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On bounded energy of convolution of fractal measures

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Abstract. For all $s \in [0,1]$ and $t \in (0,s] \cup [2-s,2)$, we find the supremum of numbers $\omega \in (0,2)$ such that

$$\mathrm{I}_\omega(\mu * \sigma) \lesssim 1,$$

where μ is any Borel measure on $B(1)$ with $\mathrm{I}_t(\mu) \le 1$ and σ is any $(s,1)$-Frostman measure on a C^2-graph with non-zero curvature. As an application, we use this to show the sharp L^6-decay of Fourier transform of σ when $s \in [\frac{2}{3}, 1]$.

Murtomittojen konvoluutioiden rajallisesta energiasta

Tiivistelmä. Jokaista arvoa $s \in [0,1]$ ja $t \in (0,s] \cup [2-s,2)$ kohti löydetään tässä työssä sellaisten lukujen $\omega \in (0,2)$ supremum, jotka toteuttavat ehdon

$$\mathrm{I}_\omega(\mu * \sigma) \lesssim 1,$$

kun μ on mikä tahansa kuulan $B(1)$ Borelin mitta, jolla $\mathrm{I}_t(\mu) \le 1$, ja σ on mikä tahansa Frostmanin $(s,1)$-mitta C^2-kuvaajalla, jonka kaarevuus on nollasta poikkeava. Tämän sovelluksena saadaan mitan σ Fourier'n muunnoksen tarkka L^6-vaimeneminen arvoilla $s \in [\frac{2}{3}, 1]$.

1. Introduction

Assume that $\psi \in C^2(\mathbb{R})$ satisfies the positive curvature condition

$$\psi''(x) > 0, \quad x \in \mathbb{R}. \tag{1.1}$$

Let Γ be the truncated graph of ψ on $[-1,1]$. For $0 < \omega < d$, let $\mathrm{I}_\omega(\mu)$ be the ω-dimensional Riesz energy of a measure μ, which is defined as

$$\mathrm{I}_\omega(\mu) := \int_{\mathbb{R}^{2d}} \frac{d\mu(x)\,d\mu(y)}{|x-y|^\omega} = c(\omega,d) \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \cdot |\xi|^{\omega-d}\,d\xi. \tag{1.2}$$

See [12, Theorem 3.10] for the proof of the second identity. We also recall the definition of Frostman measures.

Definition 1.3. Let $u \in [0,2]$ and $C_\mu \ge 1$. A Borel measure μ on $\mathbb{R}^2$ is called a (u, C_μ)-Frostman measure if $\mu(B(x,r)) \le C_\mu r^u$ for all $x \in \mathbb{R}^2$ and $r > 0$. Here C_μ is also called the Frostman constant of μ.

Of concern is the following problem.

Question 1. *Given $s \in [0,1]$ and $t \in (0,2)$, let μ be a Borel measure on $B(1)$ with $\mathrm{I}_t(\mu) \le 1$ and let σ be an $(s,1)$-Frostman measure on Γ. What is the supremum $\mathfrak{f}(s,t)$ of $\omega \in (0,2)$ such that $\mathrm{I}_\omega(\mu * \sigma) \lesssim_{\psi,s,t,\omega} 1$ for all such μ and σ?*

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The main result in this paper answers Question 1 partially. For the remaining cases $t \in [s, 3s]$, $s \in (0, \frac{1}{2}]$ and $t \in [s, 2-s]$, $s \in [\frac{1}{2}, 1]$, see Remark 4.4 for some partial results.

Theorem 1.4. *For $s \in [0,1]$ and $t \in (0,2)$, we have*

$$\mathfrak{f}(s,t) = \begin{cases} s+t, & \text{when } t \in (0,s],\ s \in (0,1], \\ s+1, & \text{when } t \in [2-s, s+1],\ s \in [\frac{1}{2},1], \\ t, & \text{when } t \in [3s, s+1],\ s \in [0,\frac{1}{2}], \\ t, & \text{when } t \in [s+1, 2),\ s \in [0,1). \end{cases} \tag{1.5}$$

Moreover, if $\mathrm{I}_t(\mu) < +\infty$ and σ is an (s, C_σ)-Frostman measure on Γ, then for $\mathfrak{f}(s,t) = s+t$ or $\mathfrak{f}(s,t) = s+1$, there exists a constant $C = C(\psi, s, t, \epsilon) > 0$ such that

$$\mathrm{I}_{\mathfrak{f}(s,t)-\epsilon}(\mu * \sigma) \le C\big(\max\{\mathrm{I}_t(\mu), 1\}\big) \cdot C_\sigma^2, \quad \forall \epsilon \in (0,1). \tag{1.6}$$

It is worth noting that curvature plays a crucial role for the first two bounds in (1.5) of Theorem 1.4. This is because curvature ensures geometric separation among translates of Γ. Specifically, when Γ has nonzero curvature, distinct translates of Γ intersect transversely, a property essential to our analysis. In contrast, if Γ is a straight line, its translates are merely parallel or overlapping lines, so that many different translation parameters lead to overlapping sets. This turns out to be a serious issue. For instance, let $\Gamma = \{(x,0)\colon x \in [-1,1]\}$, $\mu = \mathcal{H}^t|_{\mathcal{C}_t \times \{0\}}$ and $\sigma = \mathcal{H}^s|_{\mathcal{C}_s \times \{0\}}$, where $\mathcal{C}_t \subset [-\frac{1}{2}, \frac{1}{2}]$ is any t-dimensional Cantor set, and similarly for $\mathcal{C}_s$. In this case, we have $\dim_{\mathcal{H}}(\mathrm{spt}(\mu * \sigma)) \le 1$, which means $\mathrm{I}_\omega(\mu * \sigma) < +\infty$ cannot hold for $\omega > 1$ and contradicts the bound $\mathfrak{f}(s,t) = s+t$ whenever $s+t > 1$. A similar counterexample can also be constructed to contradict the bound $\mathfrak{f}(s,t) = s+1$.

Question 1 can be viewed as a measure analogue of the Furstenberg sets problem, which asks for the infimum $\gamma(s,t)$ of the Hausdorff dimensions of (s,t)-Furstenberg sets, see [17, 18, 19] for a recent solution of the Furstenberg sets problem in $\mathbb{R}^2$. However, there are important differences between the Furstenberg sets problem and Question 1. First, in the classical Furstenberg sets problem, the parameter t corresponds to the dimension of a family of lines, and points on different lines are independent. Here, the support of $\mu * \sigma$ can be expressed as a union of translates of a fixed curved set, where the set of translation parameters has dimension t, and the points on each translate are structured via translation. This structure allows us to employ Fourier analytic tools such as convolution and operator (see Section 5). There is also a variant of the Furstenberg sets problem based on this translation structure, which we take from [15].

Question 2. *Let $s \in (0,1]$ and $t \in [0,2]$. Assume that Γ is the graph of a C^3-function with non-zero curvature. Let $F \subset \mathbb{R}^2$ be a set with the following property: there exists another set $K \subset \mathbb{R}^2$ with $\dim_{\mathrm{H}} K \ge t$ such that $\dim_{\mathrm{H}}(F \cap (z+\Gamma)) \ge s$ for any $z \in K$ where $z + \Gamma$ is the translation of Γ by z. Is it true that $\dim_{\mathrm{H}}(F) \ge \min\{s+t, (3s+t)/2, s+1\}$?*

The answer to Question 2 is "Yes", and a proof will appear in a forthcoming paper [16]. We note that the translation structure is not strictly necessary: in fact, the authors can establish a curvilinear Furstenberg sets theorem without the translation framework, from which Question 2 is a direct corollary. However, a positive resolution of Question 2 does not automatically imply the remaining cases of Question 1. Seeking the connections between the two questions needs more work in the future.

Second, the constant $\mathfrak{f}(s,t)$ in our problem does not always coincide with the Furstenberg sets threshold $\gamma(s,t) = \min\{s+1, s+t, (3s+t)/2\}$. As evident from (1.5), in certain parameter ranges we have $\mathfrak{f}(s,t) = t$. This threshold is relatively straightforward to establish: by the trivial estimate $\mathrm{I}_t(\mu * \sigma) \lesssim \mathrm{I}_t(\mu) \leq 1$, we see that $\mathfrak{f}(s,t) \geq t$, while Examples 3.6 and 3.10 show that $\mathfrak{f}(s,t) \leq t$. We also note that the bound t in (1.5) does not depend on the curvature assumption as both Examples 3.6 and 3.10 can be slightly modified when Γ is replaced by a line. The first two bounds in (1.5) are more complicated and will follow from incidence estimates. When $\psi(x) = x^2$, it has been proved in [15, Theorem 6.1] that $\mathfrak{f}(s,t) \geq s+1$ if $t \in [2-s, s+1]$, $s \in [\frac{1}{2}, 1]$. In this paper, we consider more general C^2-curves with non-zero curvature.

As an application of the first two bounds in Theorem 1.4, we can show the sharp L^6-decay of Fourier transform of Frostman measures supported on Γ.

Corollary 1.7. *Let $s \in [2/3, 1]$. Assume that σ is an $(s,1)$-Frostman measure on Γ. Then*

$$\|\widehat{\sigma}\|^6_{L^6(B(R))} \leq C(\psi, s, \epsilon) R^{1-s+\epsilon}, \quad R \geq 1,\ \forall \epsilon \in (0,1). \tag{1.8}$$

Proof of Theorem 1.7. By applying (1.5) with $\mu = \sigma$, we get $\mathrm{I}_{2s-\tau}(\sigma * \sigma) \lesssim_{\psi,s,\tau} 1$ for any $\tau > 0$. A second use of Theorem 1.4 with $\mu = \sigma*\sigma$ gives $\mathrm{I}_{s+1-\tau}(\sigma*\sigma*\sigma) \lesssim_{\psi,s,\tau} 1$ when $s \geq 2/3 + \tau/3$. Letting $\tau \to 0$, we obtain that $\mathrm{I}_{s+1-\epsilon}(\sigma * \sigma * \sigma) \lesssim_{\psi,s,\epsilon} 1$ if $s \geq 2/3$. By the second identity in (1.2), we deduce

$$\|\widehat{\sigma}\|^6_{L^6(B(R))} = \|\widehat{\sigma * \sigma * \sigma}\|^2_{L^2(B(R))} \lesssim_{\psi,s,\epsilon} R^{1-s+\epsilon}, \quad \forall \epsilon \in (0,1),$$

as desired. □

The sharp exponent of the L^6-decay for all $s \in [0,1]$ was conjectured to be $2 - \min\{3s, 1+s\}$, see [13, Example 1.8]. When $s \geq 2/3$ and $\psi(x) = x^2$, (1.8) was established in [15, Theorem 1.1]. For general planar curve with non-zero curvature, see [3] for the bound "$2-2s-\beta(s)$" with $\beta(s) > 0$ a small implicit constant. Recently, Demeter–Wang [5, Theorem 1.2] improved this to "$2 - 2s - \frac{s}{4}$" when $s \in [0, \frac{1}{2}]$ by using their Szemerédi–Trotter type of incidence theorem. A further comment is that Corollary 1.7 would fail without curvature assumption (1.1). This is because curvature guarantees decay of the Fourier transform of a finite Borel measure σ supported on Γ. Precisely, when Γ has nonzero curvature, oscillatory integrals over Γ exhibit non-stationary phase behavior, resulting in cancellations that yield decay of $\widehat{\sigma}$. Concretely, one obtains $|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-\beta}$ for some $\beta > 0$ (see Lemma 5.7). In stark contrast, if Γ is merely a general C^2-curve, such as a straight line, there is no decay in directions orthogonal to Γ. Consequently, the L^6-norm of $|\widehat{\sigma}|$ cannot exhibit the same decay as in (1.8).

We do not know what the precise value of "$\mathfrak{f}(s,t)$" should be for the remaining cases of Question 1. From Remark 4.4, we know that at least $\mathfrak{f}(s,t) > t$ when $\psi(x) = x^2$. Here an natural guess is that this problem has the similar numerology to Furstenberg sets problem, which means the following conjecture may be plausible:

Conjecture 1.9. (Intermediate case) *For $s \in (0, \frac{1}{2}]$ and $t \in [s, 3s]$, or $s \in [\frac{1}{2}, 1]$ and $t \in [s, 2-s]$, there holds*

$$\mathfrak{f}(s,t) = \frac{3s+t}{2}. \tag{1.10}$$

From the proof of Corollary 1.7, it may be useful to remark that a full resolution of Conjecture 1.9 would improve Demeter–Wang's bound to "$2-2s-\frac{s}{2}$" for all $s \in [0, \frac{2}{3}]$.

At present we do not know how to solve Conjecture 1.9. One possible way is to seek the connections between Question 1 and the Furstenberg sets problem. Also, in [23] Wolff gave a connection between L^1-means of a single measure and $(s,1)$-Furstenberg sets, thus another possible direction is to consider the decay of L^p-means of $\mu*\sigma$ on the unit circle. For L^2-means of a single measure, see [2, 10, 21, 22] for more results.

1.1. Proof ideas. We will first construct examples in Section 3 to show that $\mathfrak{f}(s,t)$ cannot exceed those constants in (1.5). Also, since we have the trivial estimate $\mathrm{I}_t(\mu*\sigma)\lesssim \mathrm{I}_t(\mu)$, the threshold "$t$" will be proven once the examples have been found. Therefore, the main part of this paper will be intended to prove the inequalities

$$\mathfrak{f}(s,t)\geq \begin{cases} s+t, & \text{when } t\in(0,s],\ s\in(0,1],\\ s+1, & \text{when } t\in[2-s,s+1],\ s\in[\frac{1}{2},1]. \end{cases} \tag{1.11}$$

Instead, we prove a stronger L^2-inequality which implies (1.11):

$$\|(\mu*\sigma)_\delta\|_{L^2}^2 \lesssim_{\psi,s,t,\epsilon} \delta^{\zeta(s,t)-2-\epsilon},\quad \forall\epsilon\in(0,1), \tag{1.12}$$

where $(\mu*\sigma)_\delta=\mu*\sigma*\eta_\delta$ with $\eta_\delta(x)=\delta^{-2}\eta(\frac{x}{\delta})$ the rescaled mollifier and

$$\zeta(s,t)= \begin{cases} s+t, & \text{when } t\in(0,s],\ s\in(0,1],\\ 2s+t-1, & \text{when } t\in(0,2-s],\ s\in[0,1]. \end{cases} \tag{1.13}$$

Remark 1.14. Here we give some comments on why (1.12) implies (1.11). For $\zeta(s,t)=s+t$ when $t\in(0,s]$, there is nothing to say about the implication. For $\zeta(s,t)=2s+t-1$ when $s+t\leq 2$, we claim that this implies (1.12) with $2s+t-1$ replaced by $s+1$ when $s+t\geq 2$:

$$\|(\mu*\sigma)_\delta\|_{L^2}^2 \lesssim_{\psi,s,t,\epsilon} \delta^{s-1-\epsilon},\quad \forall\epsilon\in(0,1). \tag{1.15}$$

Indeed, if $s+t\geq 2$, we denote $\bar{t}:=2-s$, then $\mathrm{I}_{\bar{t}}(\mu)\lesssim \mathrm{I}_t(\mu)<+\infty$ and $s+\bar{t}\leq 2$. Using the result for the case $s+t\leq 2$, we readily see $\zeta(s,\bar{t})=2s+\bar{t}-1=s+1$ which proves (1.15). Then it is clear that (1.15) implies (1.11) when $t\in[2-s,s+1]$.

Next, we will reduce (1.12) to its δ-discretised version, then proving the δ-discretised version will mainly depend on incidence estimates between δ-cubes and curved δ-tubes, see Section 2 for the terminology and Section 4 for the reduction. Roughly speaking, we need to extend Fu–Ren's estimate [7, Theorem 5.2] to curvilinear case. This is easy when $\psi(x)=x^2$ because the parabola can be transformed into a line by the map

$$\Psi\colon \mathbb{R}^2\to\mathbb{R}^2,\quad \Psi(x,y):=(x,x^2-y).$$

However, if ψ is an arbitrary C^2-curve with $\psi''\neq 0$, we do not have a universal map transforming all its translates to lines, which is the main challenge in our problem. It turns out the case $t\in(0,s]$ is easier than the case $t\in(0,2-s]$ since an elementary incidence estimate can be obtained by analyzing the geometry of the intersection of two curved δ-tubes, see Proposition 6.1. As for $t\in(0,2-s]$, we will use the method in [14, Section 2]. Precisely, the following δ-incidence theorem of measures will be established, which serves as the main tool to build incidence estimates between δ-cubes and curved δ-tubes. The notations will be properly defined in Section 2.

Theorem 1.16. *Let μ and ν both be finite Borel measures with compact support on $B(1)$. Then for any $\delta\in(0,1)$, there holds*

$$\mathcal{I}_\delta(\mu,\nu)\lesssim_{\psi,t} \delta\sqrt{\mathrm{I}_{3-t}(\mu)\mathrm{I}_t(\nu)},\quad t\in(1,2). \tag{1.17}$$

The key point is that this δ-incidence estimate of measures implies an incidence estimate between cubes and tubes (Corollary 5.16) which is the main tool in extending [7, Theorem 5.2]. Theorem 1.16 will follow from the Sobolev estimates (Lemma 5.13) of an operator $\mathfrak{R}$ (Definition 5.9). We note that the idea of using operators to study incidences has also been applied in [6, 8].

1.2. Outline of the paper. Section 2 is the preliminary of this paper. In Section 3 four examples will be given to show that those constant in (1.5) is indeed sharp. In Section 4 we further reduce the first two cases in Theorem 1.4 to incidence estimates. Section 5 contains the proof of Theorem 1.4 for $t \in [2-s, s+1]$. We will mainly prove Theorem 1.16 and use it to build a weighted incidence estimate in Corollary 5.16. Once this has been built, Fu–Ren's estimate [7, Theorem 5.2] can be extended to curvilinear case whose proof we put in Appendix A for completeness. Finally, the proof of Theorem 1.4 for the case $t \in (0, s]$ will be contained in Section 6, where an improved incidence estimate can be obtained by a simple geometric argument.

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2. Preliminaries

In this section, we give all the notations and definitions needed in this paper.

Notations. We adopt the notations $\lesssim$, $\gtrsim$, $\sim$. For example, $A \lesssim B$ means $A \le CB$ for some constant $C > 0$, while $A \lesssim_r B$ stands for $A \le C(r)B$ for a positive function $C(r)$. For $\delta \in 2^{-\mathbb{N}}$, dyadic δ-cubes in $\mathbb{R}^2$ are denoted $\mathcal{D}_\delta(\mathbb{R}^2)$. For $P \subset \mathbb{R}^2$, we write $\mathcal{D}_\delta(P) := \{p \in \mathcal{D}_\delta(\mathbb{R}^d) \colon P \cap p \neq \emptyset\}$. In particular, we write $\mathcal{D}_\delta([0,1]^2) := \mathcal{D}_\delta$.

Definition 2.1. (Curved tubes) Let $\delta \in (0, 1]$ and $P \subset \mathbb{R}^2$. For each $q \in P$, let $q + \Gamma$ be the translation of Γ by q. If $q \in \mathcal{D}_\delta$, $q + \Gamma$ means the translation of Γ by the center of q. A curved δ-tube is the closed δ-neighborhood of $q + \Gamma$ for some $q \in P$. For simplicity, we let $\Gamma_q := q + \Gamma$ and write $\Gamma_q(\delta)$ as the corresponding curved δ-tube. We call P the parameter set of $\mathcal{T} := \{\Gamma_q(\delta) \colon q \in P\}$.

By definition 2.1, a set of curved δ-tubes $\mathcal{T}$ is uniquely determined by its parameter set P. In this paper, we will mainly consider curved δ-tubes with parameter set $\mathcal{P} \subset \mathcal{D}_\delta$. The next definition of Katz–Tao condition was originally introduced by Katz and Tao [9].

Definition 2.2. (Katz–Tao (δ, s, C)-set) Let $P \subset \mathbb{R}^2$ be a bounded set. Let $\delta \in (0, 1]$, $0 \le s \le 2$ and $C \ge 1$. We say that P is a *Katz–Tao (δ, s, C)-set* if

$$|P \cap B(x, r)|_\delta \le C \left(\tfrac{r}{\delta}\right)^s, \quad x \in \mathbb{R}^2,\ \delta \le r \le 1. \tag{2.3}$$

The notation $|\cdot|_\delta$ refers to the δ-covering number. If $\mathcal{P} \subset \mathcal{D}_\delta(\mathbb{R}^2)$, we say that $\mathcal{P}$ is a Katz–Tao (δ, s, C)-set if $P := \bigcup \mathcal{P}$ satisfies (2.3). A set of curved δ-tubes $\mathcal{T}$ is said to be a Katz–Tao (δ, s, C)-set if its parameter set is.

The following weighted Katz–Tao condition is only defined for dyadic elements. This condition enables us to prove a strong weighted incidence estimate in Corollary 5.16.

Definition 2.4. (Weighted Katz–Tao condition) Let $\delta \in (0, 1]$, $s \in [0, 2]$ and $C \ge 1$. Let $\mathcal{P} \subset \mathcal{D}_\delta$. For each $q \in \mathcal{P}$, there is a positive integer weight $w(q)$

associated with it. We say that $\mathcal{P}$ is a *weighted Katz–Tao* (δ, s, C)*-set* if for any $Q \in \mathcal{D}_r$ with $r \in [\delta, 1]$ there holds

$$\sum_{q \in \mathcal{P} \cap Q} w(q) \leq C \left(\tfrac{r}{\delta}\right)^s. \tag{2.5}$$

Let $\mathcal{T} := \{\Gamma_q(\delta) \colon q \in \mathcal{P} \subset \mathcal{D}_\delta\}$ be a set of curved δ-tubes, then we say $\mathcal{T}$ is a weighted Katz–Tao (δ, s, C)-set if its parameter set $\mathcal{P}$ is.

Remark 2.6. When $w \equiv 1$, a weighted Katz–Tao set $\mathcal{P} \subset \mathcal{D}_\delta$ becomes a Katz–Tao set in the sense of Definition 2.2. Since both notions are needed for us, we will always be careful and explicit in either including the word "weighted", or omitting it. Also, if $\mathcal{P}$ is a weighted Katz–Tao (δ, s, C)-set, we can deduce $w(q) \leq C$ for any $q \in \mathcal{P}$ by taking $r = \delta$ in (2.5).

Moreover, when we say $q \in \mathcal{P}$ has multiplicity $w(q)$, we simply mean that when counting incidences q appears $w(q)$ times, see the following definition.

Definition 2.7. (Weighted incidences) Let $\mathcal{F} \subset \mathcal{D}_\delta$ be a set of dyadic δ-cubes and let $\mathcal{T} = \{\Gamma_q(\delta) \colon q \in \mathcal{P} \subset \mathcal{D}_\delta\}$ be a set of curved δ-tubes. If weight functions w_1 and w_2 are associated with $\mathcal{P}$ and $\mathcal{F}$ respectively, then we define the weighted incidences between $\mathcal{F}$ and $\mathcal{T}$ by

$$\mathcal{I}_w(\mathcal{F}, \mathcal{T}) := \sum_{q \in \mathcal{P}} \sum_{p \in \mathcal{F}} w_1(q) w_2(p) \mathbf{1}_{\{p \cap \Gamma_q(\delta) \neq \emptyset\}}.$$

Finally, we introduce the δ-incidences between two finite Borel measures.

Definition 2.8. (δ-incidences) Let μ and ν both be finite Borel measures on $\mathbb{R}^2$. For any $\delta \in (0, 1)$, the δ-incidences are defined as

$$\mathcal{I}_\delta(\mu, \nu) := \mu \times \nu\big(\{(p, q) \in \mathbb{R}^2 \times \mathbb{R}^2 \colon p \in \Gamma_q(\delta)\}\big).$$

3. Examples

In this section, four examples are given to show that

$$\mathfrak{f}(s,t) = \begin{cases} t, & \text{when } t \in [3s, s+1],\ s \in [0, \frac{1}{2}], \\ t, & \text{when } t \in [s+1, 2),\ s \in [0, 1), \end{cases} \tag{3.1}$$

and

$$\mathfrak{f}(s,t) \leq \begin{cases} s+t, & \text{when } t \in (0, s],\ s \in (0, 1], \\ s+1, & \text{when } t \in [2-s, s+1],\ s \in [\frac{1}{2}, 1]. \end{cases} \tag{3.2}$$

As we mentioned before, (3.1) is due to the simple estimate $\mathrm{I}_t(\mu * \sigma) \lesssim \mathrm{I}_t(\mu)$. Therefore, the main part of this paper will be intended to prove the inequalities

$$\mathfrak{f}(s,t) \geq \begin{cases} s+t, & \text{when } t \in (0, s],\ s \in (0, 1], \\ s+1, & \text{when } t \in [2-s, s+1],\ s \in [\frac{1}{2}, 1]. \end{cases} \tag{3.3}$$

In the sequel, "$\dim_{\mathrm{H}}$" denotes Hausdorff dimension and "$\dim_{\mathrm{B}}$" denotes box counting dimension. We will use the basic fact for Borel set $A \subset \mathbb{R}^2$:

$$\dim_{\mathrm{H}}(A) = \sup\{v \colon \text{there is } \mu \in \mathcal{M}(A) \text{ such that } \mathrm{I}_v(\mu) < +\infty\},$$

where $\mathcal{M}(A)$ is the family of positive finite Borel measures with compact support in A. The proof of this identity can be found in [12, Theorem 2.8].

Example 3.4. (Case $t \in (0, s]$ and $s \in (0, 1]$) Let $\epsilon \in (0, 1)$. Choose a measure μ on $B(1)$ with

$$\mu(B(x, r)) = r^{t+\epsilon}, \quad x \in \operatorname{spt}(\mu),\ r > 0.$$

Hence $\mathrm{I}_t(\mu) \lesssim 1$. Next, choose σ on Γ such that $\sigma(B(x, r)) \sim r^s$ for any $x \in \operatorname{spt}(\sigma)$ and $r > 0$. Since both μ and σ are Ahlfors-regular, $\dim_B(\operatorname{spt}(\sigma)) = s$ and $\dim_B(\operatorname{spt}(\mu)) = t + \epsilon$, see [11, Theorem 5.7]. If $\mathrm{I}_\omega(\mu * \sigma) \lesssim_{\psi,s,t,\omega} 1$, we deduce

$$\begin{aligned} \omega &\le \dim_{\mathrm{H}}(\operatorname{spt}(\mu * \sigma)) \le \dim_{\mathrm{H}}(\operatorname{spt}(\mu) + \operatorname{spt}(\sigma)) \\ &\le \dim_B(\operatorname{spt}(\sigma)) + \dim_B(\operatorname{spt}(\mu)) = s + t + \epsilon. \end{aligned}$$

By sending $\epsilon \to 0$, we conclude that the supremum $\mathfrak{f}(s, t) \le s + t$.

Example 3.5. (Case $t \in [2 - s, s + 1]$ and $s \in [\frac{1}{2}, 1]$) We may assume $s < 1$ since otherwise $s + 1 = 2 \ge \mathfrak{f}(s, t)$ is obvious. Let $A \subset [0, 1]$ be a compact $(t - s)$-dimensional subset. Next, by [20, Example 7], for $\epsilon > 0$ with $s + \epsilon < 1$ we can find a compact subset $B \subset [0, 1]$ such that

$$\dim_B(B) = \dim_{\mathrm{H}}(B) = \dim_{\mathrm{H}}(B + B) = s + \epsilon.$$

Then $\dim_{\mathrm{H}}(A \times B) = t + \epsilon$ and $\dim_{\mathrm{H}}(\Gamma \cap (B \times \mathbb{R})) = s + \epsilon$. Let μ be a finite Borel measure supported on $(A \times B)$ with $\mathrm{I}_t(\mu) \le 1$ and let σ be an $(s, 1)$-Frostman measure supported on $\Gamma \cap (B \times \mathbb{R})$. If $\mathrm{I}_\omega(\mu * \sigma) \lesssim_{\psi,s,t,\omega} 1$, we can deduce

$$\begin{aligned} \omega &\le \dim_{\mathrm{H}}(\operatorname{spt}(\mu * \sigma)) \le \dim_{\mathrm{H}}(\operatorname{spt}(\mu) + \operatorname{spt}(\sigma)) \\ &\le \dim_{\mathrm{H}}((B + B) \times (A + \mathbb{R})) = s + 1 + \epsilon. \end{aligned}$$

By sending $\epsilon \to 0$, we conclude that $\mathfrak{f}(s, t) \le s + 1$.

Example 3.6. (Case $t \in [3s, s+1]$ and $s \in [0, \frac{1}{2}]$) Fix $t \in [3s, s+1)$ and $s \in [0, \frac{1}{2}]$ and note that the case $t = s + 1$ is contained in Example 3.10. Let $\tau \in (t, \frac{3}{2}]$. We consider the special case $\psi(x) = x^2$ and write $G(x) = (x, x^2)$.

Step 1: constructing supporting sets. Let $A = [0, 1] \cap (\delta^{\frac{\tau}{3}}\mathbb{Z})$, then the δ-neighborhood of A denoted by $A(\delta)$ is a union of $\sim \delta^{-\frac{\tau}{3}}$ equally spaced open intervals of length $2\delta > 0$ with center points in arithmetic progression. We will need the explicit form:

$$A(\delta) = \{I_k \colon 1 \le k \le n,\ n \sim \delta^{-\frac{\tau}{3}}\}, \quad \text{where } I_k = (k\delta^{\frac{\tau}{3}} - \delta, k\delta^{\frac{\tau}{3}} + \delta).$$

Since $\tau \le \frac{3}{2}$, we similarly define $B = [0, 1] \cap (\delta^{\frac{2\tau}{3}}\mathbb{Z})$ and $B(\delta) = \{J_k \colon 1 \le k \le m,\ m \sim \delta^{-\frac{2\tau}{3}}\}$, where $J_k = (k\delta^{\frac{2\tau}{3}} - \delta, k\delta^{\frac{2\tau}{3}} + \delta)$. A first observation is that

$$|A(\delta) + A(\delta)|_\delta \sim \delta^{-\frac{\tau}{3}}, \quad |B(\delta) + B(\delta)|_\delta \sim \delta^{-\frac{2\tau}{3}} \quad \text{and} \quad |A(\delta) \times B(\delta)|_\delta \sim \delta^{-\tau}. \tag{3.7}$$

Since $s < \frac{\tau}{3}$, we may assume that $\delta^s = j \cdot \delta^{\frac{\tau}{3}} + \bar{\delta}$ with $0 \le \bar{\delta} < \delta^{\frac{\tau}{3}} < \delta^s \le 1$, where $j = j(s, \tau) \ge 1$ is an integer. Hence we can choose $D \subset A$ to be

$$D := \{kj\delta^{\frac{\tau}{3}}, 1 \le k \le m\}, \quad \text{where } m \sim \frac{\delta^{-\frac{\tau}{3}}}{j} = \frac{1}{\delta^s - \bar{\delta}} \sim \delta^{-s}.$$

Then $G(D(\delta)) \subset A(\delta) \times \psi(A(\delta))$ and $|G(D(\delta))|_\delta \sim \delta^{-s}$. So far we have finished the construction of supporting sets, namely, we will construct μ and σ such that $\operatorname{spt}(\mu) \subset A(\delta) \times B(\delta)$ and $\operatorname{spt}(\sigma) \subset G(D(\delta))$. Before that, let us show the covering estimate

$$|A(\delta) \times B(\delta) + G(D(\delta))|_\delta \lesssim \delta^{-\tau}. \tag{3.8}$$

To show (3.8), we first prove that

$$|\psi(A(\delta)) + B(\delta)|_\delta \sim \delta^{-\frac{2\tau}{3}}. \tag{3.9}$$

Since $k \lesssim \delta^{-\frac{\tau}{3}}$, we observe that

$$\psi(I_k) = \left(k^2\delta^{\frac{2\tau}{3}} + \delta^2 - 2k\delta^{\frac{\tau}{3}+1}, k^2\delta^{\frac{2\tau}{3}} + \delta^2 + 2k\delta^{\frac{\tau}{3}+1}\right) \subset \left(k^2\delta^{\frac{2\tau}{3}} - 3\delta, k^2\delta^{\frac{2\tau}{3}} + 3\delta\right),$$

which means $\psi(A(\delta)) \subset B(3\delta)$. Since B is an arithmetic progression, we easily get

$$\delta^{-\frac{2\tau}{3}} \lesssim |\psi(A(\delta)) + B(\delta)|_{3\delta} \le |B(3\delta) + B(3\delta)|_{3\delta} \sim \delta^{-\frac{2\tau}{3}},$$

which implies (3.9). Now (3.8) follows easily by applying (3.7) and (3.9):

$$|A(\delta) \times B(\delta) + G(D(\delta))|_\delta \le |(A(\delta) + A(\delta)) \times (\psi(A(\delta)) + B(\delta))|_\delta \lesssim \delta^{-\tau}.$$

Step 2: constructing suitable measures. Define measures

$$\mu := \frac{\mathcal{L}^2|_{A(\delta)\times B(\delta)}}{\mathcal{L}^2(A(\delta) \times B(\delta))} \quad \text{and} \quad \sigma := G_\#\left(\frac{\mathcal{L}^1|_{D(\delta)}}{\mathcal{L}^1(D(\delta))}\right),$$

where μ is the normalized Lebesgue measure on $A(\delta) \times B(\delta)$ and σ is the pushforward of normalized Lebesgue measure on $D(\delta)$. Since both A and B are arithmetic progressions, we can verify that μ is a (τ, c)-Frostman measure and σ is an (s, c)-Frostman measure for some absolute constant $c \ge 1$. Indeed, for $\delta^{\frac{\tau}{3}} \le r \le 1$ and $x \in \mathbb{R}^2$,

$$\mu(B(x, r)) \sim \delta^{\tau-2}\mathcal{L}^2(A(\delta) \times B(\delta) \cap B(x, r)) \lesssim \delta^{\tau-2} \cdot \frac{r^2}{\delta^\tau} \cdot \delta^2 \le r^\tau.$$

For $r > 1$ and $x \in \mathbb{R}^2$, we simply have $\mu(B(x, r)) \le 1 \le r^\tau$. For $\delta \le r \le \delta^{\frac{\tau}{3}}$ and $x \in \mathbb{R}^2$, $B(x, r)$ intersects $\lesssim 1$ cubes in $A(\delta) \times B(\delta)$, thus $\mu(B(x, r)) \lesssim \delta^{\tau-2} \cdot \delta^2 \le r^\tau$. For $0 < r < \delta$ and $x \in \mathbb{R}^2$, there holds

$$\mu(B(x, r)) \lesssim \delta^{\tau-2} \cdot r^2 = r^\tau \cdot \left(\frac{r}{\delta}\right)^{2-\tau} \le r^\tau.$$

Combining the four cases above, we see that μ is (τ, c)-Frostman and also $\mathrm{I}_t(\mu) \le c_1$ for some $c_1 = c_1(t, \tau) > 0$ since $t < \tau$. The same argument shows that σ is (s, c)-Frostman. Now if we define $\bar{\mu} := \frac{1}{\sqrt{c_1}}\mu$ and $\bar{\sigma} := \frac{1}{c}\sigma$, then it is easy to verify that $\bar{\mu}$ and $\bar{\sigma}$ satisfy the assumptions in Question 1 with parameters (t, s).

We claim that $\mathrm{I}_{\tau+\epsilon}(\bar{\mu} * \bar{\sigma}) \lesssim_{\psi,s,t,\tau,\epsilon} 1$ cannot be true for any $\epsilon > 0$. Otherwise, $\mathrm{I}_{\tau+\epsilon}(\bar{\mu} * \bar{\sigma}) \lesssim_{\psi,s,t,\tau,\epsilon} 1$ implies $|\mathrm{spt}(\bar{\mu} * \bar{\sigma})|_\delta \gtrsim_{\psi,s,t,\tau,\epsilon} \delta^{-\tau-\epsilon}$ (see [13, Lemma 1.4]), which contradicts (3.8) if δ is small enough, noting that $\mathrm{spt}(\bar{\mu}) \subset A(\delta) \times B(\delta)$ and $\mathrm{spt}(\bar{\sigma}) \subset G(D(\delta))$. By the arbitrariness of ϵ, we obtain $\mathfrak{f}(s, t) \le \tau$. Letting $\tau \to t$, we get $\mathfrak{f}(s, t) \le t$. Based on the fact $\mathrm{I}_t(\bar{\mu} * \bar{\sigma}) \lesssim 1$, we conclude $\mathfrak{f}(s, t) = t$ when $t \in [3s, s + 1)$ and $s \in [0, \frac{1}{2}]$.

Example 3.10. (Case $t \in [s + 1, 2)$ and $s \in [0, 1)$) Fix $s \le t - 1 < 1$. Find a compact set $A \subset [0, 1]$ such that $\dim_{\mathrm{H}}(A) = s$ and $\dim_{\mathrm{H}}(A+A) = \dim_{\mathrm{H}}(A+A+A) = t - 1 + \epsilon$, see [20, Example 4] for the construction of such set. Then we have

$$\dim_{\mathrm{H}}((A + A) \times \mathbb{R}) = t + \epsilon, \quad \dim_{\mathrm{H}}(\Gamma \cap (A \times \mathbb{R})) = s,$$

which implies

$$\dim_{\mathrm{H}}\big((A + A) \times \mathbb{R} + \Gamma \cap (A \times \mathbb{R})\big) \le \dim_{\mathrm{H}}((A + A + A) \times \mathbb{R}) = t + \epsilon.$$

Following the same argument as in Example 3.5 and recalling $\mathrm{I}_t(\mu * \sigma) \lesssim \mathrm{I}_t(\mu)$, we conclude that $\mathfrak{f}(s,t) = t$ when $t \geq s+1$.

4. Reduction to incidence estimates

In this section, we carry on a reduction process so that Theorem 1.4 is reduced to some incidence estimates.

Due to those examples in Section 3, proving Theorem 1.4 is then equivalent to proving inequality (3.3). Moreover, by Remark 1.14, to show (3.3) it suffices to show the following L^2-smoothing estimates.

Theorem 4.1. *Let $s \in [0,1]$ and $t \in (0,2)$. Let μ be a Borel measure on $B(1)$ such that $\mathrm{I}_t(\mu) < +\infty$. Assume $C_\sigma \geq 1$ and let σ be an (s, C_σ)-Frostman measure on Γ. Then*

$$\|(\mu * \sigma)_\delta\|^2_{L^2(\mathbb{R}^2)} \lesssim_{\psi,s,t,\epsilon} \max\{\mathrm{I}_t(\mu), 1\} \cdot C_\sigma^2 \cdot \delta^{\zeta(s,t)-2-\epsilon}, \quad \epsilon \in (0,1), \tag{4.2}$$

where

$$\zeta(s,t) = \begin{cases} s+t, & \text{when } t \in (0,s],\ s \in (0,1], \\ 2s+t-1, & \text{when } t \in (0,2-s],\ s \in [0,1]. \end{cases} \tag{4.3}$$

In particular, $\mathrm{I}_{\zeta(s,t)-\epsilon}(\mu * \sigma) \lesssim_{\psi,s,t,\epsilon} \max\{\mathrm{I}_t(\mu), 1\} \cdot C_\sigma^2$ *for any* $\epsilon \in (0,1)$.

Remark 4.4. (i) If $t \in [s, 2-s]$, by (4.2) we have $\mathfrak{f}(s,t) \geq 2s+t-1$, which is a first bound we can say about the intermediate case. This is an improvement over bound "t" when $s > 1/2$ but still far from the conjectured bound (1.10). When $s \in (0, \frac{1}{2}]$ and $t \in (0, 3s)$, we can also get an improved bound over "t" for $\psi(x) = x^2$. Indeed, for any $R \geq 1$ and $\epsilon \in (0,1)$, by using [15, Theorem 1.1] and Hölder inequality,

$$\begin{aligned} \int_{B(R)} |\hat{\mu}|^2 |\hat{\sigma}|^2 &\leq \left(\int_{B(R)} |\hat{\mu}|^{2q}\right)^{1/q} \left(\int_{B(R)} |\hat{\sigma}|^{2p}\right)^{1/p} \lesssim_{s,\epsilon} \left(\int_{B(R)} |\hat{\mu}|^2\right)^{2/q} \left(\int_{B(R)} |\hat{\sigma}|^p\right)^{1/p} \\ &\lesssim_t R^{(2-t)/q} \cdot R^{(2-3s)/p+\epsilon} = R^{2-(t/q+3s/p)+\epsilon}. \end{aligned}$$

One can easily see $t/q + 3s/p > t$ if $t < 3s$. Therefore, when $\psi(x) = x^2$, we already have that $\mathfrak{f}(s,t) > t$ for intermediate case.

(ii) By Plancherel identity, Theorem 4.1 implies the sharp L^4-decay of σ, that is,

$$\|(\sigma * \sigma)_\delta\|^2_{L^2(\mathbb{R}^2)} \lesssim \delta^{2s-2-\epsilon} \Longrightarrow \|\hat{\sigma}\|^4_{L^4(B(R))} \lesssim R^{2-2s+\epsilon}, \quad R \geq 1. \tag{4.5}$$

This was proved in [13, Section 3] when $\psi(x) = x^2$ by Fourier analytic method. Our proof will be simple and depends on an incidence estimate. When $s = 1$, the L^4-estimate of $\hat{\sigma}$ also follows from the sharp restriction estimate, see for example [4, Theorem 1.14].

(iii) We include (ii) because it gives a second result for the intermediate case. For any $R \geq 1$ and $\epsilon \in (0,1)$, it follows from Cauchy–Schwarz inequality and (4.5) that

$$\begin{aligned} \int_{B(R)} |\hat{\mu}|^2 |\hat{\sigma}|^2 &\leq \left(\int_{B(R)} |\hat{\mu}|^4\right)^{1/2} \left(\int_{B(R)} |\hat{\sigma}|^4\right)^{1/2} \lesssim_{\psi,s,\epsilon} \left(\int_{B(R)} |\hat{\mu}|^2\right)^{1/2} \cdot R^{(2-2s)/2+\epsilon} \\ &\lesssim_t R^{(2-t)/2} \cdot R^{(2-2s)/2+\epsilon} = R^{2-(t/2+s)+\epsilon}. \end{aligned}$$

Although $t/2 + s$ is still far from $(3s+t)/2$, it is larger than "t" or "$2s+t-1$" in some ranges.

Before the reduction process, we take the definition of δ-measures from [15, Section 6].

Definition 4.6. (δ-measure) A collection of non-negative weights $\boldsymbol{\mu} = \{\mu(p)\}_{p\in\mathcal{D}_\delta}$ with $\|\boldsymbol{\mu}\| := \sum_p \mu(p) \le 1$ is called a δ-measure. Let $C_\mu \ge 1$. We say that a δ-measure $\boldsymbol{\mu}$ is a (δ, s, C_μ)-measure if $\mu(q) \le C_\mu r^s$ for all $q \in \mathcal{D}_r$ with $r \in [\delta, 1]$. Moreover, the s-energy of $\boldsymbol{\mu}$ is defined as

$$\mathrm{I}_s(\boldsymbol{\mu}) := 1 + \sum_{p\neq q} \frac{\mu(p)\mu(q)}{\mathrm{dist}(p,q)^s},$$

where $\mathrm{dist}(p, q)$ is the distance between the midpoints of p and q.

Now it is time to make the reductions. Namely, Theorem 4.1 can be first reduced to Proposition 4.7. This reduction is standard and we will omit the proof as one can check [15, Section 6] for the details when $\psi(x) = x^2$ (note the proof generalizes easily to our case). Here we list it for the readers' convenience.

Proposition 4.7. *Let $s \in [0, 1]$ and $t \in (0, 2)$. Assume $\mathbf{C}_\mu, \mathbf{C}_\sigma \ge 1$. Let $\boldsymbol{\mu}$ be a $(\delta, t, \mathbf{C}_\mu)$-measure. For each $q \in \mathcal{D}_\delta$, let $\boldsymbol{\sigma}_q$ be a $(\delta, s, \mathbf{C}_\sigma)$-measure supported on $\{p \in \mathcal{D}_\delta \colon p \cap \Gamma_q(\delta) \neq \emptyset\}$. Let $\zeta(s,t)$ be the same as (4.3). Then for any $\epsilon \in (0, 1)$, there holds*

$$\int \Big(\sum_{q\in\mathcal{D}_\delta} \mu(q) \sum_{p\in\mathcal{D}_\delta} \boldsymbol{\sigma}_q(p) \cdot (\delta^{-2}\mathbf{1}_p) \Big)^2 dx \lesssim_{\psi,s,t,\epsilon} (\mathbf{C}_\mu \mathbf{C}_\sigma^2 \|\boldsymbol{\mu}\|) \cdot \delta^{\zeta(s,t)-2-\epsilon} + 1. \tag{4.8}$$

Then Proposition 4.7 can be reduced to Theorem 4.9. The proof of this reduction will also be skipped since it is almost the same as the proof of [15, Proposition 6.7].

Theorem 4.9. *Let $s \in [0, 1]$ and $t \in [0, 2]$ and let $A, B \ge 1$. Assume $\mathcal{P} \subset \mathcal{D}_\delta$ is a Katz–Tao (δ, t, A)-set, and for each $q \in \mathcal{P}$ there exists a Katz–Tao (δ, s, B)-set $\mathcal{F}(q) \subset \{p \in \mathcal{D}_\delta \colon p \cap \Gamma_q(\delta) \neq \emptyset\}$. Write $\mathcal{F} := \bigcup_{q\in\mathcal{P}} \mathcal{F}(q)$. Then*

$$\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim_{\psi,s,t,\epsilon} \sqrt{\delta^{-\gamma(s,t)-\epsilon} AB |\mathcal{F}||\mathcal{P}|}, \tag{4.10}$$

where

$$\gamma(s,t) = \begin{cases} s, & \text{when } t \in [0, s],\ s \in [0, 1], \\ 1, & \text{when } t \in [0, 2-s],\ s \in [0, 1]. \end{cases} \tag{4.11}$$

When $s+t \le 2$, the incidence estimate (4.10) was first proved by Fu–Ren [7, Theorem 5.2] in linear setting, then it was proved when $\psi(x) = x^2$ in [15, Theorem 6.12]. In this paper, we can actually get a $\delta^{-\epsilon}$-free version of (4.10) when $s \in [0, 1)$ and $t \in [0, 2)$ with $s + t < 2$, see Theorem 5.1.

As a consequence, we have finished the reduction process and the remaining task of this paper is to prove Theorem 4.9 in two cases, see Section 5 and Section 6.

5. Proof of Theorem 1.4 for $t \in [2 - s, s + 1]$

In this section, we will prove Theorem 1.4 when $t \in [2 - s, s + 1]$ and $s \in [\frac{1}{2}, 1]$. Thanks to the reduction in Section 4, it suffices to prove the following incidence theorem.

Theorem 5.1. *Let $s \in [0,1)$ and $t \in [0,2)$ such that $s+t<2$ and let $A, B \geq 1$. Assume $\mathcal{P} \subset \mathcal{D}_\delta$ is a Katz–Tao (δ, t, A)-set, and for each $q \in \mathcal{P}$ there exists a Katz–Tao (δ, s, B)-set $\mathcal{F}(q) \subset \{p \in \mathcal{D}_\delta \colon p \cap \Gamma_q(\delta) \neq \emptyset\}$. Write $\mathcal{F} := \bigcup_{q\in\mathcal{P}} \mathcal{F}(q)$. Then*

$$\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim_{\psi,s,t} \sqrt{\delta^{-1}AB|\mathcal{F}||\mathcal{P}|}. \tag{5.2}$$

The incidence estimate (5.2) with a $\delta^{-\epsilon}$-error is true for any $s \in [0,1]$ and $t \in [0,2]$ with $s+t \leq 2$, but we state a $\delta^{-\epsilon}$-free version. As mentioned in Subsection 1.1, a δ-incidence theorem of measures will be first established which we restate as below. The reader may recall Definition 2.8 for δ-incidences.

Theorem 5.3. *Let μ and ν both be finite Borel measures with compact support on $B(1)$. Then for any $\delta \in (0,1)$, there holds*

$$\mathcal{I}_\delta(\mu,\nu) \lesssim_{\psi,t} \delta\sqrt{\mathrm{I}_{3-t}(\mu)\mathrm{I}_t(\nu)}, \quad t \in (1,2). \tag{5.4}$$

Remark 5.5. Although Orponen [14, Theorem 2.23] has established the same result when translates of Γ are replaced by a set of distinct lines, Theorem 5.3 does not extend to the case when Γ is a straight line. This is because translates of a line may overlap entirely, leading to arbitrarily large multiplicities in the intersection counts which may violate the incidence bound (5.4). To see this, let us choose $\Gamma' = \{(x,0) : x \in [-1,1]\}$, $\mu = \mathcal{L}^2|_{B(0,1)}$ and $\nu = \mathcal{L}^1|_{[-1/2,1/2]} \times \mathcal{H}^s|_{\mathcal{C}}$, where $\mathcal{C} \subset [-1/2,1/2]$ is an s-dimensional Cantor set with $s \in (0,1)$. Let $\{y_j\}_j^n \subset [-1/2,1/2]$ be a 10δ-separated subset, hence $n \sim \delta^{-1}$. Then for each $1 \leq j \leq n$ and each $(x,y) \in [-1/2,1/2] \times [y_j - \delta/2, y_j + \delta/2] =: I_j$, we have $I_j \subset \{z \in \mathbb{R}^2 \colon \operatorname{dist}(z, \Gamma'_{(x,y)}) \leq \delta\}$ and also

$$I_j \times I_j \subset \{(p,q) \in \mathbb{R}^2 \times \mathbb{R}^2 \colon p \in \Gamma'_q(\delta)\}.$$

Since $\{I_j\}$ are disjoint by our choice for $\{y_j\}$, this implies

$$\mathcal{I}_\delta(\mu,\nu) \gtrsim \sum_j \mu(I_j) \cdot \nu(I_j) \gtrsim \delta \cdot \delta^s \cdot \delta^{-1} = \delta^s. \tag{5.6}$$

On the other hand, if Theorem 5.3 holds, then applying estimate (5.4) with $t = 1+s/2$ gives $\mathcal{I}_\delta(\mu,\nu) \lesssim_s \delta$, which contradicts (5.6) if δ is chosen small enough in terms of s.

Moreover, if Γ is only C^2 but has regions of vanishing curvature (inflection points or flat segments), then translates of Γ may locally behave like translates of a line, leading to possible clustering and breakdown of the incidence bound (5.4). Hence the curvature condition is a genuine necessity to prevent degeneracies in overlap among the translates.

5.1. Sobolev estimates. Define a measure μ by restricting $\mathcal{H}^1$ to Γ, then

$$\hat{\mu}(\xi) = \int e^{-2\pi i w\cdot\xi}\, d\mu(w) = \int_{-1}^{1} e^{-2\pi i (x,\psi(x))\cdot\xi} \sqrt{1+\psi'(x)^2}\, dx, \quad \xi \in \mathbb{R}^2.$$

We first use the method of stationary phase to obtain the decay of $|\hat{\mu}|$, see [12, Chapter 14] or [24, Chapter 6] for some similar discussions.

Lemma 5.7. *Under assumption* (1.1), $|\hat{\mu}(\xi)| \leq C(\psi)|\xi|^{-1/2}$ *for any* $\xi \in \mathbb{R}^2$.

Proof. Write $\xi = \lambda e$ for some $\lambda > 0$ and $e \in S^1$, and $\psi_e(x) = -2\pi(xe_1 + \psi(x)e_2)$. We may assume $\lambda > 1$ since the result is trivial when $\lambda \in (0,1]$. It then suffices to show

$$|\hat{\mu}(\xi)| = |\hat{\mu}(\lambda e)| = \left| \int_{-1}^{1} e^{i\lambda\psi_e(x)} \sqrt{1+\psi'(x)^2}\, dx \right| \lesssim_\psi \lambda^{-1/2}. \tag{5.8}$$

By simple calculation, we have

$$\psi_e'(x) = -2\pi(e_1 + \psi'(x)e_2), \quad \psi_e''(x) = -2\pi\psi''(x)e_2.$$

Write $c_1 := \max_{x\in[-1,1]} |\psi'(x)| > 0$. We divide the proof into two cases.

Case 1: $|e_2| < \min\{\frac{1}{2}, \frac{1}{2c_1}\}$. Then we easily see

$$|\psi_e'(x)| \geq 2\pi(|e_1| - |\psi'(x)e_2|) \geq 2\pi(\frac{\sqrt{3}-1}{2}) > 1, \quad x \in [-1,1].$$

It follows from integrating by parts

$$\begin{aligned} |\hat{\mu}(\xi)| &= \left| \int_{-1}^{1} \frac{\sqrt{1+\psi'(x)^2}}{i\lambda\psi_e'(x)} \, de^{i\lambda\psi_e(x)} \right| \\ &\leq \frac{C(\psi)}{\lambda} + C(\psi)\lambda^{-1} \int_{-1}^{1} \left| d\frac{1}{\psi_e'(x)} \right| \leq \frac{C(\psi)}{\lambda}, \quad \text{as desired.} \end{aligned}$$

Case 2: $|e_2| \geq \min\{\frac{1}{2}, \frac{1}{2c_1}\}$. Then by (1.1) we can deduce

$$\min_{x\in[-1,1]} |\psi_e''(x)| = 2\pi|e_2| \cdot \min_{x\in[-1,1]} |\psi''(x)| \geq c_2 = c_2(\psi) > 0.$$

Without loss of generality, we assume $\psi_e'' > 0$ on $[-1,1]$. Assume that ψ_e attains its minimum on $[-1,1]$ at $x_0 \in [-1,1]$, then either $\psi_e'(x_0) = 0$ or $x_0 = \pm 1$. We first consider the case $\psi_e'(x_0) = 0$. For any $\delta > 0$, we have $\psi_e'(x) > c_2\delta$ for all $x \in [-1,1]\backslash[x_0-\delta, x_0+\delta]$, which implies

$$\begin{aligned} \left| \int_{-1}^{x_0-\delta} e^{i\lambda\psi_e(x)} \sqrt{1+\psi'(x)^2} \, dx \right| &= \left| \int_{-1}^{x_0-\delta} \frac{\sqrt{1+\psi'(x)^2}}{i\lambda\psi_e'(x)} \, de^{i\lambda\psi_e(x)} \right| \\ &\leq \frac{C(\psi)}{\lambda \cdot c_2\delta} + C(\psi)\lambda^{-1} \int_{-1}^{x_0-\delta} \left| d\frac{1}{\psi_e'(x)} \right| \leq \frac{C(\psi)}{\lambda\cdot\delta}, \end{aligned}$$

where the constant $C(\psi) > 0$ may be different from line to line. A similar argument also shows that

$$\left| \int_{x_0+\delta}^{1} e^{i\lambda\psi_e(x)} \sqrt{1+\psi'(x)^2} \, dx \right| \leq \frac{C(\psi)}{\lambda\cdot\delta}.$$

Also, by continuity we have

$$\left| \int_{x_0-\delta}^{x_0+\delta} e^{i\lambda\psi_e(x)} \sqrt{1+\psi'(x)^2} dx \right| \leq C(\psi)\cdot 2\delta.$$

Combining the three inequalities above and taking $\delta = \lambda^{-1/2}$, we get

$$|\hat{\mu}(\xi)| \leq C(\psi)\lambda^{-1/2}.$$

Here we only considered the case $-1 \leq x_0 - \delta, x_0 + \delta \leq 1$ since other cases can be calculated similarly. Regarding $x_0 = \pm 1$, it also follows from the same argument. Combining the two cases above, we thus finish the whole proof of (5.8). □

To proceed, define a convolution-type operator $\mathfrak{R}$ by

(5.9) $$\mathfrak{R}f := f * \mu, \quad \forall f \in C_c^\infty(\mathbb{R}^2),$$

then

(5.10) $$\mathfrak{R}f(x,y) = \int_{\{y-t=\psi(x-s)\}} f(s,t) \, d\mathcal{H}^1(s,t).$$

Remark 5.11. Our operator is similar to the X-ray transform which maps $g \in C_c^\infty(\mathbb{R}^2)$ to a function defined on the set of all lines in $\mathbb{R}^2$:

$$(Xg)(\ell) := \int_\ell g = \int_{\pi_\theta^{-1}\{r\}} g(z)\, d\mathcal{H}^1(z), \quad (\theta, r) \in [0,1] \times \mathbb{R},$$

where $\pi_\theta(z) = z \cdot (\cos 2\pi\theta, \sin 2\pi\theta)$ for $z \in \mathbb{R}^2$. If we replace ψ by $\widetilde{\psi}(x) = -\psi(-x)$ in our definition, then the right side of (5.10) becomes

$$\int_{(x,y)+\Gamma} f(s,t)\, d\mathcal{H}^1(s,t) =: \widetilde{\mathfrak{R}} f(x,y). \tag{5.12}$$

Hence the difference is that translations of Γ are parametrized by $(x, y) \in \mathbb{R}^2$ while the lines in X-ray transform are parametrized by $(\theta, r) \in [0,1] \times \mathbb{R}$. Note that the results in this section hold for any $\psi \in C^2(\mathbb{R}^2)$ with non-zero curvature.

For $s > -1$, let $\dot{H}^s(\mathbb{R}^2)$ be the homogeneous Sobolev space. Recall that the norm in $\dot{H}^s(\mathbb{R}^2)$ is given by

$$\|f\|_{\dot{H}^s} = \left(\int |\hat{f}(\xi)|^2 |\xi|^{2s}\, d\xi \right)^{1/2}, \quad f \in C_c(\mathbb{R}^2).$$

Next, we will apply Lemma 5.7 to obtain the Sobolev estimates of $\mathfrak{R}$.

Lemma 5.13. *For any $f \in C_c^\infty(\mathbb{R}^2)$, there exists a constant $C = C(\psi) > 0$ such that*

$$\|\mathfrak{R} f\|_{L^2} \le C \|f\|_{\dot{H}^{-1/2}} \quad \text{and} \quad \|\mathfrak{R} f\|_{\dot{H}^1} \le C \|f\|_{\dot{H}^{1/2}}.$$

Proof. By Lemma 5.7, we easily deduce

$$\begin{aligned}\|\mathfrak{R} f\|_{L^2}^2 &= \int |\widehat{f * \mu}(\xi)|^2\, d\xi = \int |\hat{f}(\xi)|^2 |\hat{\mu}(\xi)|^2\, d\xi \\ &\lesssim_\psi \int |\hat{f}(\xi)|^2 |\xi|^{-1}\, d\xi = \|f\|_{\dot{H}^{-1/2}}^2,\end{aligned}$$

and

$$\begin{aligned}\|\mathfrak{R} f\|_{\dot{H}^1}^2 &= \int |\widehat{f * \mu}(\xi)|^2 |\xi|^2\, d\xi = \int |\hat{f}(\xi)|^2 |\hat{\mu}(\xi)|^2 |\xi|^2\, d\xi \\ &\lesssim_\psi \int |\hat{f}(\xi)|^2 |\xi|^1\, d\xi = \|f\|_{\dot{H}^{1/2}}^2.\end{aligned}$$

□

The following is a corollary of Lemma 5.13 by standard interpolation argument.

Corollary 5.14. *There exists a constant $C = C(\psi) > 0$ such that*

$$\|\mathfrak{R} f\|_{\dot{H}^{s+1/2}} \le C \|f\|_{\dot{H}^s}, \quad f \in C_c^\infty(\mathbb{R}^2), \ s \in [-1/2, 1/2].$$

Proof. Take $\omega_0(\xi) = |\xi|^{-1}, \omega_1(\xi) = |\xi|$ and $v_0(x) = 1$, $v_1(x) = |x|^2$. Let $\mathfrak{F}$ be the Fourier transform operator. By Lemma 5.13, the operator $(\mathfrak{F} \circ \mathfrak{R})$ extends to a bounded operator $L^2(\mathbb{R}^2, \omega_0\, d\mathcal{L}^2) \to L^2(\mathbb{R}^2, v_0\, d\mathcal{L}^2)$ and $L^2(\mathbb{R}^2, \omega_1\, d\mathcal{L}^2) \to L^2(\mathbb{R}^2, v_1\, d\mathcal{L}^2)$. The next step is to apply Stein–Weiss L^p-interpolation (see for example [1, Theorem 5.4.1]), which gives

$$\|(\mathfrak{F} \circ \mathfrak{R})(f)\|_{L^2(v_\theta)} \le C(\psi) \|f\|_{L^2(\omega_\theta)},$$

where $\omega_\theta = \omega_0^{1-\theta} \omega_1^\theta$ and $v_\theta = v_0^{1-\theta} v_1^\theta$. After rewriting the inequality and letting $s = \theta - 1/2$, we finally get

$$\|\mathfrak{R} f\|_{\dot{H}^{s+1/2}} \lesssim_\psi \|f\|_{\dot{H}^s}.$$

□

5.2. Estimating incidences. In this subsection, the Sobolev estimates developed above are applied to establish Theorem 5.3. The following lemma builds a connection between the δ-incidences and the operator $\mathfrak{R}$ defined in (5.9).

Lemma 5.15. *Let $q=(x_q,y_q)\in B(1)$ and $\delta\in(0,1)$, then there exists a constant $c=c(\psi)>0$ such that*

$$\mathbf{1}_{\Gamma_q(\delta)}(p)\le \delta^{-1}\int_{\mathbb{R}}\mathbf{1}_{B(q,c\delta)}(x,y_p-\psi(x_p-x))\,dx,\quad \forall p=(x_p,y_p)\in B(1).$$

Proof. Fix $p\in B(1)\cap\Gamma_q(\delta)$. It suffices to show $(x,y_p-\psi(x_p-x))\in B(q,c\delta)$ for some $c(\psi)>0$ whenever $|x-x_q|\le\delta$. Since $p\in\Gamma_q(\delta)$, $dist(p,\Gamma_q)\le\delta$. Then we can find some $(x_0,y_0)\in\Gamma_q$ such that $|p-(x_0,y_0)|\le\delta$. Note $y_0=y_q+\psi(x_0-x_q)$, which implies

$$|(x_p-x_0,y_p-y_q-\psi(x_0-x_q))|\le\delta.$$

By applying the triangle inequality and noting that ψ is Lipschitz on bounded intervals, we infer

$$\begin{aligned}|y_p-y_q-\psi(x_p-x)|&\le|y_p-y_q-\psi(x_0-x_q)|+|\psi(x_0-x_q)-\psi(x_p-x)|\\&\le\delta+C(\psi)|(x-x_q)+(x_0-x_p)|\le(2C(\psi)+1)\delta.\end{aligned}$$

By choosing $c>0$ properly and recalling $|x-x_q|\le\delta$, we get $(x,y_p-\psi(x_p-x))\in B(q,c\delta)$ and complete the proof. □

We are now ready to establish the δ-incidence theorem.

Proof of Theorem 5.3. We assume $\mu\in C_c^\infty(\mathbb{R}^2)$. For general case, consider the smooth approximation of μ. Let $\eta\in C_c^\infty(\mathbb{R}^2)$ satisfy $\mathbf{1}_{B(0,1/2)}\le\eta\le\mathbf{1}_{B(0,1)}$ and $\int\eta\sim1$. Write $\eta_\delta(q)=\delta^{-2}\eta(q/\delta)$. For $\delta>0$, define $\nu_\delta:=\nu*\eta_{2c\delta}\in C_c^\infty(\mathbb{R}^2)$ where $c>0$ is the constant in Lemma 5.15. Note $\nu(B(q,c\delta))\lesssim\delta^2\nu_\delta(q)$. Fix $p\in\operatorname{spt}(\mu)\subset B(1)$, then by Lemma 5.15 and Fubini Theorem,

$$\begin{aligned}\int\mathbf{1}_{\Gamma_q(\delta)}(p)\,d\nu(q)&\le\int\delta^{-1}\int\mathbf{1}_{B(q,c\delta)}(x,y_p-\psi(x_p-x))\,dx\,d\nu(q)\\&=\delta^{-1}\int\nu(B((x,y_p-\psi(x_p-x)),c\delta))\,dx\\&\lesssim\delta\int\nu_\delta(x,y_p-\psi(x_p-x))\,dx.\end{aligned}$$

By definition of $\mathcal{I}_\delta(\mu\times\nu)$, we infer from Fubini Theorem and coarea formula

$$\begin{aligned}\delta^{-1}\mathcal{I}_\delta(\mu\times\nu)&=\delta^{-1}\int\int\mathbf{1}_{\Gamma_q(\delta)}(p)\,d\nu(q)\,d\mu(p)\\&\lesssim\int_{\mathbb{R}^2}\int_{\mathbb{R}}\nu_\delta(x,y_p-\psi(x_p-x))\,dx\,d\mu(p)\\&=\int_{\mathbb{R}}\int_{\mathbb{R}^2}\nu_\delta(x,y_p-\psi(x_p-x))\,d\mu(p)\,dx\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\nu_\delta(x,y)\int_{\{y_p-y=\psi(x_p-x)\}}\frac{\mu(p)}{\sqrt{1+\psi'(x_p-x)^2}}\,d\mathcal{H}^1(p)\,dy\,dx\\&\le\int_{\mathbb{R}^2}\nu_\delta(x,y)\int_{(x,y)+\Gamma}\mu(p)\,d\mathcal{H}^1(p)\,dx\,dy=\int_{\mathbb{R}^2}\nu_\delta(x,y)\widetilde{\mathfrak{R}}\mu(x,y)\,dx\,dy.\end{aligned}$$

Recall that $\widetilde{\mathfrak{R}}$ was defined in Remark 5.11. Next, by Plancherel identity and Cauchy–Schwarz inequality, we infer

$$\begin{aligned}\delta^{-1}\mathcal{I}_\delta(\mu\times\nu) &\lesssim \int_{\mathbb{R}^2} \widehat{\nu_\delta}(\xi)\widehat{\widetilde{\mathfrak{R}}\mu}(\xi)\,d\xi \\ &\lesssim \left(\int |\widehat{\nu}(\xi)|^2|\xi|^{t-2}\,d\xi\right)^{1/2}\left(\int |\widehat{\widetilde{\mathfrak{R}}\mu}(\xi)|^2|\xi|^{2-t}\,d\xi\right)^{1/2} \\ &\sim_t \mathrm{I}_t(\nu)^{1/2}\|\widetilde{\mathfrak{R}}\mu\|_{\dot{H}^{(2-t)/2}}.\end{aligned}$$

Since $(2-t)/2 = s+1/2$ for $s=(1-t)/2$ and $t\in(1,2)$, it follows from Corollary 5.14 (with $\mathfrak{R}$ replaced by $\widetilde{\mathfrak{R}}$) that

$$\|\widetilde{\mathfrak{R}}\mu\|_{\dot{H}^{(2-t)/2}} \lesssim_\psi \|\mu\|_{\dot{H}^s} \sim_t (\mathrm{I}_{3-t}(\mu))^{1/2}.$$

Combining the above estimates finishes the proof of (5.4). □

Theorem 5.3 yields the following weighted incidence estimate under the weighted Katz–Tao condition we defined in Definition 2.4. This weighted incidence estimate is the main tool needed to prove Theorem 5.1. See Appendix A for the details of how to deduce Theorem 5.1 from Corollary 5.16.

Corollary 5.16. *Let $s,t\in[0,2)$ with $s+t<3$ and let $A,B\geq 1$. There exists $\delta_0=\delta_0(\psi)>0$ such that the following holds for any $\delta\in(0,\delta_0]$. Assume that $\mathcal{T}:=\{\Gamma_q(\delta)\colon q\in\mathcal{P}\subset\mathcal{D}_\delta\}$ is a weighted Katz–Tao (δ,t,A)-set with weight function w_1 and $\mathcal{F}\subset\mathcal{D}_\delta$ is a weighted Katz–Tao (δ,s,B)-set with weight function w_2. Then*

$$\mathcal{I}_w(\mathcal{F},\mathcal{T}) \lesssim_{\psi,s,t} \sqrt{\delta^{-1}AB\left(\sum_{q\in\mathcal{P}} w_1(q)\right)\left(\sum_{p\in\mathcal{F}} w_2(p)\right)}. \tag{5.17}$$

Proof. Choose $u\in(s,2)$ and $v\in(t,2)$ such that $u+v=3$. Define two measures by

$$\nu := \delta^{t-2}\sum_{q\in\mathcal{P}} w_1(q)\mathbf{1}_q, \quad \mu := \delta^{s-2}\sum_{p\in\mathcal{F}} w_2(p)\mathbf{1}_p.$$

Then $\mathrm{spt}(\nu)\subset\bigcup\mathcal{P}$ and $\mathrm{spt}(\mu)\subset\bigcup\mathcal{F}$. Also, we have the useful estimates

$$\nu(q)\sim\delta^t w_1(q) \text{ if } q\in\mathcal{P}, \quad \mu(p)\sim\delta^s w_2(p) \text{ if } p\in\mathcal{F}. \tag{5.18}$$

By definition of weighted incidences and (5.18), we can deduce that

$$\begin{aligned}\mathcal{I}_w(\mathcal{F},\mathcal{T}) &\sim \delta^{-s-t}\sum_{q\in\mathcal{P}}\sum_{p\in\mathcal{F}} \nu(q)\mu(p)\mathbf{1}_{\{p\cap\Gamma_q(\delta)\neq\emptyset\}} \\ &= \delta^{-s-t}\mu\times\nu(\{(p,q)\in\mathcal{F}\times\mathcal{P}\colon p\cap\Gamma_q(\delta)\neq\emptyset\}) \\ &\leq \delta^{-s-t}\mu\times\nu(\{(x,y)\in\mathbb{R}^4\colon x\in\Gamma_y(C\delta)\}) = \delta^{-s-t}\mathcal{I}_{C\delta}(\mu\times\nu).\end{aligned} \tag{5.19}$$

Here we used the fact that there exists $C=C(\psi)>0$ such that

$$\{(p,q)\in\mathcal{F}\times\mathcal{P}\colon p\cap\Gamma_q(\delta)\neq\emptyset\}\subset\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2\colon x\in\Gamma_y(C\delta)\}.$$

Note that we choose $\delta_0>0$ small such that $C\delta\in(0,1)$ for any $\delta\in(0,\delta_0]$.

To apply Theorem 5.3, we only need to calculate the Riesz energies of ν and μ. First,

$$\begin{aligned}
\mathrm{I}_v(\nu) &= \sum_{q\in\mathcal{P}} \sum_{\substack{p\in\mathcal{P}\\ \mathrm{dist}(p,q)=0}} \int_{p\times q} \frac{d\nu(x)\,d\nu(y)}{|x-y|^v} + \sum_{\substack{p,q\in\mathcal{P}\\ \mathrm{dist}(p,q)\in[\delta,1]}} \int_{p\times q} \frac{d\nu(x)\,d\nu(y)}{|x-y|^v}\\
&\lesssim \sum_{q\in\mathcal{P}} (\delta^{t-2})^2 w_1(q) A\cdot 9 \int_{3p\times 3p} \frac{dx\,dy}{|x-y|^v} + \sum_{q\in\mathcal{P}} \sum_{j=1}^{\log\frac{1}{\delta}} \sum_{\substack{\mathrm{dist}(p,q)\\ \in[2^{-j},2^{-j+1})}} \int_{p\times q} \frac{d\nu(x)\,d\nu(y)}{|x-y|^v}\\
&\lesssim A\sum_{q\in\mathcal{P}} w_1(q)(\delta^{t-2})^2\cdot(\delta^{4-v}) + \sum_{q\in\mathcal{P}} \sum_{j=1}^{\log\frac{1}{\delta}} \sum_{\substack{\mathrm{dist}(p,q)\\ \in[2^{-j},2^{-j+1})}} 2^{jv}\nu(p)\nu(q),
\end{aligned}$$

where "$3p$" denotes the cube of side length 3δ with the same center as p. By using the weighted Katz–Tao condition of $\mathcal{P}$, then

$$\begin{aligned}
\mathrm{I}_v(\nu) &\overset{(5.18)}{\lesssim} A\delta^{2t-v} \sum_{q\in\mathcal{P}} w_1(q) + \sum_{q\in\mathcal{P}} w_1(q) \sum_{j=1}^{\log\frac{1}{\delta}} 2^{jv} \left(\sum_{p\in\mathcal{P}\cap Q_j} w_1(p) \right) \delta^{2t}\\
&\lesssim_t A\delta^{2t-v} \sum_{q\in\mathcal{P}} w_1(q),
\end{aligned}$$

where Q_j is a dyadic cube with side length $\sim 2^{-j}$ and we also used $t-v<0$. Following the same calculation, we can obtain $\mathrm{I}_u(\mu) \lesssim_s B\delta^{2s-u}\sum_{p\in\mathcal{F}} w_2(p)$.

Consequently, we infer by Theorem 5.3 that (note $v>1$ and $u+v=3$)

$$\begin{aligned}
\mathcal{I}_w(\mathcal{F},\mathcal{T}) &\overset{(5.19)}{\lesssim} \delta^{-s-t}\mathcal{I}_{C\delta}(\mu,\nu) \overset{(5.4)}{\lesssim}_{\psi,v} \delta^{-s-t+1}\sqrt{\mathrm{I}_v(\nu)\mathrm{I}_u(\mu)}\\
&\lesssim_{\psi,s,t} \delta^{-s-t+1} \sqrt{A\delta^{2t-v}\sum_{q\in\mathcal{P}} w_1(q) B\delta^{2s-u} \sum_{p\in\mathcal{F}} w_2(p)}\\
&= \sqrt{\delta^{-1}AB\sum_{q\in\mathcal{P}} w_1(q) \sum_{p\in\mathcal{F}} w_2(p)},
\end{aligned}$$

which completes the proof of (5.17). □

Remark 5.20. When $s+t=3$ or $s=2$ or $t=2$, we use the fact that $\mathcal{P}$ is a weighted Katz–Tao $(\delta, t-\epsilon, A\delta^{-\epsilon})$-set and $\mathcal{T}$ is a weighted Katz–Tao $(\delta, s-\epsilon, B\delta^{-\epsilon})$-set for any $\epsilon\in(0,1)$. Thus an incidence estimate with $\delta^{-\epsilon}$-error can be obtained by using (5.17).

6. Proof of Theorem 1.4 for $t\in(0,s]$

In this section, we prove the following proposition which implies Theorem 1.4 when $t\in(0,s]$ due to the reduction in Section 4. A subtle difference here is that the implicit constant in estimate (6.2) is irrelevant to parameter "t". This is because the proof of Proposition 6.1 uses the fact that $\mathcal{P}$ is also a Katz–Tao (δ,s)-set if $t\le s$.

Proposition 6.1. *Let $0\le t\le s\le 1$ and $A,B\ge 1$. Let $\mathcal{P}\subset\mathcal{D}_\delta$ be a Katz–Tao (δ,t,A)-set. For each $q\in\mathcal{P}$, assume that there exists a Katz–Tao (δ,s,B)-set*

$$\mathcal{F}(q)\subset\{p\in\mathcal{D}_\delta\colon p\cap\Gamma_q(\delta)\neq\emptyset\}.$$

Write $\mathcal{F} := \bigcup_{q\in\mathcal{P}} \mathcal{F}(q)$*. Then there holds*

$$\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim_{\psi,s} \log(\tfrac{1}{\delta})\sqrt{AB\delta^{-s}|\mathcal{P}||\mathcal{F}|}. \tag{6.2}$$

Proof. First, applying Cauchy–Schwarz inequality gives

$$\begin{aligned}\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| &= \sum_{p\in\mathcal{F}} |\{q\in\mathcal{P}\colon p\in\mathcal{F}(q)\}| \\ &\le |\mathcal{F}|^{1/2}\left(\sum_{p\in\mathcal{F}} |\{(q_1,q_2)\in\mathcal{P}\times\mathcal{P}\colon p\in\mathcal{F}(q_1)\cap\mathcal{F}(q_2)\}|\right)^{1/2} \\ &= |\mathcal{F}|^{1/2}\left(\sum_{q}|\mathcal{F}(q)| + \sum_{q_1\neq q_2}|\mathcal{F}(q_1)\cap\mathcal{F}(q_2)|\right)^{1/2}.\end{aligned}$$

If the first sum dominates, we get by the basic fact $|\mathcal{F}| \le B\delta^{-s}|\mathcal{P}|$ that

$$\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim |\mathcal{F}| = \sqrt{|\mathcal{F}||\mathcal{F}|} \le \sqrt{B\delta^{-s}|\mathcal{P}||\mathcal{F}|}.$$

If the "off-diagonal" sum dominates, we have

$$\begin{aligned}\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| &\lesssim |\mathcal{F}|^{1/2}\left(\sum_{q_1\neq q_2}|\mathcal{F}(q_1)\cap\mathcal{F}(q_2)|\right)^{1/2} \\ &= |\mathcal{F}|^{1/2}\left(\sum_{\substack{q_1\neq q_2\\ x_1=x_2}}|\mathcal{F}(q_1)\cap\mathcal{F}(q_2)| + \sum_{\substack{q_1\neq q_2\\ x_1\neq x_2}}|\mathcal{F}(q_1)\cap\mathcal{F}(q_2)|\right)^{1/2}.\end{aligned} \tag{6.3}$$

From now on, we use (x_i,y_i) to denote the center points of $q_i\in\mathcal{P}$. To estimate the first sum, we claim that $|y_1-y_2| \lesssim_\psi \delta$ if $\Gamma_{q_1}(\delta)\cap\Gamma_{q_2}(\delta)\neq\emptyset$ with $q_1\neq q_2$ and $x_1=x_2$. Write $\psi_i(x) := \psi(x-x_i)+y_i$ with $i=1,2$. For any $(x,y)\in\Gamma_{q_1}(\delta)\cap\Gamma_{q_2}(\delta)$, there exist $(z_i,w_i)\in\Gamma_{q_i}$ such that

$$|(x,y)-(z_i,w_i)| \le \delta, \quad i=1,\ 2.$$

Then by triangle inequality we infer that

$$\begin{aligned}|y_1-y_2| &= |\psi_1(x)-\psi_2(x)| \\ &\le |\psi_1(x)-\psi_1(z_1)| + |\psi_1(z_1)-\psi_2(z_2)| + |\psi_2(x)-\psi_2(z_2)| \\ &\le C(\psi)|x-z_1| + |w_1-w_2| + C(\psi)|x-z_2| \lesssim_\psi \delta.\end{aligned} \tag{6.4}$$

This means for a fixed q_1 there are at most $C(\psi)$ cubes q_2 such that $x_1=x_2$ and $\mathcal{F}(q_1)\cap\mathcal{F}(q_2)\neq\emptyset$. Hence we can deduce

$$\sum_{\substack{q_1\neq q_2\\ x_1=x_2}}|\mathcal{F}(q_1)\cap\mathcal{F}(q_2)| = \sum_{q_1}\sum_{\substack{q_2\neq q_1\\ x_2=x_1}}|\mathcal{F}(q_1)\cap\mathcal{F}(q_2)| \lesssim_\psi B\delta^{-s}|\mathcal{P}|, \tag{6.5}$$

using the fact $|\mathcal{F}(q)| \lesssim B\delta^{-s}$ by the Katz–Tao (δ,s,B)-condition of $\mathcal{F}(q)$.

It remains to deal with the second sum on the right side of (6.3). To get a good estimate for $|\mathcal{F}(q_1)\cap\mathcal{F}(q_2)|$, we need to study the geometry of $\Gamma_{q_1}(\delta)\cap\Gamma_{q_2}(\delta)$ when $x_1\neq x_2$. Recall that the translated graph is given by

$$\Gamma_{q_i} = \{(x,y)\colon y-y_i = \psi(x-x_i),\ x\in[-1,1]\}.$$

Take $(x,y)\in\Gamma_{q_1}\cap\Gamma_{q_2}$, then (x,y) solves the equations

$$\begin{cases} y-y_1=\psi(x-x_1),\\ y-y_2=\psi(x-x_2). \end{cases}$$

By using the mean value theorem, we can obtain

$$|y_1-y_2|=|\psi(x-x_1)-\psi(x-x_2)|\lesssim_\psi |x_1-x_2|. \tag{6.6}$$

Our main claim is that

$$\operatorname{diam}(\Gamma_{q_1}(\delta)\cap\Gamma_{q_2}(\delta))\lesssim_\psi \frac{\delta}{|x_1-x_2|}. \tag{6.7}$$

This will conclude the whole proof. Indeed, (6.7) implies that $\mathcal{F}(q_1)\cap\mathcal{F}(q_2)$ is contained in a ball with radius $\sim_\psi \frac{\delta}{|x_1-x_2|}$. Since $t\le s$, $\mathcal{P}$ is also a Katz–Tao (δ,s,A)-set. By using (6.6), (6.7) and the Katz–Tao (δ,s)-conditions of both $\mathcal{F}(q)$ and $\mathcal{P}$, we compute

$$\begin{aligned}\sum_{\substack{q_1\neq q_2\\ x_1\neq x_2}}|\mathcal{F}(p_1)\cap\mathcal{F}(p_2)| &\lesssim_\psi \sum_{\substack{q_1\neq q_2\\ x_1\neq x_2}}\frac{B}{|x_1-x_2|^s}\sim B\sum_{q_1\in\mathcal{P}}\sum_{k=1}^{\log(\frac{1}{\delta})}\sum_{\substack{q_2\in\mathcal{P}\\ |x_1-x_2|\in[2^{-k},2^{1-k})}}\frac{1}{|x_1-x_2|^s}\\ &\lesssim B\sum_{q_1\in\mathcal{P}}\sum_{k=1}^{\log(\frac{1}{\delta})}2^{ks}\left|\left\{q_2\colon |y_1-y_2|\lesssim_\psi |x_1-x_2|\in[2^{-k},2^{1-k})\right\}\right|\\ &\lesssim B\sum_{q_1\in\mathcal{P}}\sum_{k=1}^{\log(\frac{1}{\delta})}2^{ks}|\mathcal{P}\cap B_{c(\psi)2^{-k}}|\lesssim_{\psi,s}\log(\tfrac{1}{\delta})AB\delta^{-s}|\mathcal{P}|,\end{aligned}$$

as desired.

It remains to show (6.7). Let π_1 be the projection onto x-axis and recall $\psi_i(x):=\psi(x-x_i)+y_i$ with $i=1,2$. As an intermediate goal, we show that

$$\pi_1(\Gamma_{q_1}(\delta)\cap\Gamma_{q_2}(\delta))\subset\{x\colon |\psi_1(x)-\psi_2(x)|\lesssim_\psi\delta\}. \tag{6.8}$$

For any $(x,y)\in\Gamma_{q_1}(\delta)\cap\Gamma_{q_2}(\delta)$, there exist $(z_i,w_i)\in\Gamma_i$ such that

$$|(x,y)-(z_i,w_i)|\le\delta,\quad i=1,\ 2.$$

Then by using triangle inequality as (6.4) we can get $|\psi_1(x)-\psi_2(x)|\lesssim_\psi\delta$, which proves (6.8). If we write $G(x)=\psi_1(x)-\psi_2(x)$, then by assumption (1.1)

$$|G'(x)|=|\psi_1'(x)-\psi_2'(x)|=|\psi'(x-x_1)-\psi'(x-x_2)|\gtrsim_\psi |x_1-x_2|. \tag{6.9}$$

As a consequence, (6.7) follows by combining (6.8) and (6.9). This completes the proof of Proposition 6.1. □

Appendix A. An incidence estimate

In this appendix, we sketch the proof of Theorem 5.1 which we restate as follows. This is a modification of the proof of [7, Theorem 5.2], but for completeness we include here. Again, by using Remark 5.20, the estimate (A.2) with a $\delta^{-\epsilon}$-error also holds for any $s\in[0,1]$ and $t\in[0,2]$ with $s+t\le 2$.

Theorem A.1. *Let $s \in [0,1)$ and $t \in [0,2)$ such that $s+t<2$ and let $A,B \geq 1$. Assume $\mathcal{P} \subset \mathcal{D}_\delta$ is a Katz–Tao (δ,t,A)-set, and for each $q \in \mathcal{P}$ there exists a Katz–Tao (δ,s,B)-set $\mathcal{F}(q) \subset \{p \in \mathcal{D}_\delta : p \cap \Gamma_q(\delta) \neq \emptyset\}$. Write $\mathcal{F} := \bigcup_{q\in\mathcal{P}} \mathcal{F}(q)$. Then*

$$\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim_{\psi,s,t} \sqrt{\delta^{-1}AB|\mathcal{F}||\mathcal{P}|}. \tag{A.2}$$

Proof. Step 1: initial reduction. Since ψ is strictly convex, Γ can be divided into the decreasing part Γ^- and the increasing part Γ^+. For each $q \in \mathcal{P}$, let $\Gamma_q^-(\delta)$ and $\Gamma_q^+(\delta)$ be the δ-neighborhood of $\Gamma_q^- := q+\Gamma^-$ and $\Gamma_q^+ := q+\Gamma^+$ respectively, then it is easy to see

$$\Gamma_q(\delta) = \Gamma_q^-(\delta) \cup \Gamma_q^+(\delta).$$

Moreover, we define

$$\mathcal{F}^-(q) := \{p \in \mathcal{F}(q) : p \cap \Gamma_q^-(\delta) \neq \emptyset\}, \quad \mathcal{F}^+(q) := \{p \in \mathcal{F}(q) : p \cap \Gamma_q^+(\delta) \neq \emptyset\},$$

then either $\sum_q |\mathcal{F}(q)| \sim \sum_q |\mathcal{F}^-(q)|$ or $\sum_q |\mathcal{F}(q)| \sim \sum_q |\mathcal{F}^+(q)|$. Without loss of generality, we may assume the latter case and denote $\Gamma_q^+(\delta)$ and $\mathcal{F}^+(q)$ still by $\Gamma_q(\delta)$ and $\mathcal{F}(q)$ respectively.

Moreover, we divide $\mathcal{F}$ into four sub-families, say $\mathcal{F}_{i,j}$ with $i,j \in \{0,1\}$, where $\mathcal{F}_{i,j}$ is the collection of δ-cubes in $\mathcal{F}$ with upper-right vertex $(m\delta, n\delta)$ satisfying $m \equiv i, n \equiv j \pmod 2$. By translating the configuration if necessary, we may assume that every dyadic δ-cube in $\mathcal{F}$ has upper-right vertex in $(\delta(2\mathbb{Z}+1))^2$.

Step 2: constructing less concentrated pockets. Fix a dyadic number $\omega \in [\delta,1]$, we aim to construct a set of dyadic δ-cubes $\mathcal{F}_\omega$ contained in $[0,\omega]^2$. Let $\mathcal{C}$ be a standard s-dimensional Cantor set on $[0,\omega]$ and let $\mathcal{C}(\delta)$ be its δ-neighborhood. Define $\mathcal{F}_\omega$ as the set of dyadic δ-cubes with upper-right vertices $(m\delta, n\delta)$, where $m,n \in \mathbb{Z}$, $1 \leq m,n \lesssim \frac{\omega}{\delta}$ and either $m\delta$ or $n\delta$ belong to $\mathcal{C}(\delta)$. An easy observation shows that $|\mathcal{F}_\omega| \lesssim (\frac{\omega}{\delta})^{s+1}$ and for any positive number $d \in [\delta,\omega]$ we have:

$$|\{m : m\delta \in \mathcal{C}(\delta) \cap [0,d]\}| \gtrsim (\tfrac{d}{\delta})^s. \tag{A.3}$$

Step 3: fixing over-concentrated pockets. We want to replace the over-concentrated pockets in $\mathcal{F}$ by less concentrated pockets constructed in Step 2 so that $\mathcal{F}$ will be replaced by a weighted Katz–Tao $(\delta, s+1, O(B))$-set $\mathcal{F}'$. Fix $\omega \in [\delta,1] \cap 2^{-\mathbb{N}}$. Let $\mathcal{W}$ be the set of cubes in the family $\bigcup_{\delta\leq\omega\leq 1} \mathcal{D}_\omega$ that contain $\geq B(\frac{\omega}{\delta})^{1+s}$ cubes in $\mathcal{F}$. Let $\mathcal{R}$ be the maximal elements of $\mathcal{W}$, which means any $p \in \mathcal{R}$ cannot be contained in other cubes in $\mathcal{W}$. It is clear that $\mathcal{R}$ is a disjoint family of dyadic cubes. Before constructing $\mathcal{F}'$, we need to verify the following technical lemma.

Sublemma A.4. *Let $\mathcal{F}_\omega$ be the set constructed in Step 2. For any $q \in \mathcal{P}$, we have*

$$B|\{p \in \mathcal{F}_\omega : p \cap \Gamma_q(\delta) \neq \emptyset\}| \gtrsim_\psi |\mathcal{F}(q) \cap [0,\omega]^2|. \tag{A.5}$$

Proof. Fix $q \in \mathcal{P}$, we may assume that there exists $p \in \mathcal{F}(q) \cap [0,\omega]^2 \neq \emptyset$. Let d be the length of the projection of $\Gamma_q(\delta) \cap [0,\omega]^2$ onto x-axis. Note that this projection is a consecutive interval due to the reduction in Step 1. We also assume that $\Gamma_q(\delta)$ intersects the left or right edge of $[0,\omega]^2$. Otherwise, if $\Gamma_q(\delta)$ intersects the bottom or top edge of $[0,\omega]^2$ we can instead consider projection of $\Gamma_q(\delta) \cap [0,\omega]^2$ onto y-axis.

If $d \geq \delta$, then $\Gamma_q(\delta)$ intersects at least one cube in $\mathcal{F}_\omega$ with upper-right vertex $(m\delta, n\delta)$ for each $m\delta \in \mathcal{C}(\delta) \cap [0,d]$ (if $\Gamma_q(\delta)$ intersects the left edge) or $m\delta \in \mathcal{C}(\delta) \cap [\omega-d,\omega]$ (if $\Gamma_q(\delta)$ intersects the right edge). Thus we deduce by (A.3)

$$|\{p \in \mathcal{F}_\omega : p \cap \Gamma_q(\delta) \neq \emptyset\}| \gtrsim (\tfrac{d}{\delta})^s.$$

If $d < \delta$ and $\Gamma_q(\delta)$ intersects the left edge of $[0,\omega]^2$, we claim that the upper-right vertex (x_p, y_p) of p equals $(\delta, n\delta)$. Otherwise $x_p \geq 3\delta$ since (x_p, y_p) lies in $(\delta(2\mathbb{Z}+1))^2$. But this will cause $d > 2\delta$, which is a contradiction. If $\Gamma_q(\delta)$ intersects the right edge of $[0,\omega]^2$, the same argument shows that (x_p, y_p) equals $(\omega-\delta, n\delta)$ where $\omega - \delta \in \mathcal{C}(\delta) \cap \delta\mathbb{Z}$. Combining the two cases gives

$$|\{p \in \mathcal{F}_\omega \colon p \cap \Gamma_q(\delta) \neq \emptyset\}| \gtrsim \max\{1, (\tfrac{d}{\delta})^s\}.$$

On the other hand, by assumption (1.1) and simple geometric argument, we see that $\Gamma_q(\delta) \cap [0,\omega]^2$ is contained in a ball with radius $\sim_\psi d$ and $\mathcal{F}(q) \cap [0,\omega]^2$ is contained in a ball with radius $\sim_\psi (d+\delta)$. Since $\mathcal{F}(q)$ is a Katz–Tao (δ, s, B)-set, we infer

$$|\mathcal{F}(q) \cap [0,\omega]^2| \lesssim_\psi B(\tfrac{d+\delta}{\delta})^s \lesssim B\max\{1, (\tfrac{d}{\delta})^s\} \lesssim B|\{p \in \mathcal{F}_\omega \colon p \cap \Gamma_q(\delta) \neq \emptyset\}|. \quad \square$$

We are now ready to construct $\mathcal{F}'$. For each $Q \in \mathcal{R} \cap \mathcal{D}_\omega$, let $\mathcal{F}'(Q)$ be a translation of $\mathcal{F}_\omega$ placed in Q, then define

$$\mathcal{F}' := \{p \in \mathcal{F} \colon p \not\subset \cup\mathcal{R}\} \cup \bigsqcup_{Q \in \mathcal{R}} \mathcal{F}'(Q).$$

Next, we associate $\mathcal{F}'$ with a weight function

$$w(p) := \begin{cases} B, & \text{if } p \in \bigsqcup_{Q\in\mathcal{R}} \mathcal{F}'(Q), \\ 1, & \text{if } p \in \{p \in \mathcal{F} \colon p \not\subset \cup\mathcal{R}\}. \end{cases}$$

Let $\mathcal{T} := \{\Gamma_q(\delta) \colon q \in \mathcal{P}\}$. Then the following properties are easy to verify:

(P1) $\sum_{p\in\mathcal{F}'} w(p) \lesssim |\mathcal{F}|$ and $\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim \mathcal{I}_w(\mathcal{F}', \mathcal{T})$;

(P2) $\mathcal{F}'$ is a weighted Katz–Tao $(\delta, s+1, O(B))$-set.

Indeed, note (A.5) holds if $[0,\omega]^2$ is replaced by any $Q \in \mathcal{D}_\omega$ and $\mathcal{F}_\omega$ is replaced by $\mathcal{F}'(Q)$, then for any $q \in \mathcal{P}$ and $Q \in \mathcal{R}$ we have

$$\sum_{p \in \mathcal{F}'(Q) \cap \Gamma_q(\delta)} w(p) = B|\{p \in \mathcal{F}'(Q) \colon p \cap \Gamma_q(\delta) \neq \emptyset\}| \gtrsim_\psi |\mathcal{F}(q) \cap Q|,$$

which implies $\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim \mathcal{I}_w(\mathcal{F}', \mathcal{T})$. Moreover, from Step 2 we know

$$\sum_{p\in\mathcal{F}'(Q)} w(p) \lesssim B(\tfrac{\omega}{\delta})^{1+s} \lesssim |\mathcal{F} \cap Q|$$

for each $Q \in \mathcal{R}$, thus $\sum_{p\in\mathcal{F}'} w(p) \lesssim |\mathcal{F}|$ and proves (P1). Property (P2) is clear by definition of $\mathcal{F}'$. Finally, we apply Corollary 5.16 to get (recall $s+1+t < 3$)

$$\sum_{q\in\mathcal{P}} |\mathcal{F}(q)| \lesssim \mathcal{I}_w(\mathcal{F}', \mathcal{T}) \lesssim_{\psi,s,t} \sqrt{\delta^{-1}AB|\mathcal{T}| \sum_{p\in\mathcal{F}'} w(p)} \lesssim \sqrt{\delta^{-1}AB|\mathcal{P}||\mathcal{F}|}. \qquad \square$$

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