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A counting formula for extreme contractions on regular polygonal Banach spaces

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Abstract. A regular polygonal Banach space $\mathbb{X}$ is a two-dimensional real Banach space whose unit sphere is a regular polygon. We prove that the number of extreme contractions on a regular $2n$-gonal Banach space is $2n^2+4n$. A characterization of extreme contractions on $\mathbb{X}$ is also presented.

Tasakulmaisen Banachin avaruuden kutistuskuvausjoukon ääripisteiden määrä

Tiivistelmä. Tasakulmainen Banachin avaruus $\mathbb{X}$ on kaksiulotteinen reaalinen Banachin avaruus, jonka yksikköpallo on säännöllinen monikulmio. Tässä työssä todistetaan, että $2n$-tasakulmaisen Banachin avaruuden kutistuskuvausten joukolla on $2n^2 + 4n$ ääripistettä. Lisäksi annetaan yhtäpitävä ehto näille äärimmäisille kutistuskuvauksille.

1. Introduction

Let $\mathbb{X}$ be a finite-dimensional real Banach space. The space $\mathbb{X}$ is said to be polyhedral if its closed unit ball $B_{\mathbb{X}}$ has finitely many extreme points. In addition, $\mathbb{X}$ is said to be regular polygonal, if $dim\ \mathbb{X} = 2$ and its unit sphere $S_{\mathbb{X}}$ is a regular polygon. Let $E_{\mathbb{X}}$ denote the collection of all extreme points of $B_{\mathbb{X}}$ and $\mathbb{L}(\mathbb{X})$ denote the algebra of all linear operators on $\mathbb{X}$. The members of $E_{\mathbb{L}(\mathbb{X})}$ are called extreme contractions on $\mathbb{X}$.

1.1. Motivation. A remarkable result due to Kadison [5] states that extreme contractions on a Hilbert space $\mathbb{H}$ are the isometries or co-isometries on $\mathbb{H}$, see also [4, 8, 11]. However, the analogous result is not true in the Banach space setting. In fact, it appears to be difficult to build a general theory of extreme contractions [6, 7, 14, 15] on Banach spaces even in the two-dimensional case. The present article aims to provide a counting formula for extreme contractions on a regular polygonal Banach space. To be more precise, it is shown that if $\mathbb{X}$ is a regular $2n$-gonal Banach space for some natural number n, then

$$|E_{\mathbb{L}(\mathbb{X})}| = 2n^2 + 4n.$$

In the process, we also characterize the extreme contractions on $\mathbb{X}$.

1.2. Notation and terminologies. For $n \in \mathbb{N}$, $n \geq 2$, we denote

$$\begin{aligned} A_n = \{z \in \mathbb{C} \colon z^{2n} = 1\} &= \{e^{i(k-1)\frac{\pi}{n}} \colon k = 1, 2, \ldots, 2n\} \\ &= \{x_k \colon k = 1, 2, \ldots, 2n\} = \{\pm x_k \colon k = 1, 2, \ldots, n\} \\ &= \{x_k \colon k = 1, 2, \ldots, n+1\} \cup \{\overline{x_k} \colon k = 2, \ldots, n\}. \end{aligned}$$

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In what follows the symbol $\mathbb{X}$ will denote the normed space $\mathbb{X} = \mathbb{R}^2$ with $B_{\mathbb{X}} = \operatorname{co}(A_n)$, the convex hull of A_n, and $\theta = \frac{\pi}{n}$. It is clear that $E_{\mathbb{X}} = A_n$. It is not difficult to see that

$$B_{\mathbb{X}} = \left\{(x,y) \in \mathbb{R}^2 \colon \sec\left(\frac{\theta}{2}\right) \max \left| x\cos\left(\frac{(2k-1)\theta}{2}\right) + y\sin\left(\frac{(2k-1)\theta}{2}\right)\right| \leq 1, \quad 1 \leq k \leq n\right\}.$$

A face [16] of $B_{\mathbb{X}}$ is of the form $(B_{\mathbb{X}} \cap \delta M)$, where M is a closed half space [1] in $\mathbb{X}$ containing $B_{\mathbb{X}}$ and δM is the boundary of M. Thus, if F is a face of $B_{\mathbb{X}}$, then F is of the form $\{x \in S_{\mathbb{X}} \colon f(x) = 1\}$, for some norm one functional $f \in \mathbb{X}^*$. The dimension of F is defined to be the dimension of the subspace generated by the differences $u - v$ of vectors in F and F is a facet if $\dim F = 1$. It is not difficult to see that corresponding to each facet F of $B_{\mathbb{X}}$, there exists a unique hyperplane H in $\mathbb{X}$ such that $(H \cap B_{\mathbb{X}}) = F$. Thus, for any facet F of $B_{\mathbb{X}}$, there exists a unique norm one functional f on $\mathbb{X}$ that attains its norm precisely on F. For two consecutive extreme points x_k and x_{k+1} of $B_{\mathbb{X}}$, the line segment $[x_k, x_{k+1}] := \{tx_k + (1-t)x_{k+1} \colon t \in [0,1]\}$ is a facet of $B_{\mathbb{X}}$, and the unique support functional corresponding to the facet $[x_k, x_{k+1}]$ is given by

$$\begin{aligned} f_{[x_k,x_{k+1}]}(x,y) &= \sec\left(\frac{\theta}{2}\right)\left[x\cos\left(\frac{(2k-1)\theta}{2}\right) + y\sin\left(\frac{(2k-1)\theta}{2}\right)\right], \quad \forall\ (x,y) \in \mathbb{X}, \\ &= \frac{1}{\sin\theta}[x(\sin k\theta - \sin(k-1)\theta) - y(\cos k\theta - \cos(k-1)\theta)]. \end{aligned}$$

For more information regarding polyhedral spaces, readers are referred to [1, 2, 10, 17]. In the subsequent sections, all the suffixes of the extreme points have been considered in modulo $2n$. To reduce computations, we will often use the usual complex multiplication in our results: for example, the product xy of any two vectors $x = (a,b)$ and $y = (c,d)$ in $\mathbb{X}$, is given by the ordered pair $(ac - bd, ad + bc)$.

1.3. Approach. We learn from the few recent works [3, 9, 13, 16] that the study of the extreme contractions on a Banach space $\mathbb{X}$ has connections with the extremal structure of $B_{\mathbb{X}}$. On the other hand, the norm attainment sets of extreme contractions are certainly of special kinds whenever $\mathbb{X}$ is polyhedral [9, 12]. Given any $T \in \mathbb{L}(\mathbb{X})$, the symbol M_T signifies the norm attainment set of T, i.e., $M_T = \{x \in S_{\mathbb{X}} \colon \|T(x)\| = \|T\|\}$. The basic step to prove the counting formula is to study the following class of contractions

$$\mathcal{E}_{\mathbb{X}} := \{T \in S_{\mathbb{L}(\mathbb{X})} \colon \operatorname{span}(M_T \cap E_{\mathbb{X}}) = \mathbb{X}\}.$$

The reason we consider this intermediate step is the following result proved in [12, Theorem 2.2].

Theorem 1.1. *Let $\mathbb{X}$ be an n-dimensional polygonal Banach space and let $\mathbb{Y}$ be any normed linear space. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be an extreme contraction. Then span$(M_T \cap E_{\mathbb{X}}) = \mathbb{X}$. Moreover, if $(M_T \cap E_{\mathbb{X}})$ contains exactly $2n$ elements then $T(M_T \cap E_{\mathbb{X}}) \subset E_{\mathbb{Y}}$.*

The above result has been used to present a counting of extreme contractions in a specific two-dimensional case.

Theorem 1.2. [9, Theorem 2.7] *Let $\mathbb{X}$ be a two-dimensional real Banach space whose unit sphere is a regular hexagon and let $\mathbb{Y} = \ell_\infty^2$. Then $|E_{\mathbb{L}(\mathbb{X},\mathbb{Y})}| = 36$.*

Evidently, it follows from Theorem 1.1 that $\mathcal{E}_{\mathbb{X}} \supseteq E_{\mathbb{L}(\mathbb{X})}$. After finding the class $\mathcal{E}_{\mathbb{X}}$, we discard the irrelevant contractions (which are not extreme contractions) by using elementary computation based on geometric and trigonometric formulations. To serve our purpose, we further divide $\mathcal{E}_{\mathbb{X}}$ into three different classes based on the image of the norm attainment sets of its members, namely, Type - I, Type - II and Type - III:

Type - I: A linear operator $T \in \mathbb{L}(\mathbb{X})$ is said to be of Type - I if there exists $\{x, y\} \subset M_T \cap E_{\mathbb{X}}$ such that $\text{span}\{x, y\} = \mathbb{X}$ and

$$T(x) = z, \quad T(y) = w, \tag{1.1}$$

for some $z, w \in E_{\mathbb{X}}$.

Type - II: A linear operator $T \in \mathbb{L}(\mathbb{X})$ is said to be of Type - II if there exists $\{x, y\} \subset M_T \cap E_{\mathbb{X}}$ such that $\text{span}\{x, y\} = \mathbb{X}$ and

$$T(x) = z, \quad T(y) = \lambda w + (1-\lambda)x_2 w, \quad \lambda \in]0,1[, \tag{1.2}$$

for some $z, w \in E_{\mathbb{X}}$.

Type - III: A linear operator $T \in \mathbb{L}(\mathbb{X})$ is said to be of Type - III if there exists $\{x, y\} \subset M_T \cap E_{\mathbb{X}}$ such that $\text{span}\{x, y\} = \mathbb{X}$ and

$$T(x) = \mu z + (1-\mu)x_2 z, \quad T(y) = \lambda w + (1-\lambda)x_2 w, \quad \mu, \lambda \in]0,1[, \tag{1.3}$$

for some $z, w \in E_{\mathbb{X}}$.

Theorem 1.3. *Let T be in $\mathbb{L}(\mathbb{X})$. Then T is an isometry if, and only if, T is a rank two Type - I operator.*

Theorem 1.4. *For each $y \in E_{\mathbb{X}}$ and each $z \in E_{\mathbb{X}}$ there exists a rank one Type - I operator $T \in \mathbb{L}(\mathbb{X})$ such that $T(y) = T(x_2 y) = z$. Conversely, for each rank one Type - I operator T, there exist $y \in E_{\mathbb{X}}$ and $z \in E_{\mathbb{X}}$ such that $T(y) = T(x_2 y) = z$.*

Next, we characterize Type - II and Type - III operators.

Theorem 1.5. *An operator $T \in \mathbb{L}(\mathbb{X})$ is a Type - II operator if, and only if, it can be written as a convex combination of a rank one Type - I and a rank two Type - I operator.*

Theorem 1.6. *An operator $T \in \mathbb{L}(\mathbb{X})$ is a Type - III operator if, and only if, there exist $x, y \in E_{\mathbb{X}}$ with $[x, y] \subseteq S_{\mathbb{X}}$ such that one of the following holds:*

(a) $T(x) = \mu z + (1-\mu)x_2 z$,
$T(y) = \lambda z + (1-\lambda)x_2 z$,
for any $\mu, \lambda \in]0,1[$,

(b) $T(x) = \mu z + (1-\mu)x_2 z$,
$T(y) = \lambda \overline{x_2} z + (1-\lambda) z$,
for any $\mu, \lambda \in]0,1[$ *with* $\mu \geq \lambda$,

(c) $T(x) = \mu z + (1-\mu)x_2 z$,
$T(y) = \mu w + (1-\mu)x_2 w$,
where $\mu \in]0,1[$ *and* $[z, w] \subseteq S_{\mathbb{X}}$.

It turns out that if an operator T is of Type - II or Type - III, then it is not an extreme contraction. Precisely, we have the following characterization of extreme contractions on $\mathbb{X}$.

Theorem 1.7. *$T \in E_{\mathbb{L}(\mathbb{X})}$ if, and only if, T is of Type - I.*

Thus, the problem of counting extreme contraction on $\mathbb{X}$ reduces to the problem of counting Type - I operators on $\mathbb{X}$.

The proofs of Theorems 1.3, 1.4, 1.5, 1.6 and 1.7 are included in section 3. In the last section, we prove the counting formula.

2. Preparatory results

Lemma 2.1. *Let $\mathbb{Z}, \mathbb{Y}$ be normed spaces with $E_{\mathbb{Y}} \neq \emptyset$, and $T \in B_{\mathbb{L}(\mathbb{Z},\mathbb{Y})}$. Suppose there exists $A \subseteq B_{\mathbb{Z}}$ such that $\overline{\text{span}(A)} = \mathbb{Z}$ and $T(A) \subseteq E_{\mathbb{Y}}$. Then T is an extreme contraction.*

Proof. Let us suppose that there are $\alpha \in]0,1[$ and $T_1, T_2 \in B_{\mathbb{L}(\mathbb{Z},\mathbb{Y})}$ such that $T = \alpha T_1 + (1-\alpha)T_2$. Then for all $x \in A$,

$$T(x) = \alpha T_1(x) + (1-\alpha)T_2(x).$$

Since $T_1(x), T_2(x) \in B_{\mathbb{Y}}$ and $T(x) \in E_{\mathbb{Y}}$, we conclude that $T(x) = T_1(x) = T_2(x)$ for all $x \in A$ and, because $\overline{\text{span}(A)} = \mathbb{Z}$, we get $T = T_1 = T_2$ as required. □

Proposition 2.2. *Let $\mathbb{Y}$ be a finite-dimensional normed space such that $E_{\mathbb{Y}}$ is finite and let T be an isomorphism on $\mathbb{Y}$ with $T(E_{\mathbb{Y}}) \subseteq E_{\mathbb{Y}}$. Then T is an isometry.*

Proof. Since $B_{\mathbb{Y}} = \text{co}(E_{\mathbb{Y}})$ and $T(E_{\mathbb{Y}}) \subseteq E_{\mathbb{Y}}$, we get $\|T\| \leq 1$. Because $E_{\mathbb{Y}}$ is a finite set and T is injective, we have $T(E_{\mathbb{Y}}) = E_{\mathbb{Y}}$, thus $T^{-1}(E_{\mathbb{Y}}) = E_{\mathbb{Y}}$. As before, we conclude that $\|T^{-1}\| \leq 1$. Taking this into account, we obtain

$$\|x\| = \|T^{-1}(T(x))\| \leq \|T(x)\| \leq \|x\|, \quad \forall\, x \in \mathbb{Y},$$

and the proof is now complete. □

Corollary 2.3. *An operator $T \in \mathbb{L}(\mathbb{X})$ is an isometry if, and only if, there exists $z \in A_n$ such that either $T(x) = zx$ for all $x \in \mathbb{X}$ or $T(x) = z\overline{x}$ for all $x \in \mathbb{X}$.*

Proof. If $T(x) = zx$ for all $x \in \mathbb{X}$ for some $z \in A_n$, then T is an isomorphism on $\mathbb{X}$ with $T(E_{\mathbb{X}}) \subseteq E_{\mathbb{X}}$ and Proposition 2.2 gives the result. The same argument can be used if $T(x) = z\overline{x}$ for all $x \in \mathbb{X}$ and for some $z \in A_n$.

Conversely, let T be an isometry on $\mathbb{X}$, then $T(1), T(x_2) \in E_{\mathbb{X}} = A_n$. It is clear that $[1, x_2] \subseteq S_{\mathbb{X}}$. Since T is an isometry, we get $[T(1), T(x_2)] \subseteq S_{\mathbb{X}}$. There exist two possibilities: either $T(x_2) = T(1)x_2$ or $T(x_2) = T(1)\overline{x_2}$. In the former case, the operator $S(x) = T(1)x$ $(x \in \mathbb{X})$ is a linear isometry by the first part of the proof. Moreover, since $S(1) = T(1)$ and $S(x_2) = T(x_2)$, we have $T = S$ and T is an isometry. In the latter case, it can be proved that T coincides with the linear isometry S defined by $S(x) = T(1)\overline{x}$ $(x \in \mathbb{X})$. This completes the proof. □

Corollary 2.4. *Let x, y be in $E_{\mathbb{X}}$. Then there exists an isometry, S, on $\mathbb{X}$ such that $S(x) = 1$ and $S(y) = x_k$ for some $k \in \mathbb{N}$ with $k \leq n+1$.*

Proof. It is clear that $\overline{x}y \in E_{\mathbb{X}}$, therefore $\overline{x}y = x_k$ for some $k = 1, 2, \ldots, 2n$. Consider $S(z) = \overline{x}z$ if $k \leq n+1$ and $S(z) = x\overline{z}$ in case $n+2 \leq k$. Then S is as required. □

We need a couple of technical lemmas which will be used in the sequel.

Lemma 2.5. *Let $k \in \mathbb{N}$ with $k \leq n$. Then*

(1) *For an even number k,*
- $\|e^{ik\theta} + 1\| = 2\cos\left(\frac{k\theta}{2}\right)$ *and* $\|e^{ik\theta} - 1\| = 2\sin\left(\frac{k\theta}{2}\right)\|i\|$.

(2) *For an odd number k,*
- $\|e^{ik\theta} + 1\| = 2\cos\left(\frac{k\theta}{2}\right)\sec\left(\frac{\theta}{2}\right)$ *and* $\|e^{ik\theta} - 1\| = 2\sin\left(\frac{k\theta}{2}\right)\|ie^{i\frac{\theta}{2}}\|$.

Proof. An easy calculation gives us

$$e^{ik\theta}+1=2\cos\left(\frac{k\theta}{2}\right)e^{i\frac{k\theta}{2}},\quad e^{ik\theta}-1=2\sin\left(\frac{k\theta}{2}\right)ie^{i\frac{k\theta}{2}}.$$

If k is even, $\|e^{i\frac{k\theta}{2}}\|=1$ and $\|ie^{i\frac{k\theta}{2}}\|=\|i\|$, by Corollary 2.3, and this finishes the proof of (1). For (2), since $[1,e^{i\theta}]\subseteq S_{\mathbb{X}}$, we get from the first equality that $\|e^{i\frac{\theta}{2}}\|=\sec\left(\frac{\theta}{2}\right)$. If k is odd, again by Corollary 2.3, we have $\|e^{i\frac{k\theta}{2}}\|=\|e^{i\frac{\theta}{2}}\|=\sec\left(\frac{\theta}{2}\right)$ and $\|ie^{i\frac{k\theta}{2}}\|=\|ie^{i\frac{\theta}{2}}\|$, and this finishes the proof of (2). □

Lemma 2.6. *Let $k\in\mathbb{N}$, $p\in\mathbb{N}\cup\{0\}$ and $\mu,\lambda\in\,]0,1[$.*

(I) *Let*

$$a=\frac{\sin(k-1)\theta}{\sin k\theta}1+\frac{\sin\theta}{\sin k\theta}e^{ip\theta},\quad b=\frac{\sin(k-1)\theta}{\sin k\theta}1+\frac{\sin\theta}{\sin k\theta}e^{i(p+1)\theta},$$
$$c=\frac{\sin(k-1)\theta}{\sin k\theta}e^{i\theta}+\frac{\sin\theta}{\sin k\theta}e^{ip\theta},\quad d=\frac{\sin(k-1)\theta}{\sin k\theta}e^{i\theta}+\frac{\sin\theta}{\sin k\theta}e^{i(p+1)\theta}.$$

Then

(I.A) $\|\lambda a+(1-\lambda)b\|>1$, *for* $p+1\le k\le n-1$,

(I.B) $\|\mu(\lambda a+(1-\lambda)b)+(1-\mu)(\lambda c+(1-\lambda)d)\|>1$, *for* $2\le k=p\le n-1$ *and* $\mu<\lambda$,

(I.C) $\|\mu(\lambda a+(1-\lambda)b)+(1-\mu)(\lambda c+(1-\lambda)d)\|>1$, *for* $2\le p+1\le k\le n-1$.

(II) *Let*

$$a_1=-\frac{\sin\theta}{\sin k\theta}1+\frac{\sin(k+1)\theta}{\sin k\theta}e^{ip\theta},\quad b_1=-\frac{\sin\theta}{\sin k\theta}1+\frac{\sin(k+1)\theta}{\sin k\theta}e^{i(p+1)\theta},$$
$$c_1=-\frac{\sin\theta}{\sin k\theta}e^{i\theta}+\frac{\sin(k+1)\theta}{\sin k\theta}e^{ip\theta},\quad d_1=-\frac{\sin\theta}{\sin k\theta}e^{i\theta}+\frac{\sin(k+1)\theta}{\sin k\theta}e^{i(p+1)\theta}.$$

Then

(II.A) $\|\lambda a_1+(1-\lambda)b_1\|>1$, *for* $1\le k=p\le n-2$ *and* $1\le k<p\le n-1$,

(II.B) $\|\mu(\lambda a_1+(1-\lambda)b_1)+(1-\mu)(\lambda c_1+(1-\lambda)d_1)\|>1$, *for* $1\le k=p\le n-2$ *and* $\mu>\lambda$,

(II.C) $\|\mu(\lambda a_1+(1-\lambda)b_1)+(1-\mu)(\lambda c_1+(1-\lambda)d_1)\|>1$, *for* $1\le k\le p-1\le n-2$.

Proof. (I.A) Writing $e^{ip\theta}$ and $e^{i(p+1)\theta}$ in terms of the basis $\{1,e^{i\theta}\}$, we get

$$a=\frac{\sin(k-1)\theta}{\sin k\theta}1-\frac{\sin(p-1)\theta}{\sin k\theta}1+\frac{\sin p\theta}{\sin k\theta}e^{i\theta}$$

and

$$b=\frac{\sin(k-1)\theta}{\sin k\theta}1-\frac{\sin p\theta}{\sin k\theta}1+\frac{\sin(p+1)\theta}{\sin k\theta}e^{i\theta}.$$

It follows from $f_{[1,e^{i\theta}]}(1)=1=f_{[1,e^{i\theta}]}(e^{i\theta})$ that

$$f_{[1,e^{i\theta}]}(a)=\frac{\sin p\theta-\sin(p-1)\theta+\sin(k-1)\theta}{\sin k\theta}>1, \tag{2.1}$$

since for $p+1\le k\le n-1$ we have

$$\begin{aligned}&[\sin p\theta-\sin(p-1)\theta]+[\sin(k-1)\theta-\sin k\theta]\\&=2\sin\frac{\theta}{2}\left[\cos\left(\frac{(2p-1)\theta}{2}\right)-\cos\left(\frac{(2k-1)\theta}{2}\right)\right]>0.\end{aligned}$$

Similarly, $f_{[1,e^{i\theta}]}(b)\ge1$, and consequently, $\|\lambda a+(1-\lambda)b\|\ge f_{[1,e^{i\theta}]}(\lambda a+(1-\lambda)b)>1$, as $\lambda\in]0,1[$.

(I.B) Writing $e^{ip\theta}$ and $e^{i(p+1)\theta}$ in terms of the basis $\{e^{i\theta}, e^{2i\theta}\}$, we get

$$(2.2)\quad \begin{aligned} a &= \left[\frac{\sin(k-1)\theta\sin 2\theta}{\sin k\theta\sin\theta} - \frac{\sin(p-2)\theta}{\sin k\theta}\right]e^{i\theta} + \left[\frac{\sin(p-1)\theta}{\sin k\theta} - \frac{\sin(k-1)\theta}{\sin k\theta}\right]e^{2i\theta},\\ b &= \left[\frac{\sin(k-1)\theta\sin 2\theta}{\sin k\theta\sin\theta} - \frac{\sin(p-1)\theta}{\sin k\theta}\right]e^{i\theta} + \left[\frac{\sin p\theta}{\sin k\theta} - \frac{\sin(k-1)\theta}{\sin k\theta}\right]e^{2i\theta},\\ c &= \left[\frac{\sin(k-1)\theta}{\sin k\theta} - \frac{\sin(p-2)\theta}{\sin k\theta}\right]e^{i\theta} + \frac{\sin(p-1)\theta}{\sin k\theta}e^{2i\theta},\\ d &= \left[\frac{\sin(k-1)\theta}{\sin k\theta} - \frac{\sin(p-1)\theta}{\sin k\theta}\right]e^{i\theta} + \frac{\sin p\theta}{\sin k\theta}e^{2i\theta}. \end{aligned}$$

For $p = k$ we get

$$\begin{aligned} a = e^{i\theta},\quad b &= \left[\frac{\sin(k-1)\theta}{\sin k\theta}(2\cos\theta - 1)\right]e^{i\theta} + \left[1 - \frac{\sin(k-1)\theta}{\sin k\theta}\right]e^{2i\theta},\\ c &= \left[\frac{\sin(k-1)\theta}{\sin k\theta} - \frac{\sin(k-2)\theta}{\sin k\theta}\right]e^{i\theta} + \frac{\sin(k-1)\theta}{\sin k\theta}e^{2i\theta},\quad d = e^{2i\theta}. \end{aligned}$$

Now, $f_{[e^{i\theta},e^{2i\theta}]}(e^{i\theta}) = 1 = f_{[e^{i\theta},e^{2i\theta}]}(e^{2i\theta})$ gives

$$\begin{aligned} f_{[e^{i\theta},e^{2i\theta}]}(a) = 1,\quad f_{[e^{i\theta},e^{2i\theta}]}(d) = 1,\\ f_{[e^{i\theta},e^{2i\theta}]}(b+c) = \frac{2\sin(k-1)\theta\cos\theta + \sin k\theta - \sin(k-2)\theta}{\sin k\theta} = 2 \end{aligned}$$

and

$$f_{[e^{i\theta},e^{2i\theta}]}(c) = \frac{2\sin(k-1)\theta - \sin(k-2)\theta}{\sin k\theta} > 1,$$

since for $2 \le k \le n-1$,

$$\begin{aligned} &[\sin(k-1)\theta - \sin k\theta] + [\sin(k-1)\theta - \sin(k-2)\theta]\\ &= 2\sin\frac{\theta}{2}\left[\cos\left(\frac{(2k-3)\theta}{2}\right) - \cos\left(\frac{(2k-1)\theta}{2}\right)\right] > 0. \end{aligned}$$

Consequently, for $\mu, \lambda \in]0,1[$ with $\mu < \lambda$, we get

$$\begin{aligned} &\|\mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d)\|\\ &\ge f_{[e^{i\theta},e^{2i\theta}]}(\mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d))\\ &= 1 + (\lambda - \mu)(f_{[e^{i\theta},e^{2i\theta}]}(c) - 1) > 1. \end{aligned}$$

(I.C) Whenever $k = p+1$ and $2 \le p+1 \le n-1$, replacing the value of k in (2.2), we get $f_{[e^{i\theta},e^{2i\theta}]}(b) = 1$, $f_{[e^{i\theta},e^{2i\theta}]}(a) \ge 1$ and $f_{[e^{i\theta},e^{2i\theta}]}(w) > 1$ for $w \in \{c, d\}$. Thus, for $\mu, \lambda \in]0,1[$ we get,

$$\begin{aligned} &\|\mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d)\|\\ &\ge f_{[e^{i\theta},e^{2i\theta}]}(\mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d)) > 1. \end{aligned}$$

Writing $e^{ip\theta}$ and $e^{i(p+1)\theta}$ as linear combination of basis $\{1, e^{i\theta}\}$, we already have a and b at the beginning of the proof and moreover,

$$\begin{aligned} c &= \left[\frac{\sin(k-1)\theta}{\sin k\theta} - \frac{\sin(p-2)\theta}{\sin k\theta} + \frac{\sin(p-1)\theta\sin 2\theta}{\sin k\theta\sin\theta}\right]e^{i\theta} - \frac{\sin(p-1)\theta}{\sin k\theta}1,\\ d &= \left[\frac{\sin(k-1)\theta}{\sin k\theta} - \frac{\sin(p-1)\theta}{\sin k\theta} + \frac{\sin p\theta\sin 2\theta}{\sin k\theta\sin\theta}\right]e^{i\theta} - \frac{\sin p\theta}{\sin k\theta}1. \end{aligned}$$

For $2 \leq p+1 < k \leq n-1$, we get

$$f_{[1,e^{i\theta}]}(a) = f_{[1,e^{i\theta}]}(c) = \frac{\sin p\theta - \sin(p-1)\theta + \sin(k-1)\theta}{\sin k\theta} > 1.$$

Similarly, it can be shown that $f_{[1,e^{i\theta}]}(b) = f_{[1,e^{i\theta}]}(d) > 1$, and therefore,

$$\begin{aligned}&\|\mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d)\| \\ &\geq f_{[1,e^{i\theta}]}(\mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d)) > 1 \quad \forall\, \mu, \lambda \in \,]0,1[\,.\end{aligned}$$

(II) We only sketch the proof as it is similar to (I). In case of (A), whenever $1 \leq k = p \leq n-2$, writing a_1, b_1 in terms of the basis $\{e^{i(p+1)\theta}, e^{i(p+2)\theta}\}$, and considering the support functional $f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}$, we get $f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}(a_1) = 1$ and $f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}(b_1) > 1$, which proves the inequality. Whenever, $1 \leq k < p \leq n-1$, by considering the support functional $f_{[e^{ip\theta}, e^{i(p+1)\theta}]}$ and writing a_1, b_1 in terms of the basis vectors $\{e^{ip\theta}, e^{i(p+1)\theta}\}$ we get $f_{[e^{ip\theta}, e^{i(p+1)\theta}]}(a_1) = f_{[e^{ip\theta}, e^{i(p+1)\theta}]}(b_1) > 1$, and the inequality follows. In case of (B), we again write a_1, b_1, c_1, d_1 in terms of the basis $\{e^{i(p+1)\theta}, e^{i(p+2)\theta}\}$ and choose the support functional $f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}$ to get $f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}(a_1) = 1 = f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}(d_1)$, $f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}(b_1) > 1$, $f_{[e^{i(p+1)\theta}, e^{i(p+2)\theta}]}(b_1 + c_1) = 2$, and the inequality follows. In case of (C), we choose the basis $\{e^{ip\theta}, e^{i(p+1)\theta}\}$ and the functional $f_{[e^{ip\theta}, e^{i(p+1)\theta}]}$, then by the given hypothesis we get $f_{[e^{ip\theta}, e^{i(p+1)\theta}]}(a_1) = f_{[e^{ip\theta}, e^{i(p+1)\theta}]}(b_1) > 1$, $f_{[e^{ip\theta}, e^{i(p+1)\theta}]}(c_1) = f_{[e^{ip\theta}, e^{i(p+1)\theta}]}(d_1) \geq 1$. Thus, the inequality is proved and the proof of the lemma is now complete. □

Lemma 2.7. *Let T be in $\mathbb{L}(\mathbb{X})$ and $x, y, z \in E_{\mathbb{X}}$ such that $x \neq y$ and $T(x) = T(y) = z$. Then $\|T\| = 1$ if, and only if, $[x, y] \subseteq S_{\mathbb{X}}$.*

Proof. From the hypotheses, we have that $x \neq \pm y$ and, by Corollary 2.4, we can suppose that there exists $1 \leq k \leq n$ such that $T(1) = T(e^{ik\theta}) = 1$. It is easy to check that

$$T(\alpha, \beta) = \sec\left(\frac{k\theta}{2}\right)\left(\alpha\cos\left(\frac{k\theta}{2}\right) + \beta\sin\left(\frac{k\theta}{2}\right)\right), \quad \text{for every} \quad (\alpha, \beta) \in \mathbb{X}.$$

Let us suppose that $\|T\| = 1$. Then, $\|T(e^{ik\theta} + 1)\| = 2 \leq \|e^{ik\theta} + 1\|$. If k is even, Lemma 2.5 gives $2 \leq 2\cos(\frac{k\theta}{2})$, which is a contradiction. If k is odd, Lemma 2.5 gives $2 \leq 2\cos(\frac{k\theta}{2})\sec\left(\frac{\theta}{2}\right)$, that is, $\cos(\frac{\theta}{2}) \leq \cos(\frac{k\theta}{2})$, which implies that $k = 1$. As a result, $[1, e^{ik\theta}] = [1, e^{i\theta}] \subseteq S_{\mathbb{X}}$.

Conversely, let $[1, e^{ik\theta}] \subseteq S_{\mathbb{X}}$, i.e., $k = 1$. Then, $T(e^{ip\theta}) = \cos[(p - \frac{1}{2})\theta]\sec(\frac{\theta}{2})$ and consequently, $\|T(e^{ip\theta})\| < 1$ for every $p = 2, 3, \ldots, n-1$. Therefore, $\|T(e^{ip\theta})\| \leq 1$ for every $p = 0, 1, \ldots, 2n-1$ and consequently, $\|T\| = 1$. This completes the proof. □

We prove the following proposition as the main ingredient of Theorem 1.5.

Proposition 2.8. *Let $x, y, z, w \in E_{\mathbb{X}}$ with span$\{x, y\} = \mathbb{X}$ and $T \in \mathbb{L}(\mathbb{X})$ be such that*

$$T(x) = z, \quad T(y) = \lambda w + (1-\lambda)x_2 w,$$

for some $\lambda \in \,]0,1[$. Let $T_1, T_2 \in \mathbb{L}(\mathbb{X})$ be such that $T_1(x) = T_2(x) = z$, $T_1(y) = w$ and $T_2(y) = x_2 w$. Then $\|T\| = 1$ if, and only if, one of the following holds.

(a) *T_1 is a rank one Type - I and T_2 is a rank two Type - I operator.*

(b) *T_2 is a rank one Type - I and T_1 is a rank two Type - I operator.*

Proof. We first prove the sufficiency. Evidently, $\|T\| \geq 1$. However, since $T = \lambda T_1 + (1-\lambda)T_2$ and $\|T_1\| = \|T_2\| = 1$, we get $\|T\| = 1$.

We now prove the necessity. By using Corollary 2.3, composing suitable isometries, without loss of generality, we can assume that

$$T(1) = 1, \quad T(e^{ik\theta}) = \lambda e^{ip\theta} + (1-\lambda)e^{i(p+1)\theta},$$

for some $0 \leq p \leq n-1$ and $1 \leq k \leq n-1$. By the given hypothesis,

$$T_1(1) = 1, \quad T_1(e^{ik\theta}) = e^{ip\theta} \quad \text{and} \quad T_2(1) = 1, \quad T_2(e^{ik\theta}) = e^{i(p+1)\theta}.$$

It now follows from Lemma 2.7 and Theorem 1.3 that (a) holds if, and only if, $k = 1$ and $p = 0$. Similarly, (b) holds if, and only if, $k = n-1$ and $p = n-1$, by considering $T_1(-1) = T_2(-1) = -1$.

Suppose on the contrary that neither condition (a) nor condition (b) holds, i.e., $(k,p) \notin \{(1,0),(n-1,n-1)\}$. Let us now consider the following cases.

Case-I: Suppose that $k \geq p+1$ and $0 \leq p \leq n-2$ with $(k,p) \neq (1,0)$. A simple computation provides us $T(e^{i\theta}) = \lambda a + (1-\lambda)b$, where a, b are given by (I) of Lemma 2.6. Thus, applying (I.A) of Lemma 2.6, for $k \geq p+1$, we have $\|T(e^{i\theta})\| > 1$, proving $\|T\| > 1$.

Case-II: Suppose $k \leq p$ and $1 \leq p \leq n-1$ with $(k,p) \neq (n-1,n-1)$. Evidently, $T(e^{i(k+1)\theta}) = \lambda a_1 + (1-\lambda)b_1$, where a_1, b_1 are defined in (II) of Lemma 2.6. Therefore, for $k \leq p$, applying (II.A) of Lemma 2.6, we have $\|T(e^{i(k+1)\theta})\| > 1$, i.e., $\|T\| > 1$.

In all the above cases, we get $\|T\| > 1$, a contradiction. Therefore, $(k,p) \in \{(1,0),(n-1,n-1)\}$. This completes the proof. □

Fact 2.9. Without loss of generality, we meet with the following two possibilities for an operator T mentioned in the above proposition.

(A) $T(x) = z,$
$T(y) = \lambda z + (1-\lambda)x_2 z,$

(B) $T(x) = z,$
$T(y) = \lambda \overline{x_2} z + (1-\lambda)z,$

for some $\lambda \in]0,1[$ and $x, y, z \in E_{\mathbb{X}}$ with $[x,y] \subseteq S_{\mathbb{X}}$. Moreover, T is of type (A) if, and only if, ST is of type (B) for the isometry $S(x) = z^2\overline{x}$ for all $x \in \mathbb{X}$. As a consequence, every operator of type (B) is an operator of type (A) up to an isometry.

We now establish several preparatory results to prove Theorem 1.6. According to the Definition 1.3, a Type - III operator involves parameters $\mu, \lambda \in]0,1[$. The following result provides a characterization of Type - III operators with $\mu = \lambda$.

Lemma 2.10. *Let $x, y, z, w \in E_{\mathbb{X}}$ with span$\{x,y\} = \mathbb{X}$. Let $T \in \mathbb{L}(\mathbb{X})$ be such that*

$$T(x) = \lambda z + (1-\lambda)x_2 z, \quad T(y) = \lambda w + (1-\lambda)x_2 w,$$

for some $\lambda \in]0,1[$. Let $\widetilde{T}, \hat{T} \in \mathbb{L}(\mathbb{X})$ be such that $\widetilde{T}(x) = z$, $\widetilde{T}(y) = w$ and $\hat{T}(x) = x_2 z$, $\hat{T}(y) = x_2 w$. Then

(i) *T is of rank one and unit normed if, and only if, $\widetilde{T}, \hat{T}$ both are of rank one and unit normed.*

(ii) *T is of rank two and unit normed if, and only if, $\widetilde{T}, \hat{T}$ both are of rank two and unit normed.*

Proof. We first prove the sufficiency for both (i) and (ii). Observe that $T = \lambda\widetilde{T} + (1-\lambda)\hat{T}$ and it follows from the definition of $\widetilde{T}$ and $\hat{T}$ that if $\widetilde{T}$ and $\hat{T}$ are of rank one or rank two, then, T is also of rank one or rank two, respectively. Moreover, $\|T\| = 1$, since $\|\widetilde{T}\| = \|\hat{T}\| = 1$.

Necessity of (i): Suppose that T is a rank one unit normed operator of the stated form. Since T is of rank one and $\|T(x)\| = \|T(y)\| = 1$, we have $T(x) = \pm T(y)$. We first consider the case $T(x) = T(y)$. By using Corollary 2.3, composing suitable isometries, without loss of generality, we can assume that

$$T(1) = \lambda 1 + (1-\lambda)e^{i\theta}, \quad T(e^{ik\theta}) = \lambda 1 + (1-\lambda)e^{i\theta},$$

for some $1 \leq k \leq n-1$.

By the hypothesis, $\widetilde{T}(1) = \widetilde{T}(e^{ik\theta}) = 1$ and $\hat{T}(1) = \hat{T}(e^{ik\theta}) = e^{i\theta}$. Suppose on the contrary that none of the operators $\widetilde{T}$, $\hat{T}$ is of rank one and unit normed. Then it follows from Lemma 2.7 that $k \neq 1$. By writing $e^{i\theta}$ in terms of the basis $\{1, e^{ik\theta}\}$ we get

$$\|T(e^{i\theta})\| = \frac{\sin(k-1)\theta + \sin\theta}{\sin k\theta} > 1.$$

Note that $\{x, -y\}$ remains linearly independent, as $\{x, y\}$ was so. Then the case $T(x) = -T(y)$ eventually reduces to the case $T(x) = T(y)$ for some extreme points x and y. This contradicts the fact that $\|T\| = 1$. Thus, $\widetilde{T}$ and $\hat{T}$ both are rank one unit normed operator.

Necessity of (ii): By using Corollary 2.3 and Corollary 2.4, composing suitable isometries, without loss of generality, we can assume that

$$T(1) = \lambda 1 + (1-\lambda)e^{i\theta}, \quad T(e^{ik\theta}) = \lambda e^{ip\theta} + (1-\lambda)e^{i(p+1)\theta},$$

for some $0 \leq p \leq n-1$ and $1 \leq k \leq n-1$.

Then $\widetilde{T}$ and $\hat{T}$ become

$$\widetilde{T}(1) = 1, \quad \widetilde{T}(e^{ik\theta}) = e^{ip\theta} \quad \text{and} \quad \hat{T}(1) = e^{i\theta}, \quad \hat{T}(e^{ik\theta}) = e^{i(p+1)\theta}.$$

By the hypothesis, $p \neq 0$, i.e., $1 \leq p \leq n-1$. Suppose on the contrary that none of the operators $\widetilde{T}$ and $\hat{T}$ is rank two and unit normed. Then it follows from Theorem 1.3 that $k \neq p$. It is not difficult to see that

$$T(e^{i\theta}) = \lambda(\lambda a + (1-\lambda)b) + (1-\lambda)(\lambda c + (1-\lambda)d),$$
$$T(e^{i(k+1)\theta}) = \lambda(\lambda a_1 + (1-\lambda)b_1) + (1-\lambda)(\lambda c_1 + (1-\lambda)d_1),$$

where $a, b, c, d, a_1, b_1, c_1, d_1$ are given by Lemma 2.6. By putting $\mu = \lambda$ in Lemma 2.6, we obtain from (I.C) and (II.C) that $\|T(e^{i\theta})\| > 1$ for $k > p$ and $\|T(e^{i(k+1)\theta})\| > 1$ for $k < p$, respectively. As a result, if $k \neq p$ then $\|T\| > 1$, a contradiction. Therefore, if T is a rank two unit normed operator then $k = p$ and consequently, the proof follows. □

We next establish a few intermediate results to characterize the Type - III operators with $\mu \neq \lambda$.

Proposition 2.11. *Let $x, y, z, w \in E_{\mathbb{X}}$ with span$\{x, y\} = \mathbb{X}$ and $T \in \mathbb{L}(\mathbb{X})$ be such that*

$$T(x) = \mu z + (1-\mu)x_2 z, \quad T(y) = \lambda w + (1-\lambda)x_2 w,$$

for some $\mu, \lambda \in]0, 1[$ with $\mu \neq \lambda$. If $\|T\| = 1$ then at least one of the following sets $\{z, w\}$, $\{z, x_2 w\}$, $\{x_2 z, w\}$, $\{x_2 z, x_2 w\}$ is linearly dependent.

Proof. The proof is evident for $n = 2, 3$. Thus, we prove this result for $n \geq 4$. By using Corollary 2.3 and Corollary 2.4, composing suitable isometries, without loss of generality, we can assume that

$$T(1) = \mu 1 + (1-\mu)e^{i\theta}, \quad T(e^{ik\theta}) = \lambda e^{ip\theta} + (1-\lambda)e^{i(p+1)\theta}, \tag{2.3}$$

for some $0 \leq p \leq n-1$ and $1 \leq k \leq n-1$. Suppose on the contrary that all the sets $\{1, e^{ip\theta}\}$, $\{1, e^{i(p+1)\theta}\}$, $\{e^{i\theta}, e^{ip\theta}\}$, $\{e^{i\theta}, e^{i(p+1)\theta}\}$ are linearly independent, i.e., $2 \leq p \leq n-2$. We consider the following cases.

Case-I: Suppose that $k \leq p-1$. Observe that $T(x_{k+2}) = \mu(\lambda a_1 + (1-\lambda)b_1) + (1-\mu)(\lambda c_1 + (1-\lambda)d_1)$, where a_1, b_1, c_1, d_1 are given by (II) of Lemma 2.6. Therefore, by (II.C) of Lemma 2.6, for $1 \leq k \leq p-1 \leq n-3$, we have $\|T(x_{k+2})\| > 1$, proving $\|T\| > 1$.

Case-II: Let $k = p$. We claim that $\|T(x_2)\| > 1$ if $\mu < \lambda$ and $\|T(x_{k+2})\| > 1$ if $\mu > \lambda$. Evidently, $T(x_2) = \mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d)$, where a, b, c, d are given by (I) of Lemma 2.6. Therefore, by (I.B) of Lemma 2.6, for $2 \leq p \leq n-2$, $k = p$ and $\mu < \lambda$, we have $\|T(x_2)\| > 1$. Thus, $\|T\| > 1$.

Again, observe that $T(x_{k+2}) = \mu(\lambda a_1 + (1-\lambda)b_1) + (1-\mu)(\lambda c_1 + (1-\lambda)d_1)$, where a_1, b_1, c_1, d_1 are given by (II) of Lemma 2.6. Therefore, by (II.B) of the same lemma, for $2 \leq p \leq n-2$, $k = p$ and $\mu > \lambda$, we have $\|T(x_{k+2})\| > 1$, proving $\|T\| > 1$.

Case-III: Suppose that $k \geq p+1$. It is easy to see that $T(x_2) = \mu(\lambda a + (1-\lambda)b) + (1-\mu)(\lambda c + (1-\lambda)d)$, where a, b, c, d are defined in (I) of Lemma 2.6. Under the condition $3 \leq p+1 \leq k \leq n-1$, (I.C) of Lemma 2.6 ensures that $\|T(x_2)\| > 1$, consequently, $\|T\| > 1$.

From the discussion of all the above cases, we arrive at a contradiction to the fact that $\|T\| = 1$. Thus, $p \in \{0, 1, n-1\}$ and equivalently, at least one of the following sets $\{1, e^{ip\theta}\}$, $\{1, e^{i(p+1)\theta}\}$, $\{e^{i\theta}, e^{ip\theta}\}$, $\{e^{i\theta}, e^{i(p+1)\theta}\}$ has to be a linearly dependent set. This completes the proof. □

Fact 2.12. Let $x, y, z \in E_{\mathbb{X}}$ with $\text{span}\{x, y\} = \mathbb{X}$. Without loss of generality, we meet with the following three possibilities for such operators T mentioned in the above proposition.

$$(A)\ T(x) = \mu z + (1-\mu)x_2 z, \quad T(y) = \lambda z + (1-\lambda)x_2 z,$$

$$(B)\ T(x) = \mu z + (1-\mu)x_2 z, \quad T(y) = \lambda \overline{x_2} z + (1-\lambda)z,$$

$$(C)\ T(x) = \mu z + (1-\mu)x_2 z, \quad T(y) = \lambda x_2 z + (1-\lambda)x_3 z,$$

for some $\mu, \lambda \in \,]0,1[$ with $\mu \neq \lambda$. Moreover, T is of type (B) if, and only if, ST is of type (C) for the isometry $S(x) = \overline{x_2}x$ for all $x \in \mathbb{X}$. As a consequence, every operator of type (C) is an operator of type (B) up to an isometry. Therefore, we only characterize the operators of types (A) and (B).

Lemma 2.13. *Let $x, y, z \in E_{\mathbb{X}}$ with span$\{x, y\} = \mathbb{X}$ and $T \in \mathbb{L}(\mathbb{X})$ be such that*

$$T(x) = \mu z + (1-\mu)x_2 z, \quad T(y) = \lambda z + (1-\lambda)x_2 z,$$

for some $\mu, \lambda \in]0,1[$. Then $\|T\| = 1$ if, and only if, $[x, y] \subseteq S_{\mathbb{X}}$.

Proof. The proof is evident for $n = 2$. Thus, we prove the result for $n \geq 3$. By using Corollary 2.3, composing suitable isometries, without loss of generality, we can assume that

$$T(1) = \mu 1 + (1-\mu)e^{i\theta}, \quad T(e^{ik\theta}) = \lambda 1 + (1-\lambda)e^{i\theta},$$

for some $1 \leq k \leq n-1$. We first prove the necessary part. By the given hypothesis, $1 = \|T(\frac{1}{2}(1 + e^{ik\theta}))\| \leq \|\frac{1}{2}(1 + e^{ik\theta})\|$. However, $\|\frac{1}{2}(1 + e^{ik\theta})\| < 1$ for $k \neq 1$, see Lemma 2.5. Therefore, we must have $k = 1$ and consequently, $[1, e^{ik\theta}] \subseteq S_{\mathbb{X}}$.

Conversely, let $[1, e^{ik\theta}] \subseteq S_{\mathbb{X}}$. Therefore, $k = 1$. Define

$$T_1(1) = 1, \quad T_1(e^{i\theta}) = \lambda 1 + (1-\lambda)e^{i\theta}$$

and

$$T_2(1) = e^{i\theta}, \quad T_2(e^{i\theta}) = \lambda 1 + (1-\lambda)e^{i\theta}.$$

Clearly, $T = \mu T_1 + (1-\mu)T_2$. Now, Proposition 2.8 implies that $\|T_1\| = \|T_2\| = 1$. Therefore, $\|T\| = 1$. This completes the proof. □

Remark 2.14. It is worth mentioning that the conclusion of Lemma 2.13 for operators with $\mu = \lambda$ also follows directly from Lemma 2.10.

Lemma 2.15. *Let $x, y, z \in E_{\mathbb{X}}$ with span$\{x, y\} = \mathbb{X}$ and $T \in \mathbb{L}(\mathbb{X})$ be such that*

$$T(x) = \mu z + (1-\mu)x_2 z, \quad T(y) = \lambda \overline{x_2} z + (1-\lambda)z,$$

for some $\mu, \lambda \in]0, 1[$. Then:

(a) $\|T\| = 1$ *implies* $[x, y] \subseteq S_{\mathbb{X}}$.

(b) *If* $[x, y] \subseteq S_{\mathbb{X}}$, *then* $\|T\| = 1$ *if and only if,* $\mu \geq \lambda$.

Proof. The proof is evident for $n = 2$. Thus, we prove this result for $n \geq 3$. By using Corollary 2.3, composing suitable isometries, without loss of generality, we can assume that

$$T(1) = \lambda 1 + (1-\lambda)e^{i\theta}, \quad T(e^{ik\theta}) = \mu e^{i\theta} + (1-\mu)e^{2i\theta},$$

for some $1 \leq k \leq n-1$. Following a rearrangement, we get

$$T(1 + e^{ik\theta}) = e^{i\theta} + e^{2i\theta} + \lambda(1 - e^{i\theta}) + \mu(e^{i\theta} - e^{2i\theta}).$$

(a) We choose the support functional $f_{[e^{i\theta}, e^{2i\theta}]}$. Observe that $f_{[e^{i\theta}, e^{2i\theta}]}(1 - e^{i\theta}) = 2(\cos\theta - 1)$ and $f_{[e^{i\theta}, e^{2i\theta}]}(e^{i\theta} - e^{2i\theta}) = 0$, $f_{[e^{i\theta}, e^{2i\theta}]}(e^{i\theta} + e^{2i\theta}) = 2$. Therefore, by using Lemma 2.5, we have

$$\begin{aligned} 2\cos\left(\frac{k\theta}{2}\right)\sec\left(\frac{\theta}{2}\right) &\geq \|1 + e^{ik\theta}\| \geq \|T(1 + e^{ik\theta})\| \geq f_{[e^{i\theta}, e^{2i\theta}]}(T(1 + e^{ik\theta})) \\ &= 2 + 2\lambda(\cos\theta - 1). \end{aligned} \tag{2.4}$$

Consequently,

$$2\cos\left(\frac{k\theta}{2}\right)\sec\left(\frac{\theta}{2}\right) - 2\cos\theta > 2\cos\left(\frac{k\theta}{2}\right)\sec\left(\frac{\theta}{2}\right) + 2\lambda(1 - \cos\theta) - 2 \geq 0. \tag{2.5}$$

Also, $\cos\left(\frac{(k-1)\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \geq \cos\left(\frac{k\theta}{2}\right) > \cos\theta\cos\left(\frac{\theta}{2}\right)$, by (2.5). It follows that $\cos\left(\frac{(k-1)\theta}{2}\right) > \cos\theta$ and consequently, $k < 3$. However, by (2.5), $2\cos\theta + 2\lambda(1 - \cos\theta) < 2$. Thus, for $k = 2$, it follows from (2.4) that $\|T(1 + e^{2i\theta})\| \geq 2 + 2\lambda(\cos\theta - 1) > 2\cos\theta = \|1 + e^{2i\theta}\|$, by Lemma 2.5. Therefore, we have $\|T\| > 1$. As a result, $k = 1$ and consequently, $[1, e^{ik\theta}] \subseteq S_{\mathbb{X}}$.

(b) By the given hypothesis, we have the following types:

(i) $T(1) = \mu e^{i\theta} + (1-\mu)e^{2i\theta}$,
$T(e^{i\theta}) = \lambda 1 + (1-\lambda)e^{i\theta}$,

(ii) $T(1) = \lambda 1 + (1-\lambda)e^{i\theta}$,
$T(e^{i\theta}) = \mu e^{i\theta} + (1-\mu)e^{2i\theta}$.

T is of type (ii) if, and only if, TS_1S_2 is of type (i) for the isometries $S_1(x) = \overline{x}$ and $S_2(x) = xe^{-i\theta}$, for all $x \in \mathbb{X}$. As a consequence, every operator of type (i) is an operator of type (ii) up to an isometry. Therefore, it is enough to discuss the result for the type (ii) operator only.

We first prove the necessity. Suppose on the contrary that $\mu < \lambda$. In that case, we show that $\|T(e^{2i\theta})\| > 1$. Observe that $T(e^{2i\theta}) = \lambda(\mu a_1 + (1-\mu)b_1) + (1-\lambda)(\mu c_1 + (1-\mu)d_1)$, where a_1, b_1, c_1, d_1 are given by (II) of Lemma 2.6, putting $k = p = 1$. Therefore, by (II.B) of Lemma 2.6, it is clear that $\|T(e^{2i\theta})\| > 1$. Consequently, we arrive at a contradiction to the fact that $\|T\| = 1$. Thus, we must have $\mu \geq \lambda$.

We now prove the sufficiency. Suppose that $\mu \geq \lambda$. We show that for an arbitrary extreme point $e^{ik\theta}$ ($k \in \{2, \ldots, n-1\}$), $\|T(e^{ik\theta})\| \leq 1$. Consider the operator

$$T_1(1) = \lambda 1 + (1-\lambda)e^{i\theta}, \quad T_1(e^{i\theta}) = e^{i\theta}.$$

By writing $e^{ik\theta}$ in terms of the basis $\{1, e^{i\theta}\}$, we get

$$\begin{aligned} T_1(e^{ik\theta}) &= \lambda \left[\frac{\sin(1-k)\theta}{\sin\theta} 1 + \frac{\sin k\theta}{\sin\theta} e^{i\theta}\right] + (1-\lambda)\left[\frac{\sin(1-k)\theta + \sin k\theta}{\sin\theta}\right] e^{i\theta} \\ &= y_2'(\text{say}) = \lambda e^{ik\theta} + (1-\lambda)\left[\frac{\sin(1-k)\theta + \sin k\theta}{\sin\theta}\right] e^{i\theta}. \end{aligned}$$

Since

$$\left|\frac{\sin(1-k)\theta + \sin k\theta}{\sin\theta}\right| \leq 1,$$

it is clear that $\|y_2'\| < 1$. Now, if $\mu = \lambda$, then it is not difficult to see that

$$\begin{aligned} T(e^{ik\theta}) &= \lambda \left[\frac{\sin(1-k)\theta}{\sin\theta} 1 + \frac{\sin k\theta}{\sin\theta} e^{i\theta}\right] + (1-\lambda)\left[\frac{\sin(1-k)\theta}{\sin\theta} e^{i\theta} + \frac{\sin k\theta}{\sin\theta} e^{2i\theta}\right] \\ &= y_2(\text{say}) = \lambda e^{ik\theta} + (1-\lambda)e^{i(k+1)\theta}. \end{aligned}$$

If $\mu > \lambda$, let $\mu = \lambda + \varepsilon$, where $\varepsilon > 0$, then

$$\begin{aligned} T(e^{ik\theta}) &= \frac{\sin(1-k)\theta}{\sin\theta}[\lambda 1 + (1-\lambda)e^{i\theta}] \\ &\quad + \frac{\sin k\theta}{\sin\theta}[(\lambda+\varepsilon)e^{i\theta} + (1-(\lambda+\varepsilon))e^{2i\theta}]. \end{aligned} \tag{2.6}$$

A straightforward computation reveals that

$$T(e^{ik\theta}) = \frac{1-(\lambda+\varepsilon)}{(1-\lambda)} y_2 + \left[1 - \frac{1-(\lambda+\varepsilon)}{(1-\lambda)}\right] y_2'. \tag{2.7}$$

As $1 > \lambda + \varepsilon > \lambda > 0$ then $\frac{1-(\lambda+\varepsilon)}{(1-\lambda)} \in]0, 1[$. Also, since $e^{ik\theta}$ is an arbitrary extreme point, it follows from the expression (2.7) that $\|T\| \leq 1$ and consequently, $\|T\| = 1$. Thus, the proof is complete. □

Remark 2.16. Note that the conclusion of Lemma 2.15(a) for operators with $\mu = \lambda$ also follows directly from Lemma 2.10.

3. Proofs

Proof of Theorem 1.3. Let T be an isometry on $\mathbb{X}$. Then T is of rank two. Also, it follows from Corollary 2.3 that $T(E_{\mathbb{X}}) = E_{\mathbb{X}}$. Therefore, T is of Type-I. Conversely, let T be a rank two Type-I operator on $\mathbb{X}$. By using Corollary 2.3, we can compose T with suitable isometries on $\mathbb{X}$ and without loss of generality we can assume that there are $1 \leq k, m \leq n-1$ such that $T(1) = 1$ and $T(e^{ik\theta}) = e^{im\theta}$. We claim that $k = m$ and so, $T(x) = x$ for all $x \in \mathbb{X}$, and consequently, the result follows. Since $\|T\| = 1$, we obtain $\|T(e^{ik\theta} \pm 1)\| = \|e^{im\theta} \pm 1\| \leq \|e^{ik\theta} \pm 1\|$. If k, m are either both even or both odd, from Lemma 2.5 we get $\cos(\frac{m\theta}{2}) \leq \cos(\frac{k\theta}{2})$ and $\sin(\frac{m\theta}{2}) \leq \sin(\frac{k\theta}{2})$.

These two inequalities imply that $k = m$. If m is even and k is odd, we consider the following two cases.

Case-I: If n is even, then $i \in A_n = E_{\mathbb{X}}$. We have $\|i\| = 1$ and, by Corollary 2.3 and Lemma 2.5, $\|ie^{i\frac{\theta}{2}}\| = \|e^{i\frac{\theta}{2}}\| = \sec\left(\frac{\theta}{2}\right)$. Also from Lemma 2.5 we get

$$\cos\left(\frac{m\theta}{2}\right) \le \cos\left(\frac{k\theta}{2}\right)\sec\left(\frac{\theta}{2}\right), \tag{3.1}$$

$$\sin\left(\frac{m\theta}{2}\right) \le \sin\left(\frac{k\theta}{2}\right)\sec\left(\frac{\theta}{2}\right). \tag{3.2}$$

The inequality (3.1) is equivalent to

$$\begin{aligned} 0 &\le \cos\left(\frac{k\theta}{2}\right) - \cos\left(\frac{m\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ &= \cos\left(\frac{m\theta}{2}\right)\left[\cos\left(\frac{(k-m)\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\right] - \sin\left(\frac{(k-m)\theta}{2}\right)\sin\left(\frac{m\theta}{2}\right). \end{aligned}$$

If $k > m$, the two summands appearing in the last equality are negative, and we conclude that $k \le m$. In a similar way, the inequality (3.2) is equivalent to

$$\begin{aligned} 0 &\le \sin\left(\frac{k\theta}{2}\right) - \sin\left(\frac{m\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ &= \sin\left(\frac{m\theta}{2}\right)\left[\cos\left(\frac{(k-m)\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\right] + \sin\left(\frac{(k-m)\theta}{2}\right)\cos\left(\frac{m\theta}{2}\right). \end{aligned}$$

If $m > k$, the two summands appearing in the last equality are negative, and we conclude that $k = m$. This is a contradiction because m is even and k is odd.

Case-II: If n is odd, then $i = e^{i\frac{\theta}{2}}e^{i\frac{(n-1)\theta}{2}}$ and $ie^{i\frac{\theta}{2}} = e^{i\frac{(n+1)\theta}{2}}$. Then, by Corollary 2.3 and Lemma 2.5, we have $\|i\| = \|e^{i\frac{\theta}{2}}\| = \sec\left(\frac{\theta}{2}\right)$ and $\|ie^{i\frac{\theta}{2}}\| = 1$. Again from Lemma 2.5 we get

$$\cos\left(\frac{m\theta}{2}\right) \le \cos\left(\frac{k\theta}{2}\right)\sec\left(\frac{\theta}{2}\right), \tag{3.3}$$

$$\sin\left(\frac{m\theta}{2}\right)\sec\left(\frac{\theta}{2}\right) \le \sin\left(\frac{k\theta}{2}\right). \tag{3.4}$$

As before, from (3.3) we get $k \le m$. Then from (3.4), we have $\sin\left(\frac{m\theta}{2}\right) \le \sin\left(\frac{m\theta}{2}\right)\sec\left(\frac{\theta}{2}\right) \le \sin\left(\frac{k\theta}{2}\right)$. Hence $m \le k$, and we get a contradiction. Similar arguments finish the proof if m is odd and k is even. □

Proof of Theorem 1.4. Let $y \in E_{\mathbb{X}}$ and $z \in E_{\mathbb{X}}$ be arbitrary. Define $T\colon \mathbb{X} \to \mathbb{X}$ by

$$T(y) = T(x_2 y) = z.$$

Evidently, $\operatorname{span}\{y, x_2 y\} = \mathbb{X}$. Also, $\operatorname{rank}(T) = 1$. Consequently, it follows from Lemma 2.7 that $\|T\| = 1$. On the other hand, it follows from Definition 1.1 that T is of Type - I. Conversely, let T be a rank one Type - I operator. Since T is of Type - I, there exist $x, u, z, w \in E_{\mathbb{X}}$ such that $\operatorname{span}\{x, u\} = \mathbb{X}$ and $T(x) = z$ and $T(u) = w$. Since $\operatorname{rank}(T) = 1$, either $z = \pm w$. If $z = w$, then the proof follows from Lemma 2.7. Otherwise, we have $T(x) = -T(u) = z$. Since $\operatorname{span}\{x, -u\} = \mathbb{X}$, again by Lemma 2.7, we have $[x, -u] \subseteq S_{\mathbb{X}}$. Consequently, for some $y \in E_{\mathbb{X}}$, we choose $x = y$ or $x = x_2 y$, accordingly for $u = -x_2 y$ or $u = -y$, respectively. This completes the proof. □

Proof of Theorem 1.5. The result follows from Proposition 2.8. □

Proof of Theorem 1.6. The result follows from Lemma 2.10, Fact 2.12, Lemma 2.13, 2.15. □

Next, we prove Theorem 1.7. Before that we prove the following assertions.

Proposition 3.1. *No Type - II operator is an extreme contraction.*

Proof. The result follows directly from Theorem 1.5. □

Proposition 3.2. *No Type - III operator is an extreme contraction.*

Proof. All the Type - III operators are mentioned in Theorem 1.6. We now show that those are not extreme contractions.

(a) Consider $T = \mu T_1 + (1-\mu)T_2$. Here

$$T_1(x) = z, \quad T_1(y) = \lambda z + (1-\lambda)x_2 z,$$

and

$$T_2(x) = x_2 z, \quad T_2(y) = \lambda z + (1-\lambda)x_2 z,$$

for some $x, y, z \in E_{\mathbb{X}}$ with $[x, y] \subseteq S_{\mathbb{X}}$ and $\mu, \lambda \in]0,1[$. By Proposition 2.8, we have $\|T_1\| = \|T_2\| = 1$. Thus, T is not an extreme contraction.

(b) Let $\varepsilon > 0$ be chosen arbitrarily such that $\mu - \varepsilon, \mu + \varepsilon, \lambda - \varepsilon, \lambda + \varepsilon \in]0,1[$. Take $T = \mu T_1 + (1-\mu)T_2$ where

$$T_1(x) = (\mu+\varepsilon)z + (1-(\mu+\varepsilon))x_2 z, \qquad T_2(x) = (\mu-\varepsilon)z + (1-(\mu-\varepsilon))x_2 z,$$
$$T_1(y) = (\lambda+\varepsilon)\overline{x_2}z + (1-(\lambda+\varepsilon))z, \qquad T_2(y) = (\lambda-\varepsilon)\overline{x_2}z + (1-(\lambda-\varepsilon))z,$$

for some $x, y, z \in E_{\mathbb{X}}$ with $[x, y] \subseteq S_{\mathbb{X}}$ and $\mu, \lambda \in]0,1[$ with $\mu \geq \lambda$. Evidently, by Lemma 2.15 we have $\|T_1\| = \|T_2\| = 1$. Therefore, T is not an extreme contraction.

(c) It follows directly from Lemma 2.10 that T is not an extreme contraction.

Thus, none of the Type - III operators is an extreme contraction. This completes the proof. □

We obtain the following result as an immediate corollary of Lemma 2.1.

Corollary 3.3. *Every Type - I operator is an extreme contraction.*

Proof of Theorem 1.7. The result follows from Proposition 3.1, Proposition 3.2 and Corollary 3.3. □

Remark 3.4. Let $\mathbb{X} = \ell_\infty^2$ i.e., $S_{\mathbb{X}}$ is a regular 4-gon or a square. Then by Corollary 3.3 one only has to deal with Type - I operators. Since, for any Type - I operator $T \in \mathbb{L}(\mathbb{X})$, $|(M_T \cap E_{\mathbb{X}})| = 4$. Thus, by Theorem 1.1 we have $T(M_T \cap E_{\mathbb{X}}) \subseteq E_{\mathbb{X}}$.

4. Counting of extreme contractions

The purpose of this section is to prove the counting formula for the extreme contractions on $\mathbb{X}$, as mentioned earlier.

4.1.Counting formula.

Theorem 4.1. *Let $\mathbb{X}$ be a regular $2n$-gonal space. Then $|E_{\mathbb{L}(\mathbb{X})}| = 2n^2 + 4n$.*

Proof. It follows from Theorem 1.7 that extreme contractions are nothing but the Type - I operators. Firstly, the number of rank two Type - I operators are $4n$.

To see this we consider the collection

$$\tau_1 = \{T \in S_{\mathbb{L}(\mathbb{X})} \colon T(x_1), T(x_2) \in E_{\mathbb{X}}, [T(x_1), T(x_2)] \subseteq S_{\mathbb{X}}\}. \tag{4.1}$$

For any $T \in \tau_1$, T is of Type - I and thus, $T \in E_{\mathbb{L}(\mathbb{X})}$. Also, for any rank two Type - I operator A, we have $[A(x_1), A(x_2)] \subseteq S_{\mathbb{X}}$, by virtue of Theorem 1.3. Thus, $A \in \tau_1$. So, τ_1 is precisely the collection of all rank two extreme contractions on $\mathbb{X}$. Let $T \in \tau_1$. Then $T(x_1)$ has $2n$ - many choices. Also, since $[T(x_1), T(x_2)] \subseteq S_{\mathbb{X}}$, $T(x_2)$ can be x_{r+1} or x_{r-1} whenever $T(x_1) = x_r$, by Theorem 1.3. Thus, there are only two choices for $T(x_2)$. Consequently, there are $2 \times 2n$ possibilities for T and we have $|\tau_1| = 4n$.

Secondly, the number of rank one Type - I operators are $2n^2$. To see this we consider the collection

$$\tau_2 = \{T \in S_{\mathbb{L}(\mathbb{X})} \colon T(x_m), T(x_{m+1}) \in E_{\mathbb{X}}, T(x_m) = T(x_{m+1}) \text{ for some } m \in \{1, \ldots, n\}\}.$$

For any $T \in \tau_2$, T is of Type - I and thus, $T \in E_{\mathbb{L}(\mathbb{X})}$. Also, for any rank one Type - I operator A, we have $A(x_m) = A(x_{m+1})$ for some $m \in \{1, \ldots, 2n\}$ by virtue of Theorem 1.4. If $m > n$ then $m + n \in \{1, \ldots, n\}$. Also, $A(x_{m+n}) = A(x_{m+n+1})$. Thus, $A \in \tau_2$. So, τ_2 is precisely the collection of all rank one extreme contractions on $\mathbb{X}$. Fixed any $m \in \{1, \ldots, n\}$, the operator T defined by $T(x_m) = T(x_{m+1}) = x_r$ for any $x_r \in E_{\mathbb{X}}$ is a member of τ_2. In fact, we get $2n$-such members of τ_2 for this particular m, since r varies over the set $\{1, \ldots, 2n\}$. Now, varying m over $\{1, \ldots, n\}$, we get $2n^2$ members of τ_2. Also, it follows from the definition of τ_2 that for any member A of τ_2, A must be among these $2n^2$ operators. These members are distinct, since for any $T \in \tau_2$ satisfying $T(x_m) = T(x_{m+1}) = x_r$ for some $m \in \{1, \ldots, n\}$ and $x_r \in E_{\mathbb{X}}$, we have $x_m, x_{m+1} \in M_T$ and $T(E_{\mathbb{X}} \setminus (M_T \cap E_{\mathbb{X}})) \in int(B_{\mathbb{X}})$ (see the proof of the sufficient part of Lemma 2.7). Therefore, $|\tau_2| = 2n^2$. Now, the total number of extreme contractions is given by

$$|\tau_1| + |\tau_2| = 4n + 2n^2,$$

and this completes the proof. □

We end this article by explicitly finding extreme contractions on regular hexagonal Banach space with the following example.

Example 4.2. Let $\mathbb{X}$ be a regular polygonal Banach space whose unit sphere is a regular hexagon (i.e., 6-gonal space). The matrix representations of the rank two extreme contractions with respect to the standard ordered basis are given by

$$T_1 = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2 = \pm\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad T_3 = \pm\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$T_4 = \pm\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad T_5 = \pm\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_6 = \pm\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

The matrix representations of the rank one extreme contractions with respect to the standard ordered basis are given by

$$T_1 = \pm\begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix}, \quad T_2 = \pm\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad T_3 = \pm\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$T_4 = \pm\begin{pmatrix} 0 & -\frac{2}{\sqrt{3}} \\ 0 & 0 \end{pmatrix}, \quad T_5 = \pm\begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 \end{pmatrix}, \quad T_6 = \pm\begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 \end{pmatrix},$$

$$T_7 = \pm\begin{pmatrix} -1 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix}, \quad T_8 = \pm\begin{pmatrix} -\frac{1}{2} & \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad T_9 = \pm\begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

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