(\tau_3 t)|^2 t + h(\tau_4 t) h'(\tau_4 t) t \,\mathrm{d}x \\
&\quad - \mu e^{2^* s_t} \int_{\mathbb{R}^N} |h(\tau_5 t)|^{2^*-2} h(\tau_5 t) h'(\tau_5 t) t \,\mathrm{d}x,
\end{aligned}$$

for some $\tau_1, \cdots, \tau_5 \in (0,1)$. In the same manner, we get

$$\begin{aligned}
&I(s_t \star h(t)) - I(s_0 \star h(0)) \le I(s_0 \star h(t)) - I(s_0 \star h(0)) \\
&= e^{2s_0} \int_{\mathbb{R}^N} \nabla h(\tau_6 t) \nabla h'(\tau_6 t) t \,\mathrm{d}x + \big(1 - Ns_0\big) \int_{\mathbb{R}^N} h(\tau_7 t) h'(\tau_7 t) t \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} h(\tau_8 t) h'(\tau_8 t) \log |h(\tau_8 t)|^2 t + h(\tau_9 t) h'(\tau_9 t) t \,\mathrm{d}x \\
&\quad - \mu e^{2^* s_t} \int_{\mathbb{R}^N} |h(\tau_{10} t)|^{2^*-2} h(\tau_{10} t) h'(\tau_{10} t) t \,\mathrm{d}x,
\end{aligned}$$

for some $\tau_6, \cdots, \tau_{10} \in (0,1)$. Finally, we deduce that

$$\begin{aligned}
\lim_{t \to 0} \frac{J^+(h(t)) - J^+(h(0))}{t} &= \int_{\mathbb{R}^N} \nabla(s_u \star u) \nabla(s_u \star \psi) \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} (s_u \star u)(s_u \star \psi) \log |s_u \star u|^2 \,\mathrm{d}x \\
&\quad - \mu \int_{\mathbb{R}^N} |s_u \star u|^{2^*-2} (s_u \star u)(s_u \star \psi) \\
&= dI(s_u \star u)[s_u \star \psi],
\end{aligned}$$

for any $u \in S(c)$ and $\psi \in T_u S(c)$. The proof for J^- is similar. $\square$

Let $\mathcal{G}$ be the set of all singletons belongs to $S(c)$ and boundary $B = \emptyset$. It is a homotopy stable family of compact subset of $S(c)$ (without boundary) in the sense of [19, Definition 3.1]. By Lemma 4.6, we have

$$e_{\mathcal{G}}^+ := \inf_{A \in \mathcal{G}} \max_{u \in A} J^+(u) = \inf_{u \in S(c) \cap H_r^1(\mathbb{R}^N)} J^+(u) = \inf_{u \in \mathcal{P}_{r,c}^+} I(u) = \inf_{u \in \mathcal{P}_c^+} I(u),$$

$$e_{\mathcal{G}}^- := \inf_{A \in \mathcal{G}} \max_{u \in A} J^-(u) = \inf_{u \in S(c) \cap H_r^1(\mathbb{R}^N)} J^-(u) = \inf_{u \in \mathcal{P}_{r,c}^-} I(u) = \inf_{u \in \mathcal{P}_c^-} I(u).$$

Lemma 5.6. *For any $0 < c < c_0$, there exists a Palais–Smale sequence $\{u_n\}^+ \subset \mathcal{P}_c^+$ (or $\{u_n\}^- \subset \mathcal{P}_c^-$) for I restricted to $S(c) \cap H_r^1(\mathbb{R}^N)$ at level $e_{\mathcal{G}}^+$ (or $e_{\mathcal{G}}^-$, respectively).*

Proof. We first prove the case of $e_{\mathcal{G}}^+$, the one for $e_{\mathcal{G}}^-$ is almost identical. By the definition of $e_{\mathcal{G}}^+$, there exists $\{E_n\} \subset \mathcal{G}$ such that

$$\max_{u \in E_n} J^+(u) < e_{\mathcal{G}}^+ + \frac{1}{n}.$$

Define

$$\eta \colon [0,1] \times S(c) \mapsto S(c), \quad \eta(t,u) = (1 - t + ts_u) \star u.$$

By using the definition of $\mathcal{G}$, we have

$$\bar{E}_n := \eta(\{1\} \times E_n) = \{s_u \star u \colon u \in E_n\} \in \mathcal{G}.$$

It follows from Lemma 5.4 that $\bar{E}_n \subset \mathcal{P}_c^+$ for all $n \in \mathbb{N}$. Let $v \in \bar{E}_n$, i.e. $v = s_u \star u$ for some $u \in E_n$, then $J^+(v) = J^+(u)$, and finally that

$$\max_{v \in \bar{E}_n} J^+(v) = \max_{u \in E_n} J^+(u).$$

Using the terminology in [13, Definition 3.1], it means that $\bar{E}_n$ is a homotopy stable family of compact subset of $S(c) \cap H_r^1(\mathbb{R}^N)$. Therefore, $\{\bar{E}_n\}$ is another sequence sets such that $\lim_n \sup_{\bar{E}_n} J^+ = e_{\mathcal{G}}^+$. Denoting by $\|\cdot\|_*$ the dual norm of $(T_{v_n} S(c))^*$, we can apply [19, Theorem 3.2] with the minimizing sequence w_n for J^+ on $S(c)$ at level $e^+_{\mathcal{G}}$ such that

$$\begin{aligned}&\text{(i)}\quad \lim_{n\to\infty} J^+(w_n)=e^+_{\mathcal{G}},\quad \text{(ii)}\quad \lim_{n\to\infty}\|dJ^+(w_n)\|_*=0,\\ &\text{(iii)}\quad \mathrm{dist}_{W_r}(w_n,\bar{E}_n)\to 0 \ \text{ as } n\to\infty.\end{aligned}\tag{5.6}$$

Denote $v_n := s_{w_n}\star w_n \in \mathcal{P}^+_c$, we claim that $\{s_{w_n}\}$ is bounded. Notice that

$$e^{2s_{w_n}}=\frac{\|\nabla v_n\|_2^2}{\|\nabla w_n\|_2^2}.$$

In fact, $I(v_n)=J^+(w_n)\to e^+_{\mathcal{G}}=m^+(c)$, we get from that v_n is uniformly bounded in W_r. We deduce that $\{\bar{E}_n\}$ is uniformly bounded in W_r, and thus from $\mathrm{dist}_{W_r}(w_n,\bar{E}_n)$ $\to 0$ as $n\to\infty$, it gives that $\sup_n\|\nabla w_n\|_2^2<\infty$. Since $\bar{E}_n$ is compact for every $n\in\mathbb{N}$, there exists $\tilde{v}_n\in\bar{E}_n$ such that $\mathrm{dist}_{W_r}(w_n,\bar{E}_n)=\|\tilde{v}_n-w_n\|_{W_r}$, and then $\|\nabla w_n\|_2^2\ge\|\nabla\tilde{v}_n\|_2^2-\|\nabla(\tilde{v}_n-w_n)\|_2^2\ge\frac{\delta}{2}$ for a $\delta>0$. Therefore, $\{s_{w_n}\}$ is bounded.

Next, we show that $\{v_n\}\subset\mathcal{P}^+_c$ is a Palais–Smale sequence for I restricted on $S(c)$ at level $e^+_{\mathcal{G}}$. From Lemma 2.3, we get that $T_{w_n}S(c)\to T_{v_n}S(c)$ defined $\psi\to s_{w_n}\star\psi$ is an isomorphism. By (5.6), we get

$$\|dI|_{S(c)}(v_n)\|_*=\sup_{\|\psi\|\le 1,\psi\in T_uS(c)}|dI(v_n)[\psi]|=\sup_{\|\psi\|\le 1,\psi\in T_uS(c)}|dJ^+(w_n)[(-s_{w_n})\star\psi]|,$$

since s_{w_n} is bounded and $\|(-s_{w_n})\star\psi\|_{W_r}\le C$. We conclude that $\{v_n\}\subset\mathcal{P}^+_c$ is a Palais–Smale sequence for I restricted to $S(c)$ at level $e^+_{\mathcal{G}}$. □

Applying a similar arguments as the proof of Lemma 4.5. We can deduce that

Lemma 5.7. *Let $\mu>0$ and $p=2^*$. Suppose that $0<c<c_0$. Then*

$$m_c:=\inf_{u\in V_{k_0}} I(u)=m^+(c),\tag{5.7}$$

where V_{k_0} is defined in (5.3).

Lemma 5.8. *Let $\mu>0$, and $p=2^*$. Suppose that $0<c<c_0$. Then $m^+(c)$ is achieved.*

Proof. By Lemma 5.7, this follows by the same method as in Lemma 5.2. Applying Lemma 5.6, we deduce that there exists a Palais–Smale sequence $\{u_n\}\subset\mathcal{P}^+_c$ for I restricted to $S(c)$ at level $e^+_{\mathcal{G}}=m^+(c)$. Obviously,

$$\begin{aligned}I(u_n)&=\frac12\|\nabla u_n\|_2^2-\frac12\int_{\mathbb{R}^N}u_n^2\log u_n^2\,\mathrm{d}x-\frac{\mu}{2^*}\int_{\mathbb{R}^N}|u_n|^{2^*}\,\mathrm{d}x+\frac12 c^2\\ &\ge\left(\frac12-\frac{1}{2^*}\right)\|\nabla u_n\|_2^2+\frac12\int_{\mathbb{R}^N}A(|u_n|)-B(|u_n|)\,\mathrm{d}x+\frac{N}{4}c^2.\end{aligned}$$

The proof is similar to that of Lemma 4.7, we get $\{u_n\}$ is bounded in W_r, $u_n\rightharpoonup u$ weakly in W_r, $u_n\to u$ strongly in $L^2(\mathbb{R}^N)$. In fact, by the Lagrange multiplier's rule (see [10, Lemma 3]) there is $\{\lambda_n\}\subset\mathbb{R}$ such that

$$\int_{\mathbb{R}^N}\left(\nabla u_n\nabla\phi+\lambda_n u_n\phi-\mu|u_n|^{2^*-2}u_n\phi-u_n\phi\log u_n^2\right)\mathrm{d}x=o_n(1)\|\phi\|_W,\tag{5.8}$$

for each $\phi\in W$. In particular, if we take $\phi=u_n$, we then conclude λ_n is bounded. Hence, up to a subsequence, $\lambda_n\to\lambda\in\mathbb{R}$. Passing to the limit in (5.8), we deduce that u satisfies

$$-\Delta u+\lambda u=u\log u^2+\mu|u|^{2^*-2}u,\quad x\in\mathbb{R}^N.$$

It follows that $P(u) = 0$. Then

$$\begin{aligned} m^+(c) \le I(u) &\le \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \frac{N}{4}c^2 - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \, \mathrm{d}x \\ &\le \left(\frac{1}{2} - \frac{1}{2^*}\right) \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, \mathrm{d}x + \frac{1}{2} \liminf_{n\to\infty} \int_{\mathbb{R}^N} A(|u_n|) \, \mathrm{d}x \\ &\quad - \lim_{n\to\infty} \int_{\mathbb{R}^N} B(|u_n|) \, \mathrm{d}x + \frac{N}{4}c^2 \\ &\le \liminf_{n\to\infty} I(u_n) = m^+(c). \end{aligned}$$

Therefore, $I(u) = m^+(c)$. □

Lemma 5.9. *Let $p = 2^*$, $\mu > 0$. Then for $0 < c < c_0$, we have*

$$m^-(c) < m^+(c) + \frac{\mu^{-\frac{N-2}{2}}}{N}\mathcal{S}^{\frac{N}{2}}.$$

Proof. In fact, we assume that $u \in S(c)$, $I(u) = m^+(c)$. Let

$$U_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{N-2}{2}}}$$

be the positive solution of

$$-\Delta v = |v|^{2^*-2}v, \quad v \in D^{1,2}(\mathbb{R}^N).$$

Let $\overline{U}_\varepsilon = \chi(x)U_\varepsilon$ where χ is a cut-off function such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| > 2$, we have

$$\|\nabla \overline{U}_\varepsilon\|_2^2 = \mathcal{S}^{\frac{N}{2}} + O(\varepsilon^{N-2}) \quad \text{and} \quad \|\overline{U}_\varepsilon\|_{2^*}^{2^*} = \mathcal{S}^{\frac{N}{2}} + O(\varepsilon^N). \tag{5.9}$$

Define $W_{\varepsilon,t} = u(\cdot - ne_1) + t\overline{U}_\varepsilon$, where $e_1 = (1, 0, \cdots, 0)$, and $\overline{W}_{\varepsilon,t} = s^{\frac{N-2}{2}} W_{\varepsilon,t}(sx)$. Since u is a radial, non-increasing function, we know from [9, Radial Lemma A.IV] that

$$|u(z)| \le C_N |z|^{-\frac{N}{2}} c, \quad \text{for } |z| \ge 1.$$

Thus, we obtain

$$-\Delta u + q(x)u = 0, \quad u \in H^1(\mathbb{R}^N),$$

where $q(x) = \lambda - \log u^2 - \mu|u|^{2^*-2}$. Observe that $q(x) \ge \frac{1}{2}$ for $|x| > M$ if M is large enough. By [5, Lemma 3.11], there exists $a \in (0, \frac{1}{2})$ such that

$$u(x) \le Ce^{-\sqrt{1+a|x|^2}}, \quad \text{for each } |x| > M.$$

By direct calculations, we have

$$\|\nabla \overline{W}_{\varepsilon,t}\|_2^2 = \|\nabla W_{\varepsilon,t}\|_2^2, \quad \|\overline{W}_{\varepsilon,t}\|_{2^*}^{2^*} = \|W_{\varepsilon,t}\|_{2^*}^{2^*},$$

and

$$\|\overline{W}_{\varepsilon,t}\|_2^2 = s^{-2}\|W_{\varepsilon,t}\|_2^2.$$

We choose $s = \frac{\|W_{\varepsilon,t}\|_2}{c}$ such that $\overline{W}_{\varepsilon,t} \in S(c)$. By Lemma 4.3, there exists $\tau_{\varepsilon,t} \in \mathbb{R}$ such that $\tau_{\varepsilon,t} \star \overline{W}_{\varepsilon,t} \in \mathcal{P}_c^-$. Thus

$$e^{2\tau_{\varepsilon,t}}\|\nabla \overline{W}_{\varepsilon,t}\|_2^2 = \mu e^{2^*\tau_{\varepsilon,t}}\|\overline{W}_{\varepsilon,t}\|_{2^*}^{2^*} + \frac{N}{2}c^2. \tag{5.10}$$

Since $u \in \mathcal{P}_c^+$, by Lemma 4.3, we deduce $\tau_{\varepsilon,0} > 0$. By using (5.9) and (5.10), we have $\tau_{\varepsilon,t} \to -\infty$ as $t \to +\infty$ and $\varepsilon > 0$ small enough. It follows from Lemma 4.3 that

$\tau_{\varepsilon,t}$ is unique. Moreover, $\tau_{\varepsilon,t}$ is continuous for t, and then there exists t_ε such that $\tau_{\varepsilon,t_\varepsilon}=0$. Therefore, we get

$$m^-(c)\le\sup_{t\ge0}I\big(\overline{W}_{\varepsilon,t}\big).$$

There exists $t_0>0$ such that

$$I\big(\overline{W}_{\varepsilon,t}\big)<m^+(c)+\frac{1}{N}\mu^{-\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}}-\sigma,$$

for $0<t<\frac{1}{t_0}$ and $t>t_0$ with $\sigma>0$.

Since the function $\overline{U}_\varepsilon$ is compacted supported in B_2, we have that, for n large enough,

$$\int_{\mathbb{R}^N}u(x-ne_1)\overline{U}_\varepsilon\,\mathrm{d}x=O(\varepsilon^{\frac{N-2}{2}}),$$

and then

$$s^2=\frac{\|W_{\varepsilon,t}\|_2^2}{c^2}=1+\frac{2t}{c^2}\int_{\mathbb{R}^N}u(\cdot-ne_1)\overline{U}_\varepsilon\,\mathrm{d}x+\frac{t^2\|\overline{U}_\varepsilon\|_2^2}{c^2}, \tag{5.11}$$

for $\frac{1}{t_0}\le t\le t_0$. Recall the definition of $A(s)$ in Section 2, for any $0\le a,b$ and $tb\ge\frac12 a$, $A(a+tb)\le A(a)+CtA(b)$ (see [13, Lemma 1.3]). We deduce that

$$\begin{aligned}\int_{\mathbb{R}^N}|W_{\varepsilon,t}|^2\log|W_{\varepsilon,t}|^2\,\mathrm{d}x&=\int_{\mathbb{R}^N}B(|u(x-ne_1)+t\overline{U}_\varepsilon|)\,\mathrm{d}x\\&\quad-\int_{\mathbb{R}^N}A(|u(x-ne_1)+t\overline{U}_\varepsilon|)\,\mathrm{d}x\\&\ge\int_{\mathbb{R}^N}u^2\log u^2\mathrm{d}x+Ct\int_{\mathbb{R}^N}|\overline{U}_\varepsilon|^2\log|\overline{U}_\varepsilon|^2\,\mathrm{d}x.\end{aligned}$$

By using (5.11), we have

$$\begin{aligned}I\big(\overline{W}_{\varepsilon,t}\big)&=\frac12\int_{\mathbb{R}^N}|\nabla W_{\varepsilon,t}|^2\,\mathrm{d}x+\frac{s^{-2}}{2}\int_{\mathbb{R}^N}|W_{\varepsilon,t}|^2\,\mathrm{d}x-\frac{s^{-2}\log|s^{N-2}|}{2}\int_{\mathbb{R}^N}|W_{\varepsilon,t}|^2\,\mathrm{d}x\\&\quad-\frac{s^{-2}}{2}\int_{\mathbb{R}^N}|W_{\varepsilon,t}|^2\log|W_{\varepsilon,t}|^2\,\mathrm{d}x-\frac{\mu}{2^*}\int_{\mathbb{R}^N}|W_{\varepsilon,t}|^{2^*}\,\mathrm{d}x\\&\le I(u)+\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla\overline{U}_\varepsilon|^2\,\mathrm{d}x-\frac{t^{2^*}}{2^*}\mu\int_{\mathbb{R}^N}|\overline{U}_\varepsilon|^{2^*}\,\mathrm{d}x+\frac12\int_{\mathbb{R}^N}|\overline{U}_\varepsilon|^2\,\mathrm{d}x\\&\quad-Ct\int_{\mathbb{R}^N}|\overline{U}_\varepsilon|^2\log|\overline{U}_\varepsilon|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}\nabla u(\cdot-ne_1)\nabla(tU_\varepsilon)+u(\cdot-ne_1)tU_\varepsilon\,\mathrm{d}x\\&\quad-\int_{\mathbb{R}^N}|u(\cdot-ne_1)|^{2^*-1}t\overline{U}_\varepsilon\,\mathrm{d}x-\frac{t}{c^2}\int_{\mathbb{R}^N}u(\cdot-ne_1)\overline{U}_\varepsilon\,\mathrm{d}x\|W_{\varepsilon,t}\|_2^2\\&\quad+\frac{t}{c^2}\int_{\mathbb{R}^N}u(\cdot-ne_1)\overline{U}_\varepsilon\,\mathrm{d}x\int_{\mathbb{R}^N}|W_{\varepsilon,t}|^2\log|W_{\varepsilon,t}|^2\,\mathrm{d}x.\end{aligned} \tag{5.12}$$

Here, we used the fact that $f(s) = (1+s)^{2^*} \geq 1 + s^{2^*} + 2^* s^{2^*-1} \geq 0$ for all $s \geq 0$. To complete the proof it suffices to show

$$
\begin{aligned}
J_{\varepsilon,t} :=& \frac{1}{2}\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \,\mathrm{d}x - Ct\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \log|\overline{U}_\varepsilon|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \nabla u(\cdot - ne_1)\nabla(tU_\varepsilon)\,\mathrm{d}x \\
&- \int_{\mathbb{R}^N} |u(\cdot - ne_1)|^{2^*-1} t\overline{U}_\varepsilon \,\mathrm{d}x + \int_{\mathbb{R}^N} u(\cdot - ne_1) tU_\varepsilon \,\mathrm{d}x \left(1 - \frac{1}{c^2}\|W_{\varepsilon,t}\|_2^2\right) \\
&+ \frac{t}{c^2}\int_{\mathbb{R}^N} u(\cdot - ne_1)\overline{U}_\varepsilon \,\mathrm{d}x \int_{\mathbb{R}^N} |W_{\varepsilon,t}|^2 \log|W_{\varepsilon,t}|^2 \,\mathrm{d}x \\
\leq& \frac{1}{2}\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \,\mathrm{d}x - Ct\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \log|\overline{U}_\varepsilon|^2 \,\mathrm{d}x + O(\varepsilon^{\frac{N}{2}}) - O(\varepsilon^{\frac{N-2}{2}}) + O(\varepsilon^{N-2}) \\
&+ \frac{t}{c^2}\int_{\mathbb{R}^N} u(\cdot - ne_1)\overline{U}_\varepsilon \,\mathrm{d}x \int_{\mathbb{R}^N} |W_{\varepsilon,t}|^2 \log|W_{\varepsilon,t}|^2 \,\mathrm{d}x \\
\leq& \frac{1}{2}\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \,\mathrm{d}x - Ct\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \log|\overline{U}_\varepsilon|^2 \,\mathrm{d}x + O(\varepsilon^{N-2}) < 0.
\end{aligned} \tag{5.13}
$$

Here, we used the fact that $\int_{\mathbb{R}^N} u(\cdot - ne_1)\overline{U}_\varepsilon \,\mathrm{d}x \leq C\|\overline{U}_\varepsilon\|_2^2$, because u decays exponentially and $\overline{U}_\varepsilon$ has compact support.

If $N \geq 4$, we have, for $C_1, C_2 > 0$,

$$
\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \,\mathrm{d}x = \begin{cases} C_1\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2), & \text{if } N = 4, \\ C_2\varepsilon^2 + O(\varepsilon^{N-2}), & \text{if } N \geq 5. \end{cases}
$$

From [18, Lemma 3.4], we get

$$
\int_{\mathbb{R}^N} |\overline{U}_\varepsilon|^2 \log|\overline{U}_\varepsilon|^2 \,\mathrm{d}x \geq \begin{cases} C_3\varepsilon^2|\ln\varepsilon|^2 + O(\varepsilon^2), & \text{if } N = 4, \\ C_4\varepsilon^2|\ln\varepsilon| + O(\varepsilon^{N-2}), & \text{if } N \geq 5, \end{cases}
$$

where $C_3, C_4 > 0$. Therefore, $J_{\varepsilon,t} < 0$.

If $N = 3$, by the definition of $\overline{U}_\varepsilon$, for $C_5 > 0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\overline{U}_\varepsilon|^2 \log|\overline{U}_\varepsilon|^2 \,\mathrm{d}x &= \sqrt{3}\int_{B_2} \chi^2 \frac{\varepsilon}{\varepsilon^2 + |x|^2} \log\left(\sqrt{3}\chi^2 \frac{\varepsilon}{\varepsilon^2 + |x|^2}\right) \mathrm{d}x \\
&= C_5\varepsilon|\log\varepsilon| + O(\varepsilon).
\end{aligned}
$$

Moreover, $\|\overline{U}_\varepsilon\|_2^2 = O(\varepsilon)$. Therefore, $J_{\varepsilon,t} < 0$.

Combining (5.12) with (5.13), we have,

$$
I(\overline{W}_{\varepsilon,t}) < m^+(c) + \frac{1}{N}\mu^{-\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}},
$$

for $\frac{1}{t_0} \leq t_0 \leq t$. Therefore, for ε small enough, we have

$$
I(\overline{W}_{\varepsilon,t}) < m^+(c) + \frac{1}{N}\mu^{-\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}}. \qquad \square
$$

Lemma 5.10. *Let $\mu > 0$ and $p = 2^*$. Suppose that $0 < c < c_0$. Then $m^-(c)$ is achieved, there exists a second solution $u^- \in S(c)$ which satisfies*

$$
I(u^-) < m^+(c) + \frac{1}{N}\mu^{-\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}}.
$$

Proof. Similarly, by Lemma 5.6, we assume that there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_c^-$ for I restricted to $S(c)$ at level $m^-(c)$. Moreover, $\{u_n\}$ is bounded in W_r, and $u_n \rightharpoonup u$ in W_r. We need to prove that $u_n \to u$ in W_r. In fact, by the Lagrange multiplier's rule (see [10, Lemma 3]) there is $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \left(\nabla u_n \nabla \phi + \lambda_n u_n \phi - \mu |u_n|^{2^*-2} u_n \phi - u_n \phi \log u_n^2\right) \mathrm{d}x = o_n(1)\|\phi\|_W, \tag{5.14}$$

for each $\phi \in W$. In particular, if we take $\phi = u_n$, we then conclude $\{\lambda_n\}$ is bounded. Hence, up to a subsequence, we can assume $\lambda_n \to \lambda \in \mathbb{R}$. Passing to the limit in (5.14), we get that u satisfies

$$-\Delta u + \lambda u = u \log u^2 + \mu |u|^{2^*-2} u, \quad x \in \mathbb{R}^N.$$

It follows that $P(u) = 0$, we conclude that

$$\|\nabla u\|_2^2 = \mu \|u\|_{2^*}^{2^*} + \frac{N}{2} c^2.$$

Set $v_n := u_n - u$, by the Brézis–Lieb Lemma [41] and $P(u_n) = o_n(1)$, we deduce that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \, \mathrm{d}x = \mu \int_{\mathbb{R}^N} |v_n|^{2^*} \, \mathrm{d}x + o_n(1). \tag{5.15}$$

Here we distinguish the following two cases

$$\text{either} \quad \text{(i)} \quad \|v_n\|_{2^*}^{2^*} \to 0 \quad \text{or} \quad \text{(ii)} \quad \|v_n\|_{2^*}^{2^*} \to l > 0.$$

If (ii) holds, we deduce from (5.15) that

$$\|\nabla v_n\|_2^2 \ge \mu^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}} + o_n(1).$$

Thus, we have

$$\begin{aligned} m^-(c) + o_n(1) &= \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \, \mathrm{d}x - \frac{\mu}{2^*}\int_{\mathbb{R}^N} |u_n|^{2^*} \, \mathrm{d}x + \frac{1}{2}c^2 \\ &\ge \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{\mu}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} \, \mathrm{d}x + I(u) \\ &\ge \frac{1}{N}\mu^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}} + m^+(c), \end{aligned} \tag{5.16}$$

which contradicts with Lemma 5.9.

If (i) holds, then $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ converges strongly in $H_r^1(\mathbb{R}^N)$. It follows from (5.16) that $\int_{\mathbb{R}^N} u_n^2 \log u_n^2 \, \mathrm{d}x \to \int_{\mathbb{R}^N} u^2 \log u^2 \, \mathrm{d}x$, then we deduce $u_n \to u$ strongly in W_r. By Lemma 5.3, we then conclude u is a positive solution of (1.1)–(1.2), which is radially symmetric and decreasing in $r = |x|$. □

Proof of Theorem 1.3. The proof follows directly from Lemma 5.8 and Lemma 5.10. □

6. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. In order to study the asymptotic behavior of ground state as $\mu \to 0$, we first introduce some facts for equations (1.1)–(1.2) with $\mu = 0$. Let $c > 0$ be fixed, $(\lambda_0, w_0) \in \mathbb{R} \times W$ solves the following equation

$$\begin{cases} -\Delta u + \lambda u = u \log u^2, \\ \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c^2. \end{cases} \tag{6.1}$$

Define

$$m_0(c) := \inf_{u \in S(c)} I_0(u),$$

where

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \,\mathrm{d}x.$$

From [17, 13], by scaling, we obtain the unique solution of (6.1) is

$$w_0(x) := c\pi^{-\frac{N}{4}} e^{-\frac{|x|^2}{2}}, \quad \text{with} \quad \lambda_0 := -N - \log(c^{-2}\pi^{\frac{N}{2}}). \tag{6.2}$$

Therefore $m_0(c) = \frac{c^2}{2}\big(N + 1 + \log(c^{-2}\pi^{\frac{N}{2}})\big)$ and $I(w_0) = m_0(c)$.

Lemma 6.1. *Assume* (λ_μ, u_μ^+) *is the normalized ground state solution of* (1.1)–(1.2) *in Theorems 1.1–1.3, then, up to a subsequence, we have*

$$u_\mu^+ \to w_0 \quad \textit{strongly in} \ \ W,$$

and $\lambda_\mu \to \lambda_0$ *as* $\mu \to 0^+$, *where* (λ_0, w_0) *is defined by* (6.2).

Proof. Case (i): if (λ_μ, u_μ^+) is the normalized ground state solution of (1.1)–(1.2) obtained by Theorem 1.1, we then claim that $\lim_{\mu\to 0} m(c) = m_0(c)$.

Indeed, if $\mu > 0$, we have $\limsup_{\mu\to 0^+} m(c) \le m_0(c)$ since $I(w_0) < I_0(w_0) = m_0(c)$ for each $w_0 \in S(c)$. Furthermore, for any $u \in S(c)$, one has

$$m_0(c) \le I_0(u) = I(u) + \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x. \tag{6.3}$$

Letting $\mu \to 0^+$, we then get $m_0(c) \le \liminf_{\mu\to 0^+} m(c)$. Thus, the claim hold.

Therefore, as $\mu \to 0^+$, we may assume that

$$\begin{aligned} I(u_\mu^+) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_\mu^+|^2 + |u_\mu^+|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} |u_\mu^+|^2 \log |u_\mu^+|^2 \,\mathrm{d}x \\ &\quad - \frac{\mu}{p}\int_{\mathbb{R}^N} |u_\mu^+|^p \,\mathrm{d}x \le m_0(c) + 1. \end{aligned} \tag{6.4}$$

As in the proof of Lemma 3.2, (6.4) gives that u_μ^+ is bounded in W. Then, there exists $u_0 \in W$ such that $u_\mu^+ \rightharpoonup u_0$ weakly in W, $u_\mu^+ \to u_0$ strongly in $L^q(\mathbb{R}^N)$, $2 \le q < 2^*$, and $u_\mu^+ \to u_0$ a.e. $x \in \mathbb{R}^N$. Thus,

$$\begin{aligned} m_0(c) \le I_0(u_0) &\le \liminf_{\mu\to 0^+} \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u_\mu^+|^2 + |u_\mu^+|^2 + A(|u_\mu^+|)\big) \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} B(|u_0|) \,\mathrm{d}x \\ &\le \liminf_{\mu\to 0^+} I(u_\mu^+) + \frac{\mu}{p}\int_{\mathbb{R}^N} |u_\mu^+|^p \,\mathrm{d}x = m(c) + o(1). \end{aligned}$$

Since $m(c) \to m_0(c)$, $I_0(u_0) = m_0(c)$, by the uniqueness of ground state solution of (6.1), we have $u_0 = w_0$. Moreover, $u_\mu^+ \to w_0$ strongly in W, i.e. $u_\mu^+ \to w_0$ strongly in $H^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |u_\mu^+|^2 \log |u_\mu^+|^2 \,\mathrm{d}x \to \int_{\mathbb{R}^N} |w_0|^2 \log |w_0|^2 \,\mathrm{d}x$ as $\mu \to 0^+$, and then $\lambda_\mu \to \lambda_0$ as $\mu \to 0^+$.

If $\mu < 0$, we deduce that $m_0(c) = I_0(w_0) \le I(w_0)$ for $w_0 \in S(c)$, and then $m_0(c) \le \liminf\limits_{\mu\to 0^-} m(c)$. For each $u \in S(c)$, one has

$$m(c) \le I(u) = I_0(u) - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x.$$

Letting $\mu \to 0^-$, we get that $m(c) \le \liminf_{\mu\to 0^-} m_0(c)$. This proves that $\lim_{\mu\to 0^-} m(c) = m_0(c)$. In the same manner we obtain that $u_\mu^+ \to w_0$ strongly in W and $\lambda_\mu \to \lambda_0$ as $\mu \to 0^-$.

Cases (ii), if (λ_μ, u_μ^+) is the normalized ground state solution of (1.1)–(1.2) obtained by Theorems 1.2-1.3, we claim that $\lim_{\mu\to 0^+} m^+(c) = m_0(c)$. By Lemma 4.5

and Lemma 5.7, $m_c = m^+(c)$, we only need to prove $\lim_{\mu\to 0^+} m_c = m_0(c)$. From (4.3) and (5.7), $m_c = \inf_{u\in V_{k_0}} I(u)$, we then get $\limsup_{\mu\to 0} m_c \le m_0(c)$ since $w_0 \in V_{k_0}$. Moreover, for $u \in V_{k_0}$, (6.3) shows that $m_0(c) \le \liminf_{\mu\to 0^+} m(c)$ as $\mu \to 0^+$. Then,

$$I(u_\mu^+) = \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right) \|\nabla u_\mu^+\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} A(|u_\mu^+|) - B(|u_\mu^+|)\,\mathrm{d}x - \frac{p}{2(p-2)}c^2,$$

we deduce u_μ^+ is bounded in W. In the same manner, we can show that $u_\mu^+ \to w_0$ strongly in W and $\lambda_\mu \to \lambda_0$. □

Proof of Theorem 1.4. The proof directly follows from Lemma 6.1. □

7. The case $\alpha < 0$

In this section, we assume that $\alpha < 0$ and $N \ge 2$. For convenience, we assume that $\alpha = -1$. Then the energy functional is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2\,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x, \quad u \in W_r,$$

and the Pohozaev identity is gived by

$$P(u) = \|\nabla u\|_2^2 + \frac{N}{2}c^2 - \mu\gamma_p\|u\|_p^p = 0.$$

Proof of Theorem 1.5. We notice that if one of the following conditions holds

(i) $p := \bar{p} = 2 + \dfrac{4}{N}, \mu > 0$ and $c < \left(\dfrac{N+2}{N\mu}\right)^{\frac{N}{4}} \big(C(N,\bar{p})\big)^{-\frac{N+2}{2}}$,

(ii) $\mu \le 0$ and $p > 2$;

then the fiber map $\Psi_u(t) := I(t\star u)$ is strictly increasing for each $u \in S(c)$, and thus I does not have critical points on $S(c)$.

(iii) By contradiction, we suppose that there exists a positive solution $u \in W_r$, by [9, Radial Lemma A.IV], $\lim_{|x|\to\infty} u(x) = 0$. Therefore, there exists $R_0 > 0$ large enough such that

$$-\Delta u(x) = \big(-\lambda - \log u^2 + \mu|u|^{p-2}\big)u(x) \ge u(x) \quad \text{for } |x| > R_0.$$

By applying Theorem 2.1 and Theorem 2.8 of [3] with $f(s) := s$, we obtain that $-\Delta u \ge f(u)$ has no positive solution in any exterior domain of $\mathbb{R}^N$ if

$$\liminf_{s\to 0} s^{-\frac{N}{N-2}} f(s) > 0$$

for $N = 3$ or if $\lim_{s\to\infty} e^{as} f(s) = \infty$ for every $a > 0$ and $N = 2$. □

7.1. The existence of a global maximal on $\mathcal{P}_c$. Define

$$D := \left(\frac{Np\gamma_p}{2(2-p\gamma_p)}\right)^{\frac{2-p\gamma_p}{2(p-2)}} \left(\frac{\mu p\gamma_p^2}{2}C^p(N,p)\right)^{-\frac{1}{p-2}}.$$

We next show that $\mathcal{P}_c$ is not empty if $c \ge D$.

Lemma 7.1. *Let $\mu > 0$, $2 < p < 2 + \frac{4}{N}$ and $N \ge 2$, we have*

$$\mathcal{P}_c \ne \emptyset \quad \textit{if and only if} \quad c \ge D.$$

Proof. For each $u \in S(c)$, define

$$\phi_u(t) := P(t \star u) = e^{2t}\|\nabla u\|_2^2 + \frac{N}{2}c^2 - \mu\gamma_p e^{p\gamma_p t}\|u\|_p^p.$$

Therefore

$$\inf_{t\in\mathbb{R}} \phi_u(t) = -\frac{2-p\gamma_p}{p\gamma_p}\left[\frac{\mu p\gamma_p^2\|u\|_p^p}{2\|\nabla u\|_2^2}\right]^{\frac{2}{2-p\gamma_p}} \|\nabla u\|_2^2 + \frac{N}{2}c^2.$$

By the Gagliardo–Nirenberg inequality, we obtain

$$\inf_{t\in\mathbb{R}} \phi_u(t) \geq \frac{N}{2}c^2 - \frac{2-p\gamma_p}{p\gamma_p}\left(\frac{\mu p\gamma_p^2}{2}C^p(N,p)\right)^{\frac{2}{2-p\gamma_p}} c^{\frac{2p(1-\gamma_p)}{2-p\gamma_p}}.$$

If

$$c < D := \left(\frac{Np\gamma_p}{2(2-p\gamma_p)}\right)^{\frac{2-p\gamma_p}{2(p-2)}}\left(\frac{\mu p\gamma_p^2}{2}C^p(N,p)\right)^{-\frac{1}{p-2}},$$

we then get $\inf_{u\in S(c)} P(u) > 0$ and $\mathcal{P}_c = \emptyset$. Since the best constant in the Gagliardo–Nirenberg inequality is achieved, we assume that by $\bar{u} \in S(c)$, and then

$$\inf_{u\in S(c)} P(u) = P(\bar{u}) = \frac{N}{2}c^2 - \frac{2-p\gamma_p}{p\gamma_p}\left(\frac{\mu p\gamma_p^2}{2}C^p(N,p)\right)^{\frac{2}{2-p\gamma_p}} c^{\frac{2p(1-\gamma_p)}{2-p\gamma_p}}. \tag{7.1}$$

If $c > D$, it follows from (7.1) that $\inf_{u\in S(c)} P(u) < 0$. Since $\lim_{t\to+\infty}\phi_u(t) = +\infty$, by continuity of ϕ_u, there exists $\bar{t} \in \mathbb{R}$ such that $\phi_u(\bar{t}) = 0$, i.e. $\mathcal{P}_c \neq \emptyset$. Moreover, if $c = D$, then $P(\bar{u}) = 0$, and $\mathcal{P}_c \neq \emptyset$ holds. □

Lemma 7.2. *Let $2 < p < 2 + \frac{4}{N}$ and $N \geq 2$, if $c \geq D$, we have*

$$\inf_{u\in\mathcal{P}_c} I(u) = -\infty.$$

Proof. From Lemma 7.1, if $c \geq D$, then $\mathcal{P}_c \neq \emptyset$. For any $u \in S(c)$ and $P(u) \leq 0$, since $\lim_{t\to+\infty} P(t\star u) = +\infty$, by the geometry of $I(t\star u)$, we deduce there exists $t \geq 0$ such that $P(t\star u) = 0$ and $I(t\star u) \leq I(u)$. Therefore, we only need to prove that there exists a sequence $\{u_n\} \subset S(c)$ with $P(u_n) \leq 0$ and $I(u_n) \to -\infty$ as $n \to \infty$. We choose $\eta > 0$ and take $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $u \geq 0$ with $\|u\|_2^2 = c^2 - \frac{\eta}{2}$ and $P(u) < 0$. We also choose a $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $v \geq 0$ with $\|v\|_2^2 = \frac{\eta}{2}$. We define

$$u_n(x) := u(x) + \left(\frac{1}{n}\right)^{\frac{N}{2}} v\left(\frac{1}{n}(x - n^2 e_1)\right) = u(x) + v_n(x),$$

where $e_1 = (1, 0, \cdots, 0)$ and $n > 0$ is chosen sufficiently large so that the supports of u and v_n are disjoints. We then obtain

$$\begin{aligned} P(u_n) &= \int_{\mathbb{R}^N} |\nabla(u+v_n)|^2\,\mathrm{d}x + \frac{N}{2}c^2 - \mu\gamma_p\int_{\mathbb{R}^N} |u+v_n|^p\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{N}{2}c^2 - \mu\gamma_p\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x + o_n(1) \leq 0, \end{aligned}$$

since $\|\nabla v_n\|_2^2 \to 0$ and $\|v_n\|_p^p \to 0$ as $n \to \infty$. Taking into account that $\int_{\mathbb{R}^N} u_n^2 \log u_n^2\,\mathrm{d}x \to -\infty$, we then get

$$I(u_n) \to -\infty, \quad \text{as } n \to \infty.$$

Thus, the lemma is proved. □

Define
$$k_0 := \frac{p\gamma_p N}{2(2-p\gamma_p)}c^2.$$

Lemma 7.3. *Assume that $2 < p < 2+\frac{4}{N}$ and $N \geq 2$. If $P(u) \leq 0$ and $\|\nabla u\|_2^2 = k_0$, then*
$$c \geq D = \left(\frac{1}{\mu\gamma_p C^p(N,p)}\left(\frac{N}{2-p\gamma_p}\right)^{\frac{2-p\gamma_p}{2}}\left(\frac{p\gamma_p}{2}\right)^{-\frac{p\gamma_p}{2}}\right)^{\frac{1}{p-2}}.$$
Moreover, If $P(u) < 0$ and $\|\nabla u\|_2^2 = k_0$, then $c > D$.

Proof. Since $P(u) \leq 0$, by Gagliardo–Nirenberg inequality, we have
$$\|\nabla u\|_2^2 + \frac{N}{2}c^2 \leq \mu\gamma_p C^p(N,p)c^{p(1-\gamma_p)}\|\nabla u\|_2^{p\gamma_p}.$$
The above equality holds only for the best constant in the Gagliardo–Nirenberg inequality is achieved. Let $\|\nabla u\|_2^2 = \frac{p\gamma_p N}{2(2-p\gamma_p)}c^2$, we get
$$c \geq D = \left(\frac{1}{\mu\gamma_p C^p(N,p)}\left(\frac{N}{2-p\gamma_p}\right)^{\frac{2-p\gamma_p}{2}}\left(\frac{p\gamma_p}{2}\right)^{-\frac{p\gamma_p}{2}}\right)^{\frac{1}{p-2}}.$$
The lemma is completed. □

Lemma 7.4. *Let $2 < p < 2+\frac{4}{N}$ and $N \geq 2$. If $c = D$, then*
$$M(c) := \sup_{u\in\mathcal{P}_c\cap H_r^1(\mathbb{R}^N)} I(u)$$
is achieved.

Proof. From Lemma 7.1, if $c \geq D$, then $\mathcal{P}_c \neq \emptyset$. Let $u \in \mathcal{P}_c$, by Gagliardo–Nirenberg inequality, we have
$$\|\nabla u\|_2^2 + \frac{N}{2}c^2 \leq \mu\gamma_p C^p(N,p)c^{p(1-\gamma_p)}\|\nabla u\|_2^{p\gamma_p}, \tag{7.2}$$
then $\|\nabla u\|_2^2$ is bounded from above. Moreover, we deduce that
$$\begin{aligned} I(u) &= \left(\frac{1}{2}-\frac{1}{p\gamma_p}\right)\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} B(|u|) - A(|u|)\,\mathrm{d}x - \frac{p}{2(p-2)}c^2 \\ &\leq \left(\frac{1}{2}-\frac{1}{p\gamma_p}\right)\|\nabla u\|_2^2 + K_q C^q(N,q)c^{q(1-\gamma_q)}\|\nabla u\|_2^{q\gamma_q} - \frac{p}{2(p-2)}c^2, \end{aligned} \tag{7.3}$$
where $q\gamma_q \leq 2$. Therefore, I restricted to $\mathcal{P}_c$ is bounded from above, and $M(c) < +\infty$. Let $\{u_n\} \subset \mathcal{P}_c \cap H_r^1(\mathbb{R}^N)$ be a maximizing sequence for I on $M(c)$, combining (7.2) and (7.3), u_n is bounded in $H_r^1(\mathbb{R}^N)$ and $A(|u_n|)$ is bounded in $L^1(\mathbb{R}^N)$. It follows from Lemma 2.1 that u_n is bounded in W_r. From Lemma 2.2, there exists $u \in S(c) \cap H_r^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in W_r and $u_n \to u$ strongly in $L^s(\mathbb{R}^N)$, $2 \leq s < 2^*$. Moreover, $P(u) \leq 0$. By a direct calculation, if $P(u) = 0$, then $I(u) \leq M(c)$. Define
$$F(u) = I(u) - \frac{1}{p\gamma_p}P(u) = \left(\frac{1}{2}-\frac{1}{p\gamma_p}\right)\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} u^2\log u^2\,\mathrm{d}x - \frac{p}{2(p-2)}c^2.$$

Since A is lower semicontinuous for the weak convergence in W_r, we have $I(u) \geq M(c)$. By contradiction, $P(u) < 0$. Taking into account that

$$F(t \star u) = \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right) e^{2t}\|\nabla u\|_2^2 + \frac{tN}{2}\int_{\mathbb{R}^N} u^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} u^2 \log u^2 \,\mathrm{d}x - \frac{p}{2(p-2)}c^2,$$

if $\frac{d}{dt}|_{t=0}F(t \star u) = 0$, we have $\|\nabla u\|_2^2 = k_0$, which contradicts with Lemma 5.3.

We now prove $\frac{d}{dt}|_{t=0}F(t \star u) = 0$ by distinguishing two cases.

Case 1. We first assume that $\frac{d}{dt}|_{t=0}F(t\star u) < 0$. Since $\lim_{t\to-\infty} P(t\star u) = \frac{N}{2}c^2 > 0$ and $P(u) < 0$, there exists $t_0 \in \mathbb{R}$ such that $t_0 < 0$ and $t_0 \star u \in \mathcal{P}_c$. It follows that $P(t \star u) < 0$ for $t \in (t_0, 0]$. We claim that $\frac{d}{dt}F(t \star u) < 0$ for all $t \in (t_0, 0]$. In fact, by Lemma 7.3, $\frac{d}{dt}F(t \star u) \neq 0$ for $\bar{t} \in (t_0, 0]$. If $\frac{d}{dt}|_{\bar{t}}F(t \star u) > 0$ for $\bar{t} \in (t_0, 0]$, then by continuity, there exists $t_1 \in [\bar{t}, 0]$ such that $\frac{d}{dt}|_{t_1}F(t \star u) = 0$, which contradicts with Lemma 7.3. Therefore, we have

$$F(t_0 \star u) = I(t_0 \star u) > F(u) = I(u) - \frac{1}{p\gamma_p}P(u) \geq M(c),$$

this is a contradiction with the definition of $M(c)$.

Case 2. Next, we assume that $\frac{d}{dt}|_{t=0}F(t \star u) > 0$. Since $\lim_{t\to+\infty} P(t \star u) = +\infty$, there exists $t_2 > 0$ such that $P(t_2 \star u) = 0$ and $P(t \star u) < 0$ for all $t \in [0, t_2)$. Thus, similar to the proof of Case 1, we have $\frac{d}{dt}|_{\bar{t}}F(t \star u) > 0$ for $t \in [0, t_2)$. We also have $F(t_2 \star u) > M(c)$, which leads to the same contradiction.

In conclusion, we deduce that $P(u) = 0$. Therefore, we have $I(u) = M(c)$. □

Lemma 7.5. *Let $p \in (2, 2 + \frac{8}{N(N+2)}) \cup (2 + \frac{8}{N(N+2)}, 2 + \frac{4}{N})$, if $c = D$, for any maximizer of $M(c)$ is a critical point of I restricted to $S(c)$.*

Proof. We prove that $\mathcal{P}_c \cap H_r^1(\mathbb{R}^N)$ is a smooth manifold of codimension 1 in $S(c)$. By Lemma 7.4, we deduce there exists $u \in \mathcal{P}_c \cap H_r^1(\mathbb{R}^N)$ such that

$$I(u) = \max_{\mathcal{P}_c \cap H_r^1(\mathbb{R}^N)} I.$$

Then there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $dI(u) = \lambda_1 u + \lambda_2 dP(u)$, and

$$-(1 + 2\lambda_2)\Delta u + \lambda_1 u + u \log u^2 = (1 + p\gamma_p\lambda_2)\, \mu|u|^{p-2}u.$$

The Pohozaev identity for the above equation is

$$(1 + 2\lambda_2)\|\nabla u\|_2^2 + \frac{N}{2}c^2 = (1 + p\gamma_p\lambda_2)\, \mu\gamma_p\|u\|_p^p. \tag{7.4}$$

Therefore, by combining (7.4) with $P(u) = 0$, we have

$$2\lambda_2\|\nabla u\|_2^2 = p\gamma_p^2\lambda_2\mu\|u\|_p^p. \tag{7.5}$$

If $\lambda_2 \neq 0$, it follows from $P(u) = 0$ that

$$\|\nabla u\|_2^2 + \frac{N}{2}c^2 \leq \mu\gamma_p C^p(N, p)c^{p(1-\gamma_p)}\|\nabla u\|_2^{p\gamma_p}. \tag{7.6}$$

The equality in (7.6) holds only for the best constant in the Gagliardo–Nirenberg inequality is reached, from [40], u satisfies

$$\|\nabla u\|_2^2 = \|u\|_2^2. \tag{7.7}$$

Combining (7.5) with (7.7), we have $p = 2 + \frac{8}{N(N+2)}$, which is a contradiction. Therefore, the strict inequality in (7.6) holds, which contradicts with Lemma 7.3, then $\lambda_2 = 0$.

In conclusion, $\mathcal{P}_c$ is a smooth manifold of codimension 1 on $S(c)$. □

Proof of Theorem 1.6. By Lemma 7.4, we know that there exists $u \in W_r$ such that $I(u) = \sup_{\mathcal{P}_c \cap H_r^1(\mathbb{R}^N)} I$. From Lemma 7.5, u is a critical point of I restricted to $S(c)$. Moreover, it directly follows from Theorem 1.5 (iii) that u is non-positive. □

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