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Conformal and holomorphic barycenters in hyperbolic balls

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Abstract. We introduce the notions of *conformal barycenter* and *holomorphic barycenter* of a measurable set D in the hyperbolic ball. The two barycenters coincide in the disk, but they differ in multidimensional balls of $\mathbb{C}^m \cong \mathbb{R}^{2m}$. These notions are counterparts of barycenters of measures on spheres, introduced by Douady and Earle in 1986.

Konforminen ja holomorfinen painopiste hyperbolisessa kuulassa

Tiivistelmä. Tässä työssä esitellään hyperbolisen kuulan mitallisen joukon *konformisen* ja *holomorfisen painopisteen* käsitteet. Nämä kaksi painopistettä yhtyvät kiekossa, mutta eroavat moniulotteisen avaruuden $\mathbb{C}^m \cong \mathbb{R}^{2m}$ kuulassa. Nämä käsitteet ovat Douadyn ja Earlen vuonna 1986 esittelemien pallokuoren mittojen painopisteiden vastineita.

1. Introduction

The barycenter is the "center of mass" of a collection of points, weighted by their respective masses. For a set of points $\{x_i\}$ in the Euclidean space with corresponding weights $\{w_i\}$, the barycenter is their weighted average.

A conformal barycenter extends this notion to settings where we are dealing with distances and structures which are preserved under conformal maps. It is often determined as the point that minimizes a certain energy functional or as a fixed point of some iterative conformal process.

We start with the notion of a barycenter in the unit disc $\mathbb{B}^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ in the complex plane. Denote by G the group of conformal automorphisms of $\mathbb{B}^2$ and by G_+ the subgroup of orientation preserving maps. The group G_+ consists of transformations of the following form

$$g_a(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}, \quad \theta \in [0, 2\pi),\ a \in \mathbb{B}^2. \tag{1.1}$$

Denote by $d\lambda(z)$ the Lebesgue measure in the complex plane, then the hyperbolic measure reads

$$d\Lambda(z) = \frac{d\lambda(z)}{(1 - |z|^2)^2}. \tag{1.2}$$

In this paper, we prove the following theorem.

Theorem 1.1. *Let $D \subseteq \mathbb{B}^2$ be a Lebesgue-measurable set, such that $0 < \Lambda(D) < +\infty$. Then there exists a unique point $c = c(D) \in \mathbb{B}^2$ such that*

$$\int_D g_c(z)\, d\Lambda(z) = 0,$$

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where g_c is the Möbius transformation of the unit disc defined by (1.1) with arbitrary $\theta \in [0, 2\pi)$.

Definition 1.2. We say that the point c is the conformal barycenter of the set D.

For a Lebesgue-measurable set $A \subseteq \mathbb{B}^2$ consider the following function

$$H(z) = -\int_A \log \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-z\bar{\zeta}|^2}\, d\Lambda(\zeta). \tag{1.3}$$

We also prove

Theorem 1.3. *For any Lebesgue-measurable set A, such that $0 < \Lambda(A) < +\infty$ the following assertions hold:*

1. *The function $H(z)$ has a unique global minimum on $\mathbb{B}^2$.*
2. *The minimum of $H(z)$ is the conformal barycenter of A.*

Theorem 1.4. *Let $\zeta_1, \dots, \zeta_N$ be points in the unit disk $\mathbb{B}^2$. Then there exists a unique (up to a rotation) Möbius transformation of the form* (1.1)*, such that*

$$\sum_{k=1}^{N} g_c(\zeta_k) = 0.$$

Definition 1.5. The point c from the above theorem is said to be the conformal barycenter of points $\zeta_1, \dots, \zeta_N$.

To formulate the next theorem, for given points $\zeta_1, \dots, \zeta_N$ consider the following function

$$H_N(z) = -\sum_{i=1}^{N} \log \frac{(1-|z|^2)(1-|\zeta_i|^2)}{|1-\bar{z}\zeta_i|^2}.$$

Theorem 1.6. (1) *The function $H_N(z)$ has a unique global minimum on $\mathbb{B}^2$.*
(2) *The minimum of $H_N(z)$ is the conformal barycenter of points $\zeta_1, \dots, \zeta_N$.*

Theorem 1.7. *The conformal barycenter is the conformally invariant. In other words, if c is the conformal barycenter of a set D, then $g(c)$ is the conformal barycenter of $g(D)$ for any Möbius transformation $g \in G_+$.*

In the present paper, we will prove all the above results, together with their extension to the case of unit balls, and extend the definition of the conformal barycenter to higher dimensions. In particular, for even-dimensional balls, we present two extensions corresponding to two non-equivalent metrics on the ball. In the latter case, we introduce the notion of the holomorphic barycenter.

Remark 1.8. The notion of a conformal barycenter was first introduced by Douady and Earle in their seminal paper [3]. With each probability measure μ on the unit circle $\mathbb{S}^1$ they associated a vector field in $\mathbb{B}^2$ in the following way

$$\xi_\mu(w) = (1-|w|^2)\int_{\mathbb{S}^1} \frac{\zeta - w}{1-\zeta\bar{w}}\, d\mu(\zeta). \tag{1.4}$$

It is proven in [3] that for each probability measure μ which does not contain heavy atoms there is a unique point $b(\mu)$ in $\mathbb{B}^2$ at which vector field $\xi_\mu(\cdot)$ vanishes. This point is said to be the *conformal barycenter* of the measure μ and it is conformally invariant in the sense of Theorem 1.7.

The authors of [3] further exploited properties of the conformal barycenters in order to demonstrate that any quasi-symmetric homeomorphism of the circle can be extended to a homeomorphism of the unit disk disk in a conformally natural way. The definition (and the uniqueness property) of the conformal barycenter of a probability measure on the circle extends to higher dimensions (e.g. to probability measures on spheres), see [3, 2].[1] In the present paper, we build upon the idea of Douady and Earle and introduce conformal barycenters of subsets (or measures) of hyperbolic balls.

2. Preliminaries

In order to facilitate the exposition and to avoid confusion with notations, we have to start with some common facts and concepts of Riemannian geometry.

2.1. Preliminaries from Riemannian geometry. A *Riemannian metric* on a manifold M is a smoothly varying, positive-definite, symmetric bilinear form g_p on the tangent space T_pM at each point $p \in M$.

More formally, for each point $p \in M$, the metric g_p is a function:

$$g_p\colon T_pM \times T_pM \to \mathbb{R}$$

which satisfies:

- *Symmetry*: $g_p(v, w) = g_p(w, v)$ for all $v, w \in T_pM$.
- *Bilinearity*: $g_p(av+bw, u) = ag_p(v, u)+bg_p(w, u)$ for all $a, b \in \mathbb{R}$ and $v, w, u \in T_pM$.
- *Positive-definiteness*: $g_p(v, v) > 0$ for all non-zero $v \in T_pM$.

The metric provides a way of measuring the:

- *Length of a vector* $v \in T_pM$: $|v| = \sqrt{g_p(v, v)}$.
- *Angle between two vectors* $v, w \in T_pM$:

$$\cos(\theta) = \frac{g_p(v, w)}{|v||w|}.$$

A *Riemannian manifold* (M, g) is a smooth manifold M equipped with a Riemannian metric g.

Length of a Curve. Given a smooth curve $\gamma\colon [a, b] \to M$ on a Riemannian manifold (M, g), the *length of the curve* is defined as:

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}\left(\dot\gamma(t), \dot\gamma(t)\right)}\, dt,$$

where $\dot\gamma(t)$ is the velocity vector (tangent vector) of the curve at time t, and $g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))$ is the square of the speed (i.e. the norm squared of the velocity vector, measured in the Riemannian metric).

The *distance* between two points $p, q \in M$ on a Riemannian manifold is defined as the infimum of the lengths of all smooth curves connecting p and q. Formally,

$$d_M(p, q) = \inf\{L(\gamma) \mid \gamma\colon [a, b] \to M,\ \gamma(a) = p,\ \gamma(b) = q\}.$$

A *geodesic* is a curve that locally minimizes the distance between points. In a Riemannian manifold, geodesics generalize the concept of "straight lines" in Euclidean space. Formally, a geodesic $\gamma(t)$ is a curve that satisfies the *geodesic equation*, which is the second-order differential equation derived from the Riemannian metric.

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[1]However, notice that the Douady–Earle conformally natural extension of homeomorphisms on spheres are not necessarily homeomorphisms in balls.

The *sectional curvature* of a Riemannian manifold (M, g) measures how the manifold curves within two-dimensional directions (or planes) at a given point. It is associated with 2-dimensional planes inside the tangent space of the manifold at each point.

Let (M, g) be a Riemannian manifold, and let $p \in M$ be a point on the manifold. The *Riemann curvature tensor* of a Riemannian manifold (M, g) is a multilinear map that measures the failure of second covariant derivatives to commute. Given vector fields X, Y, Z, the Riemann curvature tensor R is defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where ∇ denotes the Levi–Civita connection and $[X, Y]$ is the Lie bracket of X and Y.

Consider a 2-dimensional plane $\sigma \subset T_pM$ inside the tangent space at p, spanned by two linearly independent tangent vectors $v, w \in T_pM$. The sectional curvature $K(\sigma)$ of the plane σ is a quantity defined in terms of the *Riemann curvature tensor* R as follows:

$$K(\sigma) = \frac{g_p\left(R(v,w)w, v\right)}{g_p(v,v)g_p(w,w) - g_p(v,w)^2}.$$

This quantity measures how the manifold curves along the 2-dimensional subspace spanned by v and w.

Here $R(v, w)$ is the action of the Riemann curvature tensor on the vectors v and w, while the denominator $g_p(v, v)g_p(w, w) - g_p(v, w)^2$ is the area of the parallelogram formed by the vectors v and w in the tangent space.

Two particular examples of Riemann manifolds are important for this paper. Before that, let us introduce some notation. By $\mathbb{B}^n$ and $\mathbb{B}_m$ we denote the unit balls in $\mathbb{R}^n$ and in $\mathbb{C}^m$, respectively. We will sometimes use the notation $\mathbb{B}$ to denote any ball.

The norm of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is denoted by $|x| = \sqrt{\sum_{k=1}^n x_k^2}$. The norm on of a vector $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ is $|z| = \sqrt{\langle z, z\rangle}$, where $\langle z, w\rangle = \sum_{k=1}^m z_k \overline{w_k}$.

By $d\lambda(x)$ and $d\lambda(z)$ we denote the Lebesgue measure in $\mathbb{R}^n$ and in $\mathbb{C}^m$ respectively. Then the hyperbolic measures read

$$d\Lambda(x) = \frac{d\lambda(x)}{(1 - |x|^2)^n}, \quad d\Lambda(z) = \frac{d\lambda(z)}{(1 - |z|^2)^{n+1}}. \tag{2.1}$$

We say that the set $D \subset \mathbb{B}$ is measurable if it is Lebesque-measurable and if $\Lambda(D) := \int_D d\Lambda < \infty$.

Example 2.1. Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ equipped with the metric

$$g_x(u, v) = \frac{\langle u, v\rangle}{(1 - |x|^2)}, u, v \in \mathbb{R}^n.$$

We call $(\mathbb{B}^n, g)$ the hyperbolic ball. It is well-known that the hyperbolic ball has constant negative holomorphic sectional curvature and non-positive sectional curvature.

Example 2.2. Let $\mathbb{B}_m$ be the unit ball in $\mathbb{C}^m$ equipped with the metric

$$g_z(u, v) = \langle B(z)u, v\rangle, \quad u, v \in \mathbb{C}^m, \ z \in \mathbb{B}_m.$$

Here

$$B(z) = (b(z)_{ij})_{i,j=1}^n \quad \text{and} \quad b(z)_{ij} = \frac{1}{n+1} \frac{\partial^2}{\partial \overline{z_i} \partial z_j} K(z, z),$$

where

$$K(z,w)=\frac{1}{n+1}\frac{1}{(1-\langle z,w\rangle)^{n+1}}$$

is the Bergman kernel [9].

The Riemannian manifold $(\mathbb{B}_m, g)$ is named the Bergman ball. Bergman balls have constant negative sectional curvature ([4]).

2.1.1. Poincaré distance and Möbius transformations of the unit ball. The Poincaré distance is given by

$$d_h(x,y)=\frac{1}{2}\log\frac{1+R}{1-R}, \tag{2.2}$$

where

$$R=\frac{|x-y|}{\sqrt{\rho(x,y)}} \quad \text{and} \quad \rho(x,a)=|x-a|^2+(1-|a|^2)(1-|x|^2).$$

Möbius transformations of the unit ball, up to the orthogonal transformation of the Euclidean space are given by

$$y=h_a(x)=\frac{a|x-a|^2+(1-|a|^2)(a-x)}{\rho(x,a)}. \tag{2.3}$$

It is well-known that the Poincaré metric is invariant under the action of Möbius transformations of the unit ball onto itself. Moreover $h_c^{-1}(x)=h_c(x)$ for every $c\in\mathbb{B}$.

Now if $c\in\mathbb{B}$ is arbitrary and m is any Möbius transformation preserving the unit ball, then there exists an orthogonal transformation A, such that

$$(h_{m(c)}\circ m)(x)=(A\circ h_c)(x). \tag{2.4}$$

In order to verify this, observe that

$$h_{m(c)}(m(c))=0$$

and

$$|h_{m(c)}(m(0))|=\frac{|m(c)-m(0)|}{\sqrt{\rho(m(c),m(0))}}=\frac{|c-0|}{\sqrt{\rho(c,0)}}=|c|.$$

Let $A=h_{m(c)}\circ m\circ h_c$. Since $A(0)=0$ and $|A(c)|=|c|$, we infer that A is an orthogonal transformation. This implies (2.4).

We conclude this subsection with several formulae that will be used in the remainder of the paper; see [1].

$$d_h(x,y)=\frac{1}{2}\log\frac{1+|h_a(x)|}{1-|h_a(x)|}; \tag{2.5}$$

$$(1-|h_a(x)|^2)=\frac{(1-|a|^2)(1-|x|^2)}{\rho(x,a)}. \tag{2.6}$$

Finally, the Jacobian of the mapping $y=h_a(x)$ is given by

$$J(y,x)=\frac{1-|a|^2}{\rho(a,x)^n}=\frac{(1-|y|^2)^n}{(1-|x|^2)^n}. \tag{2.7}$$

2.1.2. Bergman distance and automorphisms of the unit ball $B\subset\mathbb{C}^m$. Let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a, and let

$$Q=Q_a=I-P_a$$

be the projection onto the orthogonal complement of $[a]$. Explicitly, $P_0 = 0$ and $P = P_a(z) = \frac{\langle z,a\rangle a}{\langle a,a\rangle}$. Set $s_a = (1-|a|^2)^{1/2}$ and consider the map

$$p_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}. \tag{2.8}$$

Compositions of mappings of the form (2.8) and unitary linear mappings of the $\mathbb{C}^n$ constitute the group of holomorphic automorphisms of the unit ball $\mathbb{B}_n \subset \mathbb{C}^m$. It is easy to verify that $p_a^{-1} = p_a$. Moreover, for any automorphism q of the Bergman ball onto itself there exists a unitary transformation U such that

$$p_{q(c)} \circ q = U \circ p_c. \tag{2.9}$$

By using the representation formula [9, Proposition 1.21], we can introduce the Bergman metric as

$$d_B(z,w) = \frac{1}{2}\log\frac{1+|p_w(z)|}{1-|p_w(z)|}. \tag{2.10}$$

If $\Omega = \{z \in \mathbb{C}^n \colon \langle z, a\rangle \neq 1\}$, then the map p_a is holomorphic in Ω. It is clear that $\overline{\mathbb{B}_n} \subset \Omega$ for $|a| < 1$.

It is well-known that every automorphism q of the unit ball is an isometry w.r. to the Bergman metric, that is: $d_B(z,w) = d_B(q(z), q(w))$.

We also point out the formulae

$$(1 - |p_a(z)|^2) = \frac{(1-|z|^2)(1-|a|^2)}{|1 - \langle a, z\rangle|^2} \tag{2.11}$$

and the expression for the Jacobian

$$J(z, p_a) = \left(\frac{1-|p_a(z)|^2}{1-|z|^2}\right)^{n+1} = \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1}$$

which will be needed in the sequel.

For all the above facts we refer to monographs by Zhu [9] and Rudin [7].

3. Potentials in hyperbolic balls

Definition 3.1. Let μ be a measure defined on the Borel σ-algebra of the unit ball $\mathbb{B} \subset \mathbb{R}^n$ (or $\mathbb{C}^m$). We say that μ satisfies the *Automorphism Lusin Condition* if the following holds:

(ALC) For every Borel set $D \subseteq \mathbb{B}$, if $\mu(D) = 0$, then

$$\mu(g(D)) = 0 \quad \text{(respectively, } \mu(h(D)) = 0)$$

for every Möbius transformation g of $\mathbb{B}^n$ (respectively, every holomorphic automorphism h of $\mathbb{B}_m$).

This property ensures that μ-null sets remain null under the natural automorphisms of the domain, analogous in spirit to Lusin's classical condition (N) applied to functions.

In the present section, we prove the following

Theorem 3.2. (1) *Let μ be a measure on the unit ball $\mathbb{B}$ which satisfies assumption ALC and $A \subseteq \mathbb{B}$ a μ-measurable set, such that $0 < \mu(A) < +\infty$. The function*

$$G(x) = -\int_A \log\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2 + (1-|y|^2)(1-|x|^2)}\, d\mu(y), \quad x \in \mathbb{B}^n \tag{3.1}$$

has a unique minimum in $\mathbb{B}$.

(2) *The function*

$$L(z) = -\int_A \log \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}\, d\mu(w), \quad z \in \mathbb{B}_m \tag{3.2}$$

has a unique minimum in $\mathbb{B}$.

3.1. Auxiliary results. In order to prove the above theorem we need some results about geodesic convexity of distance functions in hyperbolic balls.

Definition 3.3. We say that a function $f\colon M \to \mathbb{R}$ is *geodesically convex* if, for every pair of points $a, b \in M$ and every geodesic $\gamma\colon [0,1] \to M$ connecting them with $\gamma(0) = a$ and $\gamma(1) = b$, the following inequality holds for all $t \in [0,1]$:

$$f(\gamma(t)) \le t f(a) + (1-t) f(b).$$

Note that the parametrization γ is assumed to be proportional to arc length.

Proposition 3.4. *Geodesically strictly convex function has no more than one local minimum in the unit ball.*

Proof. Suppose, for the sake of contradiction, that $f\colon \mathbb{B}^n \to \mathbb{R}$ is a geodesically strictly convex function and that it has two distinct local minima at points $a \neq b$ in the unit ball $\mathbb{B}^n$.

Let $\gamma\colon [0,1] \to \mathbb{B}^n$ be the geodesic connecting a and b, with $\gamma(0) = a$ and $\gamma(1) = b$. Since f is geodesically strictly convex, we have

$$f(\gamma(t)) < t f(a) + (1-t) f(b), \quad \text{for all } t \in (0,1).$$

But since a and b are local minima, we have $f(a) \le f(\gamma(t))$ and $f(b) \le f(\gamma(t))$ for all $t \in [0,1]$ sufficiently close to 0 and 1, respectively.

Therefore,

$$f(\gamma(t)) \ge \max\{f(a), f(b)\}, \quad \text{for } t \in (0,1),$$

which contradicts the strict inequality

$$f(\gamma(t)) < t f(a) + (1-t) f(b) \le \max\{f(a), f(b)\}.$$

This contradiction implies that f cannot have two distinct local minima. Hence, a geodesically strictly convex function has at most one local minimum. □

Definition 3.5. [5, Definition 3.3.5] The Hessian of a differentiable function $f\colon M \to R$ on a Riemannian manifold M is ∇df.

We have $df = \sum_{i=1}^n \frac{df}{dx^i} dx^i$ in local coordinates, hence

$$\nabla df = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^n \frac{\partial f}{\partial x^k} \Gamma_{ij}^k \right) dx^i \otimes dx^j,$$

where Γ_{ij}^k are Christoffel symbols.

Proposition 3.6. f *is strictly geodesically convex if its Hessian* ∇df *is positive definite.*

Corollary 3.7. *If* a *is a stationary point of a geodesically strictly convex function* f, *then* a *is the unique global minimum of* f.

Proof. Since a is stationary point, we have that $\frac{\partial f}{\partial x^k}(a) = 0$ for every $k = 1, \dots, n$. Hence, the matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(a)\right)_{i,j=1}^n$ is positive definite which implies that a is a local minimum of f. By Proposition 3.4, we conclude that a is a global minimum of f. □

The lemma below forms a crucial part of the proof of the main results.

Lemma 3.8. *The functions $f(x) = d_h(x,p)$ and $F(z) = d_B(z,q)$ are geodesically convex.*

Proof of Lemma 3.8. Let us prove the assertion for d_h and notice that the same proof applies to d_B. We will use the fact that the metrics d_h and d_B have negative sectional curvature.

Let $\gamma\colon [0,1] \to D$ be the geodesic line connecting a and b so that $\gamma(t)$ divides the geodesic arc ab into the ratio $1-t : t$. We need to prove that

$$d_h(p,\gamma(t)) \le (1-t)d_h(p,\gamma(0)) + t d_h(p,\gamma(1)). \tag{3.3}$$

We start from the triangle inequality

$$|d_h(p,\gamma(0)) - d_h(p,\gamma(1))| \le d_h(\gamma(0),\gamma(1)).$$

This inequality is equivalent with

$$d_h^2(p,\gamma(1)) + d_h^2(p,\gamma(0)) - d_h^2(\gamma(0),\gamma(1)) \le 2d_h(p,\gamma(1))d_h(p,\gamma(0))$$

which in turn is equivalent to

$$(1-t)d_h^2(p,\gamma(0)) + t d_h^2(p,\gamma(1)) - t(1-t)d_h^2(\gamma(0),\gamma(1))$$

$$\le ((1-t)d_h^2(p,\gamma(0)) + t d_h^2(p,\gamma(1)))^2. \tag{3.4}$$

On the other hand, the following formula holds (see [5, eq. 4.8.7])

$$d_h^2(p,\gamma(t)) \le (1-t)d_h^2(p,\gamma(0)) + t d_h^2(p,\gamma(1)) - t(1-t)d_h^2(\gamma(0),\gamma(1)). \tag{3.5}$$

By comparing inequalities (3.5) and (3) we obtain (3.3) which completes the proof. □

3.2. Proof of Theorem 3.2.

Proof. Using relations (2.5) and (2.6) we have that

$$\log \frac{(1-|x|^2)(1-|y|^2)}{\rho(y,x)} = \log \cosh^2(d_h(x,y)).$$

Hence, function (3.1) can be written as

$$G(x) = \int_A \log \cosh^2(d_h(x,y))\, d\mu(y).$$

We have already checked that $d_h(x,w)$ is a convex function of x for fixed w. Now, since $\log\cosh^2 t$ is an increasing convex function of the real variable t and its second derivative equals to $2\mathrm{sech}^2 t$, its integral is convex. Hence, the function G is strictly convex. Therefore, by Proposition 3.4, the function $G(x)$ has a unique minimum.

The second point of Theorem 3.2 for the function (3.2) can be proven in an analogous way by using Lemma 3.8 and relations (2.10) and (2.11).

□

Remark 3.9. Although Theorem 3.2 is valid for any measure μ, it is particularly meaningful in those special cases when the measure μ is conformally (or holomorphically) invariant (meaning that $\mu(D) = \mu(g(D))$ for any subset $D \subset \mathbb{B}$ and any automorphism g). These special cases of Theorem 3.2 are emphasized throughout our further exposition.

4. Barycenters in Poincaré balls

We first introduce the notion of barycenter of a set w.r. to any measure μ.

Theorem 4.1. *Let μ be a measure in the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ which satisfies assumption ALC and D a μ-measurable subset of $\mathbb{B}^n$, such that $0 < \mu(D) < +\infty$.*

(1) *There is a unique point $b = b(D) \in \mathbb{B}$, such that*

$$\int_D h_b(x)\, d\mu(x) = 0,$$

where h_b is Möbius transformation given by (2.3).

(2) *The point $b(D)$ is the unique minimizer of the function defined in* (3.1).

Proof. Consider the function

$$G(x) = -\int_A \log \frac{(1-|x|^2)(1-|y|^2)}{\rho(x,y)}\, d\mu(y).$$

By Theorem 3.2, G has a unique minimum $a \in \mathbb{B}$. Let h_a be a Möbius transformation of the unit ball onto itself so that $h_a(0) = a$ and $h_a \circ h_a = \mathrm{id}$. Then the function $G_1(x) = g(h_a(x))$ has unique minimum at $x = 0$.

Moreover,

$$\begin{aligned} G_1(x) &= -\int_A \log \frac{(1-|h_a(x)|^2)(1-|y|^2)}{\rho(h_a(x),y)}\, d\mu(y) \\ &= \int_A \log\cosh^2\left(d(h_a(x),y)\right) d\mu(y) \\ &= \int_A \log\cosh^2\left(d(x,h_a^{-1}(y))\right) d\mu(y) \\ &= \int_A \log\cosh^2\left(d(x,h_a(y))\right) d\mu(y). \end{aligned}$$

Then

$$\nabla G_1(x) = \int_A \left(\frac{2x}{1-|x|^2} + \frac{2x|h_a(y)|^2 - 2h_a(y)}{\rho(x,h_a(y))}\right) d\mu(y).$$

To justify the differentiation under the integral, observe that

$$\left|\frac{2x}{1-|x|^2} + \frac{2x|h_a(y)|^2 - 2h_a(y)}{\rho(x,h_a(y))}\right| \le \frac{2|x|}{1-|x|^2} + \frac{2+2|x|}{(1-|x|^2)^2}$$

and recall that we require $\mu(A) < \infty$.

Hence,

$$\nabla G_1(0) = -2\int_A h_a(y)\, d\mu(y).$$

Since $x = 0$ is the stationary point of G_1, it follows that

$$\int_A h_a(y)\, d\mu(y) = 0,$$

which completes the proof. □

Definition 4.2. We say that point $b(D)$ from the above theorem is the barycenter of the set D w.r. to measure μ.

4.1. Conformal barycenter in the Poincaré ball.

Definition 4.3. Barycenter of a set $D \in \mathbb{B}^n$ w.r. to the hyperbolic measure $\Lambda(x)$ defined in (2.1) is named *conformal barycenter.*

We will denote the conformal barycenter of D by $c \equiv c(D)$.

From Theorem 4.1(1) it follows that the conformal barycenter of A is minimum of the function where the measure $\mu(y)$ is replaced by $\Lambda(y)$.

The most transparent and potentially important for applications cases is when the set D is finite. In order to address this case, we apply Theorem 4.1 with μ being the counting measure:

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite;} \\ \infty & \text{if } A \text{ is infinite.} \end{cases} \tag{4.1}$$

Such a choice of μ yields the following

Corollary 4.4. *Assume that $x_1, \ldots, x_N$ are points on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$.*

(1) *There exists a unique (up to a linear isometry) Möbius transformation h of the unit ball onto itself, such that*

$$\sum_{k=1}^{N} h(x_k) = 0.$$

(2) *Decompose the Möbius transformation h as $h = A \circ h_c$ for some $c \in \mathbb{B}^n$ and a linear isometry A of the unit ball. Then point c is the unique minimum of the function*

$$G_N(y) = -\sum_{i=1}^{N} \log \frac{(1-|y|^2)(1-|x_i|^2)}{|y-x_i|^2 + (1-|x_i|^2)(1-|y|^2)}, \quad y \in \mathbb{B}^n.$$

Definition 4.5. The point c from Corollary 4.4(2) is said to be the conformal barycenter of the set $\{x_1, \ldots, x_N\}$

Theorem 4.6. *The conformal barycenter is conformally invariant. In other words, if $c = c(D)$ is the conformal barycenter of D, then $h(c)$ is the conformal barycenter of $h(D)$ for any Möbius transformation h of the unit ball.*

Proof. We aim to prove that if

$$\int_D h_c(x)\, d\Lambda(x) = 0,$$

and q is any Möbius transformation, then

$$\int_{q(D)} h_{q(c)}(y)\, d\Lambda(y) = 0.$$

Introduce the change of variables $y = g(x)$. Then by (2.7), we have $d\Lambda(y) = d\Lambda(x)$. Thus

$$\int_{q(D)} h_{q(c)}(y)\, d\Lambda(y) = \int_D h_{q(c)}(q(x))\, d\Lambda(x).$$

Now, taking into account considerations in Subsection 2.1.1, we obtain that

$$h_{q(c)}(q(x)) = Ah_c(x),$$

for an orthogonal transformation A of the unit ball onto itself. Therefore

$$\int_{q(D)} h_{q(c)}(y)\, d\Lambda(y) = A \int_D h_c \, d\Lambda(x) = 0$$

which confirms conformal invariance.

The above proof can easily be adapted to demonstrate conformal invariance of the barycenter of a finite set in the sense of Definition 4.5. □

5. Barycenters in Bergman balls

Theorem 5.1. *Let μ be a measure in $\mathbb{B}_m \subset \mathbb{C}^m$ which satisfies assumption (A1) and K a μ-measurable subset of $\mathbb{B}_m$, such that $0 < \mu(K) < +\infty$.*

(1) *There exists a unique point $a \equiv a(K) \in \mathbb{B}_m$, such that*

$$\int_K p_a(z)\, d\mu(z) = 0,$$

where p_a is the map defined by (2.8).

(2) *Point $a(K)$ is the minimum of the function 3.2.*

Proof. As before, by using (2.10) we obtain again

$$L(z) = \int_A \log\cosh^2\left(d_B(z, w)\right) d\mu(w).$$

The function L has the unique minimum a in $\mathbb{B}_m$. Let p_a be an involutive automorphism of the unit ball onto itself so that $p_a(0) = a$. Then zero is the unique minimum of $L_1(z) = g(p_a(z))$. Since automorphisms are isometries in the Bergman metric d_B, we obtain

$$L_1(z) = \int_A \log\cosh^2\left(d_B(z, p_a(w))\right) d\mu(w).$$

Moreover its gradient in $\mathbb{C}^m \cong \mathbb{R}^n$, $m = 2n$ is given by

$$\nabla L_1(z) = \int_A \left(\frac{2\left\langle z, p_a(w)\right\rangle_R + 2ip_a(w)\left\langle z, ip_a(w)\right\rangle_R - 2p_a(w)}{\left|1 - \left\langle z, p_a(w)\right\rangle\right|^2} + \frac{2z}{1 - |z|^2} \right) d\mu(w),$$

where $\langle z, w\rangle_R = \Re\left(\langle z, w\rangle\right)$ is the real inner product of vectors z and w.

By setting $z = 0$ in the above integral we get $\nabla L_1(0) = 0$, which implies that

$$\int_A p_a(w)\, d\mu(w) = 0.$$

This completes the proof. □

Definition 5.2. We say that the point a from the above theorem is the barycenter of the set A w.r. to the measure μ.

5.1. Holomorphic barycenter in the Bergman ball.

Definition 5.3. The barycenter of a set $K \subseteq \mathbb{B}_m$ with respect to the hyperbolic measure $\Lambda(z)$, defined in (2.1), is called the *holomorphic barycenter* of the set K.

From Theorem 4.1(2) it follows that the holomorphic barycenter of the set A is the minimum of the function (3.2) where the measure $\mu(w)$ is replaced by the hyperbolic measure $\Lambda(w)$.

By choosing the counting measure (4.1) in Theorem 5.1 we infer the result about the holomorphic barycenter of a finite set of points.

Corollary 5.4. *Assume that $z_1, \dots, z_N$ are points in the unit ball $\mathbb{B}_m \subset \mathbb{C}^m$.*

(1) *There exists a unique (up to a linear unitary transformation) automorphism $p(z)$ of the unit ball onto itself, such that*

$$\sum_{k=1}^{N} p(z_k) = 0.$$

(2) *Decompose automorphism p as $p = U \circ p_c$ for a certain $c \in \mathbb{B}_m$ and a linear unitary transformation U of the unit ball. Then the point c is the unique minimum of the function*

$$L_N(z) = -\sum_{i=1}^{N} \log \frac{(1-|z|^2)(1-|w_i|^2)}{|1-\langle z, w_i\rangle|^2}$$

Definition 5.5. The point c from Corollary 5.4(2) is said to be holomorphic barycenter of the set $\{z_1, \dots, z_N\}$.

Theorem 5.6. *The holomorphic barycenter is holomorphically invariant. In other words, if $c = c(K)$ is the holomorphic barycenter of K, then $p(c)$ is the holomorphic barycenter of $p(K)$ for any automorphism p of the unit ball.*

The proof of holomorphic invariance in this case is similar to the conformal case, so we skip it.

Remark 5.7. (1) Results and notions from sections 4 and 5 are equivalent for the dimension $2m = n = 2$. More precisely, results regarding both hyperbolic and Bergman balls reduce to those from Section 1 for Poincaré disk when $2m = n = 2$.

(2) From the previous point, it follows that conformal and holomorphic barycenters coincide for $n = 2$. However, they are different in the case of complex dimensions greater than one. Namely, in $\mathbb{C}^m \cong \mathbb{R}^n$, $m = 2n > 1$, Möbius self-mappings of the unit ball and holomorphic automorphisms are different mappings and their corresponding metrics d_h and d_B are not equivalent.

(3) In balls of real odd dimension, i.e. in $\mathbb{B}^3, \mathbb{B}^5, \dots$ the meaningful notion of holomorphic barycenter does not exist. In such balls one can talk about conformal barycenters only.

(4) If $D \subset \mathbb{B}$ is symmetric with respect to $z = 0$, then its barycenter (both conformal and holomorphic) is equal to zero. Namely, in that case there is a linear isometry L and a partition $\{D_1, D_2\}$ of D, with $|D_1| = |D_2|$ (of equal measure) so that $\Omega_1 = L(D_1) \subset R^n_+$ and $\Omega_2 = L(D_2) \subset R^n_-$. Then $\int_{\Omega_1} x\, d\lambda(x) = -\int_{\Omega_2} x\, d\lambda(x)$, and so

$$\int_D Lx\, d\lambda(x) = \int_{\Omega_1+\Omega_2} y\, d\lambda(y) = 0$$

where $\lambda(\cdot)$ is the Lebesgue measure. In the same way, we prove a similar statement for hyperbolic measure.

6. Examples

Example 6.1. Consider an interior of the ellipse $D = \{x + iy \colon 4x^2 + 9y^2 < 1\} \subset \mathbb{B}^2$ with semi-axes $1/2$ and $1/3$. Then both the Lebesgue and the conformal barycenter are equal to zero (see Remark 5.7(4)).

Let $D_1 = h_{1/2}(D)$, where $h_{1/2}(z) = \frac{1/2-z}{1-z/2}$ (see Figure 1). By our theorem, the conformal barycenter of D_1 with respect to the hyperbolic measure is $1/2$. However, $1/2$ is not the conformal barycenter of D_1 with respect to the Lebesgue measure. To verify this, by calculation, we have

$$\begin{aligned}\int_{D_1} h_{1/2}(z)\, d\lambda(z) &= \int_D w \frac{(1-1/4)^2}{|1-w/2|^4}\, d\lambda(w) \\ &= \frac{3\pi\left(4426\sqrt{139} - 281475 \tanh^{-1}\left[\frac{2}{\sqrt{139}}\right]\right)}{111200} \approx 0.336214 \neq 0.\end{aligned}$$

Since the last integral is not zero, $1/2$ is not the barycenter of D_1 with respect to the Lebesgue measure. Numerical methods show that the barycenter of D_1 is approximately 0.46, which is less than $0.5 = 1/2$.

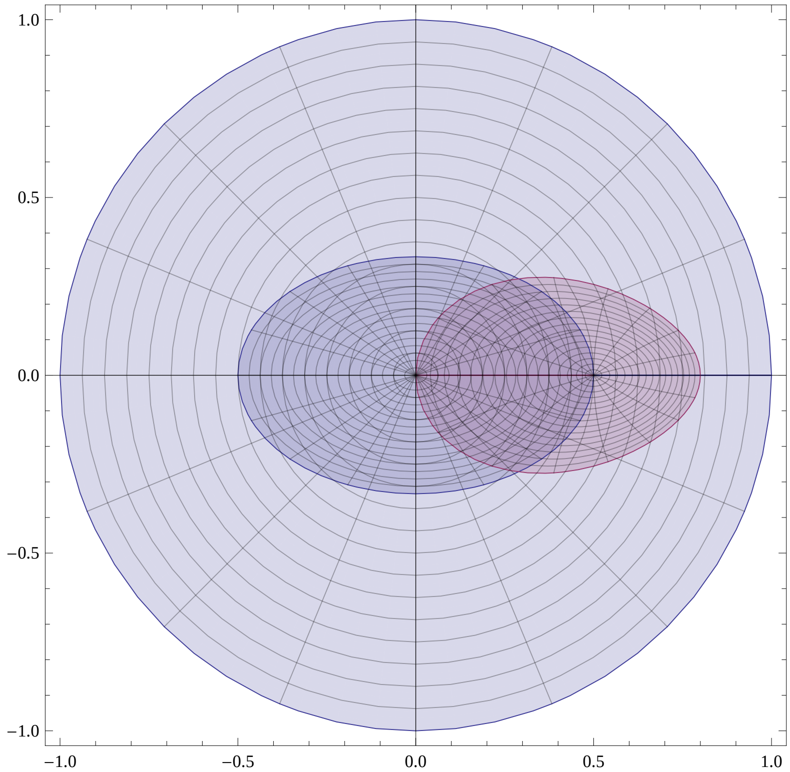

Figure 1. Interior of the ellipse D and egg-shape domain D_1 in the unit disc.

The above example demonstrates that the barycenter w.r. to Lebesgue measure is not conformally invariant.

Finally, we consider two simple examples that may provide some additional intuition on barycenters studied throughout the present paper.

Example 6.2. Let $z_1, z_2 \in \mathbb{D}$ and consider

$$g(z) = -\frac{1}{2}\sum_{k=1}^{2} \log \frac{(1-|z|^2)(1-|z_k|^2)}{|1-z\overline{z_k}|^2}.$$

Then $g_z = 0$ if and only if

$$\frac{\frac{2}{-1+z\bar{z}} + \frac{1}{1-z\overline{z_1}} + \frac{1}{1-z\overline{z_2}}}{z} = 0$$

whose solutions are

$$z_0 = 0$$

$$\hat{z} = \frac{\left(1 - |z_1 z_2|^2 - \sqrt{(1-|z_1|^2)\,(1-|z_2|^2)}|1 - z_1\overline{z_2}|\right)}{(1-|z_1|^2)\overline{z_2} + (1-|z_2|^2)\overline{z_1}}.$$

and

$$z' = \frac{\left(1 - |z_1 z_2|^2 + \sqrt{(1-|z_1|^2)\,(1-|z_2|^2)}|1 - z_1\overline{z_2}|\right)}{(1-|z_1|^2)\overline{z_2} + (1-|z_2|^2)\overline{z_1}}.$$

Moreover $|\hat{z}| < 1$ and $|z'| > 1$. Namely $|\hat{z}z'| = 1$.

Then we easily show that $z_0 = 0$ is not the minimum of g provided that $z_1 + z_2 \neq 0$, which implies that the minimum is the second stationary point $\hat{z}$, because $g(z) \to \infty$ as $|z| \to 1$. Moreover, it can be easily verified that the point $\hat{z}$ is in the midpoint of the geodesic line between z_1 and z_2.

Example 6.3. In the same way we prove that

$$a = (1+i)\left(\frac{4}{3} + \frac{5\sqrt{2}}{3\left(38636 + 1164\sqrt{1101}\right)^{1/6}} - \frac{\left(38636 + 1164\sqrt{1101}\right)^{1/6}}{3\sqrt{2}}\right)$$
$$\approx 0.156266 + 0.156266i$$

is the only stationary point of

$$g(z) = -\frac{1}{3}\sum_{k=1}^{3} \log \frac{(1-|z|^2)(1-|z_k|^2)}{|1 - z\overline{z_k}|^2},$$

where $z_1 = 1/2$, $z_2 = i/2$ and $z_3 = 0$. In this case if

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z},$$

then elementary computations yields that

$$\varphi_a(0) + \varphi_a(1/2) + \varphi_a(i/2) = 0,$$

which confirms our theorem.

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