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A dual simple proof of the classical Bernstein and Calabi–Bernstein theorems

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Abstract. In this note, we present a short and simple proof of both the Bernstein theorem and the Calabi–Bernstein theorem, which allows us to visualize both the common features and the remarkable differences between the two results. The proofs are based on the construction of conformal metrics that ensure the parabolicity of the surfaces. Consequently, they do not rely on complex analysis.

Klassisen Bernsteinin ja Calabin–Bernsteinin lauseiden yksinkertainen todistus

Tiivistelmä. Tässä työssä esitetään sekä Bernsteinin että Calabin–Bernsteinin lauseelle lyhyt ja yksinkertainen todistus, joka havainnollistaa sekä näiden tulosten yhteisiä piirteitä että niiden merkittäviä eroja. Todistukset perustuvat pintojen parabolisuuden takaavien konformimetriikoiden rakentamiseen; ne eivät käytä kompleksianalyysiä.

1. Introduction

There are two historically significant examples of non-linear elliptic partial differential equations whose origins lie in the Differential Geometry of surfaces. Let us recall the minimal surface equation in the Euclidean space $\mathbb{R}^3$. Given a smooth function $u\colon \Omega \longrightarrow \mathbb{R}$ on a domain $\Omega \subseteq \mathbb{R}^2$, the problem is described by the following non-linear differential equation in divergence form,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0, \tag{1}$$

where D and div denote the gradient and divergence operators in the Euclidean plane, respectively.

The second case is the spacelike maximal surface equation in the Lorentz–Minkowski space, $\mathbb{L}^3$. With standard Cartesian coordinates (x, y, t), the metric is given by $g_L = dx^2 + dy^2 - dt^2$, and the problem is given in the graph $t = u(x, y)$ by

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du|^2 < 1, \tag{2}$$

where D and div also denote the gradient and divergence operators in the Euclidean plane, respectively. The inequality assumption $|Du|^2 < 1$ ensures that the graph of each solution is spacelike. Moreover, this problem is elliptic due to this new condition.

In both cases, the previous equations come from the Euler–Lagrange equation of a geometrical variational problem. For each $u \in C^\infty(\Omega)$, Ω an open domain in $\mathbb{R}^2$, the 2-form $\sqrt{1+|Du|^2}\, dA$ (resp. $\sqrt{1-|Du|^2}\, dA$) on Ω represents the area

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element of the induced metric from the Euclidean space $\mathbb{R}^3$ (resp. $\mathbb{L}^3$) on the graph $\Sigma_u = \{(x, y, u(x, y))\ (x, y) \in \Omega\}$ (resp. spacelike graph). Indeed, the stationary points of the functional $u \longmapsto \int \sqrt{1 + |Du|^2}\, dA$ (resp. $u \longmapsto \int \sqrt{1 - |Du|^2}\, dA$) are characterized by Eq. (1) (resp. Eq. (2)). Note that, in the second problem the graph of u is stationary among the functions satisfying the mentioned spacelike condition.,

We are interested in entire solutions of Eq. (1) and Eq. (2), that is, solutions defined on the whole Euclidean plane $\mathbb{R}^2$. In both cases, trivial solutions are given by the affine functions (with the spacelike condition in the second case).

In a seminal result of Bernstein (1914) [4], amended by Hopf (1950) [13], it was established that

Theorem A. *The only entire solutions to Eq. (1) on the Euclidean plane are the affine functions.*

Geometrically this statement means that each entire minimal graph in the 3-dimensional Euclidean space is a plane. The proof given by Bernstein is based on the analytical theory of partial differential equations. This theorem is considered to be a key result in the global theory of minimal surfaces. A more geometrical proof making use of the Enneper–Weierstrass representation is exposed in the Osserman's book "A survey of minimal surfaces" [16]. A new simple proof was given by Chern [8] in 1969, and it was the first that did not use complex function theory. The proof of Chern is based on applications of isothermal coordinates on the surface and the Liouville theorem.

Analogously, one of the most relevant results in the context of the global geometry of spacelike surfaces in $\mathbb{L}^3$ is the Calabi–Bernstein theorem.

Theorem B. *The only entire solutions to the maximal surface equation Eq. (2) on the Euclidean plane $\mathbb{R}^2$ are affine functions.*

This result was established by Calabi [5], inspired by the classical Bernstein theorem via a duality between solutions of Eq. (1) and Eq. (2). In 1983, Kobayashi [15] gave a proof of the Calabi–Bernstein theorem through the corresponding Enneper–Weierstrass parametrization for maximal surfaces. Another derivation making use of Enneper–Weierstrass representation was given by Estudillo and Romero [12] via a local estimation of the Gaussian curvature of the surface.

Without using complex analysis, Romero [18] gave a simple proof (1996) making use of the Liouville theorem for harmonic functions on the Euclidean plane. Later, in 2010, Romero and Rubio [19] gave a new proof through a local estimation of the integral of a distinguished function defined on the annulus and the concept of parabolicity. In addition, in 2015 another proof was given by Aledo, Romero and Rubio [2] via the Bochner technique. In this work, we present two analogous simple proofs of Bernstein and Calabi–Bernstein classical theorems. Both of them are totally developed without complex analysis and are based on the use of suitable conformal metrics and the parabolicity of certain Riemannian surfaces.

Our approach shows clearly the common point, as well as the notable differences between both problems. We believe these simple proofs may be of interest and easily understood by young researchers.

We finish this introduction by making a brief mention of several extension to classical Bernstein and Calabi–Bernstein theorems.

On one hand, in the Calabi–Bernstein case, Cheng and Yau [7] extended the theorem to arbitrary dimensions.

On the other hand, the possible extension of the classical Bernstein theorem to higher dimensions is known as the *Bernstein conjecture*. It has been an amusing research topic for a long time and it has resulted in many advances on geometric analysis (see [17] for a detailed survey until 1984). In 1968, Simons [21] proved a result which in combination with theorems of De Giorgi [11] and Fleming [10] yield a proof of the Bernstein conjecture for $n \leq 7$. Moreover, there is a counterexample $u \in C^\infty(\mathbb{R}^n)$ to the Bernstein conjecture for each $n \geq 8$.

Let us point out another related scenario to the Bernstein problem, this is the Chern conjeture in affine geometry [9], namely that an affine maximal graph of a smooth, locally uniformly convex function on two dimensional Euclidean space, $\mathbb{R}^2$, must be a paraboloid. The concept of maximal hypersurface in this setting, was introduced by Calabi as a critical point of certain functional, called affine area functional [6]. Trudinger and Wang [22] verified the Chern conjecture and they extended the result to arbitrary dimension when a uniform strict convexity condition holds.

In the context of 3-dimensional Riemannian or Lorentzian product spaces $\mathbb{R} \times M^2$, where M^2 denotes a complete Riemannian surface with non-negative Gaussian curvature, we will highlight three interesting results. In 2002, Rosenberg [20] showed that an entire minimal graph in the Riemannian manifold $\mathbb{R} \times M^2$ must be totally geodesic. In [3], Alías, Dajczer and Ripoll completed Rosenberg's result showing that an entire minimal graph in $\mathbb{R} \times M^2$, with M^2 a complete Riemannian surface with non-negative Gaussian curvature K, and $K(p_0) > 0$ at some point $p_0 \in M^2$, must be a slice $\{t_0\} \times M^2$, $t_0 \in \mathbb{R}$. For the Lorentzian product $\mathbb{R} \times M^2$, Albujer and Alías [1, Th. 4.1] showed an analogous result, that is, any entire maximal graph in the Lorentzian product is totally geodesic. In additions, if $K(p_0) > 0$ at some point $p_0 \in M^2$, must be a slice $\{t_0\} \times M^2$, $t_0 \in \mathbb{R}$.

2. Preliminary formulae

Consider the Euclidean space $(\mathbb{R}^3, g)$, where $g = \langle\, , \rangle$ is the usual inner product, and (x, y, t) the standard Euclidean coordinates. Let S be a smooth surface in $\mathbb{R}^3$ endowed with the induced metric, i.e., $(S, g|_S)$ is a Riemannian surface. Suppose that S is orientable and let N be a unitary normal vector field on S. Consider $X, Y \in \mathfrak{X}(S)$ two smooth vector fields on S. The Gauss formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + II(X, Y),$$

where $\overline{\nabla}, \nabla$ denote the Levi–Civita connections of $\mathbb{R}^3$ and S respectively, and II is the second fundamental form of S. Let $A\colon \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$ be the Weingarten operator defined by $A(X) = -\overline{\nabla}_X N$. Given a point $p \in S$, if $\kappa_i(p)$, $i = 1, 2$, denote the principal curvatures, $H(p)$ the mean curvature and $K(p)$ the Gauss curvature at p, then they are given by

$$H(p) = \frac{1}{2}\,\mathrm{Trace}(A_p) = \frac{1}{2}(\kappa_1(p) + \kappa_2(p)), \quad K(p) = \det(A_p) = \kappa_1(p).\,\kappa_2(p).$$

Now, assume that S is minimal, i.e., its mean curvature vanishes identically, $H \equiv 0$. Then, the Gaussian curvature of S can be expressed as $K = -\frac{1}{2}\,\mathrm{Trace}(A^2)$.

Let us define the smooth function on S,

$$\cos(\theta) := g(N, \partial_t).$$

The following two formulas will be crucial for the proof of Bernstein's result in the next section, as long as S is *minimal*:

$$|\nabla \cos(\theta)|^2 = \frac{1}{2}\operatorname{Trace}(A^2)\sin^2(\theta), \tag{3}$$

$$\Delta \cos(\theta) = -\operatorname{Trace}(A^2)\cos(\theta). \tag{4}$$

In order to prove (3), we compute $\nabla\cos(\theta)$. For all tangent vector field to S, $X \in \mathfrak{X}(S)$,

$$\langle \nabla \cos(\theta), X\rangle = \langle \nabla_X N, \partial_t\rangle + \langle N, \nabla_X \partial_t\rangle = -\langle A(X), \partial_t^T\rangle = \langle -A(\partial_t), X\rangle,$$

where we have taken into account that ∂_t is parallel and the Weingarten operator is self-adjoint.

Now, let $\{E_1, E_2\}$ be a local orthonormal vector field basis in S such that $A(E_i) = \kappa_i E_i$, $i = 1, 2$. Then

$$|\nabla \cos(\theta)|^2 = \sum_{i=1,2}\langle \nabla \cos(\theta), E_i\rangle^2 = \sum_{i=1,2}\langle A(\partial_t^T), E_i\rangle^2 = \sum_{i=1,2}\langle \partial_t^T, A(E_i)\rangle^2.$$

If we assume that S is minimal, then $\kappa_1 = -\kappa_2 \equiv \kappa$, and

$$1 = |\partial_t|^2 = \cos^2(\theta) + \sum_{i=1,2}\langle \partial_t, E_i\rangle^2,$$

proving formula (3).

To prove formula (4), we recall that the Codazzi equation in $\mathbb{R}^3$ (and $\mathbb{L}^3$) is expressed as

$$(\nabla_X A)(Y) = (\nabla_Y A)(X), \quad \forall X, Y \in \mathfrak{X}(S).$$

Hence,

$$\Delta\cos(\theta) = -\operatorname{Trace}\{X \longmapsto \nabla_X\big(A(\partial_t^T)\big)\} = -\sum_{i=1,2}\langle \nabla_{E_i}\big(A(\partial_t^T)\big), E_i\rangle.$$

Making use of the above expression for $\nabla\cos(\theta)$, we get

$$\nabla_{E_i}\big(A(\partial_t^T)\big) = (\nabla_{E_i}A)(\partial_t^T) + A\big(\nabla_{E_i}(\partial_t^T)\big) = (\nabla_{E_i}A)(\partial_t^T) + \cos(\theta)\, A^2(E_i).$$

Finally, Codazzi's formula allow us to compute,

$$\begin{aligned}\Delta \cos(\theta) &= -\sum_{i=1}^{2}\langle(\nabla_{\partial_t^T}A)(E_i), E_i\rangle - \cos(\theta)\operatorname{Trace}(A^2)\\ &= \operatorname{Trace}\big(\nabla_{\partial_t^T}A\big) - \cos(\theta)\operatorname{Trace}(A^2).\end{aligned}$$

Since $\operatorname{Trace}(A) = 0$ and the covariant derivative commutes with the contractions, the term $\operatorname{Trace}\big(\nabla_{\partial_t^T}A\big)$ identically vanishes, and formula (4) is proved.

In the 3-dimensional Lorentz–Minkowski space, $\mathbb{L}^3 = \{(x, y, t)\ /x, y, t \in \mathbb{R}\}$, endowed with the product $g_L = dx^2 + dy^2 - dt^2$, analogous formulas to (3) and (4) may be obtained. Consider S a smooth surface in $\mathbb{L}^3$. We say that S is spacelike if, endowed with the induced metric from $\mathbb{L}^3$, $(S, g_L|_S)$ is a Riemannian surface. Note that in this case S must be non compact and orientable.

Let us take the unit timelike normal vector field N on S such that

$$\cosh(\theta) := g_L(N, \partial_t) \geq 1.$$

The tangential component of ∂_t on S is given at any point by $\partial_t^T = \partial_t + \cosh(\theta)N$. We also denote the Weingarten operator associated to N by A, it is easy to see that

$$\nabla \cosh(\theta) = -A\big(\partial_t^T\big),$$

where ∇ is again the Levi–Civita connection of $g_L|_S$. From now on, we will identify g_L and $g_L|_S$ for short.

On the other hand, when the mean curvature function of S vanishes identically, the surface is called maximal. In this case, computations similar to the Euclidean setting provide the following two formulae

$$|\nabla \cosh(\theta)|^2 = \frac{1}{2}\,\mathrm{Trace}(A^2)\sinh(\theta)^2, \tag{5}$$

$$\Delta \cosh(\theta) = \mathrm{Trace}(A^2)\cosh(\theta), \tag{6}$$

being Δ the Laplacian operator relative to (S, g_L).

3. Proof of the Bernstein theorem

Let $u \in C^\infty(\mathbb{R}^2)$ be a smooth function defined on $\mathbb{R}^2$, and consider the regular surface $\Sigma_u = \{\big(x, y, u(x, y)\big),\ x, y \in \mathbb{R}\}$ given by the graph of u on $\mathbb{R}^2$, endowed with the induced metric, which will be denoted by g. Take N_u to be the unitary normal vector field on Σ_u such that $\cos(\theta_p) := g|_p(N_u(p), \partial_t|_p) \geqslant 0$ for all $p \in \Sigma_u$. Assume Σ_u is minimal, i.e., its mean curvature function vanishes identically, $H \equiv 0$. Then, as a direct consequence, the Gaussian curvature of Σ_u may be expressed as $K_u = -\frac{1}{2}\,\mathrm{Trace}(A^2)$.

Since the entire graph Σ_u is closed in $\mathbb{R}^3$, the Hopf–Rinow theorem guarantees that (Σ_u, g) is complete.

Now, consider the following conformal metric to (Σ_u, g),

$$\widehat{g} := (1 + \cos(\theta))^2\, g. \tag{7}$$

Firstly, note that $(\Sigma_u, \widehat{g})$ is also a complete Riemannian surface, since $\widehat{g} \geq g$. On the other hand, making use of the well-known relation between the Gaussian curvatures of conformal metrics, we have

$$(1 + \cos(\theta))^2 \widehat{K} = K - \Delta\big(\log(1 + \cos(\theta))\big),$$

where $\widehat{K}$ denotes the Gaussian curvature of $(\Sigma_u, \widehat{g})$. Now, from equations (3) and (4), it is easy to check that $\widehat{K} \geqslant 0$.

Thus, $(\Sigma_u, \widehat{g})$ is a complete Riemannian surface with non-negative Gaussian curvature. We may call an Ahlfors and Blanc–Fiala–Huber result (see, for instance, [14]) which ensures that $(\Sigma_u, \widehat{g})$ is parabolic. Furthermore, using the relation between the Laplacian operators of the conformal Riemannian metrics, the function $\cos(\theta)$ satisfies the relation

$$\widehat{\Delta} \cos(\theta) = \frac{1}{(1 + \cos(\theta))^2}\Delta \cos(\theta) \leq 0.$$

As a consequence, we have $\cos(\theta)$ is a constant function, and then Σ_u is an affine plane in $\mathbb{R}^3$.

4. Calabi–Bernstein theorem

Let $v \in C^\infty(\mathbb{R}^2)$ be a smooth function. The entire graph of v,

$$\Sigma_v = \{\big(x, y, v(x, y)\big),\ x, y \in \mathbb{R}\}$$

is spacelike in $\mathbb{L}^3$ if and only if $|Dv| < 1$, where D denotes the gradient in the Euclidean plane $(\mathbb{R}^2, \langle\,,\rangle)$.

Assume (Σ_v, g_L) is a spacelike entire graph. Then, through the application $\Phi\colon \mathbb{R}^2 \longrightarrow \Sigma_v \subset \mathbb{L}^3$, $\Phi(x,y) = (x,y,v(x,y))$, an isometry is established between (Σ_v, g_L) and the Riemannian surface $(\mathbb{R}^2, g_u)$, where $g_v := \Phi^* g_L$, that is,

$$g_v = -dv^2 + dx^2 + dy^2.$$

Remark 4.1. Observe that, in opposite way to the Riemannian case, an entire graph in $\mathbb{L}^3$ is not necessarily complete. For example, consider a real function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, defined as

$$f(x) = \begin{cases} \displaystyle\int_0^{|x|} \sqrt{1-e^{-s}}\, ds & \text{if } |x| \geq 1, \\ \alpha(x) & \text{if } |x| < 1, \end{cases}$$

where $\alpha \in C^\infty(\mathbb{R})$ is a smooth function satisfying $\alpha'^{\,2} < 1$, for all $x \in (-1,1)$ and such that f is differentiable in $\mathbb{R}$. Then, the entire graph $G = \{\big(x,y,f(x)\big),\ (x,y) \in \mathbb{R}^2\}$ is a spacelike surface in $\mathbb{L}^3$ which is not complete. In order to see this, we may consider the curve $\gamma\colon \mathbb{R} \longrightarrow G$, $\gamma(s) = (s,0,f(s))$, which is clearly divergent, and its length

$$L(\gamma) = \int_\infty^\infty |\gamma'(s)|\, ds = \int_{-1}^1 \sqrt{1-\alpha'(s)^2}\, ds + 2\int_1^\infty e^{-s}\, ds < 2(1+e),$$

is finite.

Now, suppose that the graph $(\Sigma_v, g_L) \cong (\mathbb{R}^2, g_v)$ is maximal and consider the conformal metric

$$\widehat{g_v} = (1+\cosh(\theta))^2\, g_v.$$

On the one hand, using the relation between the Gaussian curvatures of the conformal metrics, it is immediate to see that the Gaussian curvature $\widehat{K_v}$ of the surface $(\mathbb{R}^2, g_v)$ vanishes identically.

On the other hand, let w be a tangent vector in $\mathbb{R}^2$. We have

$$g_v(w,w) = -\langle Dv, w\rangle^2 + \|w\|^2 \geq \big(1 - |Dv|^2\big)\|w\|^2,$$

where $\langle\,,\rangle$ denotes the Euclidean inner product in $\mathbb{R}^2$ and D its associate gradient operator.

Taking into account that $\cosh(\theta) = \frac{1}{\sqrt{1-|Dv|^2}}$ with respect to the graph (Σ_v, g_L), we get

$$\widehat{g_v}(w,w) = (1+\cosh(\theta))^2 g_v(w,w) \geq \|w\|^2.$$

Consequently, the Riemannian metric $\widehat{g_v}$ is complete and the surface $(\mathbb{R}^2, \widehat{g_v})$ is parabolic.

Finally, consider the positive function $\Lambda(\theta) := \frac{1}{\cosh(\theta)}$. From formulas (5) and (6), we may compute the Laplacian of Λ:

$$\Delta\Lambda = \frac{\operatorname{Trace}(A^2)}{\cosh(\theta)}\big(\tanh(\theta) - 1\big) \leq 0.$$

Thus $\widehat{\Delta}\Lambda \leq 0$ and $\cosh(\theta)$ is a constant function, then the entire graph (Σ_v, g_L) must be a spacelike plane in $\mathbb{L}^3$.

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References

[1] Albujer, A. L., and L. J. Alías: Calabi–Bernstein results for maximal surfaces in Lorentzian product spaces. - J. Geom. Phys. 59:5, 2009, 620–631.

[2] Aledo, J. A., A. Romero, and R. M. Rubio: The classical Calabi–Bernstein theorem revisited. - J. Math. Anal. Appl. 431, 2015, 1172–1177.

[3] Alías, L. J., M. Dajczer, and J. Ripoll: A Bernstein-type theorem for Riemannian manifolds with a Killing field. - Ann. Global Anal. Geom. 31, 2007, 363–373.

[4] Bernstein, S.: Sur une théorème de géometrie et ses applications aux équations dérivées partielles du type elliptique. - Commun. Soc. Math. Kharkov 15, 1914, 38–45.

[5] Calabi, E.: Examples of Bernstein problems for some nonlinear equations. - Proc. Symp. Pure Math. 15, 1970, 223–230.

[6] Calabi, E.: Hypersurfaces with maximal affinely invariant area. - Amer. J. Math. 104, 1982, 91–126.

[7] Cheng, S.-Y., and S.-T. Yau: Maximal space-like hypersurfaces in the Lorentz–Minkowski spaces. - Ann. of Math. (2) 104, 1976, 407–419.

[8] Chern, S.: Simple proofs of two theorems on minimal surfaces. - Enseign. Math. 15, 1969, 53–61.

[9] Chern, S.: Affine minimal hypersurfaces. In Minimal submanifolds and geodesics. - Proc. Japan-United States Sem. Tokyo, 1977, 17–30.

[10] Fleming, W. H.: On the oriented Plateau problem. - Rend. Circ. Mat. Palermo 11, 1962, 69–90.

[11] De Giorgi, E.: Una estensione del teorema di Bernstein. - Ann. Sc. Norm. Sup. Pisa 19, 1965, 79–85.

[12] Estudillo, A., and A. Romero: New proof of the Calabi–Bernstein theorem. - Geom. Dedicata 147, 2010, 173–176.

[13] Hopf, S.: On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature. - Proc. Amer. Math. Soc. 1, 1950, 80–85.

[14] Kazdan, J. L.: Parabolicity and the Liouville property on complete Riemannian manifolds. - In: Seminar on new results in nonlinear partial differential equations, Aspects Math. 10, Vieweg, Braunschweig, 1987, 153–166.

[15] Kobayashi, O.: Maximal surfaces in the 3-dimensional Minkowski space L^3. - Tokyo J. Math. 6:2, 1983, 297–309.

[16] Osserman, R.: A survey of minimal surfaces. - Dover Publications, Mineola, New York, 1969.

[17] Osserman, R.: The minimal surface equation. - In: Seminar on nonlinear partial differential equations, Math. Sci. Res. Inst. Publ. 2, Springer, New York, 1984, 237–259.

[18] Romero, A.: Simple proof of Calabi–Bernstein's theorem on maximal surfaces. - Proc. Amer. Math. Soc. 124:4, 1996, 1315–1317.

[19] Romero, A., and R. M. Rubio: New proof of the Calabi–Bernstein theorem. - Geom. Dedicata 147, 2010, 173–176.

[20] Rosenberg, H.: Minimal surfaces in $M^2 \times \mathbb{R}$. - Illinois J. Math. 46, 2002, 1177–1195.

[21] Simons, J.: Minimal varieties in Riemannian manifolds. - Ann. of Math. (2) 88, 1968, 62–105.

[22] Trudinger, N. S., and X. J. Wang: The Bernstein problem for affine maximal hypersurfaces. - Invent. Math. 140, 2000, 399–422.

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