Annales Fennici Mathematici
Volumen 50, 2025, 611–622

Generalized Schwarzians and normal families

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Abstract. We study families of analytic and meromorphic functions with bounded generalized Schwarzian derivative $S_k(f)$. We show that these families are quasi-normal. Further, we investigate associated families, such as those formed by derivatives and logarithmic derivatives, and prove several (quasi-)normality results. Moreover, we derive a new formula for $S_k(f)$, which yields a result for families $\mathcal{F} \subseteq \mathcal{H}(\mathbb{D})$ of locally univalent functions that satisfy

$$S_k(f)(z) \neq b(z) \quad \text{for some } b \in \mathcal{M}(\mathbb{D}) \text{ and all } f \in \mathcal{F},\ z \in \mathbb{C}$$

and for entire functions g with $S_k(g)(z) \neq 0$ and $S_k(g)(z) \neq \infty$ for all $z \in \mathbb{C}$. The classical Schwarzian derivative S_f is contained as the case $k = 2$.

Yleistetyt Schwarzin derivaatat ja normaalit perheet

Tiivistelmä. Tässä työssä tarkastellaan analyyttisiä ja meromorfisia funktioita, joilla on rajallinen yleistetty Schwarzin derivaatta $S_k(f)$, ja osoitetaan, että tällaisten funktioiden perheet ovat kvasinormaaleja. Lisäksi tutkitaan näistä esimerkiksi derivoimalla tai logaritmisesti derivoimalla johdettuja perheitä ja todistetaan useita kvasinormaaliustuloksia. Schwarzin derivaatalle $S_k(f)$ johdetaan uusi kaava, jonka avulla saadaan tulos sellaisten paikallisesti injektiivisten funktioiden perheille $\mathcal{F} \subseteq \mathcal{H}(\mathbb{D})$, jotka toteuttavat

$$S_k(f)(z) \neq b(z) \quad \text{jollakin } b \in \mathcal{M}(\mathbb{D}) \text{ ja kaikilla } f \in \mathcal{F},\ z \in \mathbb{C}$$

sekä kokonaisille funktioille g, joilla $S_k(g)(z) \neq 0$ ja $S_k(g)(z) \neq \infty$ kaikilla $z \in \mathbb{C}$. Klassinen Schwarzin derivativaatta S_f saadaan tapauksessa $k = 2$.

1. Introduction and main results

Throughout this paper, we denote the set of all holomorphic functions on a domain $D \subseteq \mathbb{C}$ by $\mathcal{H}(D)$. Likewise, we write $\mathcal{M}(D)$ for the set of all meromorphic functions on D. Further, we denote the open unit disk by $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$. Moreover, when referring to zeros or poles, we use "(CM)" to indicate that multiplicity is counted, and "(IM)" when it is ignored.

Let $D \subseteq \mathbb{C}$ be a domain. A family $\mathcal{F} \subseteq \mathcal{M}(D)$ is said to be *quasi-normal on* D, if for every sequence $(f_n)_n \subseteq \mathcal{F}$, there exists a subsequence $(f_{n_k})_k \subseteq (f_n)_n$ and an exceptional set $E \subseteq D$ with no accumulation point in D, such that $(f_{n_k})_k$ converges locally uniformly in $D \backslash E$ (with respect to the spherical metric).

A family $\mathcal{F} \subseteq \mathcal{M}(\mathbb{D})$ is said to be *quasi-normal at* $z_0 \in D$ if there exists a neighborhood $U \subseteq D$ around z_0, such that the restricted family $\{f|_U \colon f \in \mathcal{F}\}$ is quasi-normal.

The theory of normal families offers a multitude of different criteria to check if a given family $\mathcal{F} \subseteq \mathcal{M}(\mathbb{D})$ is normal. Probably the most notable one is Marty's theorem, which states that a family $\mathcal{F}$ is normal, if and only if the family of spherical derivatives $\left\{\frac{|f'|}{1+|f|^2} \colon f \in \mathcal{F}\right\}$ is locally uniformly bounded. This draws a connection between the normality of a family and a particular differential inequality. Similarly,

https://doi.org/10.54330/afm.176369
2020 Mathematics Subject Classification: Primary 30D45; Secondary 30D30, 30C45.
Key words: Normal family, Schwarzian derivative, quasi-normality, value distribution theory.

there are several (more or less) related results, which connect (quasi-)normality to other differential operators and inequalities, such as

$$\frac{|f^{(k)}|}{1+|f^{(j)}|^{\alpha}} \leq M, \quad M \leq \frac{|f^{(k)}|}{1+|f^{(j)}|^{\alpha}} \quad \text{or} \quad f^n(z)+f^{(k)}(z) \neq 0$$

for suitable choices of M, α, k, j, n and all admissible z (see for example [8, 7, 2, 20, 3]).

This paper seizes this idea and investigates (quasi-)normality in relation to inequalities involving a generalization of the Schwarzian derivative by Chuaqui, Gröhn and Rättyä (see [4]). This generalization is anchored around the relationship between the Schwarzian derivative and the so called Schwarzian differential equation $y'' + p_0\, y = 0$. Other generalizations of the Schwarzian derivative, which focus on different aspects of the Schwarzian derivative, can be found in [13, 22, 24].

Definition A. [4, p. 340] *Let $f \in \mathcal{M}(\mathbb{D})$ be non-constant. For $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{1\}$, we define:*

$$S_{2,n}(f) := \frac{f''}{f'} \quad \text{and} \quad S_{k+1,n}(f) := \Big(S_{k,n}(f)\Big)' - \frac{1}{n}\frac{f''}{f'} S_{k,n}(f).$$

Now, for all $k \in \mathbb{N}$, we call

$$S_k(f) := S_{k+1,k}(f)$$

the generalized Schwarzian derivative of order k.

Note that the classical Schwarzian derivative S_f is contained as the case $k = 2$, since

$$S_2(f) = S_{3,2}(f) = (S_{2,2}(f))' - \frac{1}{2}\frac{f''}{f'} S_{2,2}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = S_f.$$

For constant functions $c \in \mathcal{H}(\mathbb{D})$, we define $S_k(c) \equiv \infty$ for all $k \in \mathbb{N}$. This differs from [4], where constant functions were not mentioned at all. However, our version allows us to circumvent degenerate cases in the future. It is further motivated by the observation that $S_k(f)$ exhibits a $(k-1)$-fold pole whenever the derivative of a non-constant $f \in \mathcal{M}(\mathbb{D})$ vanishes. We will give a short proof of this result in Section 2, Proposition 2.1.

The connection between this generalization of the Schwarzian derivative and the Schwarzian differential equation becomes apparent in the following theorem.

Theorem B. [4, Lemma 3 and Lemma 5] *Let $f \in \mathcal{M}(\mathbb{D})$ and $k \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) $S_k(f) \in \mathcal{H}(\mathbb{D})$.
(b) $f' = 1/h^k$, *for some $h \in \mathcal{H}(\mathbb{D})$, which satisfies the differential equation $y^{(k)} + p_0\, y = 0$ for some $p_0 \in \mathcal{H}(\mathbb{D})$.*
(c) $f' = 1/h^k$ *for some $h \in \mathcal{H}(\mathbb{D})$, where every zero $z_0 \in \mathbb{D}$ of h with multiplicity m is also a zero of $h^{(k)}$ of multiplicity at least m.*

If condition (b) *is true, then we can specify $p_0 = S_k(f)/k$ and $S_k(f) = -k\, h^{(k)}/h$.*

Again, this result differs slightly from the one given in [4, Lemma 5]. There, it is additionally required that f' is non-vanishing. However, with our exclusion of constant functions and Proposition 2.1, we can drop this assumption.

To abbreviate our notation, we define for each $M \in \mathbb{R}$ and $k \in \mathbb{N}$:

$$\mathcal{M}_{k,M} := \{f \in \mathcal{M}(\mathbb{D}) \colon \|S_k(f)\|_\infty \leq M\} \quad \text{and} \quad \mathcal{H}_{k,M} := \mathcal{M}_{k,M} \cap \mathcal{H}(\mathbb{D}),$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{D}$. Since no constant function is contained in $\mathcal{H}_{k,M}$ or $\mathcal{M}_{k,M}$, we can also consider the following families:

$$\mathcal{M}'_{k,M} := \{f' \colon f \in \mathcal{M}_{k,M}\} \quad \text{and} \quad \mathcal{H}'_{k,M} := \{f' \colon f \in \mathcal{H}_{k,M}\},$$
$$\mathcal{M}^{f'/f}_{k,M} := \{g'/g \colon g \in \mathcal{M}_{k,M}\} \quad \text{and} \quad \mathcal{H}^{f'/f}_{k,M} := \{g'/g \colon g \in \mathcal{H}_{k,M}\},$$
$$\mathcal{M}^{f''/f'}_{k,M} := \{g''/g' \colon g \in \mathcal{M}_{k,M}\} \quad \text{and} \quad \mathcal{H}^{f''/f'}_{k,M} := \{g''/g' \colon g \in \mathcal{H}_{k,M}\}.$$

In the case of the classical Schwarzian, the quasi-normality of $\mathcal{H}_{2,M}$ can be shown for any $M \in \mathbb{R}$ as a consequence of [17, Theorem 1.1(b)]. The quasi-normality of $\mathcal{M}_{2,M}$ was later proven in [9, Theorem 1.4] for all $M \in \mathbb{R}$. There, it is also shown for any $M \in \mathbb{R}$ that $\mathcal{M}^{f''/f'}_{2,M}$ is normal and $\mathcal{M}^{f'/f'}_{2,M}$ is quasi-normal.

In this paper, we will generalize these results for all $k \in \mathbb{N}$. To achieve this, we consider families of holomorphic functions first and obtain the following theorem.

Theorem 1.1. *The following statements hold for all $k \in \mathbb{N}$ and $M \in \mathbb{R}$:*

(i) *$\mathcal{H}^{f''/f'}_{k,M}$ is locally uniformly bounded.*
(ii) *$\mathcal{H}^{f'/f}_{k,M}$ is normal and no sequence in $\mathcal{H}^{f'/f}_{k,M}$ converges to ∞.*
(iii) *$\mathcal{H}'_{k,M}$ is normal.*
(iv) *$\mathcal{H}_{k,M}$ is quasi-normal.*

Later, in Lemma 2.2, we will see that the number of poles of each $f \in \mathcal{M}_{k,M}$ is bounded by a constant that depends only on k and M. Thus, every sequence $(f_n)_n \subseteq \mathcal{M}_{k,M}$ has a subsequence with a corresponding set of poles having only isolated points of accumulation. Since (quasi-)normality is a local property, we are able to treat $\mathcal{M}_{k,M}$ mostly like its holomorphic subset $\mathcal{H}_{k,M}$ and prove the following theorem:

Theorem 1.2. *The following statements hold for all $k \in \mathbb{N}$ and $M \in \mathbb{R}$:*

(i) *$\mathcal{M}^{f''/f'}_{k,M}$ is quasi-normal and no sequence in $\mathcal{M}^{f''/f'}_{k,M}$ converges to ∞.*
(ii) *$\mathcal{M}^{f'/f}_{k,M}$ is quasi-normal and no sequence in $\mathcal{M}^{f'/f}_{k,M}$ converges to ∞.*
(iii) *$\mathcal{M}'_{k,M}$ is quasi-normal.*
(iv) *$\mathcal{M}_{k,M}$ is quasi-normal.*

The results of Theorems 1.1 and 1.2 extend to families (and the respective families of derivatives, logarithmic derivatives, or pre-Schwarzians) of the form

$$\mathcal{F} := \{f \in \mathcal{M}(\mathbb{D}) \colon |S_k(f)(z)| \le g(z) \text{ for all } z \in \mathbb{D}\} \quad \text{and} \quad \mathcal{F} \cap \mathcal{H}(\mathbb{D}),$$

where $g \colon \mathbb{D} \to \mathbb{R}^+$ is a locally bounded function. This includes bounds of hyperbolic type, such as those considered in [4, Theorem 7] or [18]. This generalization holds because each statement can be verified locally, since (quasi-)normality is a local property. More precisely, it is possible to apply Theorem 1.1 or Theorem 1.2 locally by using the invariance of the generalized Schwarzian derivative under precomposition with affine transformations.

[4, Lemma 4] shows that $S_k(f) \equiv 0$ if and only if $f' = 1/p^k$, where p is a polynomial with $\deg p \le k-1$. Thus, $S_k(f_n) \equiv 0$ for $f_n(z) = nz$, for all $k \in \mathbb{N}$, so neither $\mathcal{H}_{k,M}$ nor $\mathcal{M}_{k,M}$ are normal for $k \in \mathbb{N}$ and $M \in \mathbb{R}^+_0$.

Similarly, $\mathcal{M}'_{k,M}$ is not normal for $k \in \mathbb{N}\backslash\{1\}$ and $M \in \mathbb{R}^+_0$, since $S_k(g_n) \equiv 0$ for

$$g_n(z) := \frac{-1}{n^k(k-1)z^{k-1}} \quad \text{with} \quad g_n'(z) = \frac{1}{(nz)^k}.$$

For $k \in \mathbb{N}\backslash\{1\}$ and $M \in \mathbb{R}_0^+$, the family $\mathcal{M}_{k,M}^{f'/f}$ is not normal either. To see this, we consider the sequence

$$h_n(z) := \frac{1}{z^{k-1}} + n \quad \text{with} \quad h_n'(z) = \frac{1-k}{z^k} \quad \text{and} \quad \frac{h_n'}{h_n}(z) = \frac{1-k}{z(1+nz^{k-1})}.$$

It is unknown to the author, whether $\mathcal{M}_{k,M}^{f''/f'}$ is normal. However, by using Theorem 1.1, it is possible to show that $\mathcal{M}_{k,M}^{f''/f'}$ is normal at $z_0 \in \mathbb{D}$ if and only if there exists a neighborhood of z_0, where each $f \in \mathcal{M}_{k,M}$ has at most one pole (IM). Theorem F and the proof of [9, Theorem 1.4] show that these conditions hold for $k = 2$, but it remains open whether this is true for $k \geq 3$.

For our final results, we will regard $S_k(f)$ as a differential polynomial by showing the following formula for the generalized Schwarzian derivative.

Lemma 1.3. *Let $f \in \mathcal{M}(\mathbb{D})$ be a non-constant, meromorphic function and $k \in \mathbb{N}$. Then*

$$S_k(f) = \sum_{(n_1,\ldots,n_k)\in\Lambda} \left(\frac{-k \cdot k!}{\prod_{j=1}^k (-k \cdot j!)^{n_j} \cdot n_j!} \cdot \prod_{j=1}^k \left(\left(\frac{f''}{f'} \right)^{(j-1)} \right)^{n_j} \right), \tag{1}$$

where Λ is the set of all tuples $(n_1, \ldots, n_k)$ (with $n_r \in \mathbb{N}_0$ for all $r = 1, \ldots, k$) that satisfy $\sum_{r=1}^k r \cdot n_r = k$.

For $k \in \mathbb{N}\backslash\{1\}$, we can extract the summands where $(n_1 \ldots, n_k) = (k, 0, \ldots, 0)$ and $(n_1, \ldots, n_k) = (0, \ldots, 0, 1)$, to obtain that

$$S_k(f) = \frac{(-1)^{k+1}}{k^{k-1}} \cdot g^k + g^{(k-1)} + P[g] \tag{2}$$

for $g := f''/f'$ and some differential polynomial P.

This form is reminiscent of the well known condition $af^n(z) + f^{(m)}(z) \neq b$ for $a, b \in \mathbb{C}$, $a \neq 0$ and large enough $n, m \in \mathbb{N}$ with $n > m$. Hayman was the first to study this condition for $f \in \mathcal{M}(\mathbb{C})$ (see [11]). In the spirit of Bloch's principle, his results on value distribution were later extended to normal families: in the analytic case by Drasin in [5], and in the meromorphic case by Langley in [16]. Subsequently, Chen and Hua demonstrated in [3] that the constant value b can be replaced by an exceptional function $B \in \mathcal{M}(\mathbb{D})$. This result was further extended to more general differential polynomials (see Theorem G below) by Grahl in [6]. Now, we will show that the differential polynomial P from equation (2) fulfills the exact conditions of Theorem G. This leads to the following result:

Theorem 1.4. *Let $\mathcal{F} \subseteq \mathcal{H}(\mathbb{D})$ be a family of locally univalent functions, let $k \in \mathbb{N}\backslash\{1\}$ and $b \in \mathcal{M}(\mathbb{D})$ be a meromorphic function, such that every $f \in \mathcal{F}$ satisfies*

$$S_k(f)(z) \neq b(z) \quad \textit{for all} \quad z \in \mathbb{D}. \tag{3}$$

Then $\mathcal{F}''/\mathcal{F}' := \{f''/f' \colon f \in \mathcal{F}\}$ is a normal family.

Note that $\mathcal{F}''/\mathcal{F}'$, unlike $\mathcal{H}_{k,M}^{f''/f'}$, does not have to be locally uniformly bounded. Therefore, the family $\mathcal{F}$ does not need to be quasi-normal, as demonstrated by the family $\{z \mapsto e^{nz} \colon n \in \mathbb{N}\}$.

However, if—in addition to (3)—we know that $\mathcal{F}''/\mathcal{F}'$ is pointwise bounded in a single point, then it follows that $\mathcal{F}''/\mathcal{F}'$ is locally uniformly bounded. Now, using formula (1), we can see that the family $S_k(\mathcal{F}) := \{S_k(f) \colon f \in \mathcal{F}\}$ is locally uniformly bounded as well. This, in turn, allows us to apply Theorem 1.1 to $\mathcal{F}$. Thus, we can regard this as a "self-improving result", where we require that:

- $S_k(\mathcal{F})$ omits a function.
- $\mathcal{F}''/\mathcal{F}'$ is bounded in a single point.

and obtain that:

- $S_k(\mathcal{F})$ is locally uniformly bounded.
- $\mathcal{F}''/\mathcal{F}'$ is locally uniformly bounded.
- $\mathcal{F}'/\mathcal{F}$ and $\mathcal{F}'$ are normal, while $\mathcal{F}$ is quasi-normal.

We should note that Theorem 1.4 can not be extended to families of meromorphic functions. To see this, consider $f_n(z) := ((2z)^n - 1)^{-1}$ for $n \in \mathbb{N}$ and $z \in \mathbb{D}$ with

$$f_n'(z) = -\frac{n\,2^n\,z^{n-1}}{\big((2z)^n - 1\big)^2} \quad \text{and} \quad f_n''(z) = \frac{n\,2^n\,z^{n-2}\big(n - 1 + (n+1)(2z)^n\big)}{\big((2z)^n - 1\big)^3}.$$

Clearly, each f_n has a pole of order 1 in $z_0 = 1/2$ and is locally injective in $\mathbb{D}\backslash\{0\}$. Additionally, for $z_n := \frac{1}{2}\sqrt[n]{\frac{n-1}{n+1}}\,e^{i\pi/n}$, we have $f_n''(z_n) = 0$, while $(z_n)_n$ converges to $1/2$. Hence, the sequence $(f_n''/f_n')_n$ is not normal at $1/2$.

On the other hand, we can use the fact that the classical Schwarzian derivative is invariant under Möbius transformations, and calculate

$$S_2(f_n)(z) = S_2(z^n) = \frac{1 - n^2}{2\,z^2}.$$

Thus, $\big(S_k(f_n)\big)_{n\geq 2}$ omits the value 0, which shows that we can not extend Theorem 1.4 to families of meromorphic functions.

Still, there is a corresponding value distribution result for entire functions. Similarly to Theorem 1.4, this corollary relies heavily on the results from [6].

Corollary 1.5. *Let $f \in \mathcal{H}(\mathbb{C})$ be an entire function with*

$$S_k(f)(z) \neq 0 \quad \text{and} \quad S_k(f)(z) \neq \infty \quad \text{for all } z \in \mathbb{C}.$$

Then there are $a, b, c \in \mathbb{C}$ with $a, b \neq 0$ and $f(z) = ae^{bz} + c$.

Based on the counterexample given in [11, p. 34], we consider the locally univalent function $f(z) = \exp(\exp(cz)/c)$ with $c \in \mathbb{C}\backslash\{0\}$. A straightforward computation shows that its classical Schwarzian derivative is

$$S_2(f)(z) = -\frac{e^{2cz}}{2} - \frac{c^2}{2},$$

so the exceptional value 0 cannot be replaced by another value.

Likewise, Corollary 1.5 does not extend to meromorphic functions. For $k = 2$, this is already evident from compositions of Möbius transformations with exponential functions. Now, a natural question is whether any $f \in \mathcal{M}(\mathbb{C})$ with $S_k(f)(z) \neq 0$ and $S_k(f)(z) \neq \infty$ for all $z \in \mathbb{C}$ must have the form $f(z) = T(ae^{bz} + c)$ for some invariant function T of the k-th generalized Schwarzian derivative. However, Proposition 2.3 provides a counterexample by showing that such a generalization does not hold for $k = 2$.

2. Auxiliary lemmas and results

First, we show that non-constant functions with an analytic generalized Schwarzian derivative have non-vanishing derivatives.

Proposition 2.1. *If $f \in \mathcal{M}(\mathbb{D})$ is non-constant and f' has a zero in $z_0 \in \mathbb{D}$, then $S_k(f)$ has a k-fold pole in z_0.*

Proof. Suppose that f' has an m-fold zero in z_0. Then $S_{2,n}(f) = f''/f'$ has a simple pole in z_0 for all $n \in \mathbb{N}$. Next, we assume that $S_{k,n}(f)$ has a $(k-1)$-fold pole in z_0 for $k \geq 3$, i.e.

$$S_{k,n}(f)(z) = \frac{h(z)}{(z-z_0)^{k-1}} \quad \text{for some analytic } h \text{ in a neighborhood } U \text{ of } z_0.$$

Then we can calculate

$$S_{k+1,n}(f)(z) = \frac{1}{(z-z_0)^k}\left(\left(1-k-\frac{m}{n}\right)h(z_0)+\ldots\right) \quad \text{for } z \in U,$$

and conclude inductively that $S_{k,n}(f)$ has a k-fold pole for all $k, n \in \mathbb{N}\backslash\{1\}$. □

The proof of Theorem 1.1 relies heavily on the following theorem by Schwick.

Theorem C. [23, Theorem 5.4] *Let $(h_n)_n \subseteq \mathcal{H}(\mathbb{D})$ be a sequence of non-vanishing functions, and let $k \in \mathbb{N}$. If $(h_n^{(k)}/h_n)_n$ converges locally uniformly to some $\psi \in \mathcal{H}(\mathbb{D})$, then $(h_n'/h_n)_n$ is locally uniformly bounded in $\mathbb{D}$.*

Schwick initially stated this result in a slightly weaker form, establishing only that $(h_n'/h_n)_n$ is normal. However, his proof shows that the Nevanlinna characteristic of $(h_n'/h_n)_n$ is locally uniformly bounded, which, as shown in [23, Theorem 1.13], implies that the sequence itself is locally uniformly bounded.

Theorem C will be used to show that $\mathcal{H}_{k,M}^{f''/f'}$ is locally uniformly bounded for all $k \in \mathbb{N}$ and $M \in \mathbb{R}$. Then the following result allows us to transfer the convergence properties of sequences in $\mathcal{H}_{k,M}^{f''/f'}$ to respective subsequences in $\mathcal{H}_{k,M}^{f'/f}$ and $\mathcal{H}_{k,M}$.

Lemma D. [9, Lemma 2.4] *Let $E \subseteq \mathbb{D}$ be a set without an accumulation point in $\mathbb{D}$ and $(f_n)_n \subseteq \mathcal{M}(\mathbb{D})$ with:*

(1.) *$(f_n''/f_n')_n$ converges locally uniformly on $\mathbb{D}\backslash E$ to some $\psi \in \mathcal{H}(\mathbb{D}\backslash E)$.*

(2.) *f_n' is zero-free for all $n \in \mathbb{N}$.*

Then $(f_n)_n$ and $(f_n'/f_n)_n$ are quasi-normal on $\mathbb{D}$, and no subsequence of $(f_n'/f_n)_n$ converges to ∞. Moreover, if $E = \emptyset$, $(f_n'/f_n)_n$ is normal.

Next, we will estimate the maximal number of poles of the functions in $\mathcal{M}_{k,M}$ by using the differential equation in Theorem B(b). Here, the concept of *disconjugate differential equations* will be useful.

Definition E. *Let $D \subseteq \mathbb{C}$ be a domain, $k \in \mathbb{N}$ and $p_0, \ldots, p_{k-1} \in \mathcal{H}(D)$. We say that*

$$y^{(k)} + p_{k-1} \cdot y^{(k-1)} + p_{k-2} \cdot y^{(k-2)} + \ldots + p_0 \cdot y = 0$$

is disconjugate in D, if no non-trivial solution has more than $k-1$ zeros (CM).

A classical result concerning the Schwarzian derivative states that if $f_1, f_2 \in \mathcal{H}(\mathbb{D})$ are linearly independent solutions of the differential equation $y'' + p_0 \cdot y = 0$, then $f := f_1/f_2$ satisfies $S_2(f) = p_0/2$ (cf. [15, Theorem 6.1]). As a consequence, any linear combination $g = c_1 f_1 + c_2 f_2$ vanishes at a point z_0 if and only if $f(z_0) = -c_2/c_1$. Therefore, f attains some value n times if and only if there exists a non-trivial solution of $y'' + p_0 \cdot y = 0$ with n zeros.

This observation is due to Nehari and his paper on the Schwarzian derivative and univalence (see [18, p. 546]). Since then, numerous results provided alternative criteria for disconjugacy. One such result is the following theorem. However, with minor modifications, Theorem F and Lemma 2.2 could also be derived from [10, Theorem 2] or [19, p. 328].

Theorem F. [14, p. 723] *Let $k \in \mathbb{N}$, let $D \subseteq \mathbb{C}$ be a convex domain with $\delta := \operatorname{diam}(D) < \infty$ and let $p_0 \in \mathcal{H}(D)$ be a holomorphic function with*

$$|p_0(z)| < \frac{k!}{\delta^k} \quad \text{for all} \quad z \in D.$$

Then the differential equation $y^{(k)}(z) + p_0(z)y(z) = 0$ is disconjugate in D.

Based on this result, we can show the following lemma.

Lemma 2.2. *For $k \in \mathbb{N}$ and $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$, such that all $f \in \mathcal{M}_{k,M}$ and all $f' \in \mathcal{M}'_{k,M}$ have at most N poles (CM).*

Proof. For $f \in \mathcal{M}_{k,M}$, it suffices to show that f' has a limited number of poles. The case $M = 0$ was already established in Theorem [4, Lemma 4], so we assume $M > 0$. Due to Theorem B, we can write $f' = 1/h^k$, where $h \not\equiv 0$ is a solution of the differential equation

$$y^{(k)}(z) + \frac{S_k(f)(z)}{k} \cdot y(z) = 0. \tag{4}$$

Now, we choose $\delta > 0$ with $\delta < \left(\frac{k \cdot k!}{M}\right)^{1/k}$. This implies $\frac{\|S_k(f)\|_\infty}{k} \leq \frac{M}{k} < \frac{k!}{\delta^k}$, so Theorem F shows that (4) is disconjugate in every convex set $C_\delta \subseteq \mathbb{D}$ with diameter δ. Consequentially, h can have at most $k-1$ zeros in C_δ. By covering the unit disk $\mathbb{D}$ with $\tilde{N}$ of such convex sets, we obtain the bound $N := \tilde{N}(k-1)$ for the number of zeros of h (CM). □

For the reader's convenience, we also restate the main results of [6] with modified notation, to prevent conflicts with the notation used in this paper.

Theorem G. [6, Theorem 3] *Let $\mathcal{F} \subseteq \mathcal{H}(\mathbb{D})$, $\ell \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{1\}$. Let $a, b, a_1, \ldots, a_N \in \mathcal{M}(\mathbb{D})$ with $a \not\equiv 0$, and suppose that all poles of a have multiplicity at most $k-1$. Consider a differential polynomial of the form*

$$P[u] = \sum_{\mu=1}^{N} a_\mu \cdot \prod_{j=1}^{s_\mu} u^{(\omega_{\mu,j})},$$

where $s_\mu, \omega_{\mu,j} \in \mathbb{N}_0$ satisfy the inequality

$$(k-1) \cdot \sum_{j=1}^{s_\mu} \omega_{\mu,j} + \ell \cdot s_\mu \leq \ell \cdot k \tag{5}$$

for all $\mu = 1, \ldots, N$, where equality can only hold if $2 \leq s_\mu \leq k-1$. Suppose that for every $f \in \mathcal{F}$ and every $z \in \mathbb{D}$, the following inequality holds:

$$a(z) \cdot f^k(z) + f^{(\ell)}(z) + P[f](z) \neq b(z).$$

Then the family $\mathcal{F}$ is normal.

Theorem H. [6, Theorem 4] *Let $f \in \mathcal{H}(\mathbb{C})$, $\ell \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{1\}$. Let $a, a_1, \ldots, a_N \in \mathbb{C}$ with $a \neq 0$ and consider a differential polynomial*

$$P[u] = \sum_{\mu=1}^{N} a_\mu \cdot \prod_{j=1}^{s_\mu} u^{(\omega_{\mu,j})},$$

where $2 \le s_\mu \le k-1$ *and* $\sum_{j=1}^{s_\mu} \omega_{\mu,j} \neq 0$ *holds for all* $\mu = 1, \ldots, N$. *If the function* $a \cdot f^k + f^{(\ell)} + P[f]$ *does not vanish, then* f *must be constant.*

We conclude this section by proving the claim stated at the end of Section 1.

Proposition 2.3. *There exists some* $f \in \mathcal{M}(\mathbb{C})$ *such that* f *is not of the form* $f(z) = \frac{ae^{\alpha z}+b}{ce^{\alpha z}+d}$ *for any choice of* $a, b, c, d, \alpha \in \mathbb{C}$, *while* $S_2(f)$ *omits the values* 0 *and* ∞.

Proof. We consider the Bessel functions defined by (cf. [1, 9.1.12 and 9.1.13])

$$J_0(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \Big(\frac{z}{2}\Big)^{2k} \quad \text{and}$$
$$Y_0(z) := \frac{2}{\pi}\Big(\log \frac{z}{2} + \gamma\Big) J_0(z) + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k}{(k!)^2} \Big(\frac{z}{2}\Big)^{2k},$$

where H_k denotes the k-th harmonic number and γ is Euler's constant. These are linearly independent solutions of the Bessel differential equation (cf. [1, 9.1.1])

$$z^2 y''(z) + z y'(z) + z^2 y(z) = 0.$$

Next, we define

$$f_1(z) := J_0(e^{z/2}) \quad \text{and} \quad f_2(z) := Y_0(e^{z/2}).$$

Due to the logarithmic singularity of Y_0, we initially restrict our consideration of f_2 to the horizontal strips

$$S_n := \{z \in \mathbb{C} \colon (4n-2)\pi < \operatorname{Im}(z) < (4n+2)\pi\}.$$

Within each S_n, a direct computation yields

$$f_2''(z) + \frac{e^z}{4} f_2(z) = \frac{e^z}{4} Y_0''(e^{z/2}) + \frac{e^{z/2}}{4} Y_0'(e^{z/2}) + \frac{e^z}{4} Y_0(e^{z/2}) = 0,$$

which implies that f_2 satisfies a second-order linear differential equation with entire coefficients. Hence, f_2 is analytically extensible to an entire function (cf. [12, Satz 3.2]). Similarly, f_1 satisfies

$$f_1''(z) + \frac{e^z}{4} f_1(z) = 0.$$

Now, a classical result about the Schwarzian derivative (see [15, Theorem 6.1]) implies that the function $f := f_1/f_2$ fulfills $S_2(f)(z) = e^z/2$. In particular, $S_2(f)$ omits the values 0 and ∞.

Next, we suppose that $f(z) = \frac{ae^{\alpha z}+b}{ce^{\alpha z}+d}$ for some choice of $a, b, c, d, \alpha \in \mathbb{C}$. Then the poles and zeros of f must be periodic. However, let $j_{0,n}$ and $y_{0,n}$ denote the n-th positive real zero of J_0 and Y_0 respectively. Then asymptotically (cf. [1, 9.5.12])

$$j_{0,n} \sim \Big(n - \frac{1}{4}\Big)\pi \quad \text{and} \quad y_{0,n} \sim \Big(n - \frac{3}{4}\Big)\pi.$$

Thus, the positive real zeros of J_0 and Y_0 are almost equidistant, which shows that the zeros and poles of f are not periodic. Hence, f can not be expressed as a composition of a Möbius transformation and an exponential function. □

3. Proofs of the main results

Throughout this section, we consider $k \in \mathbb{N}$ and $M \in \mathbb{R}$ fixed.

Proof of Theorem 1.1. From Theorem B we know that the derivative of every $f \in \mathcal{H}_{k,M}$ can be written as $f' = 1/h^k$ for some zero-free function $h \in \mathcal{H}(\mathbb{D})$. Additionally we have $S_k(f) = -k\, h^{(k)}/h$. Therefore, the family $\{h^{(k)}/h \in \mathcal{H}(\mathbb{D})\colon f' = 1/h^k \text{ for some } f \in \mathcal{H}_{k,M}\}$ is locally uniformly bounded. Applying Theorem C shows that $\{h'/h \in \mathcal{H}(\mathbb{D})\colon f' = 1/h^k \text{ for some } f \in \mathcal{H}_{k,M}\}$ is locally uniformly bounded as well. Since $f''/f' = -k\, h'/h$, it follows that the family $\mathcal{H}_{k,M}^{f''/f'}$ is locally uniformly bounded.

Next, we consider a sequence $(f_n)_n \subseteq \mathcal{H}_{k,M}$. Because $(f_n''/f_n')_n$ is locally uniformly bounded and due to Proposition 2.1, we are able to apply Lemma D to a subsequence of $(f_n)_n$ with $E = \emptyset$. This implies that $\mathcal{H}_{k,M}$ is quasi-normal, that $\mathcal{H}_{k,M}^{f'/f}$ is normal, and that no sequence in $\mathcal{H}_{k,M}^{f'/f}$ converges to ∞.

For the final claim, we use the inequality $1 + x < 2\,(1 + x^2)$ for $x \in \mathbb{R}$ to obtain

$$(f')^{\#}(z) = \frac{|f''(z)|}{1 + |f'(z)|^2} < \frac{2\,|f''(z)|}{1 + |f'(z)|} < 2\left|\frac{f''(z)}{f'(z)}\right| \tag{6}$$

for all $f \in \mathcal{M}(\mathbb{D})$ and $z \in \mathbb{D}$. Since $\mathcal{H}_{k,M}^{f''/f'}$ is locally uniformly bounded, it follows that the spherical derivatives of the functions in $\mathcal{H}'_{k,M}$ are locally uniformly bounded. Now, Marty's theorem implies that $\mathcal{H}'_{k,M}$ is normal. □

Proof of Theorem 1.2. We consider a sequence $(f_n)_n \subseteq \mathcal{M}_{k,M}$. By Lemma 2.2, there exists $N \in \mathbb{N}$, such that each f_n has at most N poles. Therefore, we can find a subsequence $(f_{n_k})_{n_k}$ and a set $E \subseteq \mathbb{D}$ consisting of at most N points, such that for every $z \in \mathbb{D}\backslash E$ there exists a neighborhood U of z in which almost all f_n are analytic. Applying Theorem 1.1 locally on $\mathbb{D}\backslash E$ implies that there is a subsequence $(f''_{n_\ell}/f'_{n_\ell})_{n_\ell} \subseteq (f''_{n_k}/f'_{n_k})_{n_k}$ that converges locally uniformly on $\mathbb{D}\backslash E$ to some $F \in \mathcal{H}(\mathbb{D}\backslash E)$. Since E has no accumulation point, we conclude that $\mathcal{M}_{k,M}^{f''/f'}$ is quasi-normal and that no sequence in $\mathcal{M}_{k,M}^{f''/f'}$ converges to ∞.

Analogously to the proof of Theorem 1.1, applying Lemma D to $(f_{n_\ell})_{n_\ell}$ shows that both $\mathcal{M}_{k,M}$ and $\mathcal{M}_{k,M}^{f'/f}$ are quasi-normal, and that no subsequence of $(f'_{n_k}/f_{n_k})_{n_k}$ converges to ∞.

Finally, the convergence of $(f''_{n_\ell}/f'_{n_\ell})_{n_\ell}$ on $\mathbb{D}\backslash E$ to $F \in \mathcal{H}(\mathbb{D}\backslash E)$ together with inequality (6) show that $((f'_{n_\ell})^{\#})_{n_\ell}$ is locally uniformly bounded on $\mathbb{D}\backslash E$. Therefore, $\mathcal{M}'_{k,M}$ is quasi-normal by Marty's theorem. □

Proof of Lemma 1.3. Since f is non-constant, there exists a neighborhood $U \subseteq \mathbb{D}$ such that $S_k(f)|_U \in \mathcal{H}(U)$. By Theorem B, there exists a zero-free $h \in \mathcal{H}(U)$ with $f' = 1/h^k$ on U. Furthermore, we can find a logarithm $L \in \mathcal{H}(U)$ of h, i.e. $h = \mathrm{e}^L$. Using Faà di Bruno's formula (cf. [21, Chapter 2.8]), we have that

$$h^{(k)} = \left(\mathrm{e}^L\right)^{(k)} = \sum_{(n_1,\ldots,n_k)\,\in\,\Lambda} \left(\frac{k!}{n_1! \cdot \ldots \cdot n_k!}\, e^L \prod_{j=1}^{k} \left(\frac{L^{(j)}}{j!}\right)^{n_j} \right),$$

where Λ is the set of all natural tuples $(n_1, \ldots, n_k)$ with $\sum_{j=1}^{k} j \cdot n_j = k$.

Now, using Theorem B and $L' = h'/h = -f''/(k \cdot f')$, we have

$$S_k(f) = -k\,\frac{h^{(k)}}{h} = \sum_{(n_1,\dots,n_k)\,\in\,\Lambda} \left(\frac{-k\cdot k!}{n_1!\cdot\ldots\cdot n_k!} \prod_{j=1}^{k} \left(\frac{L^{(j)}}{j!}\right)^{n_j} \right)$$
$$= \sum_{(n_1,\dots,n_k)\,\in\,\Lambda} \left(\frac{-k\cdot k!}{n_1!\cdot\ldots\cdot n_k!} \prod_{j=1}^{k} \left(\frac{1}{-k\cdot j!}\cdot\left(\frac{f''}{f'}\right)^{(j-1)}\right)^{n_j} \right)$$
$$= \sum_{(n_1,\dots,n_k)\,\in\,\Lambda} \left(\frac{-k\cdot k!}{\prod_{j=1}^{k}\left((-k)\cdot(j!)\right)^{n_j}\cdot n_j!} \prod_{j=1}^{k} \left(\left(\frac{f''}{f'}\right)^{(j-1)}\right)^{n_j} \right).$$

Finally, by the identity theorem, this equality extends from U to $\mathbb{D}$. $\square$

Proof of Theorem 1.4. Let Λ be defined as in Lemma 1.3, and set

$$\tilde{\Lambda} \coloneqq \Lambda\backslash\{(k,0,\dots,0),(0,\dots,0,1)\}.$$

Next, we fix an enumeration μ of $\tilde{\Lambda}$, denote $N \coloneqq |\tilde{\Lambda}|$ and define for each multi-index $(n_1,\dots,n_k)\in\tilde{\Lambda}$ that

$$a_{\mu^{-1}(n_1,\dots,n_k)} \coloneqq \frac{-k\cdot k!}{\prod_{j=1}^{k}\left((-k)\cdot(j!)\right)^{n_j}\cdot n_j!}.$$

In addition, for $\nu = \mu^{-1}(n_1,\dots,n_k)$ we set:

$$s_\nu \coloneqq \sum_{r=1}^{k} n_r \quad \text{and for} \quad j = 1,\dots,s_\nu, \quad \text{we define}$$
$$\omega_{\nu,j} \coloneqq r-1 \quad \text{whenever} \quad n_1+\ldots,n_{r-1} < j \le n_1+\ldots+n_r.$$

Using Lemma 1.3, we get for all $f\in\mathcal{F}$ that

$$S_k(f) = \frac{(-1)^{k+1}}{k^{k-1}}\left(\frac{f''}{f'}\right)^k + \left(\frac{f''}{f'}\right)^{(k-1)} + \sum_{(n_1,\dots,n_k)\,\in\,\tilde{\Lambda}} a_{(n_1,\dots,n_k)} \prod_{j=1}^{k}\left(\left(\frac{f''}{f'}\right)^{(j-1)}\right)^{n_j}$$
$$(7)\qquad = \frac{(-1)^{k+1}}{k^{k-1}}\left(\frac{f''}{f'}\right)^k + \left(\frac{f''}{f'}\right)^{(k-1)} + \underbrace{\sum_{\mu=1}^{N} a_\mu \prod_{j=1}^{s_\mu}\left(\frac{f''}{f'}\right)^{(\omega_{\mu,j})}}_{=:P[f''/f']}.$$

Now, we want to apply Theorem G with $\ell \coloneqq k-1$. Note that f''/f' is analytic, since f is locally univalent. Furthermore, condition (5) holds for $\mu = 1,\dots,N$, because $\sum_{r=1}^{k} n_r\cdot r = k$ for $(n_1,\dots,n_k)\in\tilde{\Lambda}$ and therefore we get

$$(k-1)\cdot\sum_{j=1}^{s_\mu}\omega_{\mu,j} + \ell\cdot s_\mu = (k-1)\sum_{r=1}^{k} n_r\,(r-1) + (k-1)\sum_{r=1}^{k} n_r = \ell\cdot k$$

independently of μ. Additionally, we have $2\le s_\mu\le k-1$ for all $\mu = 1,\dots,N$, because we removed the multi-indices $(k,0,\dots,0)$ and $(0,\dots,0,1)$ from $\tilde{\Lambda}$. Hence, Theorem G applies and we conclude that $\mathcal{F}''/\mathcal{F}'$ is a normal family. $\square$

Proof of Corollary 1.5. We reuse the notation used in the proof of Theorem 1.4 and write $S_k(f)$ as in equation (7). By Proposition 2.1, f' does not vanish and therefore f''/f' is an entire function. Since $2\le s_\mu\le k-1$ and $\sum_{j=1}^{s_\mu}\omega_{\mu,j}\ne 0$ holds for all $\mu = 1,\dots,N$, we are able to apply Theorem H to f''/f'. Thus, $f''/f'\equiv b$ for some constant $b \in \mathbb{C}$. Notice that $b \neq 0$, since $S_k(f)$ omits the value 0. Therefore, f must be of the form $f(z) = ae^{bz} + c$ for some $a, b, c \in \mathbb{C}$ with $a, b \neq 0$. □

Acknowledgment. I am very grateful to Jürgen Grahl for several valuable remarks.

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Received 26 May 2025 • Revision received 7 October 2025 • Accepted 7 October 2025
Published online 13 October 2025

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