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Sharp Fourier decay estimates for measures supported on the well-approximable numbers

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Abstract. We construct a measure on the well-approximable numbers whose Fourier transform decays at a nearly optimal rate. This gives a logarithmic improvement on a previous construction of Kaufman.

Hyvin arvioitavien lukujen kantamien mittojen Fourier'n muunnosten tarkka vaimeneminen

Tiivistelmä. Tässä työssä rakennetaan hyvin arvioitavien lukujen kantama mitta, jonka Fourier'n muunnos vaimenee lähes parasta mahdollista vauhtia. Tämä tuottaa logaritmisen parannuksen verrattuna aiempaan Kaufmanin esimerkkiin.

1. Introduction and Background

1.1. Harmonic analysis on fractal sets. An interesting class of problems in harmonic analysis involves determining information about the Fourier transform of a compactly supported measure μ given information about the support $\operatorname{supp}\mu$ of the measure μ. A standard result in this area is Frostman's lemma, which states that if E is a set of Hausdorff dimension s, then for any $t < s$, there exists a Borel probability measure μ_t supported on E satisfying the condition that

$$\int_{\xi\in\mathbb{R}^n} |\hat{\mu}_t(\xi)|^2(1+|\xi|)^{-t} < \infty. \tag{1}$$

Frostman's lemma states that, up to an ϵ-loss in the exponent, the set E supports a measure whose Fourier transform decays like $|\xi|^{-s/2}$ in an L^2-average sense.

This version of Frostman's lemma motivates the definition of Fourier dimension. The Fourier dimension of a set $E \subset \mathbb{R}^n$ is the supremum of those values $0 \le s \le n$ such that E supports a Borel probability measure μ_s satisfying the pointwise condition

$$|\hat{\mu}_s(\xi)| \lesssim (1+|\xi|)^{-s/2}. \tag{2}$$

Observe that the condition (2) for some value of s implies equation (1) for any $t < s$. However, there is no reason to expect a converse statement to hold; in fact, if E is the usual middle-thirds Cantor set, there is no Borel probability measure μ on E such that $|\hat{\mu}(\xi)| \to 0$ as $|\xi| \to \infty$. A measure μ such that $|\hat{\mu}(\xi)| \to 0$ as $\xi \to \infty$ is called a *Rajchman measure*. On the opposite extreme, there are a number of examples of sets E of Hausdorff dimension s supporting Borel probability measures satisfying (2) for all $t < s$. Such sets are called *Salem sets*.

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If $s = n - 1$, a simple stationary phase calculation shows that the usual surface measure on the sphere satisfies the condition

$$|\hat{\mu}(\xi)| \leq (1 + |\xi|)^{-(n-1)/2}. \tag{3}$$

This well-known computation can be found in the textbooks of Wolff [16] and Mattila [12]. If $n = 1$ and $0 < s < 1$, the first examples of Salem sets were given by Salem [15] via a random Cantor set construction. A later random construction was given by Kahane [8], who shows that if $\Gamma\colon [0,1] \to \mathbb{R}^n$ is a Brownian motion and $E \subset [0,1]$ is a set of Hausdorff dimension s, then $\Gamma(E)$ will almost surely have Fourier dimension equal to $2s$. Kahane [9] also constructed Salem sets using random Fourier series whose coefficients are given by Gaussian random variables.

The first explicit, deterministic example of a Salem set of fractional dimension in $\mathbb{R}$ was given by Kaufman [11]. For an exponent τ, the well-approximable numbers $E(q^{-\tau})$ are defined by

$$E(q^{-\tau}) = \left\{ x\colon \left| x - \frac{p}{q} \right| \leq q^{-\tau} \text{ for infinitely many pairs of integers } (p,q) \right\}.$$

A classical result of Jarník [7] and Besicovitch [1] states that the Hausdorff dimension of $E(q^{-\tau})$ is equal to $\frac{2}{\tau}$. Kaufman shows that $E(q^{-\tau})$ supports a Borel probability measure μ satisfying

$$|\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-1/\tau} o(\log|\xi|).$$

Bluhm [3] provides an exposition of Kaufman's argument to prove a slightly weaker result in which the $o(\log|\xi|)$ term is replaced by $\mathcal{O}(\log|\xi|)$. More generally, given a function $\psi\colon \mathbb{N} \to [0,\infty)$, it is of interest to consider the set of ψ-approximable numbers

$$E(\psi) = \left\{ x\colon \left| x - \frac{p}{q} \right| \leq \psi(q) \text{ for infinitely many pairs of integers } (p,q) \right\}.$$

Hambrook [5] obtains lower bounds on the Fourier dimension of such sets in terms of the function ψ.

1.2. Some problems in geometric measure theory. In this paper, we will consider the question of locating sets E satisfying more precise estimates than (2) under the constraint that E has finite Hausdorff measure. As a motivating example, consider the $(n-1)$-dimensional sphere in $\mathbb{R}^n$. This set has positive and finite $(n-1)$-dimensional Hausdorff measure and supports a measure μ with Fourier transform satisfying (3). Mitsis [13] posed the following problem.

Problem 1.1. (Mitsis's problem) For which values of $0 < s < n$ does there exist a measure μ such that μ simultaneously satisfies the ball condition

$$\mu(B(x,r)) \sim r^s \quad \text{for all } x \in \operatorname{supp}\mu \text{ and all } r > 0$$

and the Fourier decay condition

$$|\hat{\mu}(\xi)| \leq |\xi|^{-s/2}?$$

We will consider a related problem. Let $0 \leq s \leq n$. Recall that a subset E of $\mathbb{R}^n$ is said to be an *s-set* if the Hausdorff measure $\mathcal{H}^s(E)$ satisfies $0 < \mathcal{H}^s(E) < \infty$.

Problem 1.2. (Fourier transform on s-sets) For which values of $0 < s < n$ does there exist an s-set E supporting a measure μ such that μ satisfies the Fourier decay condition

$$|\hat{\mu}(\xi)| \leq |\xi|^{-s/2}?$$

Of course, such a set E must be a Salem set of Hausdorff dimension s.

This problem can be extended to a question about generalized Hausdorff dimension. Recall that a positive, increasing function α is said to be a *dimension function* if $\alpha(u) \to 0$ as $u \to 0$. We will say that E is an α-set if $0 < \mathcal{H}_\alpha(\mathcal{E}) < \infty$, where $\mathcal{H}_\alpha$ is the generalized Hausdorff measure associated to α. The following question generalizes the previous one:

Problem 1.3. (Fourier transform on α-sets) For which dimension functions α does there exist an α-set E supporting a measure μ such that μ satisfies the Fourier decay condition

$$|\hat{\mu}(\xi)| \lesssim \sqrt{\alpha(1/\xi)} \quad \text{for } |\xi| \geq 1?$$

We conjecture that the only such dimension functions α are integer powers $\alpha(u) = u^{-s}$ for integers $0 \leq s \leq n$.

On the other hand, we also wish to pose the problem of determining the optimal Fourier decay estimates for measures supported on the set of well-approximable numbers $E(\psi)$.

Problem 1.4. (Fourier decay of measures supported on $E(\psi)$) Fix a function ψ. For which functions Θ does there exist a measure μ supported on $E(\psi)$ such that

$$|\hat{\mu}(\xi)| \lesssim \Theta(\xi)?$$

Although we are unable to answer Problems 1.2, 1.3, and 1.4 in this work, we are able to obtain "near"-answers to all three of these questions if the dimension function α or the approximation function ψ decay at a polynomial rate.

1.3. Notation. In this paper, constants are always allowed to depend on the parameters τ, σ, and ρ. Any dependence on these parameters will always be suppressed for simplicity of notation.

If A and B are any two quantities, we write $A = \mathcal{O}(B)$ or $A \lesssim B$ to imply that $A \leq CB$ for some constant C that does not depend on A or B (but may depend on τ, σ, or ρ). We write $B \gtrsim A$ to mean the same thing as $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. If the implicit constant in any of these inequalities is allowed to depend on some other parameter such as ϵ, we write $A \lesssim_\epsilon B$, $A \gtrsim_\epsilon B$, or $A \sim_\epsilon B$.

If $A(x)$ and $B(x)$ are functions of a variable x, we write $A(x) \lessapprox B(x)$ if $A(x) \lesssim_\epsilon x^\epsilon B(x)$ for every $\epsilon > 0$. So, for example, we write

$$x^3 \exp(\sqrt{\log x}) \log x \log\log x \lessapprox x^3.$$

If $A(x) \lessapprox B(x)$ and $B(x) \lessapprox A(x)$, we write $A(x) \approx B(x)$.

We write $A \ll B$ to mean that A is much less than B. This should be viewed as informal notation to help the reader keep in mind the sizes of the various parameters.

2. Results

First, we describe a result in the direction of Problem 1.4.

Theorem 2.1. *Let $\psi(q)$ be an arbitrary nonnegative, decreasing function satisfying the conditions*

$$2 < \lim_{q\to\infty} -\frac{\log(\psi(q))}{\log q} = \tau < \infty. \tag{4}$$

Suppose also that there exists $\sigma > 1$ such that ψ satisfies the polynomial-type decay condition

$$\frac{\psi(q_1)}{\psi(q_2)} \geq \left(\frac{q_2}{q_1}\right)^{\sigma} \quad \text{for } q_2 > q_1 \text{ sufficiently large.} \tag{5}$$

Suppose further that $1 \leq \chi(q) \leq \log q$ is a nonnegative function that satisfies

$$\sum_{\substack{q=1 \\ q \text{ prime}}}^{\infty} \frac{1}{q\chi(q)} = \infty. \tag{6}$$

and also satisfies the subpolynomial-type growth condition for any $\epsilon > 0$:

$$\frac{\chi(q_2)}{\chi(q_1)} < \left(\frac{q_2}{q_1}\right)^{\epsilon} \quad \text{for } q_1, q_2 \text{ sufficiently large depending on } \epsilon. \tag{7}$$

Then for any increasing function ω with $\lim_{\xi\to\infty} \omega(\xi) = \infty$, there exists a Borel probability measure μ supported on a compact subset of the ψ-well-approximable numbers satisfying the estimate

$$|\hat{\mu}_{\chi,\omega}(\xi)| \lesssim \frac{\omega(|\xi|)}{\psi^{-1}(1/|\xi|)\chi(\psi^{-1}(1/|\xi|))} \quad \text{for all } \xi \in \mathbb{R}. \tag{8}$$

In order to simplify our notation, we define

$$\theta(\xi) := \frac{1}{\psi^{-1}(1/\xi)\chi(\psi^{-1}(1/\xi))}. \tag{9}$$

Remark 2.2. If $\psi(q) = q^{-\tau}$, Theorem 2.1 gives estimates that improve on those of Kaufman [11]. In this case, the estimate (8) becomes

$$|\hat{\mu}_{\chi,\omega}(\xi)| \lesssim |\xi|^{-1/\tau} \frac{\omega(|\xi|)}{\chi(|\xi|^{1/\tau})}.$$

Observe that, for example, the choice $\chi(q) = \log\log q$ satisfies (6). On the other hand, ω can be taken to be any function that increases to ∞, so it is possible to choose $\omega(\xi) = \log\log\log \xi$, for example. Hence there exists a measure μ supported on the well-approximable numbers satisfying

$$|\hat{\mu}(\xi)| \lesssim |\xi|^{-1/\tau} \frac{\log\log\log|\xi|}{\log\log|\xi|} \ll |\xi|^{-1/\tau}.$$

Our next result is in the direction of Problem 1.3.

Theorem 2.3. *Let α be a dimension function with*

$$0 < \lim_{x\to 0} \frac{\log \alpha(x)}{\log x} = \nu < \infty \tag{10}$$

and for some $\rho < 1$ such that

$$\frac{\alpha(x_1)}{\alpha(x_2)} \geq \left(\frac{x_1}{x_2}\right)^{\rho} \tag{11}$$

for sufficiently small $x_1 < x_2$. Let ω be an increasing function such that $\lim_{\xi\to\infty} \omega(\xi) = \infty$. Then there exists a compact set F_α of zero α-Hausdorff measure such that there exists a measure $\mu_{\alpha,\omega}$ supported on F_α satisfying

$$|\hat{\mu}(\xi)| \lesssim \sqrt{\alpha(1/|\xi|)}\omega(|\xi|)$$

for all $\xi \neq 0$. Such a set is given by an appropriately chosen subset of the well-approximable numbers $E(\psi)$ where

$$\psi(q) = \alpha^{-1}\left(q^{-2}\right). \tag{12}$$

Remark 2.4. Although this does not provide an answer to Problem 1.3, it comes within an arbitrarily slowly growing function of answering this problem. In other words, any improvement on the estimate of Theorem 2.3 will give an answer to Problem 1.3.

Remark 2.5. Observe that the condition (10) on α implies the condition (4) on ψ for $\tau = 2/\nu$. A simple calculation also shows that the condition (11) implies the condition (5) with $\sigma = 2/\rho$. This is the only way in which the assumptions (10) and (11) will be used.

Finally, we show that, for any decreasing approximation function ψ, the set $E(\psi)$ supports a Rajchman measure. This improves a result of Bluhm [4] constructing a Rajchman measure supported on the set of Liouville numbers.

Theorem 2.6. *For an arbitrary nonnegative, decreasing function ψ there exists a Rajchman measure, μ, supported on a compact subset of the ψ-well-approximable numbers.*

In a recent work, Polasek and Rela [14] improve Bluhm's result in a different way by showing an explicit Fourier decay bound on the set of Liouville numbers. They show that if $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is any function such that

$$\limsup_{\xi\to\infty} \frac{\xi^{-\alpha}}{f(\xi)} = 0 \quad \text{for all } \alpha > 0,$$

then there exists a measure μ_f supported on the set of Liouville numbers such that $|\hat{\mu}_f(\xi)| \lesssim f(|\xi|)$ for all ξ; on the other hand, if $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is any function such that

$$\liminf_{\xi\to\infty} \frac{\xi^{-\alpha}}{f(\xi)} > 0 \quad \text{for some } \alpha > 0,$$

then there does not exist a measure μ_g supported on the set of Liouville numbers such that $|\hat{\mu}_g(\xi)| \lesssim g(|\xi|)$ for all $\xi \in \mathbb{R}$.

3. Convolution stability lemmas

The proofs of the main results of this paper rely on the construction of a sequence of functions which will approximate the measures that satisfy the statements of the theorem. The functions of the sequence are themselves a product of functions. In the frequency space, these products become convolutions and a major component of the proof is show that the Fourier decay estimates of these functions remain stable as the number of convolutions tends to infinity. The following two lemmas will be referred to when making an argument for stability by induction. This first lemma will be applied to Theorem 2.1 and Theorem 2.3.

Lemma 3.1. (Convolution Stability Lemma) *Let ψ and χ be as in Theorem 2.1, and let $\theta(\xi)$ be as in (9). Let $\omega\colon \mathbb{N} \to \mathbb{R}^+$ be a function that increases to infinity such that $\omega(t) \leq \log t$ for $t \geq 2$. Suppose that $N_1 > 0$ and suppose that that N_2 is sufficiently large depending on ω and N_1. Moreover, let $G, H\colon \mathbb{Z} \to \mathbb{C}$ be functions* *satisfying the following bounds for some large number* $N_3 > N_2$*:*

$$|G(s)| \le 1 \quad \text{for all } s \in \mathbb{Z}, \tag{13}$$

$$G(0) = 1, \tag{14}$$

$$G(s) = 0, \quad 0 < |s| \le N_2, \tag{15}$$

$$|G(s)| \lesssim \theta(|s|) \quad \text{everywhere}, \tag{16}$$

$$|G(s)| \lesssim \exp\left(-\frac{1}{2}\left|\frac{s}{2N_3}\right|^{\frac{\sigma+1}{4\sigma}}\right) \quad \text{when } |s| \ge 2N_3, \tag{17}$$

$$|H| \le 2, \tag{18}$$

$$|H(s)| \le \exp\left(-\frac{1}{2}\left|\frac{s}{8N_1}\right|^{\frac{\sigma+1}{4\sigma}}\right) \quad \text{when } |s| \ge 8N_1^2. \tag{19}$$

Then

$$|H * G(s) - H(s)| \le N_2^{-99} \quad \text{when } 0 \le |s| < N_2/4, \tag{20}$$

$$|H * G(s)| \lesssim \theta(|s|)\omega(|s|) \quad \text{when } |s| \ge N_2/4, \tag{21}$$

$$|H * G(s)| \le \exp\left(-\frac{1}{2}\left|\frac{s}{8N_3}\right|^{\frac{\sigma+1}{4\sigma}}\right) \quad \text{when } |s| \ge 8N_3^2. \tag{22}$$

A different version of this lemma will be applied to prove Theorem 2.6.

Lemma 3.2. (Convolution Stability Lemma 2) *Let* ψ *be as in Theorem 2.6. Let* $N_1 > 0$*, and let* $\delta < \frac{1}{N_1^3}$*. Suppose that* N_2 *is some number that is sufficiently large depending on* ω *and* N_1*. Moreover, let* $G, H\colon \mathbb{Z} \to \mathbb{C}$ *be functions satisfying the following bounds for some* $N_3 > N_2$*:*

$$|G(s)| \le 1 \quad \text{for all } s \in \mathbb{Z}, \tag{23}$$

$$G(0) = 1, \tag{24}$$

$$G(s) = 0, \quad 0 < |s| \le N_2, \tag{25}$$

$$|G(s)| \lesssim \delta, \quad s \ne 0, \tag{26}$$

$$|G(s)| \lesssim \exp\left(-\frac{1}{2}\left|\frac{s}{2N_3}\right|^{\frac{3}{4}}\right) \quad \text{when } |s| \ge 2N_3, \tag{27}$$

$$|H| \le 2, \tag{28}$$

$$|H(s)| \le \exp\left(-\frac{1}{2}\left|\frac{s}{8N_1}\right|^{\frac{3}{4}}\right) \quad \text{when } |s| \ge 8N_1^2. \tag{29}$$

Then

$$|H * G(s) - H(s)| \le N_2^{-99} \quad \text{when } 0 \le |s| < N_2/4, \tag{30}$$

$$|H * G(s)| \lesssim \delta^{1/3} \quad \text{when } |s| \ge N_2/4, \tag{31}$$

$$|H * G(s)| \le \exp\left(-\frac{1}{2}\left|\frac{s}{8N_3}\right|^{\frac{3}{4}}\right) \quad \text{when } |s| \ge 8N_3^2. \tag{32}$$

Before proving these lemmas, we need a preliminary estimate on θ. We will show that the function $\theta(\xi)$ decays like $\xi^{-1/\tau}$ up to an ϵ-loss in the exponent.

Lemma 3.3. *Let ψ, χ be as in Theorem 2.1, and let $\theta(\xi)$ be as in* (9). *Then $\theta(|\xi|) \approx |\xi|^{-\frac{1}{\tau}}$ for large $|\xi|$.*

Proof. Since $\psi(q) \approx q^{-\tau}$ by assumption, we have that $\psi^{-1}(t) \approx t^{-1/\tau}$. A similar argument shows that $\chi(t) \approx 1$. Hence $\chi(\psi^{-1}(1/|\xi|)) \approx 1$. Thus

$$\frac{1}{\psi^{-1}(1/|\xi|)\chi(\psi^{-1}(1/|\xi|))} \approx |\xi|^{-1/\tau}. \tag{33}$$ □

3.1. Proof of Lemma 3.1.

Proof. First, we prove (20). Assume that $0 \leq |s| \leq N_2/4$. Rewrite the expression as

$$\begin{aligned} |H * G(s) - H(s)| &= \left| \sum_{t \in \mathbb{Z}} H(s-t)G(t) - H(s) \right| \\ &= \left| H(s)G(0) - H(s) + \sum_{t \neq 0} H(s-t)G(t) \right| \\ &\leq \sum_{t \neq 0} |H(s-t)G(t)|. \end{aligned}$$

Observe that we need only consider summands such that $|t| \geq N_2$ because $G(t) = 0$ for $|t| < N_2$. The previous expression becomes

$$\sum_{|t| \geq N_2} |H(s-t)G(t)|.$$

Apply the bound (13) to $|G(t)|$. Notice that $|s-t| \geq |t|/2 \gg N_1$ when $|s| < N_2/4$. We may apply (19) with $|t|/2$ in place of s to get an upper bound given that the bounding function is decreasing. Hence

$$\sum_{|t| \geq N_2} |H(s-t)G(t)| \leq \sum_{|t| \geq N_2} \exp\left(-\frac{1}{2} \left| \frac{t}{16 N_1} \right|^{\frac{\sigma+1}{4\sigma}} \right) \leq N_2^{-99}.$$

The last inequality holds provided that N_2 is sufficiently large depending on N_1.

The next task is to prove the estimate (21). Now assume that $|s| \geq N_2/4$. We have the inequality

$$|H * G(s)| \leq \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \sum_{|t| < 8N_1^2} |H(t)G(s-t)|,$$

and

$$\mathrm{II} = \sum_{|t| \geq 8N_1^2} |H(t)G(s-t)|.$$

Beginning with the sum I, we apply (18) and observe that if N_2 is sufficiently large depending on N_1, then $|s-t| \geq |s| - |t| \geq |s|/2$ when $|s| \geq N_2/4$. Then we may apply (16) with $|s|/2$ in place of $|s|$ to get

$$\mathrm{I} \lesssim \theta(|s|/2) \sum_{|t| < 8N_1^2} 1 \lesssim \theta(|s|/2)\omega(|s|),$$

provided that N_2 is sufficiently large depending on ω and N_1 so that the final inequality holds. To complete the proof, we need to show that $\theta(|s|/2) \lesssim \theta(|s|)$. This will be shown in Lemma 4.2.

To bound the sum II, write

$$\mathrm{II} = \mathrm{A} + \mathrm{B},$$

where

$$A = \sum_{\substack{|t| \geq 8N_1^2 \\ |s-t| \leq |s|/2}} |H(t)G(s-t)|,$$

and

$$B = \sum_{\substack{|t| \geq 8N_1^2 \\ |s-t| > |s|/2}} |H(t)G(s-t)|.$$

To estimate the sum A, we apply (13) and (19). Observe that $|s-t| \leq |s|/2$ implies that $|t| \geq |s|/2$. Thus, $|t| > 8N_1^2$ when $|s| \geq N_2/4$. Therefore

$$A \lesssim \sum_{|t| \geq |s|/2} \exp\left(-\frac{1}{2}\left|\frac{t}{8N_1}\right|^{\frac{\sigma+1}{4\sigma}}\right).$$

By the integral test, we get the following upper bound for A:

$$A \lesssim \int_{|s|/2}^{\infty} \exp\left(-\frac{1}{2}\left|\frac{t}{8N_1}\right|^{\frac{\sigma+1}{4\sigma}}\right) dt.$$

Observe that the integrand is decaying nearly exponentially. From (33), we may conclude

$$A \lesssim \theta(|s|).$$

For the sum B, we apply (16) to G. Additionally, we may apply (19). Doing this, we have that

$$B \lesssim \theta(|s|) \sum_{|t| \geq 8N_1^2} \exp\left(-\frac{1}{2}\left|\frac{t}{8N_1}\right|^{\frac{\sigma+1}{4\sigma}}\right) \lesssim \theta(|s|)$$

where the last inequality is implied by

$$\sum_{|t| \geq 8N_1^2} \exp\left(-\frac{1}{2}\left|\frac{t}{8N_1}\right|^{\frac{\sigma+1}{4\sigma}}\right) \leq 1.$$

Combining the bounds for I and II completes the proof for (21). We turn now to proving (22). Assume $|s| \geq 8N_3^2$. We decompose the convolution as

$$|H * G(s)| \leq \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \sum_{|t| < |s|/2} |H(s-t)G(t)|$$

and

$$\mathrm{II} = \sum_{|t| \geq |s|/2} |H(s-t)G(t)|.$$

Starting with I, we apply (13). Then use the fact that $|s-t| \geq |s|/2$, and apply (19) with $s/2$ in place of s. Then

$$\mathrm{I} \lesssim \sum_{|t|<|s|/2} \exp\left(-\frac{1}{2}\left|\frac{s}{16N_1}\right|^{\frac{\sigma+1}{4\sigma}}\right).$$

There are at most $|s|+1$ summands in the above sum. Therefore

$$\mathrm{I} \lesssim |s| \exp\left(-\frac{1}{2}\left|\frac{s}{16N_1}\right|^{\frac{\sigma+1}{4\sigma}}\right).$$

We may absorb the linear factor and implicit constant by choosing a smaller negative power. Hence, provided that N_2 (and thus N_3) are sufficiently large depending on N_1, we have that

$$\mathrm{I} \leq \frac{1}{2} \exp\left(-\frac{1}{2}\left|\frac{s}{8N_3}\right|^{\frac{\sigma+1}{4\sigma}}\right).$$

For the sum II, apply the bounds (17) and (18) to get

$$\mathrm{II} \lesssim \sum_{|t|>|s|/2} \exp\left(-\frac{1}{2}\left|\frac{t}{2N_3}\right|^{\frac{\sigma+1}{4\sigma}}\right).$$

To bound the above sum, we use the integral test. Thus

$$\mathrm{II} \lesssim \int_{t>|s|/2} \exp\left(-\frac{1}{2}\left|\frac{t}{2N_3}\right|^{\frac{\sigma+1}{4\sigma}}\right) dt.$$

To estimate this integral, we begin with a substitution. Let

$$u = \frac{1}{2}\left|\frac{t}{2N_3}\right|^{\frac{\sigma+1}{4\sigma}}.$$

Then

$$du = \frac{\sigma+1}{16\sigma N_3}\left|\frac{t}{2N_3}\right|^{\frac{-3\sigma+1}{4\sigma}} dt.$$

The integral may be rewritten as

$$\frac{16\sigma N_3}{\sigma+1} \int_{t=|s|/2}^{\infty} \exp(-u)\,(2u)^{\frac{3\sigma-1}{\sigma+1}}\, du.$$

Integrating by parts yields

$$\left(\frac{16\sigma N_3}{\sigma+1}\right)\left(-\exp(-u)\,(2u)^{\frac{3\sigma-1}{\sigma+1}} \Big|_{t=|s|/2}^{\infty} + \frac{6\sigma-2}{\sigma+1} \int_{t=|s|/2}^{\infty} \exp(-u)\,(2u)^{\frac{3\sigma-1}{\sigma+1}-1}\, du\right).$$

It is easy to see that expression above is dominated by the first term and the integral is an error term. We consider only the first term in the estimate and evaluate the endpoints to get

$$\mathrm{II} \lesssim N_3 \exp\left(-\frac{1}{2}\left|\frac{s}{4N_3}\right|^{\frac{\sigma+1}{4\sigma}}\right)\left|\frac{s}{4N_3}\right|^{\frac{3\sigma-1}{4\sigma}}.$$

Because $|s| \geq 8N_3^2$ and because N_3 is large, We may absorb the power of s and the implicit constant by choosing a smaller negative power. Hence, if N_2 and thus N_3 is sufficiently large, then

$$\mathrm{II} \leq \frac{1}{2} \exp\left(-\frac{1}{2}\left|\frac{s}{8N_3}\right|^{\frac{\sigma+1}{4\sigma}}\right).$$

Combining the estimates on I and II completes the proof of the lemma. □

3.2. Proof of Lemma 3.2. The proof of Lemma 3.2 shares many similarities with the proof of Lemma 3.1.

Proof. Beginning with (30), assume $|s| \leq N_2/4$ and write

$$|H * G(s) - H(s)| \leq \sum_{|t| \geq N_2} |H(s-t)G(t)|.$$

If $|s| \leq N_2/4$ and $|t| \geq N_2$, then $|s-t| \geq |t|/2$. So we can apply the estimate (29) with $|t|/2$ in place of s and the estimate (26). Then

$$|H * G(s) - H(s)| \lesssim \delta \sum_{|t| \geq N_2} \exp\left(-\frac{1}{2}\left|\frac{t}{16N_1}\right|^{\frac{3}{4}}\right) \leq N_2^{-99}$$

provided that N_2 is sufficiently large depending on N_1. In order to prove the estimate (31), we assume $|s| \geq N_2/4$. Write

$$|H * G(s)| \leq \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \sum_{|t| < 8N_1^2} |H(t)G(s-t)|$$

and

$$\mathrm{II} = \sum_{|t| \geq 8N_1^2} |H(t)G(s-t)|.$$

For the sum I, apply the estimates (28) and (26). Then

$$\mathrm{I} \lesssim \delta \sum_{|t| < 8N_1^2} 1 \lesssim \delta N_1^2 < \delta^{1/3},$$

where the final inequality follows from the fact that $\delta < \frac{1}{N_1^3}$.

For the sum II, consider the term where $s = t$ separately from other summands. Write

$$\mathrm{II} = \sum_{\substack{|t| \geq 8N_1^2 \\ s \neq t}} |H(t)G(s-t)| + |H(s)G(0)|.$$

Apply the estimates (29), (26) and (24). Then

$$\mathrm{II} \lesssim \delta \sum_{\substack{|t| \geq 8N_1^2 \\ s \neq t}} \exp\left(-\frac{1}{2}\left|\frac{t}{8N_1}\right|^{\frac{3}{4}}\right) + \exp\left(-\frac{1}{2}\left|\frac{s}{8N_1}\right|^{\frac{3}{4}}\right) \lesssim \delta^{1/3}.$$

The last inequality is implied by the bounds

$$\sum_{\substack{|t| \geq 8N_1^2 \\ s \neq t}} \exp\left(-\frac{1}{2}\left|\frac{t}{8N_1}\right|^{\frac{3}{4}}\right) \lesssim 1$$

and

$$\exp\left(-\frac{1}{2}\left|\frac{s}{8N_1}\right|^{\frac{3}{4}}\right) \lesssim \delta^{1/3}.$$

For the final estimate (32), assume $|s| \geq 8N_3^2$ and write

$$|H * G(s)| \leq \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \sum_{|t|<|s|/2} |H(s-t)G(t)|$$

and

$$\mathrm{II} = \sum_{|t|\geq|s|/2} |H(s-t)G(t)|.$$

For the sum I, use the fact that $|s-t| \geq |s|/2$ and apply (23) and (29) with $|s|/2$. Then

$$\mathrm{I} \lesssim \sum_{|t|<|s|/2} \exp\left(-\frac{1}{2}\left|\frac{s}{16N_1}\right|^{\frac{3}{4}}\right) \lesssim |s| \exp\left(-\frac{1}{2}\left|\frac{s}{16N_1}\right|^{3/4}\right).$$

Because $s \geq \frac{N_2}{4} \gg N_1^2$, it follows that if N_2 (and thus N_3) are large enough depending on N_1, both the factor of $|s|$ and the implicit constant can be absorbed by changing the denominator in the exponent. Thus we have the estimate

$$\mathrm{I} \leq \frac{1}{2} \exp\left(-\frac{1}{2}\left|\frac{s}{8N_3}\right|^{\frac{3}{4}}\right).$$

For the sum II, apply (28) and (27) with $|t|$ in place of s. Then

$$\mathrm{II} \lesssim \sum_{|t|>|s|/2} \exp\left(-\frac{1}{2}\left|\frac{t}{2N_3}\right|^{\frac{3}{4}}\right).$$

By a similar argument to the one appearing in the proof of Lemma 3.1, we therefore have the estimate

$$\mathrm{II} \leq \frac{1}{2} \exp\left(-\frac{1}{2}\left|\frac{s}{8N_3}\right|^{\frac{3}{4}}\right).$$

Combining the estimates on I and II completes the proof of Lemma 3.2. □

4. Doubling functions

Definition 4.1. If $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing or eventually decreasing function, we say that f is *doubling* if $f(\xi/2) \lesssim f(\xi)$ for all sufficiently large ξ.

We will need a few basic facts about doubling functions.

Lemma 4.2. *The function $\theta(\xi)$ is doubling.*

Proof. The fact that $\theta(\xi) \approx \xi^{-1/\tau}$ implies that $\theta(\xi)$ is eventually decreasing. To see that $\theta(\xi)$ is doubling, note that for sufficiently large q_1 and q_2 with $q_1 < q_2$, we have the assumption (5), which is reproduced below for convenience.

$$\frac{\psi(q_1)}{\psi(q_2)} \geq \left(\frac{q_2}{q_1}\right)^{\sigma}.$$

Since ψ^{-1} is decreasing, we have that $\psi^{-1}(1/\xi) > \psi^{-1}(2/\xi)$. If ξ is sufficiently large that (5) applies with $q_1 = \psi^{-1}(2/\xi)$ and $q_2 = \psi^{-1}(1/\xi)$, then we have

$$\frac{\psi(q_1)}{\psi(q_2)} = \frac{2/\xi}{1/\xi} \geq \left(\frac{\psi^{-1}(1/\xi)}{\psi^{-1}(2/\xi)}\right)^{\sigma}.$$

Hence

$$\frac{\theta(\xi/2)}{\theta(\xi)} = \frac{\psi^{-1}(1/\xi)\chi(\psi^{-1}(1/\xi))}{\psi^{-1}(2/\xi)\chi(\psi^{-1}(2/\xi))} \leq \left(\frac{\psi^{-1}(1/\xi)}{\psi^{-1}(2/\xi)}\right)^{1+\epsilon} \leq 2^{(1+\epsilon)/\sigma}.$$

Hence $\theta(\xi)$ is doubling. □

Next, we show under very general conditions that a function with limit 0 must admit a decreasing, doubling majorant.

Lemma 4.3. *Suppose that $M\colon \mathbb{Z} \to \mathbb{C}$ is any function such that $|M(s)| \to 0$ as $|s| \to \infty$. Then there is a decreasing function $N\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $N(\xi) \to 0$ as $\xi \to \infty$ satisfying the doubling property such that $|M(s)| \leq N(|s|)$ for all $s \in \mathbb{Z}$.*

Proof. First, we replace M by a decreasing function $M_1\colon \mathbb{R}^+ \to \mathbb{R}^+$ as follows. For $s \in \mathbb{N}$, define

$$M_1(s) = \sup_{|t|\geq s} |M(t)|.$$

Then M_1 is decreasing on $[0,\infty)$, $|M(s)| \leq M_1(|s|)$ for all $s \in \mathbb{Z}$, and $\lim_{s\to\infty} M_1(s) = 0$.

We construct N by taking the average of M_1. For $\xi \in \mathbb{R}^+$, define

$$N(\xi) = \frac{1}{\lfloor \xi \rfloor + 1} \sum_{\substack{t\in\mathbb{N} \\ t\leq \xi}} M_1(t).$$

As N is an average of a decreasing function, it follows that N is decreasing; moreover, since $M_1(t) \to 0$ as $t \to \infty$, it follows that $N(\xi) \to 0$ as $\xi \to \infty$. Furthermore, it is easy to see that $M_1(s) \leq N(s)$ for $s \in \mathbb{N}$:

$$N(s) = \frac{1}{s+1}\sum_{t=0}^{s} M_1(t) \geq \frac{1}{s+1}\sum_{t=0}^{s} M_1(s) = \frac{1}{s+1}(s+1)M_1(s) = M_1(s),$$

So $|M(s)| \leq M_1(|s|) \leq N(|s|))$ for all $s \in \mathbb{Z}$.

It only remains to verify that $N(s)$ has the doubling property (63). We have for $s \neq 0$ that

$$\begin{aligned} N(s/2) &= \frac{1}{\lfloor s/2 \rfloor + 1} \sum_{\substack{t\leq s/2 \\ t\in\mathbb{N}}} M_1(t) \\ &\leq \frac{1}{\lfloor s/2 \rfloor + 1} \sum_{\substack{t\leq \lfloor s/2 \rfloor \\ t\in\mathbb{N}}} M_1(t) + \frac{1}{\lfloor s/2 \rfloor + 1} \sum_{\substack{\lfloor s/2 \rfloor + 1 \leq t \leq 2\lfloor s/2 \rfloor + 1 \\ t\in\mathbb{N}}} M_1(t) \\ &\leq \frac{1}{\lfloor s/2 \rfloor + 1} \sum_{\substack{t\leq \lfloor s/2 \rfloor \\ t\in\mathbb{N}}} M_1(t) + \frac{1}{\lfloor s/2 \rfloor + 1} \sum_{\substack{\lfloor s/2 \rfloor + 1 \leq t \leq s+1 \\ t\in\mathbb{N}}} M_1(t) \\ &\leq \frac{2}{2\lfloor s/2 \rfloor + 2} \sum_{\substack{t\leq s+1 \\ t\in\mathbb{N}}} M_1(t) \leq \frac{2}{s} \sum_{\substack{t\leq s+1 \\ t\in\mathbb{N}}} M_1(t) \end{aligned}$$

$$\leq \frac{2}{s}\sum_{\substack{t\leq s\\ t\in\mathbb{N}}} M_1(t) + \frac{2}{s}M_1(s+1) \leq \frac{2}{s}\sum_{\substack{t\leq s\\ t\in\mathbb{N}}} M_1(t) + \frac{2}{s}\sum_{\substack{t\leq s\\ t\in\mathbb{N}}} M_1(t)$$
$$\leq \frac{4}{s}\sum_{\substack{t\leq s\\ t\in\mathbb{N}}} M_1(t) \leq \frac{8}{s+1}\sum_{\substack{t\leq s\\ t\in\mathbb{N}}} M_1(t) = 8N(s),$$

as desired. □

5. Single-factor estimates

5.1. Single-factor estimates for Theorem 2.1 and Theorem 2.3. In this section, we construct a function g_k with its support contained in intervals centered at rational numbers with denominator close to some number M_k. Let $\psi(q)$ be a function satisfying (4) and (5). Suppose $\chi(q)$ is a function satisfying (6). In the case of Theorem 2.3, we take $\chi(q) \equiv 1$.

Let M_k be a large positive integer. We choose an integer $\beta(M_k)$ and a positive real number C_k so that

$$1 \leq \sum_{\substack{M_k\leq q\leq \beta(M_k)\\ q \text{ prime}}} \frac{1}{q\chi(q)} = C_k \leq 2.$$

The support of g_k will be contained in a family of intervals centered at rational numbers whose denominator is a prime number between M_k and $\beta(M_k)$.

We choose a nonnegative function $\phi \in C_c^\infty$ with support in the interval $[-1/2, 0]$ satisfying the conditions

$$\hat{\phi}(0) = 1 \tag{34}$$

and

$$\hat{\phi}(s) \lesssim \exp\left(-|s|^{\frac{\sigma+1}{2\sigma}}\right). \tag{35}$$

The existence of such a function is guaranteed by a result of Ingham [6].

Let

$$\phi_{p,q}(x) = \frac{1}{q^2\chi(q)\psi(q)}\phi\left(\frac{1}{\psi(q)}\left(x - \frac{p}{q}\right)\right).$$

Now define

$$g_k(x) = C_k^{-1}\sum_{\substack{M_k\leq q<\beta(M_k)\\ q \text{ prime}}}\sum_{p=1}^{q}\phi_{p,q}(x).$$

Observe that the function g_k is supported on the interval $[0, 1]$.

Lemma 5.1. *Suppose g_k is defined as above. Then we have the following estimates for $s \in \mathbb{Z}$:*

$$\hat{g}_k(0) = 1, \tag{36}$$

$$\hat{g}_k(s) = 0 \quad \text{if } 0 < |s| < M_k, \tag{37}$$

$$|\hat{g}_k(s)| \lesssim \theta(|s|) \quad \text{if } s \neq 0, \tag{38}$$

$$|\hat{g}_k(s)| \lesssim \exp\left(-\frac{1}{2}(\psi(\beta(M_k))^2|s|)^{\frac{\sigma+1}{4\sigma}}\right) \quad \text{if } |s| \geq \psi(\beta(M_k))^{-2}. \tag{39}$$

Proof. A simple calculation gives us that

$$\hat{g}_k(s) = C_k^{-1} \sum_{\substack{M_k \le q < \beta(M_k) \\ q \text{ prime}}} \frac{1}{q^2 \chi(q)} \sum_{p=1}^{q} \mathrm{e}\left(\frac{p}{q} s\right) \hat{\phi}\left(\psi(q) s\right)$$

where $\mathrm{e}(u) = \mathrm{e}^{-2\pi i u}$. The sum in p has the value

$$\sum_{p=1}^{q} \mathrm{e}\left(\frac{p}{q} s\right) = \begin{cases} q & \text{if } q \mid s, \\ 0 & \text{if } q \nmid s. \end{cases}$$

Therefore, if $s = 0$, then the above sum will be equal to 1, and if $0 < |s| < M_k$, then the above sum will vanish. This proves (36) and (37). Thus,

$$\hat{g}_k(s) = C_k^{-1} \sum_{\substack{M_k \le q < \beta(M_k) \\ q \text{ prime} \\ q \mid s}} \frac{\hat{\phi}\left(\psi(q) s\right)}{q \chi(q)}.$$

For $|s| \ge M_k$, we split the above sum into three pieces according to the size of q. We write

$$\hat{g}_k(s) = C_k^{-1}(\mathrm{I} + \mathrm{II} + \mathrm{III}),$$

where

$$\mathrm{I} = \sum_{\substack{q \ge \psi^{-1}(1/|s|) \\ q \text{ prime} \\ q \mid s}} \frac{\hat{\phi}\left(\psi(q) s\right)}{q \chi(q)},$$

$$\mathrm{II} = \sum_{\substack{\psi^{-1}(1/\sqrt{|s|}) \le q \le \psi^{-1}(1/|s|) \\ q \text{ prime} \\ q \mid s}} \frac{\hat{\phi}\left(\psi(q) s\right)}{q \chi(q)},$$

$$\mathrm{III} = \sum_{\substack{q < \psi^{-1}(1/\sqrt{|s|}) \\ q \text{ prime} \\ q \mid s}} \frac{\hat{\phi}\left(\psi(q) s\right)}{q \chi(q)}.$$

Estimate for I. For the sum I, we observe that the number of summands is $\lesssim 1$. This observation is a consequence of assumption (4) since it is implied that for a large enough q depending on ϵ we have,

$$q^{-\tau-\epsilon} \le \psi(q) \le q^{-\tau+\epsilon}$$

which gives us

$$t^{-\frac{1}{\tau+\epsilon}} \le \psi^{-1}(t) \le t^{-\frac{1}{\tau-\epsilon}},$$

since ψ is decreasing. Taking logarithms, we conclude

$$\frac{1}{\tau+\epsilon} \log|s| \le \log \psi^{-1}\left(\frac{1}{|s|}\right) \le \frac{1}{\tau-\epsilon} \log|s|. \tag{40}$$

Hence, the number of summands in the sum I is at most $\frac{\log|s|}{\log \psi^{-1}(1/|s|)} \lesssim 1$.

Apply the bound $\hat{\phi}\,(\psi(q)s) \leq 1$ to each summand to get

$$\sum_{\substack{q \geq \psi^{-1}(1/|s|) \\ q \text{ prime} \\ q|s}} \frac{\hat{\phi}\,(\psi(q)s)}{q\chi(q)} \lesssim \theta(|s|).$$

Estimate for II. For the sum II, we observe that there are $\lesssim 1$ summands by a similar argument as for the sum I. We apply the bound (35) to show that the summand is bounded above by

$$\frac{\exp(-|\psi(q)s|^{\frac{\sigma+1}{2\sigma}})}{q\chi(q)}.$$

If $q = \psi^{-1}(1/|s|)$, then

$$\frac{\exp(-|\psi(q)s|^{\frac{\sigma+1}{2\sigma}})}{q\chi(q)} \lesssim \theta(|s|). \tag{41}$$

It is enough to show for each $q < \psi^{-1}(1/|s|)$ that

$$\frac{\exp(-|\psi(q+1)s|^{\frac{\sigma+1}{2\sigma}})}{(q+1)\chi(q+1)} - \frac{\exp(-|\psi(q)s|^{\frac{\sigma+1}{2\sigma}})}{q\chi(q)} > 0. \tag{42}$$

If the inequality (42) holds for all $q < \psi^{-1}\big(\frac{1}{|s|}\big)$, then the summand is increasing in this domain, and is therefore maximized when $q = \psi^{-1}\big(\frac{1}{|s|}\big)$, establishing the bound (41) for such q.

In order to establish (42), it is enough to verify that the numerator of the difference is positive. This numerator is

$$\exp(-|\psi(q+1)s|^{\frac{\sigma+1}{2\sigma}})q\chi(q) - \exp(-|\psi(q)s|^{\frac{\sigma+1}{2\sigma}})(q+1)\chi(q+1).$$

Since the logarithm is an increasing function, it is enough to show that

$$-|\psi(q+1)s|^{\frac{\sigma+1}{2\sigma}} + \log q + \log\chi(q) > -|\psi(q)s|^{\frac{\sigma+1}{2\sigma}} + \log(q+1) + \log\chi(q+1).$$

This inequality is equivalent to

$$\log(q+1) - \log q + \log\chi(q+1) - \log\chi(q) < |s|^{\frac{\sigma+1}{2\sigma}}\big(\psi(q)^{\frac{\sigma+1}{2\sigma}} - \psi(q+1)^{\frac{\sigma+1}{2\sigma}}\big). \tag{43}$$

The Taylor series for the logarithm guarantees that $\log(q+1) - \log q = \frac{1}{q} + O\big(\frac{1}{q^2}\big)$; the subpolynomial growth condition (7) guarantees that $\log\chi(q+1) - \log\chi(q) = o\big(\frac{1}{q}\big)$. In total, the left side of inequality (43) is $\frac{1}{q} + o\big(\frac{1}{q}\big)$. On the other hand, since we are in the regime where $q < \psi^{-1}(1/|s|)$, the right side of (43) is bounded below by

$$|s|^{\frac{\sigma+1}{2\sigma}}(\psi(q)^{\frac{\sigma+1}{2\sigma}} - \psi(q+1)^{\frac{\sigma+1}{2\sigma}}) \geq 1 - \left(\frac{\psi(q+1)}{\psi(q)}\right)^{\frac{\sigma+1}{2\sigma}}.$$

By (5), we have

$$\left(\frac{\psi(q+1)}{\psi(q)}\right)^{\frac{\sigma+1}{2\sigma}} \leq \left(\frac{q}{q+1}\right)^{\frac{\sigma+1}{2}}.$$

Hence,

$$1 - \left(\frac{\psi(q+1)}{\psi(q)}\right)^{\frac{\sigma+1}{2\sigma}} \geq 1 - \left(\frac{q}{q+1}\right)^{\frac{\sigma+1}{2}} = \left(\frac{\sigma+1}{2}\right)\frac{1}{q} + O\left(\frac{1}{q^2}\right).$$

Since $\frac{\sigma+1}{2} > 1$, we see that the inequality (43) holds for $\psi^{-1}(1/\sqrt{|s|}) \leq q \leq \psi^{-1}(1/|s|)$ provided that M_k (and hence $|s|$) is sufficiently large.

Hence we have the estimate

$$\mathrm{II} \lesssim \theta(|s|).$$

Estimate for III. For the final sum, we apply the estimate (35) to $\hat{\phi}$ to get

$$\begin{aligned}
\sum_{\substack{q<\psi^{-1}(1/\sqrt{|s|}) \\ q \text{ prime} \\ q|s}} \frac{\hat{\phi}\left(\psi(q)s\right)}{q\chi(q)} &\lesssim \sum_{\substack{q<\psi^{-1}(1/\sqrt{|s|}) \\ q \text{ prime} \\ q|s}} \frac{\exp\left(-\left|\psi(q)s\right|^{\frac{\sigma+1}{2\sigma}}\right)}{q} \\
&\leq \sum_{\substack{q<\psi^{-1}(1/\sqrt{|s|}) \\ q \text{ prime} \\ q|s}} \frac{\exp\left(-\left|s\right|^{\frac{\sigma+1}{4\sigma}}\right)}{q} \\
&\leq \exp\left(-\left|s\right|^{\frac{\sigma+1}{4\sigma}}\right) \log\left(\psi^{-1}(1/\sqrt{|s|})\right) \lesssim \theta(|s|).
\end{aligned}$$

For the estimate (39), we observe that $|s|$ is sufficiently large for the estimate (35) to apply to $\hat{\phi}$ for every $q \in [M_k, \beta(M_k)]$. As such

$$|\hat{g}_k(s)| \lesssim \sum_{\substack{M_k \leq q \leq \beta(M_k) \\ q \text{ prime} \\ q|s}} \frac{\exp\left(-(\psi(q)|s|)^{\frac{\sigma+1}{2\sigma}}\right)}{\chi(q)q} \leq \sum_{\substack{M_k \leq q \leq \beta(M_k) \\ q \text{ prime} \\ q|s}} \frac{\exp\left(-(\psi(\beta(M_k))|s|)^{\frac{\sigma+1}{2\sigma}}\right)}{\chi(M_k)M_k}.$$

The inequality $|s| \geq \psi(\beta(M_k))^{-2}$ gives us $\psi(\beta(M_k)) \geq \frac{1}{\sqrt{|s|}}$. Therefore,

$$|\hat{g}_k(s)| \lesssim \sum_{\substack{M_k \leq q \leq \beta(M_k) \\ q \text{ prime} \\ q|s}} \frac{\exp\left(-|s|^{\frac{\sigma+1}{4\sigma}}\right)}{\chi(M_k)M_k}.$$

Observe that the number of summands is less than $\beta(M_k)$. Moreover, we may disregard the denominator, for large M_k, to derive an upper bound. Hence,

$$|\hat{g}_k(s)| \lesssim \beta(M_k) \exp\left(-|s|^{\frac{\sigma+1}{4\sigma}}\right).$$

Now, we need to eliminate the $\beta(M_k)$ term from the estimate, but this will be at the cost of some decay from the exponent. Rewrite the above inequality as

$$\begin{aligned}
|\hat{g}_k(s)| &\lesssim \beta(M_k) \exp\left(-\frac{1}{2}|s|^{\frac{\sigma+1}{4\sigma}}\right) \exp\left(-\frac{1}{2}|s|^{\frac{\sigma+1}{4\sigma}}\right) \\
&\leq \beta(M_k) \exp\left(-\frac{1}{2}\psi(\beta(M_k))^{-\frac{\sigma+1}{2\sigma}}\right) \exp\left(-\frac{1}{2}|s|^{\frac{\sigma+1}{4\sigma}}\right) \qquad (44)
\end{aligned}$$

when we apply $|s| \geq \psi(\beta(M_k))^{-2}$. From the equation (4), when M_k is large enough we have

$$\frac{1}{2}\tau \leq -\frac{\log \psi(\beta(M_k))}{\log \beta(M_k)} \leq 2\tau$$

which may be rewritten as

$$-\frac{1}{2}\tau \log \beta(M_k) \geq \log \psi(\beta(M_k)) \geq -2\tau \log \beta(M_k).$$

Exponentiating gives

$$\beta(M_k)^{-\frac{1}{2}\tau} \geq \psi(\beta(M_k)) \geq \beta(M_k)^{-2\tau}. \tag{45}$$

Applying the upper bound from the equation (45) to (44), we get

$$|\hat{g}_k(s)| \lesssim \beta(M_k) \exp\left(-\frac{1}{2}\beta(M_k)^{\frac{\tau(\sigma+1)}{4\sigma}}\right) \exp\left(-\frac{1}{2}|s|^{\frac{\sigma+1}{4\sigma}}\right).$$

For large M_k, we observe that the exponential term dependent on M_k is decaying much faster than $\beta(M_k)$. Hence,

$$|\hat{g}_k(s)| \lesssim \exp\left(-\frac{1}{2}|s|^{\frac{\sigma+1}{4\sigma}}\right) \lesssim \exp\left(-\frac{1}{2}(\psi(\beta(M_k))^2|s|)^{\frac{\sigma+1}{4\sigma}}\right). \qquad \square$$

5.2. Single-factor estimate for Theorem 2.6. In the case of Theorem 2.6, it is more convenient to choose the function g_k to be supported in a neighborhood of rational numbers with different denominators at very different scales. Thus, only one denominator will meaningfully contribute to the value of $|\hat{g}_k(s)|$.

As in Subsection 5.1, we begin by defining a smooth function ϕ with its support in the interval $[-1/2, 0]$ satisfying the conditions

$$\hat{\phi}(0) = 1 \tag{46}$$

and

$$\hat{\phi}(s) \lesssim \exp\left(-|s|^{3/4}\right). \tag{47}$$

Let n_k be an increasing sequence of integers to be specified later. For a given k, we choose prime numbers $q_{k,1}, \ldots, q_{k,n_k}$ as follows. First, we choose $q_{k,1}$ to be a large prime number. We choose the remaining $q_{k,j}$ so that $q_{k,2} \gg \frac{1}{\psi(q_{k,1})}$, $q_{k,3} \gg \frac{1}{\psi(q_{k,2})}, \ldots,$ $q_{k,n_k} \gg \frac{1}{\psi(q_{k,n_k-1})}$. Furthermore, we also assume that for each j, we have

$$\max\left(\frac{1}{q_{k,j}}, \psi(q_{k,j})\right) < \frac{1}{2}\psi(q_{k,j-1}). \tag{48}$$

Define

$$g_k(x) = \frac{1}{n_k}\sum_{j=1}^{n_k} \frac{1}{q_{k,j}\psi(q_{k,j})} \sum_{p=1}^{q_{k,j}} \phi\left(\frac{1}{\psi(q_{k,j})}\left(x - \frac{p}{q_{k,j}}\right)\right).$$

Then

$$\hat{g}_k(s) = \frac{1}{n_k}\sum_{j=1}^{n_k} \frac{1}{q_{k,j}} \sum_{p=1}^{q_{k,j}} \hat{\phi}(\psi(q_{k,j})s)\, \mathrm{e}\left(\frac{p}{q_{k,j}}s\right).$$

Remove any terms for which $q_{k,j}$ does not divide s to get

$$\hat{g}_k(s) = \frac{1}{n_k} \sum_{\substack{1\leq j\leq n_k \\ q_{k,j}|s}} \hat{\phi}(\psi(q_{k,j})s). \tag{49}$$

Lemma 5.2. *Suppose that g_k is defined as above. Then we have the following estimates for $s \in \mathbb{Z}$:*

$$\hat{g}_k(0) = 1, \tag{50}$$

$$\hat{g}_k(s) = 0 \quad \text{if } 0 < |s| < q_{k,1}, \tag{51}$$

$$|\hat{g}_k(s)| \lesssim \frac{1}{n_k}, \quad s \neq 0, \tag{52}$$

$$|\hat{g}_k(s)| \lesssim \exp\left(-\frac{1}{2}|\psi(q_{k,n_k})s|^{\frac{3}{4}}\right) \quad \text{if } |s| \geq \psi(q_{k,n_k})^{-1}. \tag{53}$$

Proof. First, it is clear from (46) and (49) that $\hat{g}_k(0) = 1$, establishing (50). Moreover, the sum (49) is seen to be empty if $0 < |s| < q_{k,1}$, establishing (51).

To prove (52), we split the sum (49) depending on the size of $q_{k,j}$ relative to s. Suppose $j_0(s)$ is such that $\psi(q_{k,j_0})|s| > 1$, but such that $\psi(q_{k,j_0+1})|s| \leq 1$, taking $j_0(s) = 0$ if $\psi(q_{k,1})|s| < 1$ or $j_0 = n_k$ if $\psi(q_{k,n_k})|s| > 1$,

$$|\hat{g}_k(s)| \leq \frac{1}{n_k} \sum_{\substack{j_0(s)+1 \leq j \leq n_k \\ q_{k,j}|s}} \left|\hat{\phi}(\psi(q_{k,j})s)\right| + \frac{1}{n_k} \sum_{\substack{1 \leq j \leq j_0(s) \\ q_{k,j}|s}} \left|\hat{\phi}(\psi(q_{k,j})s)\right|.$$

For the second sum, we may apply (47), the Schwartz tail for $\hat{\phi}$. Hence, using the assumption (48),

$$\begin{aligned}\frac{1}{n_k} \sum_{\substack{1 \leq j \leq j_0(s) \\ q_{k,j}|s}} \left|\hat{\phi}(\psi(q_{k,j})s)\right| &\lesssim \frac{1}{n_k} \sum_{1 \leq j \leq j_0(s)} \exp\left(-\left|\psi(q_{k,j})s\right|^{\frac{3}{4}}\right) \\ &\lesssim \frac{1}{n_k} \sum_{1 \leq j \leq j_0(s)} \exp\left(-2^{\frac{3(j_0(s)-j)}{4}}\right) \lesssim \frac{1}{n_k}.\end{aligned}$$

For the first sum, recall that j_0 is chosen so that $\psi(q_{k,j_0+1})s < 1$. Since $q_{k,j} \geq \frac{1}{\psi(q_{k,j_0+1})}$ for any $j \geq j_0 + 2$, it follows that $\frac{1}{q_{k,j}}s < 1$ for such j. This means that it is impossible for $q_{k,j}$ to divide s for $j > j_0 + 1$. Hence, the only term that can contribute to the sum is the $j = j_0 + 1$ term. To control the contribution of this term, we simply apply the bound

$$\left|\hat{\phi}(\psi(q_{k,j})s)\right| \leq 1$$

to bound the first sum by a constant times $\frac{1}{n_k}$. Thus, for any integer $s \neq 0$, we have the bound

$$|\hat{g}_k(s)| \lesssim \frac{1}{n_k}.$$

It remains to show the bound (53). For $s \geq \psi(q_{k,n_k})^{-1}$, we can in fact apply the Schwartz bound (47) for ϕ to every summand in (49). Hence

$$\begin{aligned}|\hat{g}_k(s)| &\leq \frac{1}{n_k} \sum_{\substack{1 \leq j \leq n_k \\ q_{k,j}|s}} |\hat{\phi}(\psi(q_{k,j})s)| \lesssim \frac{1}{n_k} \sum_{1 \leq j \leq n_k} \exp\left(-|\psi(q_{k,j})s|^{\frac{3}{4}}\right) \\ &\lesssim \frac{1}{n_k} \sum_{1 \leq j \leq n_k} \exp\left(-|2^{n_k-j}\psi(q_{k,n_k})s|^{\frac{3}{4}}\right) \lesssim \exp\left(-|\psi(q_{k,n_k})s|^{\frac{3}{4}}\right). \qquad \square\end{aligned}$$

6. Stability and convergence of $\hat{\mu}_{\chi,\omega}$

In order to prove Theorems 2.1, 2.3, and 2.6, we will piece together the functions g_k provided in Section 5 across multiple scales. Lemmas 3.1 and 3.2 are used to show that the Fourier transforms $\hat{g}_k$ of the functions g_k do not exhibit much interference. The construction proceeds slightly differently in the case of Theorem 2.6, as this theorem does not prescribe a specific decay rate for $\hat{\mu}$.

6.1. Construction of μ for Theorem 2.1 and Theorem 2.3. Let ψ and χ be functions satisfying the assumptions (4), (5), (6), and (7). Recall that in the case of Theorem 2.3 that we take $\chi \equiv 1$, and we showed in Remark 2.5 that ψ satisfies assumptions (4) and (5). We begin by constructing a sequence of functions $(\mu_{\chi,\omega,k})_{k\in\mathbb{N}}$ where $\mu_{\chi,\omega,k}(x)$ is the product

$$\mu_{\chi,\omega,k}(x) = \prod_{i=1}^{k} g_i(x).$$

For each g_i we choose an associated M_i such that the estimates in Lemma 5.1 apply. We further assume that the M_i's are spaced sufficiently far apart to satisfy the conditions of Lemma 3.1. We also assume that for each $i \geq 1$ we have

$$M_{i+1} \geq 8\psi(\beta(M_i))^{-4}. \tag{54}$$

Taking the Fourier transform of this sequence, we get the sequence $(\hat{\mu}_{\chi,\omega,k})_{k\in\mathbb{N}}$ where

$$\hat{\mu}_{\chi,\omega,k}(s) = \hat{g}_1 * \cdots * \hat{g}_k(s).$$

With this sequence of functions defined, the next objective is to show that the sequence is uniformly convergent and that the functions $\hat{\mu}_{\chi,\omega,i}$ satisfy a similar decay estimate (up to a constant) for all i. We begin with the latter:

Lemma 6.1. *For the sequence of functions $(\hat{\mu}_{\chi,\omega,k})_{k\in\mathbb{N}}$ defined above, we have the following statements for any integers k, l with $k > l$:*

$$|\hat{\mu}_{\chi,\omega,k}(0) - 1| \leq \frac{1}{2}, \tag{55}$$

$$|\hat{\mu}_{\chi,\omega,l}(s) - \hat{\mu}_{\chi,\omega,k}(s)| \lesssim \sum_{j=l+1}^{k} M_j^{-99} \quad \text{when } 0 \leq |s| < M_{l+1}/4, \tag{56}$$

$$|\hat{\mu}_{\chi,\omega,k}(s)| \lesssim \theta(|s|)\omega(|s|) \quad \text{when } |s| \geq M_k/4, \tag{57}$$

$$|\hat{\mu}_{\chi,\omega,k}(s)| \lesssim \exp\left(-\frac{1}{2}\left(\frac{\psi(\beta(M_k))^2|s|}{8}\right)^{\frac{\sigma+1}{4\sigma}}\right) \quad \text{if } |s| \geq 8\psi(\beta(M_k))^{-4}. \tag{58}$$

Note that since μ is a positive measure, (55) *implies that $|\hat{\mu}_{\chi,\omega,k}(s)| \leq \frac{3}{2}$ for all s.*

Proof. We prove Lemma 6.1 by induction and repeated application of Lemma 3.1. We begin with the basis by letting $k = 2$. Then $\hat{\mu}_2 = \hat{g}_1 * \hat{g}_2$. Apply Lemma 3.1 with $H = \hat{g}_1$, $G = \hat{g}_2$, $N_1 = \psi(\beta(M_1))^{-2}$, $N_2 = M_2$ and $N_3 = \psi(\beta(M_2))^{-2}$. Then the estimates (56), (57) and (58) immediately follow from (20), (21) and (22), respectively. The statement (55) can be shown by the following calculation:

$$|\hat{\mu}_{\chi,\omega,2}(0) - 1| \leq |\hat{g}_1(0) - \hat{g}_1 * \hat{g}_2(0)| \leq \mathcal{O}(M_2^{-99}) \leq \frac{1}{2}$$

where the last inequality holds if M_2 is chosen to be sufficiently large.

Now, assume (55), (56), (57), and (58) hold for $\mu_{\chi,\omega,k}$. We seek to prove these estimates for $\mu_{\chi,\omega,k+1}$. We observe that g_{k+1} and $\mu_{\chi,\omega,k}$ satisfy the conditions on G and

H in the statement of Lemma 6.1 with the choices $N_1 = \psi(\beta(M_k))^{-2}$, $N_2 = M_{k+1}$, and $N_3 = \psi(\beta(M_{k+1}))^{-2}$. Hence, Lemma 3.1 implies the estimates

$$(59)\quad |\hat{\mu}_{\chi,\omega,k+1}(s) - \hat{\mu}_{\chi,\omega,k}(s)| \le M_{k+1}^{-99} \quad \text{if } 0 \le |s| \le M_{k+1}/4,$$

$$(60)\quad |\hat{\mu}_{\chi,\omega,k+1}(s)| \lesssim \theta(|s|)\omega(|s|) \quad \text{if } |s| \ge M_{k+1}/4,$$

$$(61)\quad |\hat{\mu}_{\chi,\omega,k+1}(s)| \lesssim \exp\left(-\frac{1}{2}\left(\frac{\psi(\beta(M_k))^2 s}{8}\right)^{\frac{\sigma+1}{4\sigma}}\right) \text{ if } |s| \ge 8\psi(\beta(M_k))^{-4}.$$

The estimates (57) and (58) for $\mu_{\chi,\omega,k+1}$ follow from (60) and (61).

We now show (56) for $\mu_{\chi,\omega,k+1}$. If $l = k$, then (56) follows from (59). Now assume $l < k$ and $|s| \le M_{l+1}$. Then $|s| \le M_{k+1}$, so (59) applies. Also, we can apply the inductive assumption (56) to conclude

$$\begin{aligned}|\hat{\mu}_{\chi,\omega,k+1}(s) - \hat{\mu}_{\chi,\omega,l}(s)| &\le |\hat{\mu}_{\chi,\omega,k+1}(s) - \hat{\mu}_{\chi,\omega,k}(s)| + |\hat{\mu}_{\chi,\omega,k}(s) - \hat{\mu}_{\chi,\omega,l}(s)| \\ &\lesssim M_{k+1}^{-99} + \sum_{j=l+1}^{k} M_j^{-99} = \sum_{j=l+1}^{k+1} M_j^{-99}.\end{aligned}$$

This shows (56) for $\mu_{\chi,\omega,k+1}$.

Finally, we show that $\mu_{\chi,\omega,k+1}$ satisfies (55). We apply (56) with $l = 1$ to conclude

$$|\hat{\mu}_{\chi,\omega,k+1}(0) - 1| = |\hat{\mu}_{\chi,\omega,k+1}(0) - \hat{\mu}_{\chi,\omega,1}(0)| \lesssim \sum_{j=2}^{k+1} M_j^{-99} \le \frac{1}{2},$$

provided that the M_j are chosen to be sufficiently large. □

Turning now to proving the uniform convergence of the sequence $(\hat{\mu}_{\chi,\omega,k})_{k\in\mathbb{N}}$, we have the following lemma.

Lemma 6.2. *The sequence $(\hat{\mu}_{\chi,\omega,k})_{k\in\mathbb{N}}$ converges uniformly for all $s \in \mathbb{Z}$ to some function $M(s)$. This function $M(s)$ has the property that*

$$(62)\quad |M(s)| \lesssim \theta(|s|)\omega(|s|); \quad s \in \mathbb{Z}.$$

Proof. Let $\epsilon > 0$. There exists a k_0, depending on ϵ and ω, sufficiently large such that

$$\theta(|s|)\omega(|s|) < \epsilon/2C_1$$

when $|s| \ge M_{k_0}/4$, and so that

$$\sum_{j=k_0}^{\infty} M_j^{-99} < \epsilon/2C_2.$$

Here C_1 is taken to be the implicit constant for estimate (57) and C_2 is taken to be the implicit constant for the estimate (56).

Suppose first that $|s| \ge M_{k_0}/4$. Then there exists $l \ge k_0$ such that $M_l/4 \le |s| \le M_{l+1}/4$. If $k_0 \le k \le l$, then we apply the estimate (57) to conclude that

$$|\hat{\mu}_{\chi,\omega,k}(s)| \le C_1\theta(|s|)\omega(|s|) \le \epsilon.$$

If $k > l$, then we apply (57) and (56) to conclude that

$$|\hat{\mu}_{\chi,\omega,k}(s)| \le |\hat{\mu}_{\chi,\omega,l}(s)| + |\hat{\mu}_{\chi,\omega,k}(s) - \hat{\mu}_{\chi,\omega,l}(s)| \le C_1\theta(|s|)\omega(|s|) + C_2\sum_{j=l+1}^{k} M_j^{-99} < \epsilon.$$

Thus $|\hat{\mu}_{\chi,\omega,k}(s)| < \epsilon$ whenever $k \ge k_0$ and $|s| \ge M_{k_0}/4$.

When $0 \leq |s| \leq M_{k_0}/4$, applying the estimate (56) for any $k_0 < k < l$ gives

$$|\hat{\mu}_{\chi,\omega,k}(s) - \hat{\mu}_{\chi,\omega,l}(s)| \leq C_2 \sum_{j=n}^{\infty} M_{k+1}^{-99},$$

and the choice of m_0 guarantees that this sum is less than $\epsilon/2$.

Hence the sequence $\hat{\mu}_{\chi,\omega,k}$ has a uniform limit $M(s)$. An upper bound on $|M(s)|$ will follow from Lemma 6.1. Suppose $|s|$ is such that $\frac{M_l}{4} \leq |s| \leq \frac{M_{l+1}}{4}$.

Then the estimate (57) gives that

$$|\hat{\mu}_{\chi,\omega,l}(s)| \lesssim \theta(|s|)\omega(|s|),$$

and (56) and the triangle inequality give

$$\begin{aligned} |M(s)| &\leq |\hat{\mu}_{\chi,\omega,l}(s)| + \limsup_{k \geq l} |\hat{\mu}_{\chi,\omega,k}(s) - \hat{\mu}_{\chi,\omega,l}(s)| \\ &\lesssim \theta(|s|)\omega(|s|) + \sum_{j=l+1}^{\infty} M_j^{-99} \lesssim \theta(|s|)\omega(|s|) + |s|^{-99} \lesssim \theta(|s|)\omega(|s|), \end{aligned}$$

as desired. □

In order to show that the sequence $\mu_{\chi,\omega,k}$ converges to a weak limit μ using the convergence of the $\hat{\mu}_{\chi,\omega,k}(s)$, it is normal to appeal to a theorem such as the Lévy continuity theorem. However, this is slightly inconvenient as we only have estimates for $\hat{\mu}_{\chi,\omega,k}(s)$ at integer values s. We will provide a proof of the weak convergence below. First, we will need the following technical lemma relating the Fourier series of a measure supported on the interval $[0,1]$ to its Fourier transform. A stronger version of this lemma can be found as Lemma 1 of Chapter 17 in the book of Kahane [10].

Lemma 6.3. *Suppose that μ is a measure supported on the interval $[0,1]$ satisfying an estimate of the form*

$$|\hat{\mu}(s)| \lesssim N(|s|) \quad \textit{for all } s \in \mathbb{Z},$$

where $N\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing function satisfying the doubling property

$$N(\xi/2) \lesssim N(\xi) \quad \textit{for all } \xi \in \mathbb{R}^+. \tag{63}$$

Then $|\hat{\mu}(\xi)| \lesssim N(|\xi|)$ for all $\xi \in \mathbb{R}$.

We have already seen that $\theta(\xi)\omega(\xi)$ is a doubling function for $\xi > 0$. Thus we can apply Lemma 6.3.

Lemma 6.4. *The sequence of measures $\mu_{\chi,\omega,k}$ has a nonzero weak limit $\mu_{\chi,\omega}$. This weak limit $\mu_{\chi,\omega}$ satisfies the estimate*

$$\hat{\mu}_{\chi,\omega}(\xi) \lesssim \theta(|\xi|)\omega(|\xi|) \tag{64}$$

for all real numbers ξ.

Proof. Observe that each measure $\mu_{\chi,\omega,k}$ has total variation norm bounded by 2. We claim that the measures $\mu_{\chi,\omega,k}$ have a weak limit. First, by the Banach–Alaoglu theorem, there exists a subsequence μ_{χ,ω,k_j} that has a weak limit $\mu_{\chi,\omega}$. Since each measure μ_{χ,ω,k_j} is supported in $[0,1]$, the weak limit $\mu_{\chi,\omega}$ is supported in $[0,1]$.

In particular, since $\mu_{\chi,\omega}$ is supported in $[0,1]$, each Fourier coefficient $\hat{\mu}_{\chi,\omega}(s)$ of $\mu_{\chi,\omega}$ is obtained by integrating against a continuous, compactly supported function. Hence, for each $s \in \mathbb{Z}$, $\lim_{j\to\infty} \hat{\mu}_{\chi,\omega,k_j}(s) = M(s)$, where $M(s)$ is the limit in Lemma 6.1.

By the corollary to Theorem 25.10 of Billingsley [2], it is enough to check that each weakly convergent subsequence of $\{\mu_{\chi,\omega,k}\}$ converges weakly to $\mu_{\chi,\omega}$. Suppose $\{\nu_{\chi,\omega,k}\}$ is a subsequence of the $\mu_{\chi,\omega,k}$ with some weak limit ν. Then ν is supported on $[0,1]$, so by the same argument as in the previous paragraph, Lemma 6.1 implies that $\hat{\nu}(s) = M(s)$ for every $s \in \mathbb{Z}$. Since a measure supported on $[0,1]$ is uniquely determined by its Fourier–Stieltjes series, it follows that $\nu = \mu_{\chi,\omega}$ as desired.

Notice that the weak limit $\mu_{\chi,\omega}$ satisfies the bound

$$\hat{\mu}_{\chi,\omega}(0) = \lim_{k\to\infty} \hat{\mu}_{\chi,\omega,k}(0) \geq \frac{1}{2}$$

by (55). Therefore, it follows that $\hat{\mu}_{\chi,\omega}(0) > 0$, and therefore the weak limit $\mu_{\chi,\omega}$ is nonzero.

Finally, we verify that $\hat{\mu}_{\chi,\omega}(\xi)$ satisfies the estimate (64). This estimate holds for integer values of s by the estimate (62). Hence, Lemma 6.3 shows that $\hat{\mu}_{\chi,\omega}(\xi)$ satisfies the same estimate for $\xi \in \mathbb{R}$. □

Hence the measures $\mu_{\chi,\omega,k}$ have a weak limit supported on $[0,1]$. We now verify that this weak limit is indeed supported on the set $E(\psi)$.

Lemma 6.5. *Let $\mu_{\chi,\omega}$ be as in Lemma 6.4. Then $\mu_{\chi,\omega}$ is supported on $E(\psi)$.*

Proof. It is easy to see that

$$\operatorname{supp} \phi_{p,q} \subset \left[\frac{p}{q} - \frac{1}{2}\psi(q), \frac{p}{q} + \frac{1}{2}\psi(q)\right]$$

and therefore

$$\operatorname{supp} g_i \subset \bigcup_{\substack{M_i \leq q \leq \beta(M_i) \\ q \text{ prime}}} \bigcup_{p=1}^{q} \left[\frac{p}{q} - \frac{1}{2}\psi(q), \frac{p}{q} + \frac{1}{2}\psi(q)\right].$$

Since each $\mu_{\chi,\omega,k}$ is the product of g_i's, its support is an intersection of these supports:

$$\operatorname{supp} \mu_{\chi,\omega,k} \subset \bigcap_{i=1}^{k} \bigcup_{\substack{M_i \leq q \leq \beta(M_i) \\ q \text{ prime}}} \bigcup_{p=1}^{q} \left[\frac{p}{q} - \frac{1}{2}\psi(q), \frac{p}{q} + \frac{1}{2}\psi(q)\right].$$

Because the measure $\mu_{\chi,\omega}$ is defined as the weak limit of the measures $\mu_{\chi,\omega,k}$, we have the containment

$$\operatorname{supp} \mu_{\chi,\omega} \subset \bigcap_{i=1}^{\infty} \bigcup_{\substack{M_i \leq q \leq \beta(M_i) \\ q \text{ prime}}} \bigcup_{p=1}^{q} \left[\frac{p}{q} - \frac{1}{2}\psi(q), \frac{p}{q} + \frac{1}{2}\psi(q)\right].$$

Observe that if $x \in \operatorname{supp} \mu_{\chi,\omega}$ and $i \in \mathbb{N}$, then x must also lie in one of the intervals

$$\left[\frac{p}{q} - \frac{1}{2}\psi(q), \frac{p}{q} + \frac{1}{2}\psi(q)\right]$$

for some $M_i \leq q \leq \beta(M_i)$.

Therefore, there exists an infinite number of rational numbers $\frac{p}{q}$ which satisfy

$$\left|x - \frac{p}{q}\right| \leq \psi(q)$$

and we may conclude that $\operatorname{supp} \mu_{\chi,\omega} \subset E(\psi)$. □

The measure $\mu_{\chi,\omega}$ satisfies all of the properties required to prove Theorem 2.1. Hence, the proof of Theorem 2.1 is complete.

To show Theorem 2.3, it is also necessary to verify that the support of μ is contained in a set of generalized α-Hausdorff measure equal to zero. This will be shown in Section 7.

6.2. Construction of μ for Theorem 2.6. We now construct the measure μ described in Theorem 2.6. The biggest difference between this construction and the one in the previous subsection is that we do not state explicit quantitative estimates describing the decay of the Fourier transform of the measures.

Choose a positive integer n_1 and let M_1 be a large integer. We will choose the sequences $\{n_j\colon j \geq 2\}$ and $\{M_j\colon j \geq 2\}$ to be rapidly increasing sequences of integers satisfying a certain set of conditions below. For each j, we choose prime numbers $q_{j,1}, \ldots, q_{j,n_j}$ with $M_j \leq q_{j,1} \ll \cdots \ll q_{j,n_j}$. When we choose the M_j, we will impose the condition that $M_{j+1} \gg q_{j,n_j}$ as well. Given $q_{j,1}, \ldots, q_{j,n_j}$ we define the function g_j as in Subsection 5.2.

We define the function μ_k to be the pointwise product

$$\mu_k(x) = \prod_{j=1}^{k} g_j(x)$$

so $\hat{\mu}_1(s) = \hat{g}_1(s)$ and so that for any $k \geq 2$

$$\hat{\mu}_k(s) = \hat{g}_k(s) * \hat{\mu}_{k-1}(s).$$

We are now ready to state the main estimate on $\hat{\mu}_k$.

Lemma 6.6. (Main estimate for $\hat{\mu}_k$) *Suppose that the functions g_k are chosen as above. Then provided that the sequences n_j and M_j are chosen appropriately, the measures $\hat{\mu}_k$ satisfy the following estimates for all integers $k \geq l$. All implicit constants below are assumed to be independent of k and l.*

$$|\hat{\mu}_k(0) - 1| \leq \frac{1}{2}, \tag{65}$$

$$|\hat{\mu}_k(s) - \hat{\mu}_l(s)| \leq \sum_{j=l+1}^{k} M_j^{-99} \quad \text{if } 0 \leq |s| \leq M_{l+1}/4, \tag{66}$$

$$|\hat{\mu}_k(s)| \lesssim n_k^{-1/3} \quad \text{if } |s| \geq M_k/4, \tag{67}$$

$$|\hat{\mu}_k(s)| \lesssim \exp\left(-\frac{1}{2}\left|\frac{1}{8}\psi(q_{k,n_k})s\right|^{\frac{3}{4}}\right) \quad \text{if } |s| \geq 8\psi(q_{k,n_k})^{-2}. \tag{68}$$

Proof. Let n_1 and M_1 be positive integers, and choose prime numbers $q_{1,1}, \ldots, q_{1,n_1}$ such that $1 < q_{1,1} < q_{1,2} < \cdots < q_{1,n_1}$ satisfy the conditions of Lemma 5.2. Then $\hat{g}_1$ satisfies the estimates of Lemma 5.2 and in particular satisfies the estimates of Lemma 6.6.

Given $g_1, \ldots, g_k$ such that μ_k satisfies the four conditions above, we will describe how to choose the integers n_{k+1} and M_{k+1} and how to choose the function g_{k+1} so that μ_{k+1} will satisfy the four conditions above. Let $N_1 = \psi(\beta(M_k))^{-1}$. Lemma 3.2 requires that the quantity δ is chosen so that $\delta < \frac{1}{N_1^3}$; hence, we select $n_{k+1} = 100N_1^3$. Choose $M_{k+1} = N_2 \gg n_{k+1}$ to be a prime number that is sufficiently large to satisfy the conditions of Lemma 3.2. Take $N_2 = q_{k+1,1} < \cdots < q_{k+1,n_{k+1}}$ sufficiently well-spaced to satisfy the conditions of Lemma 5.2. Then, choose $N_3 = \frac{1}{\psi(q_{k+1,n_{k+1}})} \gg$

$q_{k+1,1}$. With these choices, we define g_{k+1} as in Subsection 5.2. Hence Lemma 5.2 implies that $\hat{g}_{k+1}$ satisfies the estimates required to serve as the function G in Lemma 3.2.

Hence, we can apply Lemma 3.2 with $H = \hat{\mu}_k, G = \hat{g}_{k+1}$, $N_1 = \psi(q_{k,n_k})^{-1}$, $N_2 = q_{k+1,1}$, and $N_3 = \psi(q_{k+1,n_{k+1}})^{-1}$, and $\delta = \frac{1}{n_{k+1}}$.

This implies the estimates

$$|\hat{\mu}_{k+1}(s) - \hat{\mu}_k(s)| \le M_{k+1}^{-99} \quad \text{if } 0 \le |s| \le \frac{M_{k+1}}{4}, \tag{69}$$

$$|\hat{\mu}_{k+1}(s)| \lesssim n_{k+1}^{-1/3} \quad \text{if } |s| \ge \frac{M_{k+1}}{4}, \tag{70}$$

$$|\hat{\mu}_{k+1}(s)| \lesssim \exp\left(-\frac{1}{2}\left|\frac{1}{8}\psi(q_{k+1,n_{k+1}})s\right|^{\frac{3}{4}}\right) \quad \text{if } |s| \ge 8\psi(q_{k+1,n_{k+1}})^{-2}. \tag{71}$$

Hence $\hat{\mu}_{k+1}$ satisfies the estimates (67) and (68). In order to check (66), assume $l < k+1$ and $|s| \le \frac{M_l}{4}$. If $l = k$, then the inequality follows from (69). If $l < k$, then applying the inductive assumption (66) to estimate the difference $\hat{\mu}_k - \hat{\mu}_l$ and applying (69) to estimate the difference $\hat{\mu}_{k+1} - \hat{\mu}_k$ gives

$$\begin{aligned} |\hat{\mu}_{k+1}(s) - \hat{\mu}_l(s)| &\le |\hat{\mu}_{k+1}(s) - \hat{\mu}_k(s)| + |\hat{\mu}_k(s) - \hat{\mu}_l(s)| \\ &\le M_{k+1}^{-99} + \sum_{j=l+1}^{k} M_j^{-99} = \sum_{j=l+1}^{k+1} M_j^{-99}. \end{aligned}$$

This establishes (66) for $\hat{\mu}_{k+1}$. Applying (66) with $l = 1$ and $s = 0$, we see that

$$|\hat{\mu}_{k+1}(0) - 1| = |\hat{\mu}_{k+1}(0) - \hat{\mu}_1(0)| \le \sum_{j=2}^{k+1} M_j^{-99} \le \frac{1}{2}$$

assuming the M_j grow sufficiently rapidly. □

Lemma 6.7. *The sequence $\hat{\mu}_k$ converges uniformly for all $s \in \mathbb{Z}$ to a function $M(s)$. This function $M(s)$ has the property that $|M(s)| \to 0$ as $|s| \to \infty$.*

Proof. The proof is similar to that of Lemma 6.2. Let $\epsilon > 0$. Because $n_k \to \infty$, there is an index k_0 such that $n_{k_0}^{-1/3} + \sum_{j=k_0+1}^{\infty} M_j^{-99} < \epsilon/2C$, where C is the implicit constant from (67).

Suppose $|s| > \frac{M_{k_0}}{4}$, and choose $l \ge k_0$ such that $\frac{M_l}{4} \le |s| < \frac{M_{l+1}}{4}$. Suppose first that $k_0 \le k \le l$. For such k, We have that $|\hat{\mu}_k(s)| \lesssim n_l^{-1/3} \le n_{k_0}^{-1/3} < \frac{\epsilon}{2}$ by (67). If, instead, $k > l$, then we have

$$|\hat{\mu}_k(s)| \le |\hat{\mu}_l(s)| + |\hat{\mu}_k(s) - \hat{\mu}_l(s)| \le \frac{\epsilon}{2} + \sum_{j=l+1}^{k} M_j^{-99} \le \frac{\epsilon}{2} + \sum_{j=k_0+1}^{\infty} M_j^{-99} \le \epsilon.$$

Hence $|\hat{\mu}_k(s)| \le \epsilon$ for all $|s| \ge \frac{M_{k_0}}{4}$ and all $k \ge k_0$.

If $|s| \le \frac{M_{k_0}}{4}$ and $k_0 \le l \le k$, then we have

$$|\hat{\mu}_k(s) - \hat{\mu}_l(s)| \le \sum_{j=l+1}^{k} M_j^{-99} \le \sum_{j=k_0+1}^{\infty} M_j^{-99} < \epsilon/2.$$

This proves that the sequence $\hat{\mu}_k(s)$ is uniformly Cauchy and hence uniformly convergent. Let $M(s)$ denote the uniform limit of this sequence.

Finally, we verify that $M(s) \to 0$ as $|s| \to \infty$. Suppose $|s|$ is such that $\frac{M_k}{4} \le |s| \le \frac{M_{k+1}}{4}$. Then we have from Lemma 6.6 that $|\hat{\mu}_k(s)| \lesssim n_k^{-1/3}$, and

$$M(s) \lesssim |\hat{\mu}_k(s)| + \limsup_{l\to\infty} |\hat{\mu}_l(s) - \hat{\mu}_k(s)| \lesssim n_k^{-1/3} + \sum_{j=k+1}^{\infty} M_j^{-99} \lesssim n_k^{-1/3},$$

since $M_{k+1} \gg n_k$. Since the sequence $n_k \to \infty$, this shows that $M(s) \to 0$ as $|s| \to \infty$, as desired. □

The rest of this proof is similar to the proof of Theorem 2.1. In order to apply Lemma 6.3, we use the fact from Lemma 4.3 that M is majorized by $N(|s|)$, where N is a doubling function. This will allow us to apply Lemma 6.3.

We are now ready to show that the measures μ_k converge to a weak limit.

Lemma 6.8. *The sequence of measures μ_k has a nonzero weak limit μ. This weak limit μ satisfies the estimate*

$$|\hat{\mu}(\xi)| \to 0 \quad \text{as } |\xi| \to \infty \text{ in } \mathbb{R}.$$

Hence μ is a Rajchman measure.

Proof. This proof is almost exactly the same as the proof of Lemma 6.4, but when we apply Lemma 6.3, we use $N(|s|)$ as the bound on $M(s)$, where $N(s)$ is the function constructed in Lemma 4.3. □

Lemma 6.9. *Let μ be the weak limit in Lemma 6.8. Then the support of μ is contained in $E(\psi)$.*

Proof. This lemma can be shown in a similar manner to Lemma 6.5; we see that if $x \in \operatorname{supp}(\mu)$ then for each k, there exists a denominator q_{k,j_k} and a numerator p_{k,j_k} such that $\left|x - \frac{p_{k,j_k}}{q_{k,j_k}}\right| \le \psi(q_{k,j_k})$; hence, x is ψ-well-approximable. □

Therefore, the measure μ satisfies all of the properties promised in the statement of Theorem 2.6. Thus the proof of Theorem 2.6 is complete.

7. A bound on the generalized Hausdorff measure

To complete the proof of Theorem 2.3, we must show that F_α, which is taken to be the support of $\mu_{k,\omega}$, has zero α-Hausdorff measure.

Lemma 7.1. *Let F_α be a closed subset of*

$$\Big\{x\colon \left|x - \frac{r}{q}\right| < \psi(q) \text{ for some integers } 0 \le r \le q-1,\ M_k \le q \le \beta(M_k),\ q \text{ prime},\ k \in \mathbb{N}\Big\}.$$

Let $\epsilon > 0$. Then there exists a cover $\mathcal{U}$ of F_α by open intervals U such that

$$\sum_{U\in\mathcal{U}} \alpha(\operatorname{diam}(U)) < \epsilon.$$

Proof. The set F_α satisfies the following containment:

$$F_\alpha \subset \bigcap_{k=1}^{\infty} \bigcup_{\substack{M_k \le q \le \beta(M_k)\\ q \text{ prime}}} \bigcup_{p=1}^{q} \left[\frac{p}{q} - \psi(q), \frac{p}{q} + \psi(q)\right].$$

For any k the following collection of closed intervals is a cover for F_α:

$$\left\{\left[\frac{p}{q}-\psi(q),\frac{p}{q}+\psi(q)\right] : M_k \le q \le \beta(M_k),\ q \text{ prime},\ 1 \le p \le q\right\}.$$

Denote this collection as $\mathcal{I}_k$. The following collection $\mathcal{U}$ is also a cover of F_α:

$$\mathcal{U}=\left\{J\bigcap\left[\frac{p}{q}-\psi(q),\frac{p}{q}+\psi(q)\right] : M_k \le q \le \beta(M_k),\ q \text{ prime},\ 1 \le p \le q;\ J\in\mathcal{I}_{k-1}\right\}.$$

Fix a prime number q with $M_k \le q \le \beta(M_k)$ and let $J \in \mathcal{I}_{k-1}$. Observe that the intersection of J with the interval $\left[\frac{p}{q}-\psi(q),\frac{p}{q}+\psi(q)\right]$ is either empty or is a closed interval of length at most $2\psi(q)$. We claim that the number of such intervals that intersect J satisfies

$$\#\left\{p\colon \left[\frac{p}{q}-\psi(q),\frac{p}{q}+\psi(q)\right]\cap J\neq\emptyset\right\}\sim |J|q,$$

where $|J|$ denotes the length of the interval J.

The interval J belongs to $\mathcal{I}_{k-1}$. Therefore, $|J|\ge\psi(\beta(M_{k-1}))^{-4}$. By the assumption (54), we know that $|J|\gg\frac{1}{M_k}$, and therefore, $|J|\gg\frac{1}{q}$.

The interval $[p/q-\psi(q),p/q+\psi(q)]$ intersects J if and only if p/q lies in a $\psi(q)$-neighborhood of J. Since $\psi(q)\approx q^{-\tau}$ by (4) and $\tau>2$, we have that $\psi(q)\ll 1/q$ if k is sufficiently large. Hence, $[p/q-\psi(q),p/q+\psi(q)]$ intersects J if and only if p/q lies in an interval J' of length $|J'|=|J|+2\psi(q)\sim|J|$.

Write $J'=[a,b]$. Then the smallest multiple of $1/q$ contained in J' is $\frac{\lceil qa\rceil}{q}$, and the largest multiple of $1/q$ contained in J' is $\frac{\lfloor qb\rfloor}{q}$. So the total number of multiples of $1/q$ contained in J' is

$$\lfloor qb\rfloor-\lceil qa\rceil+1=qb-qa+\mathcal{O}(1)=q|J'|+\mathcal{O}(1)\sim q|J|+\mathcal{O}(1).$$

Since $|J|\gg 1/q$, we have $q|J|\gg 1$. Therefore,

$$\#\left\{p\colon \left[\frac{p}{q}-\psi(q),\frac{p}{q}+\psi(q)\right]\cap J\neq\emptyset\right\}\sim |J|q,$$

as claimed.

Then

$$\begin{aligned}\sum_{\mathrm{U}\in\mathcal{U}}\alpha(\mathrm{diam(U)})&\le\sum_{J\in\mathcal{I}_{k-1}}\sum_{\substack{M_k\le q\le\beta(M_k)\\ q\text{ prime}}}\sum_{1\le p\le q}\alpha\left(\mathrm{diam}\left(J\bigcap\left[\frac{p}{q}-\psi(q),\frac{p}{q}+\psi(q)\right]\right)\right)\\ &\sim\sum_{J\in\mathcal{I}_{k-1}}|J|\sum_{\substack{M_k\le q\le\beta(M_k)\\ q\text{ prime}}}q\alpha(\psi(q)).\end{aligned}$$

From assumption (12), $\alpha(\psi(q))=q^{-2}$. Therefore,

$$\sum_{J\in\mathcal{I}_{k-1}}|J|\sum_{\substack{M_k\le q\le\beta(M_k)\\ q\text{ prime}}}q\alpha(\psi(q))=\sum_{J\in\mathcal{I}_{k-1}}|J|\sum_{\substack{M_k\le q\le\beta(M_k)\\ q\text{ prime}}}q^{-1}.$$

Recall that we chose $\beta(M_k)$ so that

$$1\le\sum_{\substack{M_k\le q\le\beta(M_k)\\ q\text{ prime}}}q^{-1}\le 2.$$

Consequently,

$$\sum_{J\in\mathcal{I}_{k-1}} |J| \sum_{\substack{M_k\le q\le\beta(M_k)\\ q \text{ prime}}} q^{-1} \sim \sum_{J\in\mathcal{I}_k} |J| \lesssim \sum_{\substack{M_{k-1}\le q\le\beta(M_{k-1})\\ q \text{ prime}}} \sum_{1\le p\le q} \psi(q)$$
$$= \sum_{\substack{M_{k-1}\le q\le\beta(M_{k-1})\\ q \text{ prime}}} q\psi(q).$$

Recall that $\psi(q) \approx q^{-\tau}$, so

$$\sum_{\substack{M_{k-1}\le q\le\beta(M_{k-1})\\ q \text{ prime}}} q\psi(q) \approx \sum_{\substack{M_{k-1}\le q\le\beta(M_{k-1})\\ q \text{ prime}}} q^{-\tau+1} \lesssim M_{k-1}^{-\tau+2}.$$

The exponent $-\tau+2<0$. Hence, if k is chosen to be sufficiently large, we have

$$\sum_{U\in\mathcal{U}} \alpha(U) < \epsilon,$$

as desired. □

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References

[1] Besicovitch, A. S.: On Kakeya's problem and a similar one. - Math. Z. 27:1, 1928, 312–320.

[2] Billingsley, P.: Probability and measure. - Wiley Ser. Probab. Stat., anniversary edition, John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[3] Bluhm, C.: Random recursive construction of Salem sets. - Ark. Mat. 34:1, 1996, 51–63.

[4] Bluhm, C. E.: Liouville numbers, Rajchman measures, and small Cantor sets. - Proc. Amer. Math. Soc. 128:9, 2000, 2637–2640.

[5] Hambrook, K.: Explicit Salem sets and applications to metrical Diophantine approximation. - Trans. Amer. Math. Soc. 371:6, 2019, 4353–4376.

[6] Ingham, A. E.: A note on Fourier transforms. - J. London Math. Soc. 9:1, 1934, 29–32.

[7] Jarník, V.: Diophantischen Approximationen und Hausdorffsches Mass. - Mat. Sborjnik 36, 1929, 371–382.

[8] Kahane, J.-P.: Images browniennes des ensembles parfaits. - C. R. Acad. Sci. Paris Sér. A-B 263, 1966, A613–A615.

[9] Kahane, J.-P.: Images d'ensembles parfaits par des séries de Fourier gaussiennes. - C. R. Acad. Sci. Paris Sér. A-B 263, 1966, A678–A681.

[10] Kahane, J.-P.: Some random series of functions. - Cambridge Stud. Adv. Math. 5, Cambridge Univ. Press, Cambridge, second edition, 1985.

[11] Kaufman, R.: On the theorem of Jarník and Besicovitch. - Acta Arith. 39:3, 1981, 265–267.

[12] Mattila, P.: Fourier analysis and Hausdorff dimension. - Cambridge Stud. Adv. Math. 150, Cambridge Univ. Press, Cambridge, 2015.

[13] Mitsis, T.: A Stein–Tomas restriction theorem for general measures. - Publ. Math. Debrecen 60:1-2, 2002, 89–99.

[14] Polasek, I., and E. Rela: The exact dimension of Liouville numbers: The Fourier side. - arXiv:2408.04148 [math.CA], 2024.

[15] Salem, R.: On singular monotonic functions whose spectrum has a given Hausdorff dimension. - Ark. Mat. 1, 1951, 353–365.

[16] Wolff, T.: Lecture notes on harmonic analysis. - Amer. Math. Soc., 2003.

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