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Two footnotes to the F. & M. Riesz theorem

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Abstract. We present a new proof of the F. & M. Riesz theorem on analytic measures of the unit circle $\mathbb{T}$ that is based the following elementary inequality: If f is analytic in the unit disc $\mathbb{D}$ and $0 \leq r \leq \varrho < 1$, then

$$\|f_r - f_\varrho\|_1 \leq 2\sqrt{\|f_\varrho\|_1^2 - \|f_r\|_1^2},$$

where $f_r(e^{i\theta}) = f(re^{i\theta})$ and where $\|\cdot\|_1$ denotes the norm of $L^1(\mathbb{T})$. The proof extends to the infinite-dimensional torus $\mathbb{T}^\infty$, where it clarifies the relationship between Hilbert's criterion for $H^1(\mathbb{T}^\infty)$ and the F. & M. Riesz theorem.

Kaksi alaviitettä F. ja M. Rieszin lauseeseen

Tiivistelmä. Työssä esitetään yksikköympyrän $\mathbb{T}$ analyyttisiä mittoja koskevalle F. ja M. Rieszin lauseelle uusi todistus, joka perustuu seuraavaan alkeelliseen epäyhtälöön: Jos f on analyyttinen yksikkökiekossa $\mathbb{D}$ ja $0 \leq r \leq \varrho < 1$, niin

$$\|f_r - f_\varrho\|_1 \leq 2\sqrt{\|f_\varrho\|_1^2 - \|f_r\|_1^2},$$

missä $f_r(e^{i\theta}) = f(re^{i\theta})$ ja merkintä $\|\cdot\|_1$ tarkoittaa avaruuden $L^1(\mathbb{T})$ normia. Todistus yleistyy ääretönulotteiseen rengaspintaan $\mathbb{T}^\infty$, missä se selkeyttää avaruuden $H^1(\mathbb{T}^\infty)$ Hilbertin ehdon sekä F. ja M. Rieszin lauseen välistä yhteyttä.

1. Introduction

A finite complex Borel measure μ on the unit circle $\mathbb{T}$ is uniquely determined by the Fourier coefficients

$$\widehat{\mu}(k) = \int_0^{2\pi} e^{-ik\theta}\, d\mu(e^{i\theta}),$$

for k in $\mathbb{Z}$. This assertion is a consequence of the fact that trigonometric polynomials are dense in $C(\mathbb{T})$ and duality in form of the Riesz representation theorem. The protagonist of the present note is the following well-known result due to F. & M. Riesz (see e.g. [12, pp. 195–212]) on analytic measures of the unit circle.

Theorem 1. *If μ is a finite complex Borel measure on $\mathbb{T}$ that satisfies $\widehat{\mu}(k) = 0$ for $k < 0$, then μ is absolutely continuous.*

There are several proofs of Theorem 1 of rather distinct flavor. The original proof of F. & M. Riesz relies on approximation (as does the short proof of Øksendal [11]), while the modern proofs use either Hilbert space techniques or the Poisson kernel. Should the reader desire a side-by-side comparison, we refer to the monograph of Koosis [9] that contains all three variants.

Our first footnote concerns a simplification to the proof based on the Poisson kernel, so let us recall the setup. The assumptions of Theorem 1 ensure that the

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Poisson extension

$$\mathfrak{P}\mu(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu(e^{i\theta})$$

is analytic (whence μ is an "analytic" measure) in the unit disc $\mathbb{D}$, since it can be represented by an absolutely convergent power series at the origin. We also get from Fubini's theorem that

$$\int_0^{2\pi} \left|\mathfrak{P}\mu(re^{i\theta})\right| \frac{d\theta}{2\pi} \leq \|\mu\|$$

for every $0 \leq r < 1$, where $\|\mu\|$ denotes the total variation of μ.

In combination, these two assertions show that the function $f = \mathfrak{P}\mu$ is in the Hardy space $H^1(\mathbb{D})$. Let us define $f_r(e^{i\theta}) = f(re^{i\theta})$ for $0 \leq r < 1$. The last step in the proof of Theorem 1 is to show that there is a function f^* in $L^1(\mathbb{T})$ such that $\|f^* - f_r\|_1 \to 0$ as $r \to 1^-$. It would follow from this that $f^* = \mu$, since they have the same Fourier coefficients. This is where our proof diverges from the standard proofs, that first use Fatou's theorem to define f^* as the boundary value function of f and then establish that f_r converges in norm to f^*. We will instead use the following result, which in particular means that Fatou's theorem is not required.

Lemma 2. *If f is analytic in $\mathbb{D}$ and $0 \leq r \leq \varrho < 1$, then*

$$\int_0^{2\pi} \left|f(re^{i\theta}) - f(\varrho e^{i\theta})\right| \frac{d\theta}{2\pi} \leq 2\sqrt{\left(\int_0^{2\pi} \left|f(\varrho e^{i\theta})\right| \frac{d\theta}{2\pi}\right)^2 - \left(\int_0^{2\pi} \left|f(re^{i\theta})\right| \frac{d\theta}{2\pi}\right)^2}.$$

Theorem 1 now follows at once. Lemma 2 shows that if f is in $H^1(\mathbb{D})$, then any sequence of functions f_r with $r \to 1^-$ forms a Cauchy sequence in $L^1(\mathbb{T})$. From this point of view, Lemma 2 should be considered a quantitative version of the qualitative assertion that $\|f^* - f_r\|_1 \to 0$ as $r \to 1^-$.

The proof of Lemma 2 is elementary: it uses only finite Blaschke products, the triangle inequality, the Cauchy–Schwarz inequality, and orthogonality. It inspired by a result of Kulikov [10, Lemma 2.1] that essentially corresponds to the case $r = 0$.

It would be interesting to know what the best constant C in the estimate appearing Lemma 2 is. Our result is that $C \leq 2$. Choosing $f(z) = 1 + \varepsilon z$ and $r = 0$, then letting $\varepsilon \to 0^+$ shows that $C \geq \sqrt{2}$. It can be extracted from the proof of the main result in [4] that $C = \sqrt{2}$ is the best constant for $r = 0$. A related problem of interest is to establish versions of Lemma 2 where $L^p(\mathbb{T})$ takes the place of $L^1(\mathbb{T})$.

Lemma 2 also contains the fact that the radial means $r \mapsto \|f_r\|_1$ are increasing. From an historical point of view, let us recall that this answers the question posed by Bohr and Landau to Hardy [6], which led to the paper that is considered to mark the starting point of the theory. Lemma 2 provides a simpler proof of this fact, which is typically established using convexity. However, the standard proofs yield the stronger assertion that $\log r \mapsto \log \|f_r\|_1$ is convex for $0 < r < 1$.

Our second footnote concerns the (countably) infinite-dimensional torus

$$\mathbb{T}^\infty = \mathbb{T} \times \mathbb{T} \times \mathbb{T} \times \cdots,$$

that forms a compact abelian group under multiplication. Its dual group is $\mathbb{Z}^{(\infty)}$, the collection of compactly supported integer-valued sequences, and its normalized Haar measure m_∞ coincides with the infinite product measure generated by the normalized Lebesgue arc length measure on $\mathbb{T}$.

The spaces $L^p(\mathbb{T}^\infty)$ contain a natural chain of subspaces that can be identified with $L^p(\mathbb{T}^d)$ for $d = 1, 2, 3, \ldots$ and *die Abschnitte* $\mathfrak{A}_d$ define bounded linear operators on $L^p(\mathbb{T}^\infty)$ that satisfy $\|\mathfrak{A}_1 f\|_p \leq \|\mathfrak{A}_2 f\|_p \leq \|\mathfrak{A}_3 f\|_p \leq \cdots \leq \|f\|_p$ for f in $L^p(\mathbb{T}^\infty)$.

It follows from this that if f is a function in $L^p(\mathbb{T}^\infty)$ and $f_d = \mathfrak{A}_d f$, then $(f_d)_{d\geq 1}$ is a bounded sequence in $L^p(\mathbb{T}^\infty)$ that enjoys the *chain property*

$$\mathfrak{A}_d f_{d+1} = f_d$$

for $d = 1, 2, 3, \ldots$. The following fundamental questions arise naturally.

(i) If f is a function in $L^p(\mathbb{T}^\infty)$, then how does $\mathfrak{A}_d f$ tend to f as $d \to \infty$?
(ii) Given a bounded sequence $(f_d)_{d\geq 1}$ in $L^p(\mathbb{T}^\infty)$ that enjoys the chain property, is there a function f in $L^p(\mathbb{T}^\infty)$ such that $f_d = \mathfrak{A}_d f$ for $d = 1, 2, 3, \ldots$?

It is not difficult to prove that if $1 \leq p < \infty$, then answer to (i) is that the sequence $(\mathfrak{A}_d f)_{d\geq 1}$ converges to f in norm (see Theorem 6 below). If $1 < p < \infty$, then a standard argument involving duality and the Banach–Alaoglu theorem shows that the answer to (ii) is affirmative. The conclusion is that in the strictly convex regime there is a one-to-one correspondence between functions in $L^p(\mathbb{T}^\infty)$ and bounded sequences in $L^p(\mathbb{T}^\infty)$ that enjoy the chain property. We refer to this type of result as *Hilbert's criterion*, as the basic idea goes back to Hilbert [8].

It is well-known that Hilbert's criterion does not hold for $L^1(\mathbb{T}^\infty)$, although we have not found this explicitly stated in the literature. Let $z = (z_1, z_2, z_3, \ldots)$ be a point in the infinite polydisc $\mathbb{D}^\infty$ and consider the sequence $(f_d)_{d\geq 1}$, where

$$f_d(\chi) = \prod_{j=1}^{d} \frac{1 - |z_j|^2}{|\chi_j - z_j|^2} \tag{1}$$

for χ on $\mathbb{T}^\infty$. It is not difficult to see that $(f_d)_{d\geq 1}$ is bounded sequence in $L^1(\mathbb{T}^\infty)$ enjoying the chain property. However, a result of Cole and Gamelin [5, Theorem 3.1] is equivalent to the assertion that there is a function f in $L^1(\mathbb{T}^\infty)$ such that $f_d = \mathfrak{A}_d f$ for $d = 1, 2, 3, \ldots$ if and only if z is in $\mathbb{D}^\infty \cap \ell^2$. Choosing therefore a point z in $\mathbb{D}^\infty \setminus \ell^2$, we see that (ii) has a negative answer for $p = 1$.

Set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and define the Hardy space $H^p(\mathbb{T}^\infty)$ as the closed subspace of $L^p(\mathbb{T}^\infty)$ consisting of the functions f whose Fourier coefficients

$$\widehat{f}(\kappa) = \int_{\mathbb{T}^\infty} f(\chi)\, \overline{\chi^\kappa}\, dm_\infty(\chi)$$

are supported on $\mathbb{N}_0^{(\infty)}$. It turns out that Hilbert's criterion holds for $H^1(\mathbb{T}^\infty)$.

Theorem 3.

(i) *If f is in $H^1(\mathbb{T}^\infty)$, then $\|f - \mathfrak{A}_d f\|_1 \to 0$ as $d \to \infty$.*
(ii) *If $(f_d)_{d\geq 1}$ is a bounded sequence in $H^1(\mathbb{T}^\infty)$ that enjoys the chain property, then there is a function f in $H^1(\mathbb{T}^\infty)$ such that $f_d = \mathfrak{A}_d f$ for $d = 1, 2, 3, \ldots$.*

Note that a more general version of Theorem 3 (ii) can be extracted from work of Bourgain [3, Section 5]. In our context, Theorem 3 was first enunciated by Aleman, Olsen and Saksman [1, Corollary 3]. To explain their approach, note that if z is in $\mathbb{D}^\infty \setminus \ell^2$, then it follows from the result of Cole and Gamelin that the sequence $(f_d)_{d\geq 1}$ with f_d as in (1) will converge weak-$*$ to a finite Borel measure μ on $\mathbb{T}^\infty$ that is not absolutely continuous (with respect to m_∞). This leads us back to the F. & M. Riesz theorem on analytic measures, which in this context can be formulated as follows.

Theorem 4. *If μ is a finite complex Borel measure on $\mathbb{T}^\infty$ whose Fourier coefficients*

$$\widehat{\mu}(\kappa) = \int_{\mathbb{T}^\infty} \chi^{-\kappa}\, d\mu(\chi)$$

are supported on $\mathbb{N}_0^{(\infty)}$, then μ is absolutely continuous.

In view of the discussion above, it is plain that Theorem 4 implies Theorem 3 (ii). A stronger version of Theorem 4 goes back to Helson and Lowdenslager [7]. The current version is as stated by Aleman, Olsen and Saksman [1, Corollary 1], who proved Theorem 4 after first establishing a version of Fatou's theorem in the infinite polydisc. The basic obstacle in this context is that the Poisson extension of μ is in general only defined on $\mathbb{D}^\infty \cap \ell^1$, and the main effort in [1] is directed at obtaining a version of Fatou's theorem where $\mathbb{T}^\infty$ is approached from $\mathbb{D}^\infty \cap \ell^1$.

Our proof of the F. & M. Riesz theorem on $\mathbb{T}$ also leads to simpler proofs of Theorem 3 and Theorem 4, since we can avoid Fatou's theorem once we have established suitable extensions of Lemma 2.

This line of reasoning also reveals that Theorem 3 only uses the case $r = 0$ of Lemma 2, while Theorem 4 requires the full result. Inspired by this and by the philosophy behind Hilbert's criterion, we find it natural to incorporate Theorem 3 in the proof of Theorem 4. Amusingly, this is the reverse direction to how the two results were established in [1].

Organization. The present note is comprised of three sections. Section 2 is devoted to the proof of Lemma 2, while Section 3 contains some expositional material and the proofs of Theorem 3 and Theorem 4.

2. Proof of Lemma 2

By continuity, it is sufficient to consider only those $0 < \varrho < 1$ such that f does not vanish on the circle $|z| = \varrho$. Since f is analytic in $\mathbb{D}$ it has only a finite number of zeros in $\varrho\mathbb{D}$. Let $(\alpha_n)_{n=1}^m$ denote these zeros (counting multiplicities) and form the finite Blaschke product

$$B(z) = \prod_{n=1}^{m} \frac{\varrho(\alpha_n - z)}{\varrho^2 - \overline{\alpha_n} z}.$$

Note that $|B(z)| = 1$ if $|z| = \varrho$. The function $F = f/B$ is analytic and non-vanishing when $|z| < \varrho + \varepsilon$ for some $\varepsilon > 0$, due to the assumption that f does not vanish on the circle $|z| = \varrho$. This means in particular that the functions $g = BF^{1/2}$ and $h = F^{1/2}$ are analytic for $|z| < \varrho + \varepsilon$ and that $f = gh$. We write

$$f_r(e^{i\theta}) = f(re^{i\theta}), \quad g_r(e^{i\theta}) = g(re^{i\theta}), \quad \text{and} \quad h_r(e^{i\theta}) = h(re^{i\theta})$$

for $0 \leq r \leq \varrho$. The triangle inequality and the Cauchy–Schwarz inequality yield that

$$\|f_r - f_\varrho\|_1 \leq \|g_r h_r - g_\varrho h_r\|_1 + \|g_\varrho h_r - g_\varrho h_\varrho\|_1 \leq \|g_r - g_\varrho\|_2 \|h_r\|_2 + \|g_\varrho\|_2 \|h_r - h_\varrho\|_2.$$

Since g and h are analytic for $|z| < \varrho + \varepsilon$, their power series at the origin converge absolutely for $|z| \leq \varrho$. We deduce from this, orthogonality, and the trivial estimate $(r^k - \varrho^k)^2 \leq \varrho^{2k} - r^{2k}$ that

$$\|g_r - g_\varrho\|_2 \leq \sqrt{\|g_\varrho\|_2^2 - \|g_r\|_2^2} \quad \text{and} \quad \|h_r - h_\varrho\|_2 \leq \sqrt{\|h_\varrho\|_2^2 - \|h_r\|_2^2}.$$

Putting together what we have done so far, we find that

$$\|f_r - f_\varrho\|_1 \leq \sqrt{\|g_\varrho\|_2^2 \|h_r\|_2^2 - \|g_r\|_2^2 \|h_r\|_2^2} + \sqrt{\|h_\varrho\|_2^2 \|g_\varrho\|_2^2 - \|g_\varrho\|_2^2 \|h_r\|_2^2}.$$

Since plainly $\|h_r\|_2^2 \le \|h_\varrho\|_2^2$ and $\|g_\varrho\|_2^2 \ge \|g_r\|_2^2$ by orthogonality, we get that

$$\|f_r - f_\varrho\|_1 \le 2\sqrt{\|g_\varrho\|_2^2\|h_\varrho\|_2^2 - \|g_r\|_2^2\|h_r\|_2^2}.$$

We use that $|B(z)| = 1$ for $|z| = \varrho$ to infer that $\|g_\varrho\|_2^2 = \|f_\varrho\|_1$ and $\|h_\varrho\|_2^2 = \|f_\varrho\|_1$, and the Cauchy–Schwarz inequality to infer that $\|g_r\|_2^2\|h_r\|_2^2 \ge \|f_r\|_1^2$. □

3. Hilbert's criterion

We find it necessary to begin with some expository material in order to properly set the stage for the proofs of Theorem 3 and Theorem 4.

If K is a finite subset of $\mathbb{Z}^{(\infty)}$, then we say that the function

$$T(\chi) = \sum_{\kappa\in K} a_\kappa \chi^\kappa \tag{2}$$

is a *trigonometric polynomial* on $\mathbb{T}^\infty$. It follows from the definition of $\mathbb{Z}^{(\infty)}$ that there is for each trigonometric polynomial T a positive integer d such that T only depends on a subset of the variables $\chi_1, \chi_2, \ldots, \chi_d$.

We let $L^p(\mathbb{T}^d)$ stand for the closed subspace of $L^p(\mathbb{T}^\infty)$ obtained as the closure of the set of such trigonometric polynomials. If f is in $L^p(\mathbb{T}^d)$, then the Fourier coefficients of f are plainly supported on sequences in $\mathbb{Z}^{(\infty)}$ of the form

$$(\kappa_1, \kappa_2, \ldots, \kappa_d, 0, 0, \ldots). \tag{3}$$

For $d = 1, 2, 3, \ldots$, die Abschnitte $\mathfrak{A}_d f$ are formally defined as replacing the Fourier coefficient $\widehat{f}(\kappa)$ by 0 whenever κ is not of the form (3). The following result can be obtained from density and the mean value property of trigonometric polynomials. The proof is not difficult and we omit it.

Lemma 5. *Let $1 \le p < \infty$. For $d = 1, 2, 3, \ldots$, die Abschnitte $\mathfrak{A}_d$ extend to bounded linear operators from $L^p(\mathbb{T}^\infty)$ to $L^p(\mathbb{T}^d)$ satisfying*

$$\|\mathfrak{A}_1 f\|_p \le \|\mathfrak{A}_2 f\|_p \le \|\mathfrak{A}_3 f\|_p \le \cdots \le \|f\|_p$$

for every f in $L^p(\mathbb{T}^\infty)$.

We are now in a position to establish Hilbert's criterion for $L^p(\mathbb{T}^\infty)$, which in particular covers the assertion (i) of Theorem 3.

Theorem 6. *Suppose that $1 \le p < \infty$. If f is in $L^p(\mathbb{T}^\infty)$, then*

$$\lim_{d\to\infty} \|f - \mathfrak{A}_d f\|_p = 0.$$

Proof. Fix $\varepsilon > 0$. By density, we can find a trigonometric polynomial T such that $\|f - T\|_p \le \varepsilon/2$. Since T is a trigonometric polynomial, there is a positive integer d_0 such that T is in $L^p(\mathbb{T}^{d_0})$. It now follows from the triangle inequality and Lemma 5 that if $d \ge d_0$, then

$$\|f - \mathfrak{A}_d f\|_p \le \|f - T\|_p + \|T - \mathfrak{A}_d f\|_p = \|f - T\|_p + \|\mathfrak{A}_d(T - f)\|_p \le \varepsilon. \qquad \square$$

We will use a weaker and less attractive version of Lemma 2 in the proofs of Theorem 3 (ii) and Theorem 4. We retain the notation $f_r(e^{i\theta}) = f(re^{i\theta})$ for analytic functions f in $\mathbb{D}$ and $0 \le r < 1$, but write $\|\cdot\|_{L^1(\mathbb{T})}$ to distinguish the norm of $L^1(\mathbb{T})$ from the norm of $L^1(\mathbb{T}^\infty)$.

Lemma 7. *If f is analytic in $\mathbb{D}$ and $0 \le r \le \varrho < 1$, then*

$$\|f_r - f_\varrho\|_{L^1(\mathbb{T})} \le 2\sqrt{2}\sqrt{\|f_\varrho\|_{L^1(\mathbb{T})}}\sqrt{\|f_\varrho\|_{L^1(\mathbb{T})} - \|f_r\|_{L^1(\mathbb{T})}}.$$

Proof. Use Lemma 2 and the fact that $b^2 - a^2 \leq 2b(b-a)$ for $0 \leq a \leq b$. $\square$

A *polynomial* P on $\mathbb{T}^\infty$ is a trigonometric polynomial (2) where the index set K is a subset of $\mathbb{N}_0^{(\infty)}$. Polynomials on $\mathbb{T}^\infty$ are nothing more than classical polynomials in, say, d variables restricted to $(\chi_1, \chi_2, \ldots, \chi_d)$. This means we can extend polynomials on $\mathbb{T}^\infty$ to $\mathbb{C}^\infty$ in the obvious way. In particular, if $d_1 \leq d$, then

$$\mathfrak{A}_{d_1} P(\chi) = P(\chi_1, \chi_2, \ldots, \chi_{d_1}, 0, 0, \ldots, 0).$$

For the proof of Theorem 3 (ii), we will use the following consequence of Lemma 7. The basic idea to embed a *slice* of the disc in a polydisc is from Rudin [13, p. 44].

Lemma 8. *If f is in $H^1(\mathbb{T}^\infty)$ and if $d_1 \leq d_2$ are positive integers, then*

$$\|\mathfrak{A}_{d_1} f - \mathfrak{A}_{d_2} f\|_1 \leq 2\sqrt{2}\sqrt{\|\mathfrak{A}_{d_2} f\|_1}\sqrt{\|\mathfrak{A}_{d_2} f\|_1 - \|\mathfrak{A}_{d_1} f\|_1}.$$

Proof. By density and Lemma 5, it is sufficient to establish the stated estimate for polynomials P in $L^p(\mathbb{T}^{d_2})$. In this case, we define

$$F(\chi, z) = P(\chi_1, \chi_2, \ldots, \chi_{d_1}, \chi_{d_1+1} z, \chi_{d_1+2} z, \ldots, \chi_{d_2} z)$$

for χ on $\mathbb{T}^\infty$ and z in $\mathbb{C}$. If χ is fixed, then $f(z) = F(\chi, z)$ is a polynomial and it is permissible to use Lemma 7 with $r = 0$ and $\varrho = 1$. We next integrate over χ on $\mathbb{T}^\infty$, then finally use the Cauchy–Schwarz inequality to infer that

$$\begin{aligned}&\int_{\mathbb{T}^\infty} \|F(\chi,\cdot) - F(\chi,0)\|_{L^1(\mathbb{T})}\, dm_\infty(\chi)\\ &\leq 2\sqrt{2}\sqrt{\int_{\mathbb{T}^\infty} \|F(\chi,\cdot)\|_{L^1(\mathbb{T})}\, dm_\infty(\chi)}\sqrt{\int_{\mathbb{T}^\infty} \big(\|F(\chi,\cdot)\|_{L^1(\mathbb{T})} - |F(\chi,0)|\big)\, dm_\infty(\chi)}.\end{aligned}$$

The stated estimate follows from this after using that $F(\chi, 0) = \mathfrak{A}_{d_1} P(\chi)$ twice, then using Fubini's theorem with the rotational invariance of m_∞ thrice. $\square$

Lemma 8 is the key ingredient in our proof of Theorem 3 (ii). The idea to establish Hilbert's criterion via a result such as Lemma 8 is from [2, Section 2.2].

Proof of Theorem 3 (ii). If $(f_d)_{d\geq 1}$ is a bounded sequence in $H^1(\mathbb{T}^\infty)$ that enjoys the chain property, then it follows from Lemma 8 that $(f_d)_{d\geq 1}$ is a Cauchy sequence in $H^1(\mathbb{T}^\infty)$. Hence it must converge to some function f in $H^1(\mathbb{T}^\infty)$. Fourier coefficients are preserved under convergence in $L^1(\mathbb{T}^\infty)$, so that $\mathfrak{A}_d f = f_d$ for $d = 1, 2, 3, \ldots$. $\square$

In preparation for the proof of Theorem 4, we recall that a result of Cole and Gamelin [5, Theorem 4.1] asserts that the infinite product

$$\prod_{j=1}^{\infty} \frac{1 - |z_j|^2}{|\chi_j - z_j|^2}$$

converges to a bounded function on $\mathbb{T}^\infty$ if and only if z is in $\mathbb{D}^\infty \cap \ell^1$. This means that the Poisson extension

$$\mathfrak{P}\mu(z) = \int_{\mathbb{T}^\infty} \prod_{j=1}^{\infty} \frac{1 - |z_j|^2}{|\chi_j - z_j|^2}\, d\mu(\chi)$$

of a finite complex Borel measure μ on $\mathbb{T}^\infty$ can in general only be defined in $\mathbb{D}^\infty \cap \ell^1$.

Our final preparation for the proof of Theorem 4 is to recall that finite complex Borel measures on $\mathbb{T}^\infty$ are uniquely determined by their Fourier coefficients. As in the classical setting, this is a direct consequence of the Riesz representation theorem and the fact that trigonometric polynomials are dense in $C(\mathbb{T}^\infty)$.

Proof of Theorem 4. If χ is on $\mathbb{T}^\infty$, z is in $\mathbb{D}$, and d is a positive integer, then the point $(\chi_1 z, \chi_2 z, \ldots, \chi_d z, 0, 0, \ldots)$ is plainly in $\mathbb{D}^\infty \cap \ell^1$. We can therefore define

$$F(\chi, z, d) = \mathfrak{P}\mu(\chi_1 z, \chi_2 z, \ldots, \chi_d z, 0, 0, \ldots).$$

Using Fubini's theorem as in the classical setting discussed in the introduction, we get that $\|F(\cdot, \varrho, d)\|_1 \le \|\mu\|$. If χ and d are fixed, then this and the assumption on the support of the Fourier coefficients of μ ensure that $F(\cdot, z, d)$ is in

$$H^1(\mathbb{T}^d) = H^1(\mathbb{T}^\infty) \cap L^1(\mathbb{T}^d).$$

This assumption also ensures that if χ and d are fixed, then $f(z) = F(\chi, z, d)$ is analytic in $\mathbb{D}$. Arguing as in the proof of Lemma 8, we infer from Lemma 7 that

$$\|F(\cdot, r, d) - F(\cdot, \varrho, d)\|_1 \le 2\sqrt{2}\sqrt{\|F(\cdot, \varrho, d)\|_1}\sqrt{\|F(\cdot, \varrho, d)\|_1 - \|F(\cdot, r, d)\|_1}$$

for $0 \le r \le \varrho < 1$. We infer from this that there is a function f_d in $H^1(\mathbb{T}^d)$ with $\|f_d\|_1 \le \|\mu\|$ such that

$$\lim_{r \to 1^-} \|f_d - F(\cdot, r, d)\|_1 = 0.$$

It follows that $(f_d)_{d \ge 1}$ is a bounded sequence in $H^1(\mathbb{T}^\infty)$ that enjoys the chain property, so by Theorem 3 (ii) there is a function f in $H^1(\mathbb{T}^\infty)$ such that $f_d = \mathfrak{A}_d f$ for $d = 1, 2, 3, \ldots$ and so $f = \mu$ by Theorem 3 (i). □

It is possible to give a slightly different proof of Theorem 4 that does not use Hilbert's criterion. The idea (from [1]) is to consider the Poisson extensions of μ to the points $(\chi_1 z, \chi_2 z^2, \chi_3 z^3, \ldots)$, which are in $\mathbb{D}^\infty \cap \ell^1$ for χ on $\mathbb{T}^\infty$ and z in $\mathbb{D}$, and then use Lemma 7 as above.

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