

Finite dimensionality of Besov spaces and potential-theoretic decomposition of metric spaces

TAKASHI KUMAGAI, NAGESWARI SHANMUGALINGAM
and RYOSUKE SHIMIZU

Abstract. In the context of a metric measure space (X, d, μ) , we explore the potential-theoretic implications of having a finite-dimensional Besov space. We prove that if the dimension of the Besov space $B_{p,p}^\theta(X)$ is $k > 1$, then X can be decomposed into k number of irreducible components (Theorem 1.1). Note that θ may be bigger than 1, as our framework includes fractals. We also provide sufficient conditions under which the dimension of the Besov space is 1. We introduce critical exponents $\theta_p(X)$ and $\theta_p^*(X)$ for the Besov spaces. As examples illustrating Theorem 1.1, we compute these critical exponents for spaces X formed by glueing copies of n -dimensional cubes, the Sierpiński gaskets, and of the Sierpiński carpet.

Besovin avaruuksien äärellisulotteisuus ja metristen avaruuksien potentiaalteoreettinen hajottaminen

Tiivistelmä. Tässä työssä tarkastellaan Besovin avaruuden äärellisulotteisuuden potentiaalteoreettisia seurauksia metrisessä mitta-avaruudessa (X, d, μ) . Työssä osoitetaan, että jos Besovin avaruuden $B_{p,p}^\theta(X)$ ulottuvuusluku on $k > 1$, voidaan avaruus X hajottaa k -lukumäärään pelkistymättömiä osia (lause 1.1). Suure θ voi olla suurempi kuin 1, koska tarkasteltu asetelma sisältää myös murtojoukkoja. Lisäksi annetaan riittävät ehdot Besovin avaruuden yksiulotteisuudelle. Työssä esitellään kaksi Besovin avaruuden kriittistä eksponenttia, $\theta_p(X)$ ja $\theta_p^*(X)$. Lausetta 1.1 havainnollistetaan laskemalla nämä kriittiset eksponentit avaruuksille, jotka muodostuvat yhteen liimatuista n -ulotteisista kuutioista, Sierpińskin kolmioista ja Sierpińskin matoista.

1. Introduction

Given a compact metric space (X, d) equipped with a doubling measure μ , a viable theory of local Dirichlet-type energy forms is obtained by considering the Newton–Sobolev class $N^{1,p}(X)$ of functions on X if we know that (X, d, μ) supports a p -Poincaré inequality for some $1 \leq p < \infty$. However, when no Poincaré type inequality is available on (X, d, μ) , a more natural local energy form is given by the so-called Korevaar–Schoen space $KS_p^1(X)$, see for instance [20]. We are interested in the function-classes $B_{p,p}^\theta(X)$ (Besov), $B_{p,\infty}^\theta(X)$, and $KS_p^\theta(X)$ (Korevaar–Schoen). These are spaces of functions in $L^p(X)$ for which the following respective energies

<https://doi.org/10.54330/afm.163110>

2020 Mathematics Subject Classification: Primary 31E05, 28A80; Secondary 46E36, 31C25.

Key words: Besov spaces, Korevaar–Schoen spaces, fractal, irreducible p -energy form, Newton–Sobolev spaces, p -Poincaré inequality, Sierpiński fractals, decomposition.

T.K.’s work is partially supported by JSPS KAKENHI Grant Numbers 22H00099 and 23KK0050. N.S.’s work is partially supported by the NSF (U.S.A.) grant DMS #2054960. R.S.’s work (JSPS Research Fellow-PD) is partially supported by JSPS KAKENHI Grant Number 23KJ2011. The authors thank the referee for the kind suggestions that helped improve the exposition of the paper.

© 2025 The Finnish Mathematical Society

are finite:

$$\begin{aligned} \|u\|_{B_{p,p}^\theta(X)}^p &:= \int_X \int_X \frac{|u(y) - u(x)|^p}{d(x,y)^{\theta p} \mu(B(x,d(x,y)))} d\mu(y) d\mu(x) \\ &\approx \int_0^{\text{diam}(X)} \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\theta p}} d\mu(y) d\mu(x) \frac{dt}{t}; \\ \|u\|_{B_{p,\infty}^\theta(X)}^p &:= \sup_{t>0} \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\theta p}} d\mu(y) d\mu(x); \\ \|u\|_{KS_p^\theta(X)}^p &:= \limsup_{t \rightarrow 0^+} \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\theta p}} d\mu(y) d\mu(x), \end{aligned}$$

where, by $F \approx H$ we mean that there is a constant $C \geq 1$, independent of the parameters F and H depend on (in the above it would be u), so that $C^{-1} \leq F/H \leq C$. (For the equivalence on $\|u\|_{B_{p,p}^\theta(X)}^p$ under the volume doubling property, see [13, Theorem 5.2].) While the energy $\|u\|_{KS_p^\theta(X)}$ is local, the energy $\|u\|_{B_{p,\infty}^\theta(X)}$ is not. In general we do not know that the two norms $\|u\|_{B_{p,\infty}^\theta(X)}$ and $\|u\|_{KS_p^\theta(X)}$ are comparable, but because μ is doubling, we have that as sets, $B_{p,\infty}^\theta(X) = KS_p^\theta(X)$, see Lemma 2.6 below.

The goal of this paper is to investigate what the potential-theoretic implications are of knowing that $B_{p,p}^\theta(X)$ has finite dimension. The following two critical exponents $\theta_p(X)$ and $\theta_p^*(X)$ for the Besov space will play important roles. Throughout the paper, we assume that X has infinitely many points. Inspired by the ground-breaking result of Bourgain, Brezis and Mironescu [6], we define

$$\begin{aligned} \theta_p(X) &:= \theta_p := \sup\{\theta > 0: B_{p,p}^\theta(X) \text{ contains non-constant functions}\}; \\ \theta_p^*(X) &:= \theta_p^* := \sup\{\theta > 0: B_{p,p}^\theta(X) \text{ is dense in } L^p(X)\}. \end{aligned}$$

Note that $\theta_p(X) \geq 1$ if (X, d, μ) is a doubling metric measure space (see Lemma 2.3), and that $\theta_p(X) \geq \theta_p^*(X)$. When the measure on X is doubling and supports a p -Poincaré inequality for all function-upper gradient pairs as in (2.2), then we must have $\theta_p = 1$. If the dimension of $B_{p,p}^\theta(X)$ is 1, then $B_{p,p}^\theta(X)$ consists solely of constant functions and $\theta_p(X) \leq \theta$. The following theorem tells us that if the dimension of $B_{p,p}^\theta(X)$ is finite but larger than 1, then X can be decomposed into as many pieces as the dimension of $B_{p,p}^\theta(X)$ so that there is no potential-theoretic communication between different pieces.

Theorem 1.1. *Let (X, d, μ) be a doubling metric measure space as in (2.1) and $\theta > 0$. Suppose that the dimension of $B_{p,p}^\theta(X)$ is finite. Then either $\mu(X) = \infty$ and $B_{p,p}^\theta(X) = \{0\}$ (in which case $\theta \geq \theta_p(X)$), or there exist measurable sets $E_1, \dots, E_k$, with k the dimension of $B_{p,p}^\theta(X)$, such that the following hold:*

- (1) $0 < \mu(E_i) < \infty$ for $i = 1, \dots, k$,
- (2) If $\mu(X) < \infty$, then $\mu(X \setminus \bigcup_{i=1}^k E_i) = 0$,
- (3) $\chi_{E_i} \in B_{p,p}^\theta(X)$ for $i = 1, \dots, k$, and $\{\chi_{E_i}: i = 1, \dots, k\}$ forms a basis for $B_{p,p}^\theta(X)$.
- (4) $B_{p,p}^\theta(X) = \oplus_{i=1}^k B_{p,p}^\theta(E_i) := \{f \in L^p(X): f|_{E_i} \in B_{p,p}^\theta(E_i), i = 1, \dots, k\}$ as sets. Moreover, the dimension of $B_{p,p}^\theta(E_i)$ is 1 for all $i = 1, \dots, k$.
- (5) $\|\chi_{E_i}\|_{KS_p^\theta(X)} = 0$ for $i = 1, \dots, k$.

(6) If $u \in KS_p^\theta(X) \cap L^\infty(X)$, then for $j = 1, \dots, k$ we have

$$\|u \chi_{E_j}\|_{KS_p^\theta(X)}^p = \limsup_{r \rightarrow 0^+} \int_{E_j} \int_{B(x,r) \cap E_j} \frac{|u(y) - u(x)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x).$$

(7) $\theta \leq \theta_p(X)$ if $k > 1$ or $\mu(X) = \infty$ with $k = 1$, and $\theta \geq \theta_p(X)$ if $\mu(X) < \infty$ and $k = 1$.

In Condition (6) above, we do not know whether we can remove the requirement that $u \in L^\infty(X)$.

As a consequence of the above theorem, if $k > 1$, we have a decomposition of X into measurable pieces E_i , $i = 1, \dots, k$ (up to a null-measure set) so that there is no potential theoretic communication between different pieces; this is encoded in the claim $\|\chi_{E_i}\|_{KS_p^\theta(X)} = 0$. Moreover, Condition (4) also encodes the property that $\mu(E_i \cap E_j) = 0$ when $i, j \in \{1, \dots, k\}$ with $i \neq j$.

We now introduce the notion of *irreducible p -energy form* for convenience.

Definition 1.2. (Irreducible p -energy form) Assume that $\mu(X) < \infty$. Let $\mathcal{F}_p$ be a linear subspace of $L^p(X, \mu)$ and let $\mathcal{E}_p: \mathcal{F}_p \rightarrow [0, \infty)$ be such that $\mathcal{E}_p(\cdot)^{1/p}$ is a seminorm on $\mathcal{F}_p$. We say that $(\mathcal{E}_p, \mathcal{F}_p)$ is a *irreducible p -energy form* on (X, μ) if whenever $u \in \mathcal{F}_p$ with $\mathcal{E}_p(u) = 0$, we must have that u is a constant function (μ -a.e.). Otherwise, we say $(\mathcal{E}_p, \mathcal{F}_p)$ is a *reducible p -energy form*.

Remark 1.3. The above definition is inspired by the theory of symmetric Dirichlet forms (i.e. $p = 2$ case). See [11, Theorem 2.1.11] for other (equivalent) formulations of the irreducibility of recurrent symmetric Dirichlet forms.

By Theorem 1.1 (5), we have the following; if the dimension of $B_{p,p}^\theta(X)$ is finite and larger than 1, then $(\|\cdot\|_{KS_p^\theta(X)}, KS_p^\theta(X))$ is reducible. Note that if the dimension of $B_{p,p}^\theta(X)$ is 1 and $\mu(X) < \infty$, then clearly $(\|\cdot\|_{B_{p,p}^\theta(X)}^p, B_{p,p}^\theta(X))$ is irreducible, and only constant functions are in $B_{p,p}^\theta(X)$. Next we provide a sufficient condition regarding the behaviors of $\|\cdot\|_{B_{p,p}^\theta(X)}$ and of $\|\cdot\|_{KS_p^\theta(X)}$ under which the dimension of $B_{p,p}^\theta(X)$ is 1.

Definition 1.4. We say that X satisfies the *weak maximality property*, or $(w\text{-max})_{p,\theta}$ property, for $B_{p,\infty}^\theta(X)$ if there is a constant $C \geq 1$ such that for each $u \in B_{p,\infty}^\theta(X)$ we have that

$$(w\text{-max})_{p,\theta} \quad \|u\|_{B_{p,\infty}^\theta(X)} \leq C \|u\|_{KS_p^\theta(X)}.$$

Theorem 1.5. We fix $1 < p < \infty$ and $\theta > 0$. If (X, d, μ) is a doubling metric measure space that satisfies the $(w\text{-max})_{p,\theta}$ property for $B_{p,\infty}^\theta(X)$, then the dimension of $B_{p,p}^\theta(X)$ is at most 1, and $\theta_p(X) \leq \theta$.

In the spirit of [7] we prove the following theorem, which also gives a sufficient condition for the dimension of $B_{p,p}^\theta(X)$ to be at most 1. For $p = 2$, a similar result was proved in [23] under certain estimates on the heat kernel, in particular, the cases of Sierpiński gasket and the Sierpiński carpet are included in [23].

Theorem 1.6. Let $1 < p < \infty$ and (X, d, μ) be a doubling metric measure space. Assume that (X, d, μ) supports the following Sobolev-type inequality: there

exist positive real numbers θ, C such that for any $u \in B_{p,p}^\theta(X)$,

$$(1.7) \quad \int_X |u - u_X|^p d\mu \leq C \liminf_{t \rightarrow 0^+} \int_X \int_{B(x,t)} \frac{|u(x) - u(y)|^p}{t^{\theta p}} d\mu(y) d\mu(x).$$

Then for that choice of θ we have that $B_{p,p}^\theta(X)$ has dimension at most 1.

In the case that (X, d, μ) supports a p -Poincaré inequality for function-upper gradient pairs, it is known that $N^{1,p}(X) = KS_p^1(X)$ (see, e.g., [20, Section 4] or [15, Section 10.4, Theorem 10.4.3, and Corollary 10.4.6]) and that $\theta_p(X) = 1$ (see [1, Theorem 5.1]). These facts, along with Theorem 1.6, imply the following corollary.

Corollary 1.8. *Suppose that $1 < p < \infty$ and (X, d, μ) is a doubling metric measure space that supports a p -Poincaré inequality for function-upper gradient pairs (see (2.2)). Then $\theta_p(X) = 1$ and $B_{p,p}^1(X)$ has at most dimension 1.*

We emphasize that, in Theorems 1.1, 1.5, and 1.6, we do not confine ourselves to the case $0 < \theta \leq 1$ in view of some recent studies of Sobolev spaces on fractals; see, e.g., [1, 18, 19, 22, 24]. For example, in the case that X is the Sierpiński carpet, Murugan and the third-named author [22] proposed a way to define the $(1, p)$ -Sobolev space $\mathcal{F}_p$ on X through discrete approximations of X , and demonstrated that $\mathcal{F}_p = KS_p^{d_{w,p}/p}(X)$ (see [22, Theorem 7.1]) with $d_{w,p} > p$ (see [24, Theorem 2.27]). Hence a Korevaar–Schoen space $KS_p^\theta(X)$ with $\theta > 1$ appears as a function space playing the role of a $(1, p)$ -Sobolev space on a fractal space. Here the parameter $d_{w,p}$ is called the p -walk dimension of the Sierpiński carpet, X , given by $d_{w,p} := \log(8\rho_p)/\log 3$, where $\rho_p \in (0, \infty)$ is a value called the p -scaling factor of X as defined in [22, Theorem 8.5 and Definition 8.7], 3 is the reciprocal of the common contraction ratio of the family of similitudes associated with X and 8 is the number of similitudes in this family. (For $X = [0, 1]^n$, we can decompose X into 3^n cubes with side lengths $1/3$ and then see that the p -scaling factor with respect to this decomposition is given by 3^{p-n} . Hence $d_{w,p} = \log(3^n \cdot 3^{p-n})/\log 3 = p$.) In the case $p = 2$, $(\rho_2)^{-1}$ coincides with the *resistance scaling factor* of X . As a connection with quasiconformal geometry, it is known that $\rho_p > 1$ if and only if $p > d_{\text{ARC}}$, where d_{ARC} is the Ahlfors regular conformal dimension of the Sierpiński carpet. See [22, Definitions 1.6] and [10] for further details on d_{ARC} .

When μ is doubling and $0 < \theta < 1$, the corresponding space $B_{p,p}^\theta(X)$ can be seen as the trace space of a strongly local energy form on a larger space (Ω, ν) with $X = \partial\Omega$ and μ and ν are related in a co-dimensional manner, as demonstrated in [4]. From the viewpoint of trace theorems on fractals, a Besov space $B_{p,p}^\theta(X)$ with $\theta \geq 1$ can appear as indicated in [16, Theorem 2.5 and 2.6] for the case $p = 2$.

In some circumstances the reason for $\theta_p(X) > 1$ may be due to X being obtained as a gluing of smaller metric measure spaces along sets that are too small to allow communication between these component spaces via the gluing set, as seen in Example 3.1 below, where the gluing set of two n -dimensional hypercubes is discussed. In this case, when $1 < p < n$, we have that $\theta_p(X) = n/p > 1$, but once we have decomposed X into the two constituent component cubes E and $X \setminus E$, we have that $\theta_p(E) = \theta_p(X \setminus E) = 1$, and $B_{p,p}^\theta(X)$ is well-understood when $0 < \theta < 1$ as trace of a larger local process, and when $1 \leq \theta < \theta_p(X)$ as piecewise constant functions. Our main theorem, Theorem 1.1, gives a way of identifying this possibility. However, there are many situations where the need for $\theta \geq 1$ is more integral to the space, as is the case of the Sierpiński gasket and the Sierpiński carpet, as explained in the

previous paragraph. For these spaces, typically, $B_{p,p}^\theta(X)$ has either infinite dimension or dimension 1.

We conclude the introduction by reviewing some concrete examples discussed in this paper. In Example 3.1, for $n \in \mathbb{N}$ with $n \geq 2$, as mentioned above we consider the metric measure space X obtained as the union of two n -dimensional hypercubes glued at a vertex, and observe that the dimension of $B_{p,p}^1(X)$ is 2 when $1 < p < n$. Note that each cubical component of X supports a p -Poincaré inequality for any $p \geq 1$, while X does not support a p -Poincaré inequality when $1 < p \leq n$. Similar observations will be made in the case X is the union of two copies of the Sierpiński carpet glued at a vertex in Example 3.10; indeed, the dimension of $B_{p,p}^{d_{w,p}/p}(X)$ is 2 when $1 < p < d_{\text{ARC}}$. Note that the Ahlfors regular conformal dimension d_{ARC} and the p -walk dimension of the n -dimensional hypercube are n and p respectively. In both examples mentioned above, the two critical exponents $\theta_p(X)$ and $\theta_p^*(X)$ turn out to be different when $1 < p < d_{\text{ARC}}$. Namely, the following holds, where d_f is the Hausdorff dimension of X .

Theorem 1.9. *Let X be one of the glued metric measure spaces as in Examples 3.1 and 3.10. Then $\theta_p(X) = \frac{1}{p} \max\{d_f, d_{w,p}\}$ and $\theta_p^*(X) = \frac{d_{w,p}}{p}$.*

By [5, Corollary 3.7] and [10, Corollary 1.4], we know that $d_{w,p} > d_f$ if and only if $p > d_{\text{ARC}}$, that $d_{w,p} < d_f$ if and only if $p < d_{\text{ARC}}$, and that $d_{w,p} = d_f$ for $p = d_{\text{ARC}}$ for these examples. This result suggests that the case $1 < p < d_{\text{ARC}}$ requires a careful treatment of the “potential-theoretic decomposability” of the underlying example spaces. See also [8] for a few examples of self-similar sets that have a similar spirit, and [3] for the validity/invalidity of Poincaré type inequalities on a general *bow-tie*, which is obtained by gluing two metric spaces at a point. Note that $d_{\text{ARC}} = 1$ for the standard Sierpiński gaskets (see, e.g., [17, Theorem B.8]), so the case $1 < p < d_{\text{ARC}}$ does not occur in this example. If X is the space obtained by gluing two copies of the Sierpiński gasket, then $\theta_p(X) = \theta_p^*(X)$ holds for any $p \in (1, \infty)$; see Example 3.5 and Theorem 3.9.

2. Background and general results

2.1. Background. Throughout this paper, the triple (X, d, μ) is a separable metric space (X, d) , equipped with a Borel measure μ ; we require in this paper that X has infinitely many points and that $0 < \mu(B(x, r)) < \infty$ for each $x \in X$ and $r > 0$, where $B(x, r)$ denotes the set of all points $y \in X$ such that $d(x, y) < r$. We also fix $p \in (1, \infty)$. Note that μ is σ -finite in this setting.

We say that (X, d, μ) is a *doubling metric measure space*, or μ is a *doubling measure* on (X, d) , if there exists a constant C_D such that

$$(2.1) \quad 0 < \mu(B(x, 2r)) \leq C_D \mu(B(x, r)) < \infty \quad \text{for all } x \in X, r > 0.$$

Without loss of generality, we may assume that $C_D > 1$ if needed.

In this paper the primary function-spaces of interest are the Besov spaces and the Korevaar–Schoen spaces $B_{p,p}^\theta(X)$, $B_{p,\infty}^\theta(X)$, and $KS_p^\theta(X)$, as described at the beginning of Section 1 above. In addition, the Newton–Sobolev class $N^{1,p}(X)$ will play an auxiliary role, and we describe this class next.

A function $f: X \rightarrow [-\infty, \infty]$ is said to have a Borel function $g: X \rightarrow [0, \infty]$ as an *upper gradient* if we have

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g \, ds$$

whenever $\gamma: [a, b] \rightarrow X$ is a rectifiable curve with $a < b$. (We interpret the inequality as also meaning that $\int_\gamma g ds = \infty$ whenever at least one of $f(\gamma(a)), f(\gamma(b))$ is not finite.) We say that $f \in \widetilde{N^{1,p}}(X)$ if

$$\|f\|_{N^{1,p}(X)} := \left(\int_X |f|^p d\mu \right)^{1/p} + \inf_g \left(\int_X g^p d\mu \right)^{1/p}$$

is finite, where the infimum is over all upper gradients g of f . Then one can see that $\widetilde{N^{1,p}}(X)$ is a vector space. For $f_1, f_2 \in \widetilde{N^{1,p}}(X)$, we say that $f_1 \sim f_2$ if $\|f_1 - f_2\|_{N^{1,p}(X)} = 0$. Now the *Newton–Sobolev class* $N^{1,p}(X)$ is defined as the collection of the equivalence classes with respect to $\sim$, i.e., $N^{1,p}(X) := \widetilde{N^{1,p}}(X) / \sim$. For more on this space we refer the interested reader to [15].

We say that (X, d, μ) supports a *p -Poincaré inequality* (with respect to upper gradients) if there are constants $C > 0$ and $\lambda \geq 1$ such that for every measurable function f on X and every upper gradient g of f and ball $B(x, r)$,

$$(2.2) \quad \int_{B(x,r)} |f - f_{B(x,r)}| d\mu \leq Cr \left(\int_{B(x,\lambda r)} g^p d\mu \right)^{1/p}.$$

From [20, Theorem 4.1] or [15, Section 10.4] we know that if $u \in L^p(X)$ such that there is a non-negative function $g \in L^p(X)$ with (u, g) satisfying the p -Poincaré inequality (2.2), then $u \in KS_p^1(X)$. In [20] the space $KS_p^1(X)$ is denoted by $\mathcal{L}^{1,p}(X)$. Moreover, from [15, Theorems 10.5.1 and 10.5.2] we know that $KS_p^1(X) \subset N^{1,p}(X)$ even if X does not support a p -Poincaré inequality, and that when X supports a p -Poincaré inequality in addition, we also have $KS_p^1(X) = N^{1,p}(X)$. Thus the index $\theta = 1$ plays a key role in the theory of Sobolev spaces in nonsmooth analysis.

2.2. General results. We present some lemmata on Besov spaces $B_{p,p}^\theta(X)$, $B_{p,\infty}^\theta(X)$ and the Korevaar–Schoen space $KS_p^\theta(X)$.

Lemma 2.3. *Suppose that μ is a doubling measure. Then $\theta_p(X) \geq 1$.*

Proof. Fix positive $\theta < 1$ and $x_0 \in X$. We fix a positive number $R_0 < \frac{1}{2} \text{diam}(X)$ so that $B(x_0, R_0)$ has at least two points, and set $u: X \rightarrow \mathbb{R}$ by

$$u(x) = \max\{1 - d(x_0, x)/R_0, 0\}.$$

Note that u is $1/R_0$ -Lipschitz continuous on X , $0 \leq u \leq 1$ on X , and is zero outside the bounded set that is $B := B(x_0, R_0)$. Now

$$\begin{aligned} \|u\|_{B_{p,p}^\theta(X)}^p &= \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x, y)^{\theta p} \mu(B(x, d(x, y)))} d\mu(y) d\mu(x) \\ &\leq \int_{2B} \int_{2B} \frac{d(x, y)^p}{R_0^p d(x, y)^{\theta p} \mu(B(x, d(x, y)))} d\mu(y) d\mu(x) \\ &\quad + 2 \int_B \int_{X \setminus 2B} \frac{1}{d(x, y)^{\theta p} \mu(B(x, d(x, y)))} d\mu(y) d\mu(x). \end{aligned}$$

For each non-negative integer j and $x \in X$, we set $A_j(x) := B(x, 2^{j+1}R_0) \setminus B(x, 2^j R_0)$. Since $X \setminus 2B \subset X \setminus B(x, R_0)$ for $x \in B$, we see that

$$\begin{aligned} & \int_B \int_{X \setminus 2B} \frac{1}{d(x, y)^{\theta p} \mu(B(x, d(x, y)))} d\mu(y) d\mu(x) \\ & \leq \int_B \sum_{j=0}^{\infty} \int_{A_j(x)} \frac{1}{d(x, y)^{\theta p} \mu(B(x, d(x, y)))} d\mu(y) d\mu(x) \\ & \leq \int_B \sum_{j=0}^{\infty} \int_{A_j(x)} \frac{1}{(2^j R_0)^{\theta p} \mu(B(x, 2^j R_0))} d\mu(y) d\mu(x) \\ & \leq \frac{\mu(B)}{R_0^{\theta p}} \sum_{j=0}^{\infty} 2^{-j\theta p} \frac{\mu(B(x, 2^{j+1} R_0))}{\mu(B(x, 2^j R_0))} \\ & \leq \frac{C_D}{1 - 2^{-\theta p}} \frac{\mu(B)}{R_0^{\theta p}} < \infty. \end{aligned}$$

Moreover, setting $E_k(x) := B(x, 2^{-k+2}R_0) \setminus B(x, 2^{-k+1}R_0)$ for non-negative integers k and $x \in X$, we have

$$\begin{aligned} & \int_{2B} \int_{2B} \frac{d(x, y)^p}{R_0^p d(x, y)^{\theta p} \mu(B(x, d(x, y)))} d\mu(y) d\mu(x) \\ & \leq R_0^{-p} \int_{2B} \int_{B(x, 4R_0)} \frac{d(x, y)^{(1-\theta)p}}{\mu(B(x, d(x, y)))} d\mu(y) d\mu(x) \\ & \leq R_0^{-p} 2^{2(1-\theta)p} \int_{2B} \sum_{k=0}^{\infty} \int_{E_k(x)} \frac{2^{[-k(1-\theta)p]} R_0^{p(1-\theta)}}{\mu(B(x, 2^{-k+1}R_0))} d\mu(y) d\mu(x) \\ & \leq R_0^{-\theta p} \mu(2B) C_D \sum_{k=-2}^{\infty} 2^{-kp(1-\theta)} < \infty. \end{aligned}$$

It follows that $u \in B_{p,p}^{\theta}(X)$. □

A function v is called a normal contraction of a function u if the following holds for all $x, y \in X$:

$$|v(x) - v(y)| \leq |u(x) - u(y)| \quad \text{and} \quad |v(x)| \leq |u(x)|.$$

Examples of normal contractions include functions v of the form $v(x) = \max\{0, u(x) - a_0\}$ for any non-negative number a_0 . In the case $a_0 = 0$, we define $u_+(x) := \max\{0, u(x)\}$. The following lemma is easy to check by the definition of $B_{p,p}^{\theta}(X)$. Note that if $a \in \mathbb{R}$, $u \in B_{p,p}^{\theta}(X)$ and $\mu(X) < \infty$, then $u + a$ is also in $B_{p,p}^{\theta}(X)$.

Lemma 2.4. *Let $u \in B_{p,p}^{\theta}(X)$ and v be a normal contraction of u . Then $v \in B_{p,p}^{\theta}(X)$ and $\|v\|_{B_{p,p}^{\theta}(X)}^p \leq \|u\|_{B_{p,p}^{\theta}(X)}^p$. As a consequence, we also have that if $u \in B_{p,p}^{\theta}(X)$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq 0 \leq \beta$, then $w_{\alpha,\beta} := \max\{\alpha, \min\{u, \beta\}\}$ is also in $B_{p,p}^{\theta}(X)$ with $\|w_{\alpha,\beta}\|_{B_{p,p}^{\theta}(X)} \leq \|u\|_{B_{p,p}^{\theta}(X)}$.*

The following lemma is also immediate from the definition of $B_{p,p}^{\theta}(X)$.

Lemma 2.5. *Let $u, v \in B_{p,p}^{\theta}(X) \cap L^{\infty}(X)$. Then $uv \in B_{p,p}^{\theta}(X)$ with*

$$\|uv\|_{B_{p,p}^{\theta}(X)} \leq \|u\|_{L^{\infty}(X)} \|v\|_{B_{p,p}^{\theta}(X)} + \|v\|_{L^{\infty}(X)} \|u\|_{B_{p,p}^{\theta}(X)}.$$

Lemma 2.6. Suppose that μ is a doubling measure on X and that $\theta > 0$.

- (1) $B_{p,\infty}^\theta(X) = KS_p^\theta(X)$ as sets and as vector spaces.
(2) For any $0 < \delta < \theta$, $B_{p,p}^\theta(X) \subset B_{p,\infty}^\theta(X) \subset B_{p,p}^{\theta-\delta}(X)$.

Proof. The assertions (1) and (2) are proved in [1, Lemma 3.2] and [12, Proposition 2.2] respectively, but we give the proof for the reader's convenience.

(1): It is direct that $B_{p,\infty}^\theta(X) \subset KS_p^\theta(X)$, and so it suffices to show the reverse inclusion. To this end, let $u \in KS_p^\theta(X)$. Then there is some $r_u > 0$ such that

$$(2.7) \quad \sup_{0 < r \leq r_u} \int_X \int_{B(x,r)} \frac{|u(x) - u(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x) \leq \|u\|_{KS_p^\theta(X)}^p + 1.$$

For $r > r_u$ we have that

$$(2.8) \quad \begin{aligned} & \int_X \int_{B(x,r)} \frac{|u(x) - u(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x) \\ &= \int_X \frac{\mu(B(x, r_u))}{\mu(B(x, r))} \int_{B(x, r_u)} \frac{|u(x) - u(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x) \\ & \quad + \int_X \frac{1}{\mu(B(x, r))} \int_{B(x,r) \setminus B(x, r_u)} \frac{|u(x) - u(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x) \\ & \leq \|u\|_{KS_p^\theta(X)}^p + 1 \\ & \quad + \int_X \frac{2^p}{\mu(B(x, r))} \int_{B(x,r)} \frac{|u(y)|^p + |u(x)|^p}{r_u^{\theta p}} d\mu(y) d\mu(x). \end{aligned}$$

Note that

$$\begin{aligned} & \int_X \frac{2^p}{\mu(B(x, r))} \int_{B(x,r)} \frac{|u(y)|^p + |u(x)|^p}{r_u^{\theta p}} d\mu(y) d\mu(x) \\ &= \frac{2^p}{r_u^{\theta p}} \int_X |u(x)|^p d\mu(x) + \frac{2^p}{r_u^{\theta p}} \int_X \int_X \frac{|u(y)|^p \chi_{B(x,r)}(y)}{\mu(B(x, r))} d\mu(y) d\mu(x) \\ & \leq \frac{2^p}{r_u^{\theta p}} \|u\|_{L^p(X)}^p + \frac{2^p C}{r_u^{\theta p}} \int_X |u(y)|^p \int_X \frac{\chi_{B(y,r)}(x)}{\mu(B(y, r))} d\mu(x) d\mu(y) \\ &= \frac{2^p(1+C)}{r_u^{\theta p}} \|u\|_{L^p(X)}^p, \end{aligned}$$

where we have used the doubling property of μ and Tonelli's theorem in the penultimate step. Now from (2.8) and (2.7) above we see that for each $r > 0$ we have

$$\int_X \int_{B(x,r)} \frac{|u(x) - u(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x) \leq \|u\|_{KS_p^\theta(X)}^p + 1 + \frac{2^p(1+C)}{r_u^{\theta p}} \|u\|_{L^p(X)}^p,$$

and as the right-hand side of the above inequality is independent of r , it follows that $u \in B_{p,\infty}^\theta(X)$.

(2): The inclusion $B_{p,p}^\theta(X) \subset B_{p,\infty}^\theta(X)$ follows from Lemma 2.9 below together with claim (1) above, and so we prove $B_{p,\infty}^\theta(X) \subset B_{p,p}^{\theta-\delta}(X)$ here. Let $u \in B_{p,\infty}^\theta(X)$

and fix a choice of α satisfying $0 < \alpha < \text{diam}(X)$. Then we see that

$$\begin{aligned}
& \int_0^{\text{diam}(X)} \int_X \int_{B(x,t)} \frac{|u(x) - u(y)|^p}{t^{(\theta-\delta)p}} d\mu(y) d\mu(x) \frac{dt}{t} \\
&= \int_0^\alpha \int_X \int_{B(x,t)} \frac{|u(x) - u(y)|^p}{t^{(\theta-\delta)p}} d\mu(y) d\mu(x) \frac{dt}{t} \\
&\quad + \int_\alpha^{\text{diam}(X)} \int_X \int_{B(x,t)} \frac{|u(x) - u(y)|^p}{t^{(\theta-\delta)p}} d\mu(y) d\mu(x) \frac{dt}{t} \\
&\leq \|u\|_{B_{p,\infty}^\theta(X)}^p \int_0^\alpha t^{\delta p-1} dt + 2^{p-1} \left(\int_\alpha^{\text{diam}(X)} \frac{\|u\|_{L^p(X)}^p}{t^{(\theta-\delta)p+1}} dt \right. \\
&\quad \left. + \int_\alpha^{\text{diam}(X)} \int_X \int_X \frac{|u(y)|^p \chi_{B(x,t)}(y)}{t^{(\theta-\delta)p+1} \mu(B(x,t))} d\mu(y) d\mu(x) dt \right) \\
&\leq \frac{\alpha^{\delta p}}{\delta p} \|u\|_{B_{p,\infty}^\theta(X)}^p + \frac{2^{p-1}}{(\theta-\delta)p} \left[\frac{1}{\alpha^{(\theta-\delta)p}} - \frac{1}{\text{diam}(X)^{(\theta-\delta)p}} \right] \|u\|_{L^p(X)}^p \\
&\quad + 2^{p-1} C_D \int_\alpha^{\text{diam}(X)} \int_X \int_X \frac{|u(y)|^p \chi_{B(y,t)}(x)}{t^{(\theta-\delta)p+1} \mu(B(y,t))} d\mu(x) d\mu(y) dt \\
&\leq \frac{\alpha^{\delta p}}{\delta p} \|u\|_{B_{p,\infty}^\theta(X)}^p + \frac{2^{p-1} (1 + C_D)}{(\theta-\delta)p} \left[\frac{1}{\alpha^{(\theta-\delta)p}} - \frac{1}{\text{diam}(X)^{(\theta-\delta)p}} \right] \|u\|_{L^p(X)}^p,
\end{aligned}$$

where we have used the doubling property of μ and Tonelli's theorem in the third inequality. Note if X is unbounded, then $\frac{1}{\text{diam}(X)^{(\theta-\delta)p}} = 0$. This estimate shows that $u \in B_{p,p}^{\theta-\delta}(X)$. $\square$

In general, unlike the energy related to $B_{p,\infty}^\theta(X)$, the energy $\|u\|_{KS_p^\theta(X)}$ is zero whenever $u \in B_{p,p}^\theta(X)$.

Lemma 2.9. *Let μ be a doubling measure on X and $\theta > 0$. Then $B_{p,p}^\theta(X) \subset KS_p^\theta(X)$ with $\|u\|_{KS_p^\theta(X)} = 0$ whenever $u \in B_{p,p}^\theta(X)$.*

Proof. Let $u \in B_{p,p}^\theta(X)$. Then we have that

$$\int_0^{\text{diam} X} \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\theta p}} d\mu(y) d\mu(x) \frac{dt}{t} < \infty.$$

For $t > 0$ we set

$$\mathcal{E}_\theta(u, t) := \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\theta p}} d\mu(y) d\mu(x).$$

Let $k_* \in \mathbb{Z} \cup \{\infty\}$ be the maximum of all the positive integers k such that $2^{k-1} < \text{diam} X$. By the doubling property of μ we have

$$\begin{aligned}
\int_0^{\text{diam} X} \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\theta p}} d\mu(y) d\mu(x) \frac{dt}{t} &\geq \sum_{i=-\infty}^{k_*-2} \int_{2^i}^{2^{i+1}} \mathcal{E}_\theta(u, t) \frac{dt}{t} \\
&\approx \sum_{i=-\infty}^{k_*-2} \mathcal{E}_\theta(u, 2^i).
\end{aligned}$$

Since the left-most expression is finite, it follows that the series on the right-hand side of the above estimate is also finite, and therefore

$$\lim_{i \rightarrow -\infty} \mathcal{E}_\theta(u, 2^i) = 0.$$

By the doubling property of μ we also have that for positive real numbers $t < \text{diam}(X)$,

$$\frac{1}{C} \mathcal{E}_\theta(u, 2^{i-1}) \leq \mathcal{E}_\theta(u, t) \leq C \mathcal{E}_\theta(u, 2^i) \quad \text{whenever } 2^{i-1} \leq t \leq 2^i.$$

It follows that

$$\limsup_{t \rightarrow 0^+} \mathcal{E}_\theta(u, t) \leq C \lim_{i \rightarrow -\infty} \mathcal{E}_\theta(u, 2^i) = 0,$$

completing the proof. $\square$

3. Examples

The following examples show that even though the two vector spaces considered in Lemma 2.9 are the same as sets, their energy norms can be incomparable.

Example 3.1. In this example we consider X to be the union of two n -dimensional hypercubes glued at the vertex $o = (0, \dots, 0)$, given by

$$X = [0, 1]^n \cup [-1, 0]^n,$$

equipped with the Euclidean metric and the n -dimensional Lebesgue measure $\mathcal{L}^n$. Here, with $u := \chi_E$ where $E = [0, 1]^n$, we see that $u \in B_{p,p}^\theta(X)$ precisely when $p\theta < n$, but we have $\|u\|_{B_{p,\infty}^\theta(X)} > 0$ (see (3.2) for a detailed calculation) but from Lemma 2.9 we also have that $\|u\|_{KS_p^\theta(X)} = 0$. To see that $u \in B_{p,p}^\theta(X)$ when $p\theta < n$, we decompose the two pieces E and $X \setminus E$ into dyadic annuli given by $L_i := \{(x_1, \dots, x_n) \in E : 2^{-i-1}R < \sqrt{x_1^2 + \dots + x_n^2} \leq 2^{-i}R\}$ and $R_i = \{(x, y) \in X \setminus E : 2^{-i-1}R < \sqrt{x_1^2 + \dots + x_n^2} \leq 2^{-i}R\}$ with $R = \sqrt{n}$, we have that

$$\begin{aligned} & \int_X \int_X \frac{|\chi_E(x) - \chi_E(y)|^p}{d(x, y)^{n+\theta p}} d\mathcal{L}^n(y) d\mathcal{L}^n(x) \\ & \approx \sum_{i,j \in \mathbb{N} \cup \{0\}} \int_{L_i} \int_{R_j} \frac{|\chi_E(x) - \chi_E(y)|^p}{d(x, y)^{n+\theta p}} d\mathcal{L}^n(y) d\mathcal{L}^n(x) \\ & \approx \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \int_{L_i} \int_{R_j} \frac{1}{d(x, y)^{n+\theta p}} d\mathcal{L}^n(y) d\mathcal{L}^n(x) \\ & \approx \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \frac{2^{-ni} R^n 2^{-nj} R^n}{(2^{-i} + 2^{-j})^{n+\theta p} R^{n+\theta p}} \\ & \approx \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} 2^{i\theta p} 2^{-nj} \approx \sum_{i=0}^{\infty} 2^{-i(n-\theta p)}. \end{aligned}$$

The above sum is finite if and only if $\theta p < n$. Thus $\chi_E \in B_{p,p}^\theta(X)$ if and only if $\theta p < n$, and so $\chi_E \in KS_p^\theta(X)$ with $\|u\|_{KS_p^\theta(X)} = 0$ whenever $\theta p < n$.

In addition, in computing $\int_{B(x,r)} \frac{|\chi_E(x) - \chi_E(y)|^p}{r^{p\theta}} d\mathcal{L}^n(y)$ for $x \in E$, we need only consider $x = (x_1, \dots, x_n) \in E$ for which $\sqrt{x_1^2 + \dots + x_n^2} < r$, and so by restricting

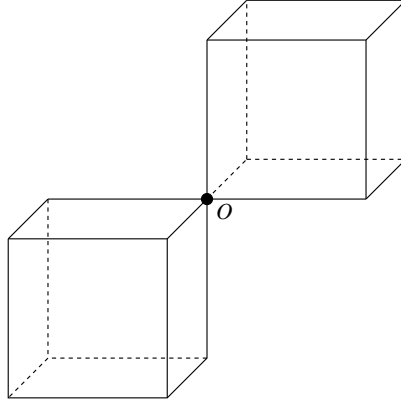


Figure 1. Gluing of two unit cubes at the origin.

our attention to the slices L_j for which $2^{-j}R \lesssim r$, we obtain

$$(3.2) \quad \int_X \int_{B(x,r)} \frac{|\chi_E(x) - \chi_E(y)|^p}{r^{p\theta}} d\mathcal{L}^n(y) d\mathcal{L}^n(x) \approx r^{n-p\theta}.$$

Hence $\chi_E \in KS_p^\theta(X)$ whenever $p\theta \leq n$; note that $\|u\|_{KS_p^\theta(X)} = 0$ if $p\theta < n$.

The following proposition states a relation between $KS_n^1(X)$ and $N^{1,n}(X)$. Set $E_1 := [0, 1]^n$, $E_2 := [-1, 0]^n$ and $o := (0, \dots, 0) \in E_1 \cap E_2$ for simplicity. In what follows, if u is a function defined on a set $E \subset X$, then the zero-extension of u to $X \setminus E$ is denoted by $u\chi_E$.

Proposition 3.3. *In the above setting $X = [0, 1]^n \cup [-1, 0]^n$, it follows that*

$$(1) \quad KS_n^1(X) = \left\{ u_1\chi_{E_1} + u_2\chi_{E_2} \mid u_i \in N^{1,n}(E_i), i \in \{1, 2\}, I_{KS}(u_1, u_2) < \infty \right\},$$

where

$$I_{KS}(u_1, u_2) := \limsup_{r \rightarrow 0^+} \int_{E_1 \cap B(o,r)} \int_{E_2 \cap B(o,r)} \frac{|u_1(x) - u_2(y)|^n}{r^{2n}} d\mathcal{L}^n(y) d\mathcal{L}^n(x).$$

$$(2) \quad KS_n^1(X) \subsetneq N^{1,n}(X).$$

Proof. We first note that the n -modulus of the all rectifiable curves in X through o is 0 by [15, Corollary 5.3.11], and that $KS_n^1(X) \subset N^{1,n}(X)$ by [15, Theorem 10.5.1] and [21, Corollary 6.5]. As a consequence, we have

$$N^{1,n}(X) = \{u_1\chi_{E_1} + u_2\chi_{E_2} \mid u_i \in N^{1,n}(E_i) \text{ for } i = 1, 2\}.$$

In addition, $KS_n^1(E_i) = N^{1,n}(E_i)$ with comparable norms by [15, Theorem 10.5.2]. When $u \in KS_n^1(X)$, necessarily $u\chi_{E_i} \in KS_n^1(E_i)$. This is because when $x \in E_i$ and $0 < r < 1$, we must have that $\mathcal{L}^n(B(x, r)) \approx r^n \approx \mathcal{L}^n(B(x, r) \cap E_i)$.

Proof of (1): Let $u_i \in N^{1,n}(E_i)$ for $i = 1, 2$, and set $u = u_1\chi_{E_1} + u_2\chi_{E_2}$. We define

$$\mathcal{E}_r^{KS}(v; A_1, A_2) := \int_{A_1} \int_{A_2 \cap B(x,r)} \frac{|v(x) - v(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x),$$

for $v \in L^n(A_1 \cup A_2)$ and Borel sets A_i of X . Observe that

$$\begin{aligned} & \int_X \int_{B(x,r)} \frac{|u(x) - u(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x) \\ & \approx \frac{1}{r^n} \left(\mathcal{E}_r^{KS}(u_1; E_1, E_1) + \mathcal{E}_r^{KS}(u_2; E_2, E_2) \right. \\ & \quad \left. + \mathcal{E}_r^{KS}(u; E_1, E_2) + \mathcal{E}_r^{KS}(u; E_2, E_1) \right). \end{aligned}$$

Since

$$\limsup_{r \rightarrow 0^+} \frac{\mathcal{E}_r^{KS}(u_i; E_i, E_i)}{r^n} \approx \int_{E_i} |\nabla u_i(x)|^n d\mathcal{L}^n(x)$$

it suffices to prove that $u \in KS_n^1(X)$ if and only if $I_{KS}(u_1, u_2) < \infty$.

Given the above discussion, we know that $u \in KS_n^1(X)$ if and only if

$$(3.4) \quad \limsup_{r \rightarrow 0^+} \frac{1}{r^n} \left(\mathcal{E}_r^{KS}(u; E_1, E_2) + \mathcal{E}_r^{KS}(u; E_2, E_1) \right) < \infty.$$

Let us focus our attention on $\mathcal{E}_r^{KS}(u; E_1, E_2)$, with the second term above being handled in a similar manner. Note that

$$\mathcal{E}_r^{KS}(u; E_1, E_2) = \int_{E_1} \int_{E_2 \cap B(x,r)} \frac{|u_1(x) - u_2(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x),$$

and so in order for $E_2 \cap B(x, r)$ to be non-empty when $x \in E_1$, it must be the case that $x \in B(o, r)$. Thus

$$\begin{aligned} \mathcal{E}_r^{KS}(u; E_1, E_2) &= \int_{E_1 \cap B(o,r)} \int_{E_2 \cap B(x,r)} \frac{|u_1(x) - u_2(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x) \\ &\leq \int_{E_1 \cap B(o,r)} \int_{E_2 \cap B(o,r)} \frac{|u_1(x) - u_2(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x), \end{aligned}$$

and moreover,

$$\begin{aligned} \mathcal{E}_r^{KS}(u; E_1, E_2) &= \int_{E_1 \cap B(o,r)} \int_{E_2 \cap B(x,r)} \frac{|u_1(x) - u_2(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x) \\ &\geq \int_{E_1 \cap B(o,r/4)} \int_{E_2 \cap B(o,r/4)} \frac{|u_1(x) - u_2(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x). \end{aligned}$$

Similarly, we also see that

$$\begin{aligned} \mathcal{E}_r^{KS}(u; E_2, E_1) &\leq \int_{E_1 \cap B(o,r)} \int_{E_2 \cap B(o,r)} \frac{|u_1(x) - u_2(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x), \\ \mathcal{E}_r^{KS}(u; E_2, E_1) &\geq \int_{E_1 \cap B(o,r/4)} \int_{E_2 \cap B(o,r/4)} \frac{|u_1(x) - u_2(y)|^n}{r^n} d\mathcal{L}^n(y) d\mathcal{L}^n(x). \end{aligned}$$

It follows that (3.4) holds if and only if

$$I_{KS}(u_1, u_2) = \limsup_{r \rightarrow 0^+} \int_{E_1 \cap B(o,r)} \int_{E_2 \cap B(o,r)} \frac{|u_1(x) - u_2(y)|^n}{r^{2n}} d\mathcal{L}^n(y) d\mathcal{L}^n(x) < \infty.$$

These complete the proof of (1).

Proof of (2): It suffices to find $u \in N^{1,n}(X) \setminus KS_n^1(X)$; note that $u \in N^{1,n}(X)$ if and only if $u|_{E_i} \in N^{1,n}(E_i)$ for $i = 1, 2$. By direct computation or by [14], we know

that the function $v(x) := \log(-\log|x|)$ for $x \in E_1 \setminus \{o\}$ belongs to $N^{1,n}(E_1)$. Note that

$$\lim_{r \rightarrow 0^+} \operatorname{ess\,inf}_{E_1 \cap B(o,r)} |v| = \infty.$$

Now we define $u \in N^{1,n}(X)$ by $u(x) := v(x)$ for $x \in E_1$ and $u(x) := 0$ for $x \in E_2 \setminus \{o\}$. Then we easily see that

$$\int_{E_1 \cap B(o,r)} \int_{E_2 \cap B(o,r)} |u(x) - u(y)|^n d\mathcal{L}^n(y) d\mathcal{L}^n(x) \geq \left(\operatorname{ess\,inf}_{E_1 \cap B(o,r)} |v| \right)^n,$$

and so $u \notin KS_n^1(X)$ though $u \in N^{1,n}(X)$, since $\operatorname{ess\,inf}_{E_1 \cap B(o,r)} |v| \rightarrow \infty$ as $r \rightarrow 0^+$. $\square$

Note that the dimension of $B_{p,p}^1(X)$ is 2 when $1 < p < n$. Moreover, thanks to [6] applied to each of the two n -dimensional hypercubes of X and (3.2), we know that $\theta_p(X) = n/p$, in particular, $\theta_p(X) > 1$ when $1 < p < n$.

Proof of Theorem 1.9 for the glued hypercubes. Note that $d_f = n$ and $d_{w,p} = p$ in this case. As already mentioned, $\theta_p(X) = n/p = d_f/p$ when $p < n$. The estimate (3.2), along with the fact that $\theta_p([0,1]^n) = 1$, shows $\theta_p(X) = 1 = d_{w,p}/p$ when $p \geq n$. Moreover, for $B_{p,p}^\theta(X)$ to be dense in $L^p(X)$ it is necessary to have that $B_{p,p}^\theta([0,1]^n)$ be dense in $L^p([0,1]^n)$, and this requires $\theta < 1$. It follows that $\theta_p^*(X) \leq 1$. On the other hand, when $\theta < 1$, $B_{p,p}^\theta(X)$ is dense in $L^p(X)$ due to the results of [4] because the class of Lipschitz continuous functions forms a dense subclass of both spaces. Hence we have $\theta_p^*(X) = 1 = d_{w,p}/p$. $\square$

A similar example can be considered by gluing two copies of the Sierpiński gasket, but the resultant example has dramatically different phenomena in comparison to Example 3.1 above. Precisely, for any $p \in (1, \infty)$, $\theta_p(X) = \theta_p^*(X)$ for this example as shown in Theorem 3.9 below. In comparison, in Example 3.1 we have that $\theta_p(X) = n/p$. However, when $\theta \geq 1$ we necessarily have that any function $u \in B_{p,p}^\theta(X)$ must be constant on each of the two cubes $[0,1]^n$ and $[-1,0]^n$, thanks to the results in [6]. Therefore $\theta_p^*(X) = 1$ in Example 3.1.

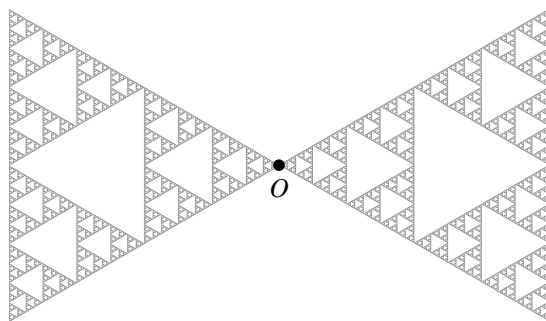


Figure 2. Gluing of two copies of the Sierpiński gasket.

Example 3.5. (Gluing copies of the Sierpiński gasket) In this example, we consider X to be the union of two copies of the n -dimensional standard Sierpiński gasket glued at a point. Let $n \in \mathbb{N}$ with $n \geq 2$, let K be the standard n -dimensional Sierpiński gasket, rotated so that it is symmetric about the x_n -axis in $\mathbb{R}^n$ and located in the half-space $\{x_n \geq 0\}$ and has a vertex at $o := (0, 0, \dots, 0)$, $K^+ := K$ and K^- the reflection of K in the hyperplane $\{x_n = 0\}$, and then set $X = K^+ \cup K^-$ (see

Figure 2 for the case $n = 2$). Let d be the Euclidean metric (restricted to X) and μ be the d_f -dimensional Hausdorff measure on X , where $d_f := \log(n+1)/\log 2$. Then μ is Ahlfors d_f -regular on X , i.e., there exists $c_1 \geq 1$ such that

$$(3.6) \quad c_1^{-1} r^{d_f} \leq \mu(B(x, r)) \leq c_1 r^{d_f} \quad \text{for any } x \in X, \quad 0 < r < \text{diam}(X).$$

Now let us focus on the following Besov-type energy functional of χ_{K^+} :

$$\int_X \int_{B(x, r)} \frac{|\chi_{K^+}(x) - \chi_{K^+}(y)|^p}{r^{p\theta}} d\mu(y) d\mu(x), \quad r > 0.$$

Note that if $x \in K^-$ and $B(x, r) \cap K^+ \neq \emptyset$, then $o \in B(x, r)$ and hence $B(x, r) \subset B(o, 2r)$. Therefore,

$$(3.7) \quad \begin{aligned} & \int_X \int_{B(x, r)} \frac{|\chi_{K^+}(x) - \chi_{K^+}(y)|^p}{r^{p\theta}} \mu(dy) \mu(dx) \\ & \leq c_1 r^{-d_f} \int_{B(o, 2r) \cap K^-} \int_{B(o, 2r) \cap K^+} \frac{|\chi_{K^+}(x) - \chi_{K^+}(y)|^p}{r^{p\theta}} \mu(dy) \mu(dx) \\ & \leq c_1 r^{-d_f - p\theta} \mu(B(o, 2r))^2 \leq c_1^3 r^{d_f - p\theta}. \end{aligned}$$

Since $\mu(B(o, r/4) \cap K^\pm) \geq c_2 r^{d_f}$, we also have

$$(3.8) \quad \begin{aligned} & \int_X \int_{B(x, r)} \frac{|\chi_{K^+}(x) - \chi_{K^+}(y)|^p}{r^{p\theta}} \mu(dy) \mu(dx) \\ & \geq c_1^{-1} r^{-d_f} \int_{B(o, r/4) \cap K^-} \int_{B(o, r/4) \cap K^+} \frac{|\chi_{K^+}(x) - \chi_{K^+}(y)|^p}{r^{p\theta}} \mu(dy) \mu(dx) \\ & \geq c_1 r^{-d_f - p\theta} \mu(B(o, r/4) \cap K^-) \mu(B(o, r/4) \cap K^+) \geq c_1^{-1} c_2^2 r^{d_f - p\theta}. \end{aligned}$$

Hence $\chi_{K^+} \in B_{p,p}^\theta(X)$ if and only if $0 < \theta < d_f/p$, and $\chi_{K^+} \in KS_p^\theta(X)$ if and only if $0 < \theta \leq d_f/p$. Moreover, $\|\chi_{K^+}\|_{KS_p^\theta(X)} = 0$ for $\theta \in (0, d_f/p)$, and $\|\chi_{K^+}\|_{KS_p^{d_f/p}(X)} > 0$.

In particular, the p -energy form $(\|\cdot\|_{KS_p^\theta(X)}^p, KS_p^\theta(X))$ is reducible when $\theta \in (0, d_f/p)$.

Let $d_{w,p}$ be the p -walk dimension of the n -dimensional standard Sierpiński gasket K^+ , i.e., $d_{w,p} = \log((n+1)\rho_p)/\log 2$ where ρ_p is the p -scaling factor of K^+ used in constructing the analog of the Sobolev space $\mathcal{F}_p$ on the gasket (see [17, Subsection 9.2] for further details on the p -walk dimension of Sierpiński gaskets). From [18, Theorems 5.16, 5.26, Corollary 5.27, Proposition 5.28] and Lemma 2.6(2) above, we know that $\theta_p(K^\pm) = \theta_p^*(K^\pm) = d_{w,p}/p$. It is known that $d_{w,p} > p$ and $d_{w,p} > d_f$ for any $p \in (1, \infty)$; see [17, Theorems 9.13, B.8 and (8.39)] and [19, Proposition 3.3]. In the next theorem we determine $\theta_p(X)$ and $\theta_p^*(X)$ (note that the Ahlfors regular conformal dimension of the n -dimensional standard Sierpiński gasket is 1; see, e.g., [17, Theorem B.8]).

Theorem 3.9. *In the above setting of $X = K^+ \cup K^-$, where each $K^\pm$ is the n -dimensional Sierpiński gasket, we have $\theta_p(X) = \theta_p^*(X) = \frac{d_{w,p}}{p}$ for $1 < p < \infty$.*

Proof. We first show that $\theta_p(X) = d_{w,p}/p$. Since $B_{p,\infty}^{d_{w,p}/p}(K^\pm) \subset C(K^\pm)$ and $B_{p,\infty}^{d_{w,p}/p}(K^\pm)$ is dense in $C(K^\pm)$ by [17, Corollary 9.11] and [18, Theorem 5.26], we have $\theta_p(X) \geq d_{w,p}/p$. Indeed, by this density we can find a non-constant function $u \in B_{p,\infty}^{d_{w,p}/p}(K^+)$, and then its reflection v given by

$$v(x) = \begin{cases} u(x) & \text{if } x \in K^+, \\ u(-x) & \text{if } x \in K^-, \end{cases}$$

belongs to $B_{p,\infty}^{d_{w,p}/p}(X)$, and so we have a non-constant function in $B_{p,\infty}^{d_{w,p}/p}(X)$.

For any $\theta > d_{w,p}/p$ and $u \in B_{p,p}^\theta(X)$, we have from Lemma 2.6 (2) that $u|_{K^\pm} \in B_{p,\infty}^\theta(K^\pm)$. Then $u|_{K^+}$ and $u|_{K^-}$ must be constant functions since $\theta_p(K^\pm) = d_{w,p}/p$. Since $\chi_{K^+} \notin B_{p,p}^\theta(X)$ by the discussion preceding the statement of the theorem being proved here, and since $\theta > d_{w,p}/p > d_f/p$, the function u has to be constant on X . Hence, $\theta_p(X) \leq d_{w,p}/p$. The proof of $\theta_p(X) = d_{w,p}/p$ is completed.

Next we prove that $\theta_p^*(X) = d_{w,p}/p$. It suffices to show that $B_{p,\infty}^{d_{w,p}/p}(X)$ is dense in $C(X)$; indeed, if this is true, then we have from Lemma 2.6 (2) and the fact that $C(X)$ is dense in $L^p(X)$ that $B_{p,p}^\theta(X)$ is dense in $L^p(X)$ for any $\theta < d_{w,p}/p$ and hence $\theta_p^*(X) \geq d_{w,p}/p$. (Recall that $\theta_p^*(X) \leq \theta_p(X) = d_{w,p}/p$.)

To show that $B_{p,\infty}^{d_{w,p}/p}(X)$ is dense in $C(X)$, let $u \in C(X)$. We can assume that $u(o) = 0$ by adding a constant function. Recall that $u_+(x) := \max\{0, u(x)\}$ and set $u_- := u_+ - u$. Since $B_{p,\infty}^{d_{w,p}/p}(K^\pm)$ is dense in $C(K^\pm)$, for any $\varepsilon > 0$ there exist four continuous functions $u_{\pm,\varepsilon}^{K^+} \in B_{p,\infty}^{d_{w,p}/p}(K^+)$, $u_{\pm,\varepsilon}^{K^-} \in B_{p,\infty}^{d_{w,p}/p}(K^-)$ such that

$$\sup_{x \in K^+} |u_\pm(x) - u_{\pm,\varepsilon}^{K^+}(x)| \leq \varepsilon \quad \text{and} \quad \sup_{x \in K^-} |u_\pm(x) - u_{\pm,\varepsilon}^{K^-}(x)| \leq \varepsilon.$$

We can also assume that $u_{\pm,\varepsilon}^{K^+}$ and $u_{\pm,\varepsilon}^{K^-}$ are nonnegative. Since $u(o) = 0$ and $u_{\pm,\varepsilon}^{K^+}, u_{\pm,\varepsilon}^{K^-}$ are continuous, there exists $\delta > 0$ such that

$$\sup_{x \in B(o,\delta) \cap K^+} |u_{\pm,\varepsilon}^{K^+}(x)| \leq 2\varepsilon \quad \text{and} \quad \sup_{x \in B(o,\delta) \cap K^-} |u_{\pm,\varepsilon}^{K^-}(x)| \leq 2\varepsilon.$$

Now we set

$$u_\varepsilon := [(u_{+, \varepsilon}^{K^+} - 2\varepsilon)_+ - (u_{-, \varepsilon}^{K^+} - 2\varepsilon)_+] \chi_{K^+} + [(u_{+, \varepsilon}^{K^-} - 2\varepsilon)_+ - (u_{-, \varepsilon}^{K^-} - 2\varepsilon)_+] \chi_{K^-}.$$

Then $u_\varepsilon \in C(X)$. Note that $u_\varepsilon = 0$ on $B(o, \delta)$ and that $\|u - u_\varepsilon\|_{\sup} \leq 3\varepsilon$. We conclude that $u_\varepsilon \in B_{p,\infty}^{d_{w,p}/p}(X)$ by using the “locality” of $\|\cdot\|_{KS_p^{d_{w,p}/p}(X)}$; indeed,

$$\|u_\varepsilon\|_{KS_p^{d_{w,p}/p}(X)}^p \leq \|u_\varepsilon|_{K^+}\|_{KS_p^{d_{w,p}/p}(K^+)}^p + \|u_\varepsilon|_{K^-}\|_{KS_p^{d_{w,p}/p}(K^-)}^p.$$

Therefore, $B_{p,\infty}^{d_{w,p}/p}(X)$ is dense in $C(X)$. $\square$

Example 3.10. (Gluing copies of the Sierpiński carpet) In this example, we consider X to be the union of two isometric copies of the planar standard Sierpiński carpet glued at a point. We confine ourselves to the planar case unlike in Examples 3.1 and 3.5, because the construction of a self-similar p -energy form and its corresponding Sobolev analog $\mathcal{F}_p$ for all $1 < p < \infty$ including the case where p is less than or equal to the Ahlfors regular conformal dimension (denoted by d_{ARC} below) is currently known only for the planar carpet.

Let K be the standard Sierpiński carpet, rotated so that it is symmetric about the line $\{y = x\}$ in $\mathbb{R}^2$ and located in the quadrant $\{x \leq 0, y \leq 0\}$ and has a vertex at $o := (0, 0)$, $K^+ := K$ and K^- be the reflection of K in the line $\{y = -x\}$, and then set $X = K^+ \cup K^-$ (see Figure 3). Let d be the Euclidean metric (restricted on X) and μ be the d_f -dimensional Hausdorff measure on X , where $d_f := \log 8 / \log 3$. Then μ is Ahlfors d_f -regular on X , i.e., (3.6) holds. Similar to (3.7) and (3.8), we can estimate

$$(3.11) \quad \int_X \int_{B(x,r)} \frac{|\chi_{K^+}(x) - \chi_{K^+}(y)|^p}{r^{p\theta}} \mu(dy) \mu(dx) \approx r^{d_f - p\theta}.$$

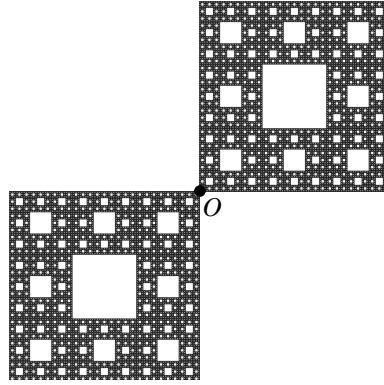


Figure 3. Gluing of two copies of the Sierpiński carpet.

Hence $\chi_{K^+} \in B_{p,p}^\theta(X)$ if and only if $\theta \in (0, d_f/p)$, and $\chi_{K^+} \in KS_p^\theta(X)$ if and only if $\theta \in (0, d_f/p]$. Also, we have $\|\chi_{K^+}\|_{KS_p^\theta(X)} = 0$ for $\theta \in (0, d_f/p)$ and $\|\chi_{K^+}\|_{KS_p^{d_f/p}(X)} > 0$. In particular, $(\|\cdot\|_{KS_p^\theta(X)}, KS_p^\theta(X))$ is reducible when $\theta \in (0, d_f/p)$.

Similar to Example 3.5, from [22, Theorems 1.1 (ii), 1.4 and proof of Theorem 7.1] and Lemma 2.6 (2), we know that $\theta_p(K^\pm) = \theta_p^*(K^\pm) = d_{w,p}/p$ where $d_{w,p}$ is the p -walk dimension of the Sierpiński carpet. By [24, Theorem 2.24] or [17, Theorem 9.8], we have $d_{w,p} > p$ for any $p \in (1, \infty)$. Next let us recall a relation with the Ahlfors regular conformal dimension d_{ARC} of the Sierpiński carpet that is discussed in the end of introduction. From [5, Corollary 3.7] and [10, Corollary 1.4] (see also [8, Proof of Proposition 1.7]), we know that $d_{w,p} > d_f$ if and only if $p > d_{\text{ARC}}$, that $d_{w,p} < d_f$ if and only if $p < d_{\text{ARC}}$, and that $d_{w,p} = d_f$ for $p = d_{\text{ARC}}$. Also, $d_{\text{ARC}} \geq 1 + \frac{\log 2}{\log 3}$ by [2, Remark 1]. We can determine $\theta_p(X)$ and $\theta_p^*(X)$ as in Theorem 1.9, in particular, there is a gap between $\theta_p(X)$ and $\theta_p^*(X)$ when $1 < p < d_{\text{ARC}}$.

Proof of Theorem 1.9 for the glued Sierpiński carpets. By [22, Theorems 1.1 and 1.4], $B_{p,\infty}^{d_{w,p}/p}(K^\pm) \cap C(K^\pm)$ is dense in $C(K^\pm)$ for any $p \in (1, \infty)$. Hence we can show $\theta_p(X) = d_{w,p}/p$ when $d_{w,p} > d_f$ in the same way as Theorem 3.9. Assume that $d_{w,p} \leq d_f$. Since $\chi_{K^+} \in B_{p,p}^\theta(X)$ if and only if $\theta < d_f/p$, we have $\theta_p(X) \geq d_f/p$. To see that $\theta_p(X) \leq d_f/p$, let $\theta > d_f/p \geq d_{w,p}/p$ and let $u \in B_{p,p}^\theta(X)$. Then by Lemma 2.9 we know that $u \in KS_p^\theta(X)$ and so by Lemma 2.6(2) we also have that $u \in B_{p,p}^{d_{w,p}/p}(X)$. Note that then $u|_{K^\pm} \in B_{p,p}^{d_{w,p}/p}(K^\pm)$. Now by Lemma 2.9 again, we know that $\|u|_{K^+}\|_{KS_p^{d_{w,p}/p}(K^+)} = \|u|_{K^-}\|_{KS_p^{d_{w,p}/p}(K^-)} = 0$. Hence we have from [22, Theorems 1.1 and 1.4] that $u|_{K^+}$ and $u|_{K^-}$ are constant. Since $\chi_{K^+} \notin B_{p,p}^\theta(X)$, u has to be a constant function, whence it follows that $\theta_p(X) \leq d_f/p$.

Next we prove that $\theta_p^*(X) = d_{w,p}/p$. Since $B_{p,\infty}^{d_{w,p}/p}(K^\pm) \cap C(K^\pm)$ is dense in $C(K^\pm)$, we can show that $\theta_p^*(X) \geq d_{w,p}/p$ in the same manner as in the proof of Theorem 3.9. Since $B_{p,\infty}^\theta(K^+)$ and $B_{p,\infty}^\theta(K^-)$ have only constant functions when $\theta > d_{w,p}/p$, $B_{p,p}^\theta(X)$ can not be dense in $L^p(X, \mu)$ for such θ . Hence, by Lemma 2.6 (2), $B_{p,p}^\theta(X)$ is not dense in $L^p(X, \mu)$ for any $\theta > d_{w,p}/p$, from which it follows that $\theta_p^*(X) \leq d_{w,p}/p$. $\square$

The following proposition is an analog of Proposition 3.3 where now X is the glued Sierpiński carpet. In this case, when p is the Ahlfors regular conformal dimension d_{ARC} of the carpet, we must have $\theta_p(X) = \theta_p^*(X)$.

Proposition 3.12. *Let X be the glued Sierpiński carpet and let $p = d_{\text{ARC}}$. Set $E_1 := K^+$ and $E_2 := K^-$ for ease of notation.*

(1) *It follows that*

$$KS_p^{\theta_p}(X) = \left\{ u_1 \chi_{E_1} + u_2 \chi_{E_2} \mid \begin{array}{l} u_i \in L^p(X, \mu), \ u_i|_{E_i} \in KS_p^{\theta_p}(E_i), \\ i \in \{1, 2\}, \ I_{KS}(u_1, u_2) < \infty \end{array} \right\},$$

where

$$I_{KS}(u_1, u_2) := \limsup_{r \rightarrow 0^+} \int_{E_1 \cap B(o, r)} \int_{E_2 \cap B(o, r)} \frac{|u_1(x) - u_2(y)|^p}{r^{d_f + p\theta_p}} dy dx.$$

(2) $KS_p^{\theta_p}(X) \subsetneq \{u_1 \chi_{E_1} + u_2 \chi_{E_2} \mid u_i \in L^p(X, \mu), \ u_i|_{E_i} \in KS_p^{\theta_p}(E_i), \ i \in \{1, 2\}\}$.

Proof. The proof of (1) can be obtained via minor modifications of the proof of Proposition 3.3 (1), and we leave it to the interested reader to verify. By [9, Proof of Theorem 2.7], [22, Theorem 1.4] and the fact that $d_{w,p} = d_f$ when $p = d_{\text{ARC}}$ (see [22, Remark 9.17]), there exists $v \in KS_p^{\theta_p}(K^+)$ such that $\lim_{r \rightarrow 0^+} \text{ess inf}_{K^+ \cap B(o, r)} |v| = \infty$. Once we obtain such a discontinuous function, then using the zero-extension u of such a function v to K^- , the proof of Proposition 3.3 verbatim tells us that $u \notin KS_p^{d_{w,p}/p}(X)$. The proof of (2) is now completed. $\square$

4. Proof of Theorem 1.1

We now prove Theorem 1.1; the proof is broken down step by step by the following lemmata.

Lemma 4.1. *Let μ be a doubling measure on X . Suppose that $B_{p,p}^\theta(X)$ is k -dimensional for some $k \in \mathbb{N}$ as a vector space (hence $B_{p,p}^\theta(X) \neq \{0\}$). Then the following hold.*

- (i) *Every function in $B_{p,p}^\theta(X)$ is bounded.*
- (ii) *Every function $f \in B_{p,p}^\theta(X)$ is a simple function. Moreover, if $\mu(X) < \infty$ and $k = 1$, then f is necessarily constant, and if $\mu(X) < \infty$ and $k > 1$ or $\mu(X) = \infty$ and $k \geq 1$, then outside of a set of measure zero, f takes on at most $k + 1$ values.*
- (iii) *Suppose $k > 1$. Then there is a collection of measurable subsets E_i , $i = 1, \dots, k$, of X such that the collection $\{\chi_{E_i} : 1 \leq i \leq k\}$ forms a basis for $B_{p,p}^\theta(X)$ and in addition, $0 < \mu(E_i) < \infty$ for each $i = 1, \dots, k$, $\mu(E_i \cap E_j) = 0$ whenever $i \neq j$, and if in addition we have that $\mu(X) < \infty$, then $\mu(X \setminus \bigcup_{j=1}^k E_j) = 0$.*
- (iv) *$B_{p,p}^\theta(X) = \bigoplus_{i=1}^k B_{p,p}^\theta(E_i)$ as sets. Moreover, the dimension of $B_{p,p}^\theta(E_i)$ is 1 for all $i = 1, \dots, k$.*

Proof. *Proof of (i):* Suppose that the dimension of $B_{p,p}^\theta(X)$ is finite and that there is an unbounded function $f \in B_{p,p}^\theta(X)$. By considering f_+, f_- separately, we may consider without loss of generality that $f \geq 0$ (note that if $f \in B_{p,p}^\theta(X)$, then $f_+, f_- \in B_{p,p}^\theta(X)$ by Lemma 2.4). Then we can find a strictly increasing sequence of positive integers $(n_i)_{i \in \mathbb{N}}$ such that $\mu(f^{-1}((n_i, n_{i+1}])) > 0$ for each $i \in \mathbb{N}$. Set

$$f_i(x) := \max\{f(x) - n_i, 0\},$$

then $f_i \in B_{p,p}^\theta(X)$ by Lemma 2.4.

Note that f_1 is not a linear combination of any of up to ℓ many choices of functions $f_{i_1}, \dots, f_{i_\ell}$ with $i_1, \dots, i_\ell$ distinct from 1, for all such linear combinations will vanish on the set $f^{-1}((n_1, n_2])$ where f_1 is nonzero. Note also that f_2 cannot be a linear combination of f_1 and other f_j , $j \neq 2$, either, as on the set $f^{-1}((n_2, n_3])$ the functions f_j , $j \geq 3$, vanish and so if f_2 were to be such a linear combination, on that set we must have $f_2 = af_1$ for some $a \neq 0$. This also is not possible as f_1 is nonzero on the set $f^{-1}((n_1, n_2])$ and f_2 and all f_j , $j > 2$, vanish there. Hence f_1 and f_2 are linearly independent of each other and of all the other f_j , $j \geq 3$. We have also proved that $\sum_{j=1}^2 a_j f_j = 0$ on $f^{-1}((n_1, n_3])$ implies that $a_1 = a_2 = 0$.

Now we proceed by induction. Suppose we have shown that $f_1, \dots, f_i$ are linearly independent of each other and of all the other f_j , $j \geq i+1$ and that $\sum_{j=1}^i a_j f_j = 0$ on $f^{-1}((n_1, n_{i+1}])$ implies that $a_j = 0$ for $j = 1, \dots, i$. We wish to show that f_{i+1} is also independent of the other functions f_j , $j \neq i+1$. Indeed, if it is not, then by considering the set $f^{-1}((n_1, n_{i+2}])$, we see that on this set we must have $f_{i+1} = \sum_{j=1}^i a_i f_i$ with at least one of a_i nonzero. But then, on the set $f^{-1}((n_1, n_{i+1}])$ we have that $\sum_{j=1}^i a_j f_j = 0$, which then indicates that each $a_j = 0$ for $j = 1, \dots, i$. That is, f_{i+1} cannot be a linear combination of the other functions f_j , $j \neq i$. It follows that the collection $\{f_i : i \in \mathbb{N}\}$ is a linearly independent subcollection of $B_{p,p}^\theta(X)$, violating the finite dimensionality of $B_{p,p}^\theta(X)$. Thus f must be bounded.

Proof of (ii): Let $f \in B_{p,p}^\theta(X)$ such that f is not the zero function. Then both f_+ and f_- are in $B_{p,p}^\theta(X)$, and so we first focus on the possibility that $f \geq 0$ with $f \not\equiv 0$. We want to prove that there are positive real numbers $b_1, b_2, \dots, b_l$ with $l \leq k$ and $b_i < b_{i+1}$ for $i = 1, \dots, l-1$ such that

$$\mu(X \setminus f^{-1}(\{b_1, \dots, b_l, 0\})) = 0,$$

or equivalently the support of the push-forward measure $f_*\mu$, $\text{supp}(f_*\mu)$, is contained in $\{b_1, \dots, b_l, 0\}$. (Note that $\text{supp}(f_*\mu) \subseteq [0, \|f\|_{L^\infty}]$ since f is bounded due to (i).) We prove this by contradiction. Suppose the above claim fails. Then recalling that $f \geq 0$,

$$\#\{\alpha \in \mathbb{R} \mid \alpha \geq 0 \text{ and } \mu(\{|f - \alpha| < \varepsilon\}) > 0 \text{ for all } \varepsilon > 0\} \geq k+2$$

and we can find positive numbers $a_2, \dots, a_{k+2}$ with $a_1 = 0$, $a_{k+2} = \|f\|_{L^\infty}$ and $a_i < a_{i+1}$ for $i = 1, \dots, k+1$, such that $\mu(f^{-1}((a_i, a_{i+1}])) > 0$ for $i = 1, \dots, k+1$.

As in the proof of (i), we consider the functions f_i , $i = 1, \dots, k+1$, given by

$$f_i(x) = \max\{f(x) - a_i, 0\}.$$

Since $a_i \geq 0$, it follows from Lemma 2.4 that $f_i \in B_{p,p}^\theta(X)$. Now a repeat of the proof of (i) tells us that the collection $\{f_1, \dots, f_{k+1}\} \subset B_{p,p}^\theta(X)$ is linearly independent, violating the hypothesis that the dimension of $B_{p,p}^\theta(X)$ is k . The claim now follows for non-negative functions that are not identically zero. In particular, for such functions, we can set $E_i := f^{-1}(\{b_i\})$ for $i = 1, \dots, l$, and see that

$$f = \sum_{i=1}^l b_i \chi_{E_i}.$$

We now set $b_0 := 0$, and by Lemma 2.4, note that for $i = 1, \dots, l$, the function h_i given by $h_i(x) = \max\{0, \min\{f(x) - b_{i-1}, b_i - b_{i-1}\}\}$ belongs to $B_{p,p}^\theta(X)$ with $h_i = (b_i - b_{i-1})\chi_{F_i}$, where $F_i := \bigcup_{j=i}^l E_j$. It follows that $\chi_{F_i} = (b_i - b_{i-1})^{-1} h_i \in B_{p,p}^\theta(X)$

and hence $\chi_{F_i} \in B_{p,p}^\theta(X)$. It follows that $\chi_{E_i} \in B_{p,p}^\theta(X)$ as well for $i = 1, \dots, l$. Note that $\mu(E_i \cap E_j) = 0$ when $i \neq j$.

If f is not non-negative and not identically zero, then we apply the above conclusion to f_+ and f_- separately, and so we have distinct positive numbers $a_1, \dots, a_j$ and distinct positive numbers $b_1, \dots, b_l$ with $j, l \leq k$, and measurable sets $E_1, \dots, E_j$ and $F_1, \dots, F_l$ such that

$$f = f_+ - f_- = \sum_{i=1}^j a_i \chi_{E_i} - \sum_{m=1}^l b_m \chi_{F_m}.$$

We can also ensure that $\mu(E_i \cap F_m) = 0$ for all (i, m) . Moreover, as $f \in L^p(X)$, we must have $\mu(E_i)$ and $\mu(F_m)$ are finite whenever $1 \leq i \leq j$ and $1 \leq m \leq l$. Thus the collection $\{\chi_{E_i}, \chi_{F_m} : i \in \{1, \dots, j\}, m \in \{1, \dots, l\}\}$ is a linearly independent collection of functions in $B_{p,p}^\theta(X)$, and hence we must have that $m + l \leq k$, that is, there are at most k non-zero real numbers $c_1, \dots, c_n$ such that

$$\mu(X \setminus f^{-1}(\{c_1, \dots, c_n, 0\})) = 0.$$

Proof of (iii): Let $\{f_1, \dots, f_k\}$ be a basis for $B_{p,p}^\theta(X)$. By (ii), we know that for each $j = 1, \dots, k$ there are measurable subsets $E_{j,1}, \dots, E_{j,N_j}$ of X with $\chi_{E_{j,i}} \in B_{p,p}^\theta(X)$ and *distinct* non-zero real numbers $a_{j,1}, \dots, a_{j,N_j}$ such that

$$f_j = \sum_{i=1}^{N_j} a_{j,i} \chi_{E_{j,i}}.$$

We can make this simple-function decomposition of f_j so that $\mu(E_{j,i} \cap E_{j,k}) = 0$ for $i, k \in \{1, \dots, N_j\}$ with $i \neq k$ and in addition we require that $\mu(E_{j,i}) > 0$ for each $i = 1, \dots, N_j$.

Next, we break the sets $E_{j,i}$, $j = 1, \dots, k$ and $i = 1, \dots, N_j$ into pairwise disjoint subsets as follows. Observing that $\mu(E_{j,i} \cap E_{j,n}) = 0$ if $i \neq n$, it suffices to consider pairs of sets $E_{j,i}$ and $E_{m,n}$ with $j \neq m$. Since $\chi_{E_{j,i}}$ and $\chi_{E_{m,n}}$ are in $B_{p,p}^\theta(X)$, it follows from Lemma 2.5 that the function $\chi_{E_{j,i} \cap E_{m,n}} = \chi_{E_{j,i}} \chi_{E_{m,n}}$ is also in $B_{p,p}^\theta(X)$. If $\mu(E_{j,i} \cap E_{m,n}) > 0$ and $\mu(E_{j,i} \Delta E_{m,n}) > 0$, then we can replace $E_{j,i}$ and $E_{m,n}$ with $E_{j,i} \cap E_{m,n}$, and $E_{j,i} \setminus E_{m,n}$ if $\mu(E_{j,i} \setminus E_{m,n}) > 0$ and $E_{m,n} \setminus E_{j,i}$ if $\mu(E_{m,n} \setminus E_{j,i}) > 0$ (note that in the case considered here, we must have at least one of $\mu(E_{m,n} \setminus E_{j,i})$ and $\mu(E_{j,i} \setminus E_{m,n})$ is positive).

Since the collection $\{E_{j,i} : j = 1, \dots, k, i = 1, \dots, N_j\}$ is a finite collection of sets, the above procedure involving each pair of sets from this collection needs to be done only finitely many times; thus we obtain the collection of sets E_i , $i = 1, \dots, N$ such that

$$(4.2) \quad \mu(E_i \cap E_j) = 0 \quad \text{whenever } i \neq j.$$

As each f_j is a linear combination of the characteristic functions of $E_{j,i}$, $i = 1, \dots, N_j$, it follows that f_j is a linear combination of the characteristic functions χ_{E_i} , $i = 1, \dots, N$. Because the collection $\{f_1, \dots, f_k\}$ spans $B_{p,p}^\theta(X)$, the collection $\{\chi_{E_i} : i = 1, \dots, N\}$ spans $B_{p,p}^\theta(X)$ as well. Moreover, by (4.2) this collection of functions is also linearly independent; hence $N = k$, and this collection forms a basis for $B_{p,p}^\theta(X)$.

Finally, note that when $\mu(X) < \infty$, the constant function $u \equiv 1$ is in $B_{p,p}^\theta(X)$, and so necessarily $u = \sum_{j=1}^k \chi_{E_j}$, that is, $\mu(X \setminus \bigcup_{j=1}^k E_j) = 0$.

Proof of (iv): By (iii), it is enough to show that $B_{p,p}^\theta(E_i)$ consists only of constant functions (i.e. the dimension of $B_{p,p}^\theta(E_i)$ is 1) for all $i = 1, \dots, k$. Now suppose there is $i \in \{1, \dots, k\}$ and a non-constant $g \in B_{p,p}^\theta(E_i)$. By Lemma 2.4, we may assume that g is bounded. Since $\chi_{E_i} \in B_{p,p}^\theta(X)$, we have

$$(4.3) \quad \begin{aligned} \|\chi_{E_i}\|_{B_{p,p}^\theta(X)}^p &= \int_{E_i^c} \int_{E_i} \frac{1}{d(x,y)^{\theta p} \mu(B(x,d(x,y)))} d\mu(y) d\mu(x) \\ &\quad + \int_{E_i} \int_{E_i^c} \frac{1}{d(x,y)^{\theta p} \mu(B(x,d(x,y)))} d\mu(y) d\mu(x) < \infty. \end{aligned}$$

Now define $\tilde{g}: X \rightarrow \mathbb{R}$ by $\tilde{g} = g_i \chi_{E_i}$, that is, $\tilde{g}|_{E_i} = g$ and $\tilde{g}|_{E_i^c} = 0$. Then $\|\tilde{g}\|_{L^p(X)}^p = \|g\|_{L^p(E_i)}^p < \infty$ and

$$\begin{aligned} \|\tilde{g}\|_{B_{p,p}^\theta(X)}^p &\leq \|g\|_{B_{p,p}^\theta(E_i)}^p + \int_{E_i^c} \int_{E_i} \frac{|g(y)|^p}{d(x,y)^{\theta p} \mu(B(x,d(x,y)))} d\mu(y) d\mu(x) \\ &\quad + \int_{E_i} \int_{E_i^c} \frac{|g(x)|^p}{d(x,y)^{\theta p} \mu(B(x,d(x,y)))} d\mu(y) d\mu(x) \\ &\leq \|g\|_{B_{p,p}^\theta(E_i)}^p + \|g\|_{L^\infty(X)}^p \|\chi_{E_i}\|_{B_{p,p}^\theta(X)}^p < \infty, \end{aligned}$$

where the last inequality is due to (4.3). It follows that $\tilde{g} \in B_{p,p}^\theta(X)$, and so by (iii) there are real numbers $a_1, \dots, a_k$ such that $\tilde{g} = \sum_{j=1}^k a_j \chi_{E_j}$, which in turn means that $\tilde{g}$ (and hence g) is constant μ -a.e. in E_i , contradicting the non-constant nature of g . It follows that every function in $B_{p,p}^\theta(E_i)$ must be constant. $\square$

Remark 4.4. Lemma 4.1 proves claims (1), (2), (3) and (4) of Theorem 1.1. Lemma 2.9 verifies claim (5) of Theorem 1.1. Claim (7) of Theorem 1.1 follows consequently from the definition of $\theta_p(X)$.

Lemma 4.5. *Under the hypotheses of Lemma 4.1 above, and with the sets E_i , $i = 1, \dots, k$, as constructed in that lemma, we have that $u\chi_{E_i} \in KS_p^\theta(X)$ whenever $u \in KS_p^\theta(X)$ is bounded.*

Proof. The claim follows immediately from combining Lemma 2.5 and the fact that $\chi_{E_i} \in B_{p,p}^\theta(X)$. $\square$

Finally, the next lemma verifies (6) of Theorem 1.1 and completes the proof of Theorem 1.1.

Lemma 4.6. *Under the setting of Theorem 1.1, claim (6) holds true.*

Proof. Let $u \in KS_p^\theta(X)$ such that $\|u\|_{L^\infty(X)} =: M$ is bounded. Then

$$\begin{aligned} &\int_X \int_{B(x,r)} \frac{|u(x)\chi_{E_j}(x) - u(y)\chi_{E_j}(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x) \\ &= \int_{E_j} \int_{B(x,r) \cap E_j} \frac{|u(y) - u(x)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\ &\quad + \int_{E_j} \int_{B(x,r) \setminus E_j} \frac{|u(x)\chi_{E_j}(x)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\ &\quad + \int_{X \setminus E_j} \int_{B(x,r) \cap E_j} \frac{|u(y)\chi_{E_j}(y)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x). \end{aligned}$$

Note that

$$\begin{aligned}
& \int_{E_j} \int_{B(x,r) \setminus E_j} \frac{|u(x)\chi_{E_j}(x)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\
& + \int_{X \setminus E_j} \int_{B(x,r) \cap E_j} \frac{|u(y)\chi_{E_j}(y)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\
& \leq M^p \int_{E_j} \int_{B(x,r) \setminus E_j} \frac{|\chi_{E_j}(x)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\
& + M^p \int_{X \setminus E_j} \int_{B(x,r) \cap E_j} \frac{|\chi_{E_j}(y)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\
& = M^p \int_{E_j} \int_{B(x,r) \setminus E_j} \frac{|\chi_{E_j}(x) - \chi_{E_j}(y)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\
& + M^p \int_{X \setminus E_j} \int_{B(x,r) \cap E_j} \frac{|\chi_{E_j}(x) - \chi_{E_j}(y)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x) \\
& \leq M^p \int_X \int_{B(x,r)} \frac{|\chi_{E_j}(x) - \chi_{E_j}(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x),
\end{aligned}$$

and thanks to (5) of Theorem 1.1 (verified above), the last expression above tends to 0 as $r \rightarrow 0^+$. It follows that

$$\begin{aligned}
\|u\chi_{E_j}\|_{KS_p^\theta(X)}^p &= \limsup_{r \rightarrow 0^+} \int_X \int_{B(x,r)} \frac{|u(x)\chi_{E_j}(x) - u(y)\chi_{E_j}(y)|^p}{r^{\theta p}} d\mu(y) d\mu(x) \\
&= \limsup_{r \rightarrow 0^+} \int_{E_j} \int_{B(x,r) \cap E_j} \frac{|u(y) - u(x)|^p}{r^{\theta p} \mu(B(x,r))} d\mu(y) d\mu(x),
\end{aligned}$$

completing the proof. $\square$

5. Proof of Theorem 1.5 and Theorem 1.6

In this section we provide a proof of the remaining two main results of this paper.

Proof of Theorem 1.5. It suffices to show that any function in $B_{p,p}^\theta(X)$ is a constant function, in particular, the dimension of $B_{p,p}^\theta(X)$ is 1 if $\mu(X) < \infty$, and $B_{p,p}^\theta(X) = \{0\}$ if $\mu(X) = \infty$. Suppose there is a non-constant function $g \in B_{p,p}^\theta(X)$. Since g is non-constant, at least one of g_+ and g_- is non-constant; hence, without loss of generality, we may assume that $g \geq 0$ on X . Then there is a positive real number a such that $\mu(g^{-1}([a, \infty))) > 0$ and $\mu(g^{-1}([0, a))) > 0$. We can then find a positive real number $\delta < a$ such that $\mu(g^{-1}([0, a - \delta])) > 0$ as well. Now by Lemma 2.4 and Lemma 2.9, we know that $g_{a,\delta} := \max\{0, \min\{g - (a - \delta), \delta\}\} \in B_{p,p}^\theta(X) \subset KS_p^\theta(X)$ with $\|g_{a,\delta}\|_{KS_p^\theta(X)} = 0$. On the other hand, the choices of a and δ mean that $\|g_{a,\delta}\|_{B_{p,\infty}^\theta(X)} > 0$, violating condition (w-max) $_{p,\theta}$. Thus no such g exists. $\square$

Proof of Theorem 1.6. In [12, Theorem 1.5], a condition called property (NE) is assumed in addition; however, the proof of inequality (2.8) in the proof of that theorem in [12] does not need this property, and so we can use [12, (2.8)] verbatim in our setting. Now, by [12, (2.8)] and by [13, Theorem 5.2], there exists $C \geq 1$ such that for any $u \in B_{p,\infty}^\theta(X)$,

$$\liminf_{t \rightarrow 0^+} \int_X \int_{B(x,t)} \frac{|u(x) - u(y)|^p}{t^{p\theta}} d\mu(y) d\mu(x) \leq C \liminf_{\theta' \rightarrow \theta^-} (\theta - \theta') \|u\|_{B_{p,p}^{\theta'}(X)}^p.$$

Now suppose that there is a non-constant function $u \in B_{p,p}^\theta(X)$. Then we have by the Lebesgue dominated convergence theorem that

$$\lim_{\theta' \rightarrow \theta^-} \|u\|_{B_{p,p}^{\theta'}(X)}^p = \|u\|_{B_{p,p}^\theta(X)}^p > 0,$$

but then

$$\liminf_{\theta' \rightarrow \theta^-} (\theta - \theta') \|u\|_{B_{p,p}^{\theta'}(X)}^p = 0,$$

whence it follows from (1.7) that $\int_X |u - u_X|^p d\mu = 0$. Hence u must be constant on X , which is a contradiction of the supposition that u is non-constant on X . Therefore $B_{p,p}^\theta(X)$ consists only of constant functions. $\square$

Proof of Corollary 1.8. Under the hypotheses of Corollary 1.8, we obtain $\theta_p(X) = 1$ and (1.7) by [1, Theorem 5.1] and [15, Theorem 10.5.2], so we can apply Theorem 1.6. $\square$

Acknowledgements. The authors are grateful to the anonymous referee for helpful comments that improved the exposition of this paper.

References

- [1] BAUDOUIN, F.: Korevaar–Schoen-Sobolev spaces and critical exponents in metric measure spaces. - Ann. Fenn. Math. 49:2, 2024, 487–527.
- [2] BISHOP, C. J., and J. TYSON: Locally minimal sets for conformal dimension. - Ann. Acad. Sci. Fenn. Math. 26:2, 2001, 361–373.
- [3] BJÖRN, A., J. BJÖRN, and A. CHRISTENSEN: Poincaré inequalities and A_p weights on bow-ties. - J. Math. Anal. Appl. 539:1, 2024, Paper No. 128483, 28 pp.
- [4] BJÖRN, A., J. BJÖRN, and N. SHANMUGALINGAM: Extension and trace results for doubling metric measure spaces and their hyperbolic fillings. - J. Math. Pures Appl. (9) 159, 2022, 196–249.
- [5] BOURDON, M., and B. KLEINER: Combinatorial modulus, the combinatorial Loewner property, and Coxeter groups. - Groups Geom. Dyn. 7:1, 2013, 39–107.
- [6] BOURGAIN, J., H. BREZIS, and P. MIRONESCU: Another look at Sobolev spaces. - In: Optimal control and partial differential equations (eds. J.L. Menaldi et al.), IOS Press, Amsterdam, 2001, 439–455.
- [7] BREZIS, H.: How to recognize constant functions. A connection with Sobolev spaces. - Uspekhi Mat. Nauk 57:4, 2002, 59–74; Engl. transl. in: Russian Math. Surveys 57:4, 2002, 693–708.
- [8] CAO, S., and Z.-Q. CHEN: Whether p -conductive homogeneity holds depends on p . - J. Fractal Geom. 12:1-2, 2025, 93–104.
- [9] CAO, S., Z.-Q. CHEN, and T. KUMAGAI: On Kigami’s conjecture of the embedding $\mathcal{W}^p \subset C(K)$. - Proc. Amer. Math. Soc. 152:8, 2024, 3393–3402.
- [10] CARRASCO PIAGGIO, M.: On the conformal gauge of a compact metric space. - Ann. Sci. Éc. Norm. Supér. (4) 46:3, 2013, 495–548.
- [11] CHEN, Z.-Q., and M. FUKUSHIMA: Symmetric Markov processes, time change, and boundary theory. - London Math. Soc. Monogr. Ser. 35, Princeton Univ. Press, Princeton, NJ, 2012.
- [12] GAO, J., Z. YU, and J. ZHANG: Heat kernel-based p -energy norms on metric measure spaces. - arXiv:2303.10414 [math.FA], 2023.
- [13] GOGATISHVILI, A., P. KOSKELA, and N. SHANMUGALINGAM: Interpolation properties of Besov spaces defined on metric spaces. - Math. Nachr. 283:2, 2010, 215–231.
- [14] GÓRKA, P., and A. ŚLABUSZEWSKI: A discontinuous Sobolev function exists. - Proc. Amer. Math. Soc. 147:2, 2019, 637–639.

- [15] HEINONEN, J., P. KOSKELA, N. SHANMUGALINGAM, and J. TYSON: Sobolev spaces on metric measure spaces. An approach based on upper gradients. - New Math. Monogr. 27, Cambridge Univ. Press, Cambridge, 2015.
- [16] HINO, M., and T. KUMAGAI: A trace theorem for Dirichlet forms on fractals. - J. Funct. Anal. 238:2, 2006, 578–611.
- [17] KAJINO, N., and R. SHIMIZU: Contraction properties and differentiability of p -energy forms with applications to nonlinear potential theory on self-similar sets. - arXiv:2404.13668v2 [math.FA], 2024.
- [18] KAJINO, N., and R. SHIMIZU: Korevaar–Schoen p -energy forms and associated energy measures on fractals. - Tohoku Series in Mathematical Sciences, Springer (to appear).
- [19] KIGAMI, J.: Conductive homogeneity of compact metric spaces and construction of p -energy. - Mem. Eur. Math. Soc. 5, 2023.
- [20] KOSKELA, P., and P. MACMANUS: Quasiconformal mappings and Sobolev spaces. - Studia Math. 131:1, 1998, 1–17.
- [21] LAHTI, P., A. PINAMONTI, and X. ZHOU: A characterization of BV and Sobolev functions via nonlocal functionals in metric spaces. - Nonlinear Anal. 241, 2024, Paper No. 113467, 14 pp.
- [22] MURUGAN, M., and R. SHIMIZU: First-order Sobolev spaces, self-similar energies and energy measures on the Sierpiński carpet. - Comm. Pure Appl. Math. (to appear).
- [23] PIETRUSKA-PALUBA, K.: Heat kernels on metric spaces and a characterisation of constant functions. - Manuscripta Math. 115:3, 2004, 383–399.
- [24] SHIMIZU, R.: Construction of p -energy and associated energy measures on Sierpiński carpets. - Trans. Amer. Math. Soc. 377:2, 2024, 951–1032.

Received 2 September 2024 • Revision received 1 June 2025 • Accepted 17 June 2025

Published online 27 June 2025

Takashi Kumagai
Waseda University
Department of Mathematics
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555
Japan
t-kumagai@waseda.jp

Nageswari Shanmugalingam
University of Cincinnati
Department of Mathematical Sciences
P.O. Box 210025, Cincinnati, OH 45221-0025
U.S.A.
shanmun@uc.edu

Ryosuke Shimizu
Waseda University
Waseda Research Institute for Science and Engineering
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
Current address: Kyoto University
Graduate School of Informatics
Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
r.shimizu@acs.i.kyoto-u.ac.jp