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# Limits of manifolds with a Kato bound on the Ricci curvature. II

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**Abstract.** We prove that metric measure spaces obtained as limits of closed Riemannian manifolds with Ricci curvature satisfying a uniform Kato bound are rectifiable. In the case of a non-collapsing assumption and a strong Kato bound, we additionally show that for any $\alpha \in (0,1)$ the regular part of the space lies in an open set with the structure of a $\mathcal{C}^\alpha$-manifold.

**Riccin kaarevuuden Katon rajaa noudattavien monistojen raja-arvot. II**

**Tiivistelmä.** Tässä työssä osoitetaan, että metrinen mitta-avaruus on suoristuva, jos se saadaan Riccin kaarevuuden tasaista Katon rajaa noudattavien suljettujen Riemannin monistojen raja-arvona. Jos lisätään luhistumattomuusoletus ja vahva Katon raja, näytetään lisäksi, että avaruuden säännöllinen osa sisältyy avoimeen joukkoon, jolla on $\mathcal{C}^\alpha$-moniston rakenne millä tahansa $\alpha \in (0,1)$.

## 1. Introduction

In this paper, we establish new geometric and analytic properties of Kato limit spaces, i.e. measured Gromov–Hausdorff limits of closed Riemannian manifolds with Ricci curvature satisfying a uniform Kato bound. Our work continues the study began in [CMT24] where we introduced these spaces.

For a closed Riemannian manifold $(M^n, g)$ of dimension $n \geq 2$, define

$$\mathrm{k}_t(M^n, g) := \sup_{x\in M} \int_0^t \int_M H(s,x,y)\mathrm{Ric}_-(y)\,\mathrm{d}\nu_g(y)\,\mathrm{d}s$$

for any $t > 0$, where $H$ is the heat kernel of $M$, $\nu_g$ is the Riemannian volume measure and $\mathrm{Ric}_-\colon M \to \mathbb{R}_+$ is the lowest non-negative function such that for any $x \in M$,

$$\mathrm{Ric}_x \geq -\mathrm{Ric}_-(x) g_x.$$

Equivalently, $\mathrm{Ric}_-$ is the negative part of the smallest eigenvalue of the Ricci tensor.

For the whole article, we keep a positive number $T$ and a function $f\colon (0,T] \to \mathbb{R}_+$ fixed, so that $f$ is non-decreasing and

$$\lim_{t\to 0} f(t) = 0 \quad \text{and} \quad f(T) \leq \frac{1}{16n}. \tag{1}$$

We let $\mathcal{K}(n,f)$ be the set of isometry classes of $n$-dimensional closed Riemannian manifolds $(M^n, g)$ satisfying the Kato bound

$$\mathrm{k}_t(M^n, g) \leq f(t), \quad \forall t \in (0,T]. \tag{K}$$

This bound is implied, for instance, by a lower bound on the Ricci curvature, or by a suitable uniform $L^p$ estimate on $\mathrm{Ric}_-$ [RS17].

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For $c>0$ fixed throughout the article, let $\mathcal{K}_{\mathsf{m}}(n,f,c)$ be the set of quadruples $(M^n,\mathsf{d}_g,\mu,o)$ where $(M^n,g)\in\mathcal{K}(n,f)$, $o\in M$, $\mathsf{d}_g$ is the Riemannian distance associated with $g$ and $\mu$ is a multiple of $\nu_g$ satisfying

$$c\le\mu(B_{\sqrt{T}}(o))\le c^{-1}.$$

As proved in [Car19, CMT24], elements in $\mathcal{K}(n,f)$ satisfy a uniform doubling condition. As a consequence, Gromov's precompactness theorem ensures that the set $\mathcal{K}_{\mathsf{m}}(n,f,c)$ is precompact in the pointed measured Gromov–Hausdorff topology. We call Kato limit space any element in the closure $\overline{\mathcal{K}_{\mathsf{m}}(n,f,c)}$ with respect to this topology. Observe that Ricci limit spaces, that is limits of manifolds with a uniform Ricci lower bound [CC97, CC00a, CC00b, Che01], are Kato limit spaces.

Our first result is the rectifiability of Kato limit spaces. This was shown for Ricci limit spaces in [CC00b, Theorem 5.7].

**Theorem 1.1.** *Let $(X,\mathsf{d},\mu,o)$ be a Kato limit space. Then $(X,\mathsf{d},\mu)$ is rectifiable as a metric measure space, in the sense that there exists a countable collection $\{(k_i,V_i,\phi_i)\}_i$ where $\{V_i\}$ are Borel subsets covering $X$ up to a $\mu$-negligible set, $\{k_i\}$ are positive integers, and $\phi_i\colon V_i\to\mathbb{R}^{k_i}$ is a bi-Lipschitz map such that $(\phi_i)_\#(\mu\llcorner V_i)\ll\mathcal{H}^{k_i}$ for any $i$, where $\mathcal{H}^{k_i}$ is the $k_i$-dimensional Hausdorff measure.*

Consider now the non-collapsing case, that is, there exists $v>0$ such that for some $o\in M$

$$\nu_g(B_{\sqrt{T}}(o))\ge vT^{\frac{n}{2}}. \tag{NC}$$

Assume that $f$ additionally satisfies

$$\int_0^T\frac{\sqrt{f(t)}}{t}\,\mathrm{d}t<+\infty. \tag{SK}$$

In this case, we say that $(M^n,g)\in\mathcal{K}(n,f)$ satisfies a strong Kato bound. Let $\mathcal{K}(n,f,v)$ be the set of isometry classes of pointed closed $n$-dimensional manifolds $(M^n,g,o)$ satisfying a strong Kato bound and the non-collapsing assumption. We call non-collapsed strong Kato limit space any element in the closure $\overline{\mathcal{K}(n,f,v)}$ with respect to the pointed Gromov–Hausdorff topology. Notice that we do not need to consider measured Gromov–Hausdorff topology, because, thanks to the volume continuity proved in [CMT24, Theorem 7.1], Riemannian volumes converge to the $n$-dimensional Hausdorff measure.

Our second main result is the bi-Hölder regularity of the regular set of non-collapsed strong Kato limit spaces. This was proved for non-collapsed Ricci limit spaces in [CC97, Theorem 5.14].

**Theorem 1.2.** *Let $(X,\mathsf{d},o)$ be a non-collapsed strong Kato limit space. Then for any $\alpha\in(0,1)$ the regular set*

$$\mathcal{R}:=\{x\in X\colon(\mathbb{R}^n,\mathsf{d}_e,0)\in\mathrm{Tan}(X,x)\}$$

*is contained in an open $\mathcal{C}^\alpha$ manifold $\mathcal{U}_\alpha\subset X$. Here $\mathsf{d}_e$ is the Euclidean distance and $\mathrm{Tan}(X,x)$ is the set of metric tangent cones of $X$ at $x$, see Definition 2.1.*

In [CMT24, Theorem 6.2] we also showed that non-collapsed strong Kato limit spaces admit a stratification. By combining this with volume continuity and arguments from [CC97, Theorem 6.1] (see also [Che01, Theorem 10.22]), we then prove that the singular set $\mathcal{S}:=X\setminus\mathcal{R}$ of any $(X,\mathsf{d},o)\in\overline{\mathcal{K}(n,f,v)}$ has codimension two. For the sake of completeness, we provide a proof in the Appendix.

Our proofs of Theorem 1.1 and Theorem 1.2 strongly rely on the existence of splitting maps on Kato limit spaces. These are harmonic maps with a suitable $W^{2,2}$-estimate which realize a Gromov–Hausdorff approximation between a small ball around $x$ and a Euclidean ball of same radius. In Section 3, we give conditions for the existence of such maps, and establish some of their properties, relying on the analysis performed in [CMT24].

In order to prove Theorem 1.1, we start by observing that almost splitting maps exist around any point $x$ of a Kato limit space admitting a Euclidean tangent cone. After that, by means of a suitable propagation property of these maps, we adapt arguments from [BPS21] which built upon [GP21] to provide a proof of the rectifiability of $\mathrm{RCD}(K, N)$ spaces [MN19] via almost splitting maps. Let us point out that, unlike the uniform lower Ricci bound considered in [CC97], the Kato bound (K) does not provide a directionally restricted relative volume comparison on the limit space, so that the proof of rectifiability by Cheeger and Colding, based on a suitable control on the volume deformation of pseudo-cubes through pseudo-translations, do not carry out.

To prove Theorem 1.2, a key tool is the following almost monotone quantity, which we introduced in [CMT24] to get information on the infinitesimal geometry of non-collapsed strong Kato limits. For $X \in \overline{\mathcal{K}(n, f, v)}$, $x \in X$, $t > 0$, consider

$$\theta(t, x) := (4\pi t)^{n/2} H(t, x, x)$$

where $H$ is the heat kernel of $X$. In case $(M^n, g)$ is a Riemannian manifold with non-negative Ricci curvature, the Li–Yau inequality implies that the function $t \mapsto \theta(t, x) \in [1, +\infty[$ is non-decreasing for all $x \in M$. When $(M^n, g)$ satisfies a strong Kato bound, we showed in [CMT24] that this function is almost non-decreasing everywhere. In particular, its limit as $t$ goes to zero is well-defined, not less than one, and coincides with the inverse of the volume density at $x$. In the present paper, we prove that under (SK) the regular set of $X$ is given by points where the limit of $\theta$ equals one:

$$\mathcal{R} = \left\{ x \in X \colon \lim_{t \to 0} \theta(t, x) = 1 \right\}.$$

We also establish that if $\theta(t, x)$ is close enough to 1 for some $t > 0$ and $x \in X$, then any ball centered around $x$ with small radius is Gromov–Hausdorff close to a Euclidean ball with same radius. More precisely, we prove the following Reifenberg regularity statement, where $\mathsf{d}_{\mathrm{GH}}$ denotes the Gromov–Hausdorff distance.

**Theorem 1.3.** *Assume that* (SK) *holds. Then for any $\varepsilon > 0$ there exists $\delta > 0$ depending on $n$, $f$ and $\varepsilon$ such that for any $(X, \mathsf{d}, o) \in \overline{\mathcal{K}(n, f, v)}$, if $x \in X$ and $t \in (0, \delta T)$ satisfy*

$$\theta(t, x) \leq 1 + \delta \tag{2}$$

*then for any $y \in B_{\sqrt{t}}(x)$ and $s \in (0, \sqrt{t}]$,*

$$\mathsf{d}_{\mathrm{GH}}\left(B_s(y), \mathbb{B}^n_s\right) \leq \varepsilon s,$$

*where $\mathbb{B}^n_s$ is the Euclidean ball of radius $s$ centered at $0 \in \mathbb{R}^n$.*

In addition to the almost monotonicity of $\theta$ and the appropriate Li–Yau inequality for Kato limit spaces (see Proposition 2.9), a salient ingredient in our proof of Theorem 1.3 is the heat kernel rigidity result obtained in [CT22], which allows for a suitable contradiction argument.

From Theorem 1.3 we could immediately appeal on the intrinsic Reifenberg theorem of Cheeger and Colding [CC97, Theorem A.1.1] and get the conclusion of Theorem 1.2. We prefer to provide an explicit construction of a bi-Hölder homeomorphism obtained from almost splitting maps through a Transformation Theorem, in the spirit of [CJN21]. One key new point in our approach is an almost-rigidity statement implying that for sufficiently small $\delta$, if a point $x$ in a non-collapsed strong Kato limit space satisfies

$$\theta(t,x) < 1+\delta,$$

then an almost splitting map realizing a GH-isometry exists from $B_{\sqrt{t}}(x)$ to an Euclidean ball of radius $\sqrt{t}$. We next prove a Transformation Theorem that eventually provides a better regularity on such harmonic maps: these are bi-Hölder homeomorphisms. The proof of the Transformation Theorem is a direct one and uses some results of [CMT24] about convergence of harmonic functions together with the refinements that we develop in Section 3.

We conclude this introduction by pointing out that our recent work [CMT25] allows us to extend all of the previous results to limits of *complete* manifolds. Moreover, we improve the result of Theorem 1.1 and obtain the rectifiability of a Kato limit of complete manifolds with all the dimension $k_i$ being equal to a unique $k \in \{0, \dots, n\}$. As for Theorems 1.2 and 1.3, we obtain them under the weaker assumption that the integral between 0 and $T$ of $t \mapsto f(t)/t$ is finite, instead of the hypothesis (SK).

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## 2. Preliminaries

In a metric space $(X, \mathsf{d})$ we denote by $B_r(x)$ the open ball of radius $r$ centered at $x \in X$. Letting $B = B_r(x)$, for any $\lambda > 0$ we denote by $\lambda B$ the re-scaled ball centered at $x$ of radius $\lambda r$. We call metric measure space any triple $(X, \mathsf{d}, \mu)$ where $(X, \mathsf{d})$ is a geodesic and proper metric space and $\mu$ is a fully supported Borel measure such that $\mu(B_r(x))$ is strictly positive and finite for any $x \in X$ and $r > 0$.

The Cheeger energy of $(X, \mathsf{d}, \mu)$

$$\mathsf{Ch}\colon L^2(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

is defined as the lower semi-continuous envelope of the functional:

$$f \in \mathrm{Lip}_c(X) \mapsto \int_X \mathrm{lip}^2(f)\, \mathrm{d}\mu,$$

where $\mathrm{lip}(f)$ denotes the local Lipschitz constant of $f$. Following [Gig15, Gig18b] we say that $(X, \mathsf{d}, \mu)$ is infinitesimally Hilbertian if $\mathsf{Ch}$ is quadratic, in which case the closure of $\mathsf{Ch}$, still denoted by $\mathsf{Ch}$, is a Dirichlet form with domain denoted by $H^{1,2}(X, \mathsf{d}, \mu)$. We write $L$ for the associated non-positive, self-adjoint operator and $\{e^{-tL}\}_{t>0}$ for the Markov semi-group generated by $L$. For any $f \in H^{1,2}(X, \mathsf{d}, \mu)$ there exists a unique $|df| \in L^2(X, \mu)$ called minimal relaxed slope of $f$ such that

$$\mathsf{Ch}(f) = \int_X |df|^2\, \mathrm{d}\mu.$$

Moreover, $\mathsf{Ch}$ is strongly local and regular, and its carré du champ is given by

$$\mathrm{d}\Gamma(u,v) = \frac{1}{4}(|d(u+v)|^2 - |d(u-v)|^2)\,\mathrm{d}\mu =: \langle du, dv\rangle\,\mathrm{d}\mu$$

for any $u, v \in H^{1,2}(X, \mathsf{d}, \mu)$. For any open set $\Omega \subset X$ we also set

$$H^{1,2}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mu) := \{f \in L^2_{\mathrm{loc}}(\Omega, \mu) \colon \phi f \in H^{1,2}(X, \mathsf{d}, \mu) \text{ for any } \phi \in \mathrm{Lip}_c(\Omega)\}.$$

We say that $f \in H^{1,2}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mu)$ is harmonic in $\Omega$ if for any $\phi \in \mathrm{Lip}_c(\Omega)$,

$$\int_\Omega \langle df, d\phi\rangle\,\mathrm{d}\mu = 0.$$

If $(M^n, g)$ is a smooth and connected Riemannian manifold, the Cheeger energy of $(M, \mathsf{d}_g, \nu_g)$ coincides with its usual Dirichlet energy. We often implicitly identify a Riemannian manifold $(M^n, g)$ with its isometry class or with the metric measure space $(M, \mathsf{d}_g, \nu_g)$.

For any positive integer $k$, we denote by $\mathbb{B}^k_r$ the Euclidean ball of radius $r$ centered at the origin of $\mathbb{R}^k$, and we write $\mathbb{B}^k_r(p) = p + \mathbb{B}^k_r$ for any $p \in \mathbb{R}^k$.

**2.1. Notions of convergence.** We assume the reader to be familiar with the various notions of Gromov–Hausdorff convergence; we refer to [HKST15, Section 11], for instance, if this is not the case. We simply recall that a map $\phi\colon (X, \mathsf{d}_X) \to (Y, \mathsf{d}_Y)$ is called an $\varepsilon$-GH isometry if $|\mathsf{d}_X(x, x') - \mathsf{d}_Y(\phi(x), \phi(x'))| \le \varepsilon$ for any $x, x' \in X$ and for any $y \in Y$ there exists $x \in X$ such that $\mathsf{d}_Y(\phi(x), y) \le \varepsilon$. If $\{(X_\alpha, \mathsf{d}_\alpha, o_\alpha)\}, (X, \mathsf{d}, o)$ are pointed metric spaces such that $(X_\alpha, \mathsf{d}_\alpha, o_\alpha) \to (X, \mathsf{d}, o)$ in the pointed Gromov–Hausdorff topology, we denote by $x_\alpha \in X_\alpha \to x \in X$ a convergent sequence of points, following [CMT24, Characterization 1] and the definition soon after.

**2.1.1. Tangent cones.** Let us recall the classical definitions of tangent cones.

**Definition 2.1.** (1) Let $(X, \mathsf{d})$ be a metric space. For any $x \in X$, we call metric tangent cone of $(X, \mathsf{d})$ at $x$ any pointed metric space $(Y, \mathsf{d}_Y, x)$ obtained as a limit point in the pointed Gromov–Hausdorff topology of the family of rescalings $\{(X, r^{-1}\mathsf{d}, x)\}_{r>0}$ as $r \downarrow 0$. Note that by a slight abuse of notation, we identically denote the base point $x$ in the tangent cone $(Y, \mathsf{d}_Y, x)$ and the point $x \in X$. These are strictly speaking not the same points as they belong to different spaces. We denote by $\mathrm{Tan}(X, x)$ the set of metric tangent cones of $(X, \mathsf{d})$ at $x$.

(2) Let $(X, \mathsf{d}, \mu)$ be a metric measure space. For any $x \in X$, we call metric measured tangent cone of $(X, \mathsf{d}, \mu)$ at $x$ any pointed metric measure space $(Y, \mathsf{d}_Y, \mu_Y, x)$ obtained as a limit point in the pointed measured Gromov–Hausdorff topology of the family of rescalings $\{(X, r^{-1}\mathsf{d}, \mu(B_r(x))^{-1}\mu, x)\}_{r>0}$ as $r \downarrow 0$. We denote by $\mathrm{Tan}_{\mathsf{m}}(X, x)$ the set of metric measured tangent cones of $(X, \mathsf{d}, \mu)$ at $x$.

We are especially interested in tangent cones which split off a Euclidean factor. Let us recall the definition.

**Definition 2.2.** Let $k$ be a positive integer.

(1) We say that a pointed metric space $(X, \mathsf{d}, x)$ splits off an $\mathbb{R}^k$ factor if there exist a pointed metric space $(Z, \mathsf{d}_Z, z)$ and an isometry $\phi\colon X \to \mathbb{R}^k \times Z$ such that $\phi(x) = (0, z)$.

(2) We say that a pointed metric measure space $(X, \mathsf{d}, \mu, o)$ splits off an $\mathbb{R}^k$ factor if there exist a pointed metric measure space $(Z, \mathsf{d}_Z, \mu_Z, z)$ and an isometry $\phi\colon X \to \mathbb{R}^k \times Z$ such that $\phi(x) = (0, z)$ and $\phi_\# \mu = \mathcal{H}^k \otimes \mu_Z$.

Here and in the sequel the space $\mathbb{R}^k \times Z$ is implicitly equipped with the classical Pythagorean product distance.

**2.1.2. Convergence of functions.** Let us recall now some notions of convergence for functions defined on varying spaces. We refer to [CMT24, Section 1.4] and the references therein for a more exhaustive presentation.

**Definition 2.3.** Let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_\alpha, (X, \mathsf{d}, \mu, o)$ be infinitesimally Hilbertian metric measure spaces such that $(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha) \to (X, \mathsf{d}, \mu, o)$ in the pointed measured Gromov–Hausdorff topology.

(1) Let $\varphi_\alpha \in \mathcal{C}_c(X_\alpha)$ for any $\alpha$ and $\varphi \in \mathcal{C}_c(X)$ be given. We say that $\{\varphi_\alpha\}$ converges uniformly on compact sets to $\varphi$, if there exists $R > 0$ such that $\operatorname{supp}\varphi \subset B_R(o)$ and $\operatorname{supp}\varphi_\alpha \subset B_R(o_\alpha)$ for any $\alpha$, and $\varphi_\alpha(x_\alpha) \to \varphi(x)$ whenever $x_\alpha \in X_\alpha \to x \in X$. We write $\varphi_\alpha \xrightarrow{\mathcal{C}_c} \varphi$ if this convergence holds.

(2) Let $f_\alpha \in L^2(X_\alpha, \mu_\alpha)$ for any $\alpha$ and $f \in L^2(X, \mu)$ be given.

- We say that $\{f_\alpha\}$ converges to $f$ weakly in $L^2$ if $\sup_\alpha \|f_\alpha\|_{L^2} < +\infty$ and
$$\int_{X_\alpha} \varphi_\alpha f_\alpha \, \mathrm{d}\mu_\alpha \to \int_X \varphi f \, \mathrm{d}\mu$$
whenever $\varphi_\alpha \xrightarrow{\mathcal{C}_c} \varphi$; we write $f_\alpha \overset{L^2}{\rightharpoonup} f$ if this convergence holds.
- We say that $\{f_\alpha\}$ converges to $f$ strongly in $L^2$ if $f_\alpha \overset{L^2}{\rightharpoonup} f$ and $\lim_\alpha \|f_\alpha\|_{L^2} = \|f\|_{L^2}$; we write $f_\alpha \overset{L^2}{\to} f$ if this convergence holds.

(3) Let $f_\alpha \in H^{1,2}(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha)$ for any $\alpha$ and $f \in H^{1,2}(X, \mathsf{d}, \mu)$ be given.

- We say that $\{f_\alpha\}$ converges to $f$ weakly in energy if $f_\alpha \overset{L^2}{\rightharpoonup} f$ and $\sup_\alpha \mathsf{Ch}_\alpha(f_\alpha) < +\infty$; we write $f_\alpha \overset{\mathrm{E}}{\rightharpoonup} f$ if this convergence holds.
- We say that $\{f_\alpha\}$ converges to $f$ strongly in energy if $f_\alpha \overset{\mathrm{E}}{\rightharpoonup} f$ and $\lim_\alpha \mathsf{Ch}_\alpha(f_\alpha) = \mathsf{Ch}(f)$; we write $f_\alpha \overset{\mathrm{E}}{\to} f$ if this convergence holds.

**2.2. Kato bound and Kato limits.** Recall that $T$, $f$ are fixed and satisfy (1). The following has been proved in [CMT24, Proposition 2.3].

**Proposition 2.4.** *There exists $\mathsf{\kappa} \geq 1$ and $\mathsf{\lambda} > 0$ depending only on $n$ such that any $(M^n, g) \in \mathcal{K}(n, f)$ satisfies*

1. *a uniform volume estimate: for any $x \in M$ and $0 < s < r \leq \sqrt{T}$,*
$$\frac{\nu_g(B_r(x))}{\nu_g(B_s(x))} \leq \mathsf{\kappa} \left(\frac{r}{s}\right)^{e^2 n}, \tag{3}$$
2. *a uniform local Poincaré inequality: for any ball $B \subset M$ with radius $r \leq \sqrt{T}$ and any $\varphi \in \mathcal{C}^1(B)$,*
$$\int_B \left(\varphi - \fint_B \varphi \, \mathrm{d}\nu_g\right)^2 \mathrm{d}\nu_g \leq \mathsf{\lambda} r^2 \int_B |d\varphi|^2 \, \mathrm{d}\nu_g. \tag{4}$$

**Remark 2.5.** Note that (3) implies a so-called doubling condition:
$$\nu_g(B_{2r}(x)) \leq A(n)\nu_g(B_r(x)) \tag{5}$$
for any $x \in X$ and $r \in (0, \sqrt{T}/2]$, where $A(n) := \mathsf{\kappa} 2^{e^2 n}$. We shall often use the following consequence of the doubling condition: for any $\lambda \in (0, 1)$ there exists a

constant $C(n,\lambda) \geq 1$ such that for any ball $B \subset M$ and any locally integrable $\phi\colon B \to \mathbb{R}$,

$$\fint_{\lambda B} |\phi| \, \mathrm{d}\nu_g \leq C(n,\lambda) \fint_B |\phi| \, \mathrm{d}\nu_g. \tag{6}$$

The next proposition collects estimates on the heat kernel of $(M^n, g) \in \mathcal{K}(n,f)$.

**Proposition 2.6.** *There exists a constant $\gamma \geq 1$ depending only on $n$ such that for any $(M^n, g) \in \mathcal{K}(n,f)$, for all $x, y \in M$ and $t \in (0,T)$,*

i) $\dfrac{\gamma^{-1}}{\nu_g(B_{\sqrt{t}}(x))} e^{-\gamma \frac{\mathsf{d}_g^2(x,y)}{t}} \leq H(t,x,y) \leq \dfrac{\gamma}{\nu_g(B_{\sqrt{t}}(x))} e^{-\frac{\mathsf{d}_g^2(x,y)}{5t}}$,

ii) $\left| \dfrac{\partial}{\partial t} H(t,x,y) \right| \leq \dfrac{\gamma}{t \nu_g(B_{\sqrt{t}}(x))} e^{-\frac{\mathsf{d}_g{}^2(x,y)}{5t}}$,

iii) $|d_x H(t,x,y)| \leq \dfrac{\gamma}{\sqrt{t} \nu_g(B_{\sqrt{t}}(x))} e^{-\frac{\mathsf{d}_g{}^2(x,y)}{5t}}$.

*Proof.* The first estimate i) was established in [Car19], see also [CMT24, Proposition 2.3]. The second estimate ii) is a consequence of i), see e.g. [Gri95, Corollary 3.1]. The third estimate iii) is a consequence of the Li–Yau inequality [Car19, Proposition 3.3]:

$$e^{-2} |d_x H(t,x,y)|^2 \leq \frac{e^2 n}{2t} H^2(t,x,y) + H(t,x,y) \left| \frac{\partial}{\partial t} H(t,x,y) \right|,$$

together with i) and ii). □

Let us now recall a couple of results from [CMT24] about Kato limit spaces.

**Proposition 2.7.** *Any $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n,f,c)}$ is an infinitesimally Hilbertian space satisfying the doubling condition* (5) *and the local Poincaré inequality* (4)*. Moreover, for any $x \in X$, any $(Y, \mathsf{d}_Y, \mu_Y, x) \in \mathrm{Tan}_{\mathfrak{m}}(X,x)$ is a pointed* $\mathrm{RCD}(0,n)$ *space.*

Metric measure spaces satisfying an $\mathrm{RCD}(0,n)$ bound have, in a synthetic sense, non-negative Ricci curvature and dimension less than $n$. We refer to [Gig18a] for a survey about their properties.

From [CMT24], we also know that the following hold.

**Proposition 2.8.** *Let $\{(M^n_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\} \subset \mathcal{K}_{\mathfrak{m}}(n,f,c)$ be converging to $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n,f,c)}$ in the pointed measured Gromov–Hausdorff topology. Let $H_\alpha$ be the heat kernel of $(M^n_\alpha, g_\alpha)$ for any $\alpha$. Then $X$ admits a locally Lipschitz heat kernel, that is to say a map $H\colon (0,+\infty) \times X \times X \to (0,+\infty)$ such that*

$$\left(e^{-tL} f\right)(x) = \int_X H(t,x,y) f(y) \, \mathrm{d}\mu(y)$$

*for any $f \in L^2(X,\mu)$, any $t > 0$ and $\mu$-a.e. $x \in X$. Moreover, $H$ is analytic with respect to $t$ and it satisfies the three estimates in Proposition 2.6. Furthermore, the following convergence results hold.*

- *For any $t > 0$ and $x_\alpha \in M_\alpha \to x \in X$, $y_\alpha \in M_\alpha \to y \in X$,*

$$H_\alpha(t, x_\alpha, y_\alpha) \to H(t,x,y) \quad \text{and} \quad \frac{\partial}{\partial t} H_\alpha(t, x_\alpha, y_\alpha) \to \frac{\partial}{\partial t} H(t,x,y). \tag{7}$$

- *For any $t > 0$ and $x_\alpha \in M_\alpha \to x \in X$,*

$$H_\alpha(t, x_\alpha, \cdot) \xrightarrow{L^2} H(t,x,\cdot). \tag{8}$$

As an important consequence, we derive in the next statement a Li–Yau inequality for Kato limit spaces.

**Proposition 2.9.** *Consider* $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n, f, c)}$. *Set* $\gamma(t) = \exp\left(8\sqrt{nf(t)}\right)$ *for any* $t \in (0, T]$. *Then for any* $x \in X$ *and* $t \in (0, T]$, *the Li–Yau inequality*

$$\gamma^{-1}(t)\,|dH(t,x,\cdot)|^2 - H(t,x,\cdot)\frac{\partial}{\partial t}H(t,x,\cdot) \le \frac{n\gamma(t)}{2t}H^2(t,x,\cdot) \tag{9}$$

*holds* $\mu$*-a.e. on* $X$.

*Proof.* Let $\{(M^n_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\} \subset \mathcal{K}_{\mathfrak{m}}(n, f, c)$ be converging to $(X, \mathsf{d}, \mu, o)$ in the pointed measured Gromov–Hausdorff topology. By [Car19, Proposition 3.3], for any $x, y \in M_\alpha$ and $t \in (0, T]$,

$$\gamma^{-1}(t)\,|d_y H_\alpha(t,x,y)|^2 - H_\alpha(t,x,y)\frac{\partial}{\partial t}H_\alpha(t,x,y) \le \frac{n\gamma(t)}{2t}H^2_\alpha(t,x,y). \tag{10}$$

Take $x_\alpha \in M_\alpha \to x \in X$ and set $u_\alpha(y) = H_\alpha(t, x_\alpha, y)$ for any $y \in M_\alpha$ and any $\alpha$. The $L^2$ heat kernel convergence (8) yields

$$u_\alpha \xrightarrow{L^2} u := H(t,x,\cdot).$$

Moreover, the semi-group property implies

$$\begin{aligned}\int_{M_\alpha} |du_\alpha|^2 \,\mathrm{d}\nu_{g_\alpha} &= \int_{M_\alpha} u_\alpha \Delta_{g_\alpha} u_\alpha \,\mathrm{d}\nu_{g_\alpha} = -\frac{1}{2}\frac{\partial}{\partial t}H_\alpha(2t, x_\alpha, x_\alpha)\\ &= -\frac{\partial H_\alpha}{\partial t}(2t, x_\alpha, x_\alpha),\end{aligned} \tag{11}$$

hence by Proposition 2.6.ii) the sequence $\{u_\alpha\}$ is bounded in energy, hence $u_\alpha \overset{E}{\rightharpoonup} u$ by definition. Since the semi-group property also implies (11) with $u$, $H$ and $x$ in place of $u_\alpha$, $H_\alpha$ and $x_\alpha$ respectively, the convergence (7) yields $\lim_\alpha \|du_\alpha\|_{L^2} = \mathsf{Ch}(u)$, hence by definition $u_\alpha \xrightarrow{E} u$. Proposition 2.6.iii) implies that the sequence $\{|du_\alpha|\}$ is locally bounded in $L^\infty$ hence with [CMT24, Proposition E.7] we can conclude that

$$|du_\alpha| \overset{L^2}{\rightharpoonup} |du|.$$

This convergence, together with (7) and (10), implies (9). ☐

**Remark 2.10.** If there exists $\tau \in (0, T]$ such that

$$\lim_\alpha \mathrm{k}_\tau(M_\alpha, g_\alpha) = 0,$$

then for any $x \in X$ and $t \in (0, \tau]$, the Li–Yau inequality

$$|dH(t,x,\cdot)|^2 - H(t,x,\cdot)\frac{\partial}{\partial t}H(t,x,\cdot) \le \frac{n}{2t}H^2(t,x,\cdot)$$

holds $\mu$-a.e. on $X$.

## 3. Almost splittings maps and consequences of GH-closedness on functions

In this section, we define $(k, \varepsilon)$-splitting maps on Kato limits and prove some relevant properties. Such maps were introduced in [CC96, Col97, CC97] for the study of Ricci limit spaces and extensively used later in the study of limit spaces and $\mathrm{RCD}(K, N)$ spaces, see for instance [CN15, Bam20, CJN21, BPS21].

From now on, for any positive integer $k$, we let $\mathcal{M}_k(\mathbb{R})$ be the space of $k \times k$ matrices with real entries, $\mathcal{S}_k(\mathbb{R}) \subset \mathcal{M}_k(\mathbb{R})$ be the subspace made of symmetric matrices, and we denote by $\|\cdot\|$ the matrix norm induced by the Euclidean norm $|\cdot|$, meaning that $\|A\|^2 := \sup\{{}^t(A\xi)A\xi : \xi \in \mathbb{R}^k \text{ such that } {}^t\xi\xi = 1\}$ for any $A \in \mathcal{M}_k(\mathbb{R})$. We denote by $\mathrm{Id}_k$ the identity matrix in $\mathcal{M}_k(\mathbb{R})$. Then the following holds.

**Lemma 3.1.** *Assume that $A \in \mathcal{S}_k(\mathbb{R})$ is positive definite. Then there exists a unique lower triangular matrix $T \in \mathcal{M}_k(\mathbb{R})$ such that*

$$TA^tT = \mathrm{Id}_k. \tag{12}$$

*Moreover, if there exists $\varepsilon \in (0, 1/2)$ such that $A \in \mathcal{S}_k(\mathbb{R})$ satisfies*

$$\|A - \mathrm{Id}_k\| < \varepsilon, \tag{13}$$

*then for some $C_k$ depending only on $k$, the matrix $T$ satisfies*

$$\|T - \mathrm{Id}_k\| < C_k\varepsilon. \tag{14}$$

**Remark 3.2.** The matrix ${}^tT$ is obtained by applying the Gram–Schmidt process.

**3.1. Almost splitting maps.** For any infinitesimally Hilbertian metric measure space $(X, \mathsf{d}, \mu)$, whenever a map $u = (u_1, \ldots, u_k) \colon B \to \mathbb{R}^k$ satisfies $u_i \in H^{1,2}(B, \mathsf{d}, \mu)$ for any $i \in \{1, \ldots, k\}$ we define the Gram matrix map of $u$ as the $\mathcal{S}_k(\mathbb{R})$-valued map

$$G_u := [G_{i,j}] \quad \text{where } G_{i,j} := \langle du_i, du_j\rangle \text{ for any } 1 \le i, j \le k,$$

and we set $|dG_u|^2 := \sum_{1 \le i,j \le k} |dG_{i,j}|^2$. Note that if $T$ is a lower triangular $k \times k$ matrix and $\tilde{u} := T \circ u$, then

$$G_{\tilde{u}} = TG_u{}^tT \quad \mu\text{-a.e. in } B. \tag{15}$$

**Definition 3.3.** Let $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n, f, c)}$. Let $B \subset X$ be a ball of radius $r > 0$, $k \in \{1, \ldots, n\}$ and $\varepsilon > 0$.

(1) We call $(k, \varepsilon)$-splitting of $B$ any harmonic map $u \colon B \to \mathbb{R}^k$ such that $\|du\|_{L^\infty(B)} \le 2$ and

$$\fint_B \|G_u - \mathrm{Id}_k\| \,\mathrm{d}\mu < \varepsilon. \tag{16}$$

(2) We say that a $(k, \varepsilon)$-splitting $u$ of $B$ is reinforced if

$$\fint_B (\|G_u - \mathrm{Id}_k\| + r^2|dG_u|^2) \,\mathrm{d}\mu < \varepsilon.$$

(3) We say that a (possibly reinforced) $(k, \varepsilon)$-splitting $u$ of $B$ is balanced if

$$\fint_B G_u \,\mathrm{d}\mu = \mathrm{Id}_k.$$

**Remark 3.4.** Assumption $\|du\|_{L^\infty(B)} \le 2$ implies

$$\sup_{1 \le i,j \le k} |G_{i,j}(y)| \le 4 \quad \text{for } \mu\text{-a.e. } y \in B.$$

**Remark 3.5.** Condition (16) implies that the symmetric matrix $A_u = \fint_B G_u \,\mathrm{d}\mu$ is $\varepsilon$-close to the identity $\mathrm{Id}_k$. As a consequence of Lemma 3.1 applied with $A = A_u$, for any $\varepsilon \in (0, 1/2)$ and any $(k, \varepsilon)$-splitting $u \colon B \to \mathbb{R}^k$ there exists a lower triangular matrix $T$ with $\|T\| \le 1 + C_k\varepsilon$ such that the map $\tilde{u} = T \circ u \colon B \to \mathbb{R}^k$ satisfies

$$\fint_B G_{\tilde{u}} \,\mathrm{d}\mu = \mathrm{Id}_k \quad \text{and} \quad \fint_B \|G_{\tilde{u}} - \mathrm{Id}_k\| \,\mathrm{d}\mu < (1 + C_k\varepsilon)^2\varepsilon. \tag{17}$$

**Remark 3.6.** The definition of reinforced splitting is just a technical convenience. Indeed, by means of Bochner's formula and of the Hessian bound given in [CMT24, Proposition 3.5], one can prove that any splitting on a Riemannian manifold with a Kato bound is a reinforced splitting on a ball with smaller radius, and then show that this property for manifolds with a uniform Kato bound is stable under pointed measured Gromov–Hausdorff convergence. This implies, in particular, that if $u$ is a reinforced splitting of a ball $B$ in a Kato limit space, then the coefficients of the Gram matrix map $G_u$ all belong to $H^{1,2}_{loc}(B, \mathsf{d}, \mu)$.

The next result provides an improvement of the local Lipschitz constant for splittings.

**Proposition 3.7.** *Let $(M^n, g)$ be a closed Riemaniann manifold, $B \subset M$ a ball of radius $r > 0$, $k \in \{1, \dots, n\}$, $\eta \in (0,1)$, $L > 1$ and $u\colon B \to \mathbb{R}^k$ a harmonic map such that $\|du\|_{L^\infty(B)} \leq L$. Let $G_u$ be the Gram matrix map of $u$. Assume that there exists $\delta \in (0, 1/16n]$ such that*

$$\mathrm{k}_{r^2}(M^n, g) < \delta, \qquad \fint_B \|G_u - \mathrm{Id}_k\| \, \mathrm{d}\nu_g < \delta.$$

*Then there exists $C(n, \eta, L) > 0$ such that $\|du\|_{L^\infty(\eta B)} \leq 1 + C(n, \eta, L)\delta$.*

*Proof.* In the proof of [CMT24, Proposition 7.5], use the gradient bound iii) in Proposition 2.6 to get $II \leq C\delta$ instead of $II \leq C\delta^{1/2}$. Apply the resulting statement to any function $u_\xi := \langle \xi, u \rangle$ with $\xi \in \mathbb{R}^k$ satisfying $|\xi| = 1$, and conclude by taking $\xi = du/|du|$ pointwise. □

**3.2. GH-closedness and harmonic functions.** In the setting of uniform lower Ricci bounds, existence of almost splitting maps is closely related to mGH-closedness of a ball to a Euclidean ball. We show below that the same relation actually holds for Kato limit spaces.

Throughout this subsection, we let $k \in \{1, \dots, n\}$ be fixed. We denote by $\|\cdot\|_1$ the $L_{1,1}$ matrix norm, namely $\|M\|_1 = \sum_{i,j=1}^k |m_{i,j}|$ for any $M \in \mathcal{M}_k(\mathbb{R})$. Note that $\|\cdot\| \leq \|\cdot\|_1$. We denote by $\mathsf{d}_{\mathrm{mGH}}$ a distance associated to the measured Gromov–Hausdorff topology.

**Theorem 3.8.** *For all $\varepsilon, \eta, \lambda \in (0,1)$ such that $\lambda < \eta$ there exists $\nu$ depending only on $\varepsilon, \eta, \lambda, n, f, c$ such that if $(X, \mathsf{d}, \mu, o), (X', \mathsf{d}', \mu', o') \in \overline{\mathcal{K}_{\mathsf{m}}(n, f, c)}$, $x \in X$, $x' \in X'$ and $r \in (0, \sqrt{T}]$, are such that*

$$\mathsf{d}_{\mathrm{mGH}}(B_r(x), B_r(x')) < \nu r,$$

*if $h\colon B_r(x) \to \mathbb{R}^k$ is a harmonic function satisfying $\|dh\|_{L^\infty(B_r(x))} \leq L$ for some $L > 1$, then there exists a harmonic function $h'\colon B_{\eta r}(x') \to \mathbb{R}^k$ satisfying $\|dh'\|_{L^\infty(B_{\eta r}(x'))} \leq LC(n, \eta)$ for some $C(n, \eta) \geq 1$ and:*

1. *$\|h' \circ \Phi - h\|_{L^\infty(B_{\eta r}(x))} < \varepsilon r$, where $\Phi$ is a $(\nu r)$-GH isometry between $B_r(x)$ and $B_r(x')$;*
2. *for all $s \in [\lambda r, \eta r]$*

$$\left\| \fint_{B_s(x)} G_h \, \mathrm{d}\mu - \fint_{B_s(x')} G_{h'} \, \mathrm{d}\mu' \right\| < \varepsilon,$$

3. *for all $A \in \mathcal{M}_k(\mathbb{R})$ and $s \in [\lambda r, \eta r]$,*

$$\left| \fint_{B_s(x)} \|G_h - A\|_1 \, \mathrm{d}\mu - \fint_{B_s(x')} \|G_{h'} - A\|_1 \, \mathrm{d}\mu' \right| \leq \varepsilon.$$

The previous is a consequence of the analysis made in [CMT24, Appendix A]. For the sake of completeness, we provide a proof in Appendix B.

Theorem 3.8 has the following direct consequence about existence of reinforced almost splittings.

**Proposition 3.9.** *For any $\varepsilon, \eta \in (0,1)$ there exists $\delta > 0$ depending on $n, f, c, \varepsilon$ and $\eta$ such that if $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n,f,c)}$, $x \in X$ and $r \in (0, \sqrt{T}]$ satisfy*

$$f(r^2) \le \delta \quad \textit{and} \quad \mathsf{d}_{\mathrm{mGH}}(B_r(x), \mathbb{B}^k_r) < \delta r,$$

*then there exists a reinforced $(k,\varepsilon)$-splitting of $B_{\eta r}(x)$.*

*Proof.* By density and approximation, it is enough to show this proposition for $(M^n, \mathsf{d}_g, \mu = \lambda \nu_g, o) \in \mathcal{K}_{\mathfrak{m}}(n,f,c)$. Let $\varepsilon, \eta \in (0,1)$. We are going to show that if $\delta > 0$ is chosen sufficiently small (depending on $n, f, c, \varepsilon$ and $\eta$), if

$$f(r^2) \le \delta \text{ and } \mathsf{d}_{\mathrm{mGH}}(B_r(x), \mathbb{B}^k_r) < \delta r,$$

then the conclusion holds. Consider the identity map from $\mathbb{B}^k_r$ to $\mathbb{R}^k$ which is a harmonic isometry. Then, according to Theorem 3.8, for any $\tau \in (0,1)$, if $\delta$ is smaller than $\nu(\tau\varepsilon, \sqrt{\eta}, \eta, n, f, c)$, there exists a harmonic map $h = (h_1, \ldots, h_k) \colon B_{\sqrt{\eta} r}(x) \to \mathbb{R}^k$ satisfying $\|dh\|_{L^\infty(B_{\sqrt{\eta}r}(x))} \le C(n,\eta)$ and such that for any $s \in [\eta r, \sqrt{\eta} r]$ we have

$$\fint_{B_s(x)} \|G_h - \mathrm{Id}_k\|_1 \,\mathrm{d}\mu \le \tau\varepsilon.$$

Then according to Proposition 3.7, we know that

$$\|dh\|_{L^\infty(B_{\eta r}(x))} \le 1 + C(n,\eta)\left(\tau\varepsilon + \delta\right).$$

Hence if $\tau$ and $\delta$ are additionally chosen so that

$$\tau \le \frac{1}{2C(n,\eta)} \quad \text{and} \quad \delta \le \frac{1}{2C(n,\eta)}$$

then

$$\|dh\|_{L^\infty(B_{\eta r}(x))} \le 2.$$

An easy variation on the proof of the Hessian bound given in [CMT24, Proposition 3.5] provides the following estimate

$$\begin{aligned}(\eta r)^2 \fint_{B_{\eta r}(x)} |\nabla dh|^2 \,\mathrm{d}\mu &\le C(n,\eta,f) \fint_{B_s(x)} \|G_h - \mathrm{Id}_k\|_1 \,\mathrm{d}\mu + 4f(r^2) \\ &\le C(n,\eta,f)\tau\varepsilon + 4\delta\end{aligned}$$

and because

$$|\nabla \langle dh_i, dh_j\rangle| \le 2\left(|\nabla dh_i| + |\nabla dh_j|\right)$$

we also get

$$(\eta r)^2 \fint_{B_{\eta r}(x)} |\nabla G_h|^2 \,\mathrm{d}\mu \le 4C(n,\eta,f)\tau\varepsilon + 16\delta.$$

Hence the conclusion holds provided that $\tau$ and $\delta$ are moreover chosen so that

$$\tau \le \frac{1}{8C(n,\eta,f)} \quad \text{and} \quad \delta \le \frac{\varepsilon}{32}.$$ □

Moreover, Theorem 3.8 implies that almost splittings are GH-isometries under the appopriate assumptions.

**Proposition 3.10.** *For any $\varepsilon, \eta \in (0,1)$ there exist $\delta > 0$ depending on $n, f, c, \varepsilon$ and $\eta$ and a constant $C(n,\eta) > 0$, such that for all $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n,f,c)}$, if $u\colon B_r(x) \to \mathbb{R}^k$ is a $(k,\varepsilon)$-splitting and*

$$\mathsf{d}_{\mathrm{mGH}}(B_r(x), \mathbb{B}^k_r) < \delta r,$$

*then $u$ is a $(C(n,\eta)\sqrt{\varepsilon} r)$-GH isometry between $B_{\eta r}(x)$ and $\mathbb{B}^k_{\eta r}(u(x))$.*

The proof of this proposition relies on the following Euclidean result.

**Lemma 3.11.** *If $v\colon \mathbb{B}^k \to \mathbb{R}^k$ is a harmonic map such that*

$$\fint_{\mathbb{B}^k} \|G_v - \mathrm{Id}_k\| \le \varepsilon,$$

*then $v\colon \mathbb{B}^k_\eta \to \mathbb{R}^k$ is a $(C(n,\eta)\sqrt{\varepsilon})$-GH isometry between $\mathbb{B}_\eta$ and $\mathbb{B}^k_\eta(v(0))$.*

*Proof.* We will assume that $\eta \ge 1/2$. Consider a cut-off function $\chi$ equal to 1 on $\frac{1+\eta}{2}\mathbb{B}^k$ and vanishing outside $\frac{3+\eta}{4}\mathbb{B}^k$, with

$$\|\Delta\chi\|_{L^\infty} \le C(k,\eta).$$

By the Bochner formula we have that

$$|\mathrm{Hess}\, v|^2 + \frac{1}{2}\Delta(\mathrm{Tr}(G_v - \mathrm{Id}_k)) = 0$$

where Tr is the trace function for matrices. Then

$$\begin{aligned}\int_{\frac{1+\eta}{2}\mathbb{B}^k} |\mathrm{Hess}\, v|^2 \le \int_{\mathbb{B}^k} \chi |\mathrm{Hess}\, v|^2 &= -\frac{1}{2}\int_{\mathbb{B}^k} (\Delta\chi)\mathrm{Tr}(G_v - \mathrm{Id}_k) \\ &\le C(k,\eta)\int_{\mathbb{B}^k} \|G_v - \mathrm{Id}_k\| \le C(k,\eta)\varepsilon.\end{aligned}$$

Using classical elliptic estimate, we obtain a $\mathcal{C}^2$ estimate on $v$:

$$\|\mathrm{Hess}\, v\|_{L^\infty(\eta\mathbb{B}^k)} \le C(k,\eta)\sqrt{\varepsilon}.$$

With Taylor formula, we get that for any $x \in \eta\mathbb{B}^k$,

$$|v(x) - v(0) - dv(0)(x)| \le C(k,\eta)\sqrt{\varepsilon} \quad \text{and} \quad |dv(0) - dv(x)| \le C(k,\eta)\sqrt{\varepsilon}.$$

But we also have

$$\fint_{\eta\mathbb{B}^k} \|G_v - \mathrm{Id}_k\| \le \eta^{-k}\fint_{\mathbb{B}^k} \|G_v - \mathrm{Id}_k\| \le 2^k\varepsilon.$$

Hence we find a point $x_o \in \eta\mathbb{B}^k$ such that

$$\|G_v(x_o) - \mathrm{Id}_k\| \le 2^k\varepsilon.$$

Using the polar decomposition of $dv(x_o)$ we obtain a linear isometry $g \in \mathrm{O}(k)$ such

$$|dv(x_o) - g| \le C(k)\varepsilon.$$

Introducing the affine isometry $\iota := v(0) + g$ we get that for any $x \in \eta\mathbb{B}^k$,

$$|v(x) - \iota(x)| \le C(k,\eta)\sqrt{\varepsilon}.$$

Setting $C'(n,\eta) = \max_{1\le k\le n} C(k,\eta)$ eventually leads to the desired result. □

*Proof of Proposition 3.10.* We let $\varepsilon, \eta \in (0,1)$. We will assume that $\eta \ge 1/2$. With Theorem 3.8, we find some $\delta(n,\varepsilon,\eta,f,c) > 0$ such that if $(X,\mathsf{d},\mu,o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n,f,c)}$, if $u\colon B_r(x) \to \mathbb{R}^k$ is a $(k,\varepsilon)$-splitting and

$$\mathsf{d}_{\mathrm{mGH}}(B_r(x), \mathbb{B}^k_r) < \delta r,$$

then there is some harmonic map

$$v\colon \mathbb{B}^k_{(1+\eta)\frac{r}{2}} \to \mathbb{R}^k$$

and some $\delta r$-GH isometry $\Phi\colon B_r(x) \to \mathbb{B}^k_r$ such that

$$\|v \circ \Phi - u\|_{L^\infty(B_{(1+\eta)\frac{r}{2}}(x))} \le \varepsilon r \tag{18}$$

and

$$\left| \fint_{B_{(1+\eta)\frac{r}{2}}(x)} \|G_u - \mathrm{Id}_k\|_1 \,\mathrm{d}\mu - \fint_{\mathbb{B}^k_{(1+\eta)\frac{r}{2}}} \|G_v - \mathrm{Id}_k\|_1 \right| \le \varepsilon.$$

Observe that the doubling condition and the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_1$ yield

$$\fint_{B_{(1+\eta)\frac{r}{2}}(x)} \|G_u - \mathrm{Id}_k\|_1 \,\mathrm{d}\mu \le A(n) \fint_{B_r(x)} \|G_u - \mathrm{Id}_k\|_1 \,\mathrm{d}\mu \le C(n)\varepsilon$$

for some $C(n)$ only depending on $n$. Since $\|\cdot\| \le \|\cdot\|_1$, we get

$$\fint_{\mathbb{B}^k_{(1+\eta)\frac{r}{2}}} \|G_v - \mathrm{Id}_k\| \le (1 + C(n))\varepsilon.$$

Hence according to the previous lemma, we know that $v$ is a $(C(n,\eta)\sqrt{\varepsilon} r)$-GH isometry between $\mathbb{B}^k_{\eta r}$ and itself. Using (18), we obtain the desired conclusion about the restriction of $u$ to $B_{\eta r}(x)$. □

**3.3. Propagation of reinforced almost splittings.** The next result is an important propagation property of reinforced splittings.

**Proposition 3.12.** (Propagation of reinforced splittings) *Consider $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n, f, c)}$. There exists $C > 0$ depending only on $n$ such that for any $k \in \{1, \dots, n\}$ and $\varepsilon \in (0,1)$, if $u$ is a reinforced $(k,\varepsilon)$-splitting of a ball $B_r(x) \subset X$ with $r \in (0, \sqrt{T})$, then there exists a Borel set $\Omega_\varepsilon \subset B_{r/2}(x)$ such that:*

- (A) $\mu(B_{r/2}(x) \backslash \Omega_\varepsilon) \le C\sqrt{\varepsilon}\mu(B_{r/2}(x))$,
- (B) *the restriction of $u$ to $B_s(y)$ is a reinforced $(k, \sqrt{\varepsilon})$-splitting for any $y \in \Omega_\varepsilon$ and $s \in (0, r/2)$,*
- (C) *for $\mu$-a.e. $y \in \Omega_\varepsilon$, for any $\xi \in \mathbb{R}^k$,*

$$(1 - \sqrt{\varepsilon})|\xi|^2 \le {}^t\xi G_u(y)\xi \le (1 + \sqrt{\varepsilon})|\xi|^2, \tag{19}$$

- (D) *any $y \in \Omega_\varepsilon$ is such that any $(Y, \mathsf{d}_Y, \mu_Y, y) \in \mathrm{Tan}_{\mathfrak{m}}(X, y)$ splits off an $\mathbb{R}^k$ factor.*

*Proof.* Let $x \in X$ and $r \in (0, \sqrt{T})$. Assume that $u\colon B_r(x) \to \mathbb{R}^k$ is a reinforced $(k,\varepsilon)$-splitting. Set

$$\Omega_\varepsilon := \{y \in B_{r/2}(x)\colon M_{r/2}v(y) \le \sqrt{\varepsilon}\}$$

where

$$v := \|G_u - \mathrm{Id}_k\| + r^2|dG_u|^2$$

and

$$M_{r/2}v(y) := \sup_{s \in (0,r/2)} \fint_{B_s(y)} v \,\mathrm{d}\mu.$$

The definition of $\Omega_\varepsilon$ is made so that (B) is satisfied. Let us prove (A). For any $y \in B_{r/2}(x) \backslash \Omega_\varepsilon$ there exists $s_y \in (0, r/2)$ such that $\mu(B_{s_y}(y)) < (\sqrt{\varepsilon})^{-1} \int_{B_{s_y}(y)} v \,\mathrm{d}\mu$. By the Vitali covering lemma, there exists a countable family of points $\{y_i\} \subset B_{r/2}(x) \backslash \Omega_\varepsilon$

such that the balls $\{B_{s_{y_i}}(y_i)\}$ are pairwise disjoint and $B_{r/2}(x)\backslash\Omega_\varepsilon \subset \bigcup_i B_{5s_{y_i}}(y_i)$. Then, with a constant $C$ depending only on $n$ which may change from line to line,

$$\begin{aligned}\mu(B_{r/2}(x)\backslash\Omega_\varepsilon) &\leq \sum_i \mu(B_{5s_{y_i}}(y_i)) \leq C\sum_i \mu(B_{s_{y_i}}(y_i)) < C\frac{1}{\sqrt{\varepsilon}}\sum_i \int_{B_{s_{y_i}}(y_i)} v\,\mathrm{d}\mu \\ &\leq C\frac{1}{\sqrt{\varepsilon}}\int_{B_r(x)} v\,\mathrm{d}\mu \leq C\sqrt{\varepsilon}\mu(B_r(x)) \leq C\sqrt{\varepsilon}\mu(B_{r/2}(x))\end{aligned}$$

where we have used the doubling condition to get the second and the last inequalities, and the fact that $u$ is a reinforced $(k,\varepsilon)$-splitting of $B_r(x)$ to get the fifth one. This shows (A).

Let us prove (C). It follows from the Lebesgue differentiation theorem for doubling metric measure spaces (see e.g. [Hei01]) that the set of Lebesgue points of $v$ has full measure in $\Omega_\varepsilon$. At any Lebesgue point $y \in \Omega_\varepsilon$ of $v$ we know that

$$\|G_u(y) - \mathrm{Id}_k\| \leq v(y) = \lim_{s\downarrow 0} ⨍_{B_s(y)} v\,\mathrm{d}\mu \leq M_{r/2}v(y) \leq \sqrt{\varepsilon},$$

which yields (19).

We are left with proving (D) namely that for any $y \in \Omega_\varepsilon$, any $(Y, \mathsf{d}_Y, \mu_Y, y) \in \mathrm{Tan}_{\mathfrak{m}}(X, y)$ splits off an $\mathbb{R}^k$ factor. To this aim, we are going to build a harmonic map $\tilde{u}_\infty\colon Y \to \mathbb{R}^k$ such that $G_{\tilde{u}_\infty}(z) = \mathrm{Id}_k$ for $\mu_Y$-a.e. $z \in Y$. For any $s \in (0, r/2)$, set

$$\overline{G}_s := ⨍_{B_s(y)} G_u\,\mathrm{d}\mu.$$

Following a classical argument (see [Che99, (4.21)], for instance) involving Hölder's inequality, the doubling condition, and the local Poincaré inequality,

$$\begin{aligned}\|\overline{G}_s - \overline{G}_{s/2}\| &\leq ⨍_{B_{s/2}(y)} \|G_u - \overline{G}_s\|\,\mathrm{d}\mu \leq A(n) ⨍_{B_s(y)} \|G_u - \overline{G}_s\|\,\mathrm{d}\mu \\ &\leq A(n)\left(⨍_{B_s(y)} \|G_u - \overline{G}_s\|^2\,\mathrm{d}\mu\right)^{1/2} \\ &\leq A(n)\lambda^{1/2}s\left(⨍_{B_s(y)} |dG_u|^2\,\mathrm{d}\mu\right)^{1/2} \leq A(n)\lambda^{1/2}s\frac{\varepsilon^{1/4}}{r}\cdot\end{aligned}$$

This shows that $\{\overline{G}_s\}_{0<s<r/2}$ is a Cauchy sequence, hence it admits a limit $\overline{G}$ as $s \downarrow 0$. Since

$$\|\overline{G} - \mathrm{Id}_k\| = \lim_{s\to 0}\|\overline{G}_s - \mathrm{Id}_k\| \leq \lim_{s\to 0} ⨍_{B_s(y)} \|G_u - \mathrm{Id}_k\|\,\mathrm{d}\mu \leq \sqrt{\varepsilon},$$

we know from Remark 3.5 that there exists a lower triangular $k\times k$ matrix $T$ such that $T\overline{G}{}^tT = \mathrm{Id}_k$ and $\|T\| \leq C(n)$ for some generic constant $C(n)$ only depending on $n$. Moreover, for any $s \in (0, r/2)$, the previous computation yields

$$⨍_{B_s(y)} \|G_u - \overline{G}_s\|\,\mathrm{d}\mu \leq A(n)\lambda^{1/2}s\frac{\varepsilon^{1/4}}{r},$$

and a telescopic argument gives

$$\|\overline{G}_s - \overline{G}\| \leq C(n)s\frac{\varepsilon^{1/4}}{r},$$

hence $\tilde{u} := T \circ u$ satisfies

$$\fint_{B_s(y)} \|G_{\tilde{u}} - \mathrm{Id}_k\| \, \mathrm{d}\mu \leq C(n) s \frac{\varepsilon^{1/4}}{r} \cdot \tag{20}$$

Now we let $\{s_\alpha\} \subset (0, +\infty)$ be such that $s_\alpha \downarrow 0$ and $\{(X, \mathsf{d}_\alpha := s_\alpha^{-1}\mathsf{d}, \mu_\alpha := \mu(B_{s_\alpha}(y))^{-1}\mu, y)\}$ converges to $(Y, \mathsf{d}_Y, \mu_Y, y)$ in the pointed measured Gromov–Hausdorff topology. Then the maps

$$\tilde{u}_\alpha := \frac{1}{s_\alpha}(\tilde{u} - \tilde{u}(y)) : B^{\mathsf{d}_\alpha}_{r/2s_\alpha}(y) \to \mathbb{R}^k$$

are all harmonic and locally 2-Lipschitz. By [CMT24, Proposition E.10], up to extracting a subsequence we may assume that $\{\tilde{u}_\alpha\}$ converges uniformly on compact sets and locally strongly in energy to some harmonic map

$$\tilde{u}_\infty \colon Y \to \mathbb{R}^k.$$

Then the local strong convergence in energy and (20) imply that for any $R > 0$,

$$\begin{aligned} \fint_{B^{\mathsf{d}_Y}_R(y)} \|G_{\tilde{u}_\infty} - \mathrm{Id}_k\| \, \mathrm{d}\mu_Y &= \lim_\alpha \fint_{B^{\mathsf{d}_\alpha}_R(y)} \|G_{\tilde{u}_\alpha} - \mathrm{Id}_k\| \, \mathrm{d}\mu_\alpha \\ &= \lim_\alpha \fint_{B_{Rs_\alpha}(y)} \|G_{\tilde{u}} - \mathrm{Id}_k\| \, \mathrm{d}\mu = 0. \end{aligned}$$

Since $(Y, \mathsf{d}_Y, \mu_Y)$ is an $\mathrm{RCD}(0, n)$ space, the Functional Splitting Lemma [ABS19, Lemma 1.21] then yields the conclusion. □

**Remark 3.13.** The choice of $r/2$ in the previous proof is arbitrary: we can replace it with $\sigma r$ for $\sigma \in (0, 1)$ and get the same result.

## 4. Rectifiability of Kato limits

Let us begin this section with recalling the definitions of bi-Lipschitz map and bi-Lipschitz chart.

**Definition 4.1.** Let $(X, \mathsf{d})$ be a metric space, $k$ a positive integer, and $\varepsilon \in (0, 1)$. We say that a map $\phi \colon X \to \mathbb{R}^k$ is:

(1) bi-Lipschitz onto its image if there exists $C \geq 1$ such that $C^{-1}\mathsf{d}(x, y) \leq |\phi(x) - \phi(y)| \leq C\mathsf{d}(x, y)$ for any $x, y \in X$,

(2) $(1+\varepsilon)$-bi-Lipschitz onto its image if $(1+\varepsilon)^{-1}\mathsf{d}(x, y) \leq |\phi(x) - \phi(y)| \leq (1+\varepsilon)\mathsf{d}(x, y)$ for any $x, y \in X$.

Moreover, we call $(1+\varepsilon)$-bi-Lipschitz chart from $X$ to $\mathbb{R}^k$ any couple $(V, \phi)$ where $V$ is a Borel set of $X$ and $\phi \colon V \to \mathbb{R}^k$ is a $(1+\varepsilon)$-bi-Lipschitz map onto its image.

We now provide a definition of rectifiability for metric measure spaces which is a natural variant of the one introduced in [CC00b, Definition 5.3] and which has notably been used in the setting of $\mathrm{RCD}(K, N)$ spaces [DPMR17, KM18, GP21].

**Definition 4.2.** We say that a metric measure space $(X, \mathsf{d}, \mu)$ is rectifiable if there exists a countable collection $\{(k_i, V_i, \phi_i)\}_i$ where $\{V_i\}$ are Borel subsets covering $X$ up to a $\mu$-negligible set, $\{k_i\}$ are positive integers, and $\phi_i \colon V_i \to \mathbb{R}^{k_i}$ is a bi-Lipschitz map such that $(\phi_i)_\#(\mu \llcorner V_i) \ll \mathcal{H}^{k_i}$ for any $i$.

According to this definition, our goal in this section is to prove that Kato limit spaces are rectifiable. Actually, we prove a more precise result which involves the so-called $k$-regular sets.

**Definition 4.3.** For any $k \in \{1, \dots, n\}$, we define the $k$-regular set of a space $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n, f, c)}$ as

$$\mathcal{R}_k := \{x \in X \colon \operatorname{Tan}_{\mathfrak{m}}(X, x) = \{(\mathbb{R}^k, \mathsf{d}_e, \mathcal{H}^k, 0\}\}.$$

Our main result in this section is the following.

**Theorem 4.4.** *Let $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathfrak{m}}(n, f, c)}$. Then the following hold.*

(A) *Up to a negligible set, the space $X$ coincides with the union of its $k$-regular sets:*

$$\mu\left(X \setminus \bigcup_{k=1}^{n} \mathcal{R}_k\right) = 0. \tag{21}$$

(B) *For any $k \in \{1, \dots, n\}$ and $\varepsilon \in (0, 1)$, there exists a countable family of $(1+\varepsilon)$-bi-Lipschitz charts $\{(V_i^\varepsilon, \phi_i^\varepsilon)\}$ from $X$ to $\mathbb{R}^k$ such that*

$$\mu\left(\mathcal{R}_k \setminus \bigcup_i V_i^\varepsilon\right) = 0$$

*and $(\phi_i^\varepsilon)_\#(\mu \llcorner V_i^\varepsilon) \ll \mathcal{H}^k$ for any $i$.*

We call (21) the essential decomposition of $X$. Rectifiability of Kato limit spaces as stated in Theorem 1.1 is then an obvious corollary of Theorem 4.4.

The rest of this section is devoted to proving Theorem 4.4. Our proof is inspired by [GP21, BPS21] but contains some simplifications over the arguments presented there. To keep the notations short, we write $Y \in \operatorname{Tan}_{\mathfrak{m}}(X, x)$ instead of $(Y, \mathsf{d}_Y, \mu_Y, x) \in \operatorname{Tan}_{\mathfrak{m}}(X, x)$.

**4.1. Essential decomposition.** In this subsection, we prove (A) in Theorem 4.4.

*Proof of* (A) *in Theorem 4.4.* First observe that the doubling condition implies the iterated tangent property, meaning that there exists a Borel set $E$ such that $\mu(X \setminus E) = 0$ and for any $x \in E$, any $Y \in \operatorname{Tan}_{\mathfrak{m}}(X, x)$ and any $y \in Y$, it holds

$$\operatorname{Tan}_{\mathfrak{m}}(Y, y) \subset \operatorname{Tan}_{\mathfrak{m}}(X, x). \tag{22}$$

This property goes back to the pioneering work of Preiss [Pre87], who showed it for iterated tangents of measures in the Euclidean space, and was later adapted to metric doubling spaces by Le Donne [LD11] and by Gigli–Mondino–Rajala in our setting [GMR15].

Take $x \in E$ and assume that for some $l \in \{0, \dots, n\}$ there exists a pointed $\mathrm{RCD}(0, n-l)$ space $Z$ such that $\mathbb{R}^l \times Z \in \operatorname{Tan}_{\mathfrak{m}}(X, x)$. If $Z$ is not reduced to a singleton, Gigli's splitting theorem [Gig] ensures that there exists $z \in Z$ such that any $Z_z \in \operatorname{Tan}_{\mathfrak{m}}(Z, z)$ splits off an $\mathbb{R}$ factor, so that (22) implies that there exists a pointed $\mathrm{RCD}(0, n-l-1)$ space $Z'$ such that $\mathbb{R}^{l+1} \times Z' \in \operatorname{Tan}_{\mathfrak{m}}(X, x)$. Then

$$\mathbb{R}^{d(x)} \in \operatorname{Tan}_{\mathfrak{m}}(X, x)$$

where

$$d(x) := \max\{1 \le l \le n \colon \text{there exists a pointed } \mathrm{RCD}(0, n) \text{ space } Z \text{ such that } \mathbb{R}^l \times Z \in \operatorname{Tan}_{\mathfrak{m}}(X, x)\}.$$

Setting

$$i(x) := \min\{1 \le l \le n\colon \text{there exists a pointed RCD}(0,n) \text{ space } Z \\ \text{which splits off no } \mathbb{R} \text{ such that } \mathbb{R}^l \times Z \in \mathrm{Tan}_{\mathsf{m}}(X,x)\},$$

we obtain (A) in Theorem 4.4 as a consequence of

$$i(x) = d(x) \quad \text{for } \mu\text{-a.e. } x \in E. \tag{23}$$

Let us prove (23) by contradiction, assuming

$$\mu(\{x \in E\colon i(x) < d(x)\}) > 0.$$

Set

$$\mathfrak{J}_k := \{x \in E\colon d(x) = k \text{ and } i(x) < k\}$$

for any $1 \le k \le n$, and note that these sets are measurable as can be proved following the arguments of [MN19, Lemma 6.1]. Since

$$\{x \in E\colon i(x) < d(x)\} = \bigcup_{1\le k\le n} \mathfrak{J}_k$$

there exists $k \in \{1,\dots,n\}$ such that

$$\mu(\mathfrak{J}_k) > 0.$$

Then $\mathfrak{J}_k$ admits a point with density 1, that is to say a point $x \in \mathfrak{J}_k$ such that

$$\lim_{r\downarrow 0} \frac{\mu(B_r(x) \cap \mathfrak{J}_k)}{\mu(B_r(x))} = 1. \tag{24}$$

Since $\mathbb{R}^k \in \mathrm{Tan}_{\mathsf{m}}(X,x)$, there exist two infinitesimal sequences $\{\varepsilon_i\}$ and $\{r_i\}$ such that for any $i$ there exists a $(k,\varepsilon_i)$-splitting $u_i$ of $B_{2r_i}(x)$. By propagation of splittings given in Proposition 3.12, for any $i$ there exists a Borel set $\Omega_i \subset B_{r_i}(x)$ such that

$$\frac{\mu(B_{r_i}(x)\backslash\Omega_i)}{\mu(B_{r_i}(x))} \le C\sqrt{\varepsilon_i} \tag{25}$$

and for any $y \in \Omega_i$ any $Y \in \mathrm{Tan}_{\mathsf{m}}(X,y)$ splits off an $\mathbb{R}^k$ factor. As a consequence $i(y) \ge k$. This yields $\Omega_i \cap \mathfrak{J}_k = \emptyset$ and (25) implies

$$\lim_{i\to\infty} \frac{\mu(\Omega_i)}{\mu(B_{r_i}(x))} = 1,$$

hence we get a contradiction with (24). □

**4.2. Rectifiability of the regular sets: our key result.** In this subsection, with a view to proving (B) in Theorem 4.4, we establish the next key technical proposition, where we make use of the almost $k$-regular sets $(\mathcal{R}_k)_{\delta,r} \subset X$, defined as

$$(\mathcal{R}_k)_{\delta,r} := \left\{x \in X\colon\ \mathsf{d}_{\mathrm{mGH}}(B_s(x), \mathbb{B}^k_s) \le \delta s \text{ for any } s \in (0,r]\right\}$$

for any $\delta, r > 0$. Note that each $(\mathcal{R}_k)_{\delta,r}$ is a closed set. We also define

$$(\mathcal{R}_k)_\delta := \bigcup_{r>0} (\mathcal{R}_k)_{\delta,r} \subset \left\{x \in X\colon\ \mathsf{d}_{\mathrm{mGH}}(B^Y_1(x), \mathbb{B}^k_1) \le \delta \text{ for any } Y \in \mathrm{Tan}_{\mathsf{m}}(X,x)\right\}$$

for any $\delta > 0$, and we point out that for any $0 < \delta' < \delta$,

$$(\mathcal{R}_k)_\delta \supset \left\{x \in X\colon\ \mathsf{d}_{\mathrm{mGH}}(B^Y_1(x), \mathbb{B}^k_1) \le \delta' \text{ for any } Y \in \mathrm{Tan}_{\mathsf{m}}(X,x)\right\}.$$

Moreover, we have

$$\mathcal{R}_k = \bigcap_\delta (\mathcal{R}_k)_\delta.$$

**Proposition 4.5.** *Let $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_{\mathsf{m}}(n, f, c)}$, $k \in \{1, \dots, n\}$ and $\varepsilon \in (0, 1/2)$ be given. Then there exists $\delta > 0$ such that for any $x \in (\mathcal{R}_k)_{\delta,16r}$ with $r \leq \sqrt{T}/16$ and $f(256r^2) \leq \delta$ and any $s \in (0, r]$ there exist a function $u\colon B_{2s}(x) \to \mathbb{R}^k$ and a Borel set $V \subset B_s(x)$ such that:*

i) *$u$ is a $(k, \varepsilon)$-splitting of $B_{2s}(x)$;*
ii) *$\mu(B_s(x)\backslash V) \leq \varepsilon\mu(B_s(x))$;*
iii) *$u$ is an $(\varepsilon\,\sigma)$-GH isometry between $B_\sigma(y)$ and $u(y) + \mathbb{B}^k_\sigma$ for any $y \in V \cap (\mathcal{R}_k)_{\delta,16r}$ and any $\sigma \leq s/2$;*
iv) *$u$ is $(1 + \varepsilon)$-bi-Lipschitz on $V \cap (\mathcal{R}_k)_{\delta,16r}$;*
v) *$u_\# \left(\mathbf{1}_{V\cap(\mathcal{R}_k)_{\delta,16r}}\, \mathrm{d}\mu\right) \ll \mathcal{H}^k$.*

In the proof of the last point of this proposition, we use a fundamental result of De Philippis and Rindler [DPR16, Corollary 1.12] which requires the terminology of currents. For the interested reader, we refer to [Fed14] or [Sim14].

Roughly speaking a current in $\mathbb{R}^k$ is a differential form whose coefficients are distributions. To be more precise, let $d$ be a positive integer. A $d$-dimensional current $T$ on $\mathbb{R}^k$ is a continuous linear map

$$T\colon \mathcal{C}^\infty_0 \left(\mathbb{R}^k, \Lambda^d(\mathbb{R}^k)^*\right) \to \mathbb{R}.$$

The differential of a $d$-dimensional current $T$ is the $(d-1)$-dimensional current $dT$ defined by

$$dT(\omega) := T(d\omega)$$

for any $\omega \in \mathcal{C}^\infty_0 \left(\mathbb{R}^k, \Lambda^{d-1}(\mathbb{R}^k)^*\right)$. Here we consider only currents with finite mass, that is to say differential forms whose coefficents are finite Radon measures. Any one dimensional current with finite mass admits a canonical decomposition

$$T(\cdot) = \int_{\mathbb{R}^k} \langle \cdot, \vec{T}\rangle\, \mathrm{d}\|T\| \tag{26}$$

where $\|T\|$ is a Radon measure and $\vec{T}$ is a $\|T\|$-integrable unitary vector field. In this regard, we shall make use of the following easy lemma, whose proof is omitted for brevity.

**Lemma 4.6.** *Let $\nu$ be a Radon measure on $\mathbb{R}^d$ and $\vec{V}$ a square $\nu$-integrable vector field such that $|\vec{V}(x)| > 0$ for $\nu$-a.e. $x \in \mathbb{R}^k$. Let $T$ be the one-dimensional current on $\mathbb{R}^k$ defined by*

$$T(\omega) = \int_{\mathbb{R}^k} \langle \omega, \vec{V}\rangle\, \mathrm{d}\nu$$

*for any $\omega \in \mathcal{C}^\infty_0 \left(\mathbb{R}^k, \Lambda^1(\mathbb{R}^k)^*\right)$. Then $\|T\|$ is absolutely continuous with respect to $\nu$ with density $|\vec{V}|$ and $\vec{T}(x) = \vec{V}(x)/|\vec{V}(x)|$ for $\nu$-a.e. $x \in \mathbb{R}^k$.*

A current $T$ with finite mass such that $dT$ has finite mass too is called a normal current. We recall the result due to De Philippis and Rindler that we shall use [DPR16, Corollary 1.12].

**Theorem 4.7.** *Let $\nu$ be a Radon measure on $\mathbb{R}^k$, and let $\{T_i\}_{1\leq i\leq k}$ be normal one-dimensional currents on $\mathbb{R}^k$ such that $\nu \ll \|T_i\|$ for any $i$, and the vectors $\{\vec{T}_i(x)\}_{1\leq i\leq k}$ are independent for $\nu$-a.e. $x \in \mathbb{R}^k$. Then $\nu \ll \mathcal{H}^k$.*

We are now in a position to prove Proposition 4.5.

*Proof.* We first prove the first three assertions which are direct consequences of the propagation property of splittings we established in Section 3. Let us set

$$\tau(n) := \min\left\{1, \frac{1}{4}C^{-1}(n,1/2), (A(n)\sqrt{C'(n)})^{-1}\right\}$$

where $A(n)$ is given by the doubling condition (5), $C(n,1/2)$ is given by Proposition 3.10, and $C'(n)$ is given by Proposition 3.12. According to Proposition 3.10, there is some $\delta_1$ such that when $y \in (\mathcal{R}_k)_{\delta_1,16r}$, $\sigma \in (0,4r]$ and $v\colon B_{4\sigma}(y) \to \mathbb{R}^k$ is a $(k,[\tau(n)\varepsilon]^2)$-splitting of $B_{4\sigma}(x)$ then $v$ is an $(\varepsilon\,\sigma)$-GH isometry between $B_{2\sigma}(y)$ and $v(y) + \mathbb{B}^k_{2\sigma}$.

According to Proposition 3.9, there is a $\delta \le \delta_1$ such that if $x \in (\mathcal{R}_k)_{\delta,16r}$ and $s \le r$ then there is $u\colon B_{8s}(x) \to \mathbb{R}^k$ a reinforced $(k,[\tau(n)\varepsilon]^4)$-splitting of $B_{8s}(x)$.

Now let $x \in (\mathcal{R}_k)_{\delta,16r}$ and let $s \in (0,r]$ and $u\colon B_{8s}(x) \to \mathbb{R}^k$ be a reinforced $(k,[\tau(n)\varepsilon]^4)$-splitting of $B_{8s}(x)$. With Proposition 3.12, we find $\Omega \subset B_{4s}(x)$ such that

$$\mu(B_{4s}(x)\backslash\Omega) \le C'(n)\tau^2(n)\varepsilon^2\,\mu(B_{4s}(x))$$

such that for any $y \in \Omega$ and any $\sigma \le s$ then $u$ is a $(k,[\tau(n)\varepsilon]^2)$-splitting of $B_{4\sigma}(y)$. If furthermore $y \in (\mathcal{R}_k)_{\delta,16r}$ then $u$ is an $(\varepsilon\,\sigma)$-GH isometry between $B_{2\sigma}(y)$ and $u(y) + \mathbb{B}^k_{2\sigma}$.

We set $V := \Omega \cap B_s(x)$. Then

$$\begin{aligned}\mu(B_s(x)\backslash V) &\le \mu(B_{4s}(x)\backslash\Omega) \le C'(n)\tau^2(n)\varepsilon^2\,\mu(B_{4s}(x))\\ &\le A^2(n)C'(n)\tau^2(n)\varepsilon^2\,\mu(B_s(x)) \le \varepsilon\mu(B_s(x)).\end{aligned}$$

The fourth assertion is a consequence of the third one. Indeed, if $y,z \in V \cap (\mathcal{R}_k)_{\delta,16r}$, define $2\sigma := \mathsf{d}(y,z) \le 2s$. Then, since $u$ is an $(\varepsilon\,\sigma)$-GH isometry between $B_{2\sigma}(y)$ and $u(y) + \mathbb{B}^k_{2\sigma}$, we get

$$\left||u(y)-u(z)| - \mathsf{d}(y,z)\right| \le \varepsilon\sigma = \varepsilon\frac{\mathsf{d}(y,z)}{2}$$

from which follows the desired result.

In order to prove the last point we only need to show that if $K$ is a compact subset of $V \cap (\mathcal{R}_k)_{\delta,16r} \subset B_s(x)$ with $\mu(K) > 0$ then

$$u_\#\left(\mathbf{1}_K\,\mathrm{d}\mu\right) \ll \mathcal{H}^k.$$

*Step 1.* To prepare the application of Theorem 4.7, let us introduce a series of Radon measures and discuss some properties of these measures. Set $B := B_{2s}(x)$. Choose $\{\chi_\ell\} \subset \mathrm{Lip}_c(B,[0,1])$ such that $\chi_\ell \downarrow \mathbf{1}_K$ : for instance for any $\ell$ we may choose $\chi_\ell(\cdot) := \left(1 - \ell\mathsf{d}(K,\cdot)\right)_+$ which has support $K_\ell = \left\{\mathsf{d}(K,\cdot) \le \frac{1}{\ell}\right\}$. For convenience we also set $\chi_\infty := \mathbf{1}_K$. We define the following Radon measures on $\mathbb{R}^k$:

$$\nu^\ell_{i,j} := u_\#\left(\chi_\ell\Gamma(u_i,u_j)\right) \quad \text{and} \quad \nu^\ell := u_\#\left(\chi_\ell\mu\right)$$

for $i,j \in \{1,\dots,k\}$ and $\ell \in \mathbb{N}\cup\{\infty\}$ and we also set

$$\nu := u_\#\left(\mathbf{1}_B\mu\right).$$

Notice that the measures $\nu^\ell_{i,j}$ are signed Radon measures. The coefficients

$$\frac{\mathrm{d}\Gamma(u_i,u_j)}{\mathrm{d}\mu} = \langle du_i, du_j\rangle, \quad i,j \in \{1,\dots,k\},$$

of the Gram matrix map of $u = (u_1, \dots, u_k)$ are bounded Borel functions, hence there exist bounded Borel functions such that for any $i, j \in \{1, \dots, k\}$ and $\ell \in \mathbb{N} \cup \{\infty\}$,

$$\mathrm{d}\nu^\ell_{i,j} = \rho^\ell_{i,j}\,\mathrm{d}\nu^\ell.$$

There are also bounded Borel functions $J^\ell$ such that

$$\mathrm{d}\nu^\ell = J^\ell\,\mathrm{d}\nu$$

and $J^{\ell+1} \le J^\ell \le 1$ for any $\ell \in \mathbb{N} \cup \{\infty\}$. Moreover, recalling that $J^\ell\,\mathrm{d}\nu = \mathrm{d}\nu^\ell = u_\#\,(\chi_\ell\,\mathrm{d}\mu)$ and $J^\infty\,\mathrm{d}\nu = \mathrm{d}\nu^\infty = u_\#\,(\chi_\infty\,\mathrm{d}\mu)$, one gets

$$\|J^\ell - J^\infty\|_{L^1(\mathrm{d}\nu)} = \int_{\mathbb{R}^k} (J^\ell - J^\infty)\,\mathrm{d}\nu = \int_B (\chi_\ell - \chi_\infty)\,\mathrm{d}\mu \le \mu(K_\ell \setminus K) \to 0,$$

so that

$$\lim_{\ell\to+\infty} \|J^\ell - J^\infty\|_{L^1(\mathrm{d}\nu)} = 0. \tag{27}$$

*Step 2.* For any $\ell \in \mathbb{N} \cup \{\infty\}$, let $\lambda^\ell$ be the lowest eigenvalue of the symmetric matrix $\left(\rho^\ell_{i,j}\right)$. Our goal is now to establish

$$\lim_{\ell\to+\infty} \|\lambda^\ell - \lambda^\infty\|_{L^1(\mathrm{d}\nu^\infty)} = 0 \tag{28}$$

and for $\nu^\infty$-a.e. $p \in \mathbb{R}^k$,

$$\lambda^\infty(p) \ge 1 - \epsilon. \tag{29}$$

For any $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$ such that ${}^t\xi\xi = 1$ and any $\ell \in \mathbb{N} \cup \{\infty\}$, we introduce

$$\rho^\ell_\xi := \sum_{i,j} \xi_i \xi_j \rho^\ell_{i,j}.$$

Setting

$$u_\xi := \langle \xi, u\rangle$$

we have

$$\rho^\ell_\xi = \frac{\mathrm{d}u_\#\,(\chi_\ell \Gamma(u_\xi, u_\xi))}{\mathrm{d}\nu^\ell}.$$

In particular, $\{\rho^\ell_\xi(p)\}_\ell$ is a non negative non increasing sequence for $\nu^\infty$-a.e. $p \in \mathbb{R}^k$. Arguing as we did to get (27) yields

$$\lim_{\ell\to+\infty} \|J^\ell \rho^\ell_\xi - J^\infty \rho^\infty_\xi\|_{L^1(\mathrm{d}\nu)} = 0.$$

Since

$$\begin{aligned}\|\rho^\ell_\xi - \rho^\infty_\xi\|_{L^1(\mathrm{d}\nu^\infty)} &= \int_{\mathbb{R}^k} (\rho^\ell_\xi - \rho^\infty_\xi) J^\infty\,\mathrm{d}\nu \\ &= \int_{\mathbb{R}^k} (J^\ell \rho^\ell_\xi - J^\infty \rho^\infty_\xi)\,\mathrm{d}\nu - \int_{\mathbb{R}^k} (J^\ell - J^\infty)\rho^\ell_\xi\,\mathrm{d}\nu\end{aligned}$$

we also get

$$\lim_{\ell\to+\infty} \|\rho^\ell_\xi - \rho^\infty_\xi\|_{L^1(\mathrm{d}\nu^\infty)} = 0.$$

Using that $\xi \mapsto \rho^\ell_\xi$ is quadratic, by polarization we deduce that for any $i, j$,

$$\lim_{\ell\to+\infty} \|\rho^\ell_{i,j} - \rho^\infty_{i,j}\|_{L^1(\mathrm{d}\nu^\infty)} = 0.$$

Up to extraction of a subsequence we can assume that there exists a set $C$ of full $\nu^\infty$ measure such that for any $i,j\in\{1,\dots,k\}$ and $p\in C$,

$$\lim_{\ell\to+\infty}\rho^\ell_{i,j}(p)=\rho^\infty_{i,j}(p).$$

Then for $\nu^\infty$-a.e. $p\in\mathbb{R}^k$,

$$\lim_{\ell\to+\infty}\lambda^\ell(p)=\lambda^\infty(p) \tag{30}$$

and thus we get (28).

For $\nu^\infty$-a.e. $p\in u(K)$, we have

$$\rho^\infty_\xi(p)=\lim_{\sigma\to 0}\frac{\int_{K\cap u^{-1}\left(\mathbb{B}^k_\sigma(p)\right)}\mathrm{d}\Gamma(u_\xi,u_\xi)}{\mu\left(K\cap u^{-1}\left(\mathbb{B}^k_\sigma(p)\right)\right)}.$$

Since $\mu$-a.e. on $B$ we have

$$\frac{\mathrm{d}\Gamma(u_\xi,u_\xi)}{\mathrm{d}\mu}={}^t\xi G_u\xi,$$

from (19) in Proposition 3.12 we get $\mu$-a.e. on $K$:

$$1-\varepsilon\le\frac{\mathrm{d}\Gamma(u_\xi,u_\xi)}{\mathrm{d}\mu}\le 1+\varepsilon.$$

Thus for $\nu^\infty$-a.e. $p\in u(K)$,

$$1-\epsilon\le\rho^\infty_\xi(p)\le 1+\epsilon, \tag{31}$$

from which follows (29).

*Step 3.* Recall that our final goal is to prove that $\nu^\infty\ll\mathcal{H}^k$. To this aim, we will apply Theorem 4.7 for any finite $\ell$ to the currents

$$T^\ell_i=\sum_{j=1}^k\nu^\ell_{i,j}dx_j=\sum_{j=1}^k\rho^\ell_{i,j}\nu^\ell dx_j.$$

These are indeed normal currents as for $\psi\in\mathcal{C}^\infty_0(\mathbb{R}^k)$,

$$\begin{aligned}
dT^\ell_i(\psi)&=\sum_{j=1}^k\int_B\chi_\ell\,\frac{\partial\psi}{\partial x_j}\circ u\ \mathrm{d}\Gamma(u_i,u_j)\\
&=\int_B\chi_\ell\ \mathrm{d}\Gamma(\psi\circ u,u_i)\quad\text{using the chain rule}\\
&=-\int_B\psi\circ u\ \mathrm{d}\Gamma(\chi_\ell,u_i)\quad\text{by the fact that } u_i \text{ is harmonic,}
\end{aligned}$$

hence

$$T^\ell_i=-u_\#\left(\Gamma(\chi_\ell,u_i)\right)$$

is a finite Radon measure. Moreover, by Lemma 4.6, the decomposition (26) of $T^\ell_i$ is given by

$$\vec{T}^\ell_i=\left(\rho^\ell_i\right)^{-1}\left(\rho^\ell_{i,1},\dots,\rho^\ell_{i,k}\right)$$

with

$$\rho^\ell_i=\left(\sum_{j=1}^k\left(\rho^\ell_{i,j}\right)^2\right)^{\frac12}\quad\text{and}\quad\|T^\ell_i\|=\rho^\ell_i\nu^\ell.$$

Notice that $\rho_i^\ell \geq \rho_{i,i}^\ell$ hence

$$\rho_{i,i}^\infty \nu^\infty = \nu_{i,i}^\infty \ll \nu_{i,i}^\ell = \rho_{i,i}^\ell \nu^\ell \ll \|T_i^\ell\|,$$

and inequality (31) implies that $\nu^\infty$-a.e. $\rho_{i,i}^\infty \geq 1-\sqrt{\varepsilon}$ so that

$$\nu^\infty \ll \|T_i^\ell\|.$$

We remark that for any $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$ unitary it holds

$$\left\langle \left( \sum_{i=1}^k \xi_i \rho_i^\ell \vec{T}_i^\ell \right), \xi \right\rangle = \rho_\xi^\ell.$$

We set

$$\mathcal{B}_\ell := \{p \in \mathbb{R}^k \colon \lambda^\ell(p) \leq (1-\varepsilon)/2\}.$$

Since $\rho_i^\ell$ are bounded functions, we deduce that if $p \in \mathbb{R}^k \setminus \mathcal{B}_\ell$ then $\vec{T}_1^\ell(p), \dots, \vec{T}_k^\ell(p)$ is a basis of $\mathbb{R}^k$. Applying Theorem 4.7 we get

$$\mathbf{1}_{\mathbb{R}^k \setminus \mathcal{B}_\ell} \nu^\infty \ll \mathcal{H}^k.$$

But the convergence (28) and the lower bound (29) yield

$$\lim_{\ell \to \infty} \nu^\infty(\mathcal{B}_\ell) = 0,$$

hence we get $\nu^\infty \ll \mathcal{H}^k$. □

**4.3. Rectifiability of the regular sets: end of the proof.** To get (B) in Theorem 4.4 from Proposition 4.5, we use the following definition, introduced in [BPS21].

**Definition 4.8.** Let $(X, \mathsf{d}, \mu)$ be a metric measure space, $k$ a positive integer and $\varepsilon \in (0,1)$. We call $(\mu, k, \varepsilon)$-rectifiable any Borel set $\Omega \subset X$ for which there exists a countable family of $(1+\varepsilon)$-bi-Lipschitz charts $\{(V_i^\varepsilon, \phi_i^\varepsilon)\}$ from $X$ to $\mathbb{R}^k$ such that $\mu(\Omega \setminus \bigcup_i V_i^\varepsilon) = 0$.

According to the previous definition, we are left with establishing the following.

**Proposition 4.9.** *Let $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_\mathfrak{m}(n, f, c)}$, $k \in \{1, \dots, n\}$ and $\varepsilon \in (0,1)$. Then $\mathcal{R}_k$ is $(\mu, k, \varepsilon)$-rectifiable.*

To this aim, we prove a lemma which is a consequence of our key Proposition 4.5.

**Lemma 4.10.** *Let $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_\mathfrak{m}(n, f, c)}$ and $k \in \{1, \dots, n\}$. Then for any $p \in X$, $R > 0$ and $\varepsilon \in (0,1)$, there exists a $(\mu, k, \varepsilon)$-rectifiable set $\Omega_\varepsilon \subset \mathcal{R}_k \cap B_R(p)$ such that $\mu([\mathcal{R}_k \cap B_R(p)] \setminus \Omega_\varepsilon) \leq \varepsilon$.*

*Proof.* Let $(X, \mathsf{d}, \mu, o) \in \overline{\mathcal{K}_\mathfrak{m}(n, f, c)}$, $k \in \{1, \dots, n\}$, $p \in X$, $R > 0$ and $\varepsilon > 0$ be given. Set $\varepsilon' := \varepsilon / \mu(\mathcal{R}_k \cap B_R(p))$. Let $\delta > 0$ be given by Proposition 4.5 applied to $\varepsilon'$. For any $x \in \mathcal{R}_k$ there exists $r(x) > 0$ such that $x \in (\mathcal{R}_k)_{\delta, 16 r(x)}$. Apply the Vitali covering lemma for doubling metric measure spaces [Hei01, Theorem 1.6] to the set $\mathcal{R}_k \cap B_R(p)$ and the collection of balls $A := \{B_r(x)\}_{x \in \mathcal{R}_k \cap B_R(p), 0 < r \leq r(x)}$. Then there exist countably many pairwise disjoint balls $\{B_{r_{x_i}}(x_i)\} \subset A$ such that $\mu([\mathcal{R}_k \cap B_R(p)] \setminus \bigcup_i B_{r_{x_i}}(x_i)) = 0$. By Proposition 4.5 for any $i$ there exists a Borel set $V_i \subset B_{r_{x_i}}(x_i)$ which is the domain of a bi-Lipschitz chart and such that $\mu(B_{r_{x_i}}(x_i) \setminus V_i) \leq$

$\varepsilon'\mu(B_{r_{x_i}}(x_i))$. Set $\Omega_\varepsilon = \bigcup_i V_i \cap \mathcal{R}_k$. Then $\Omega_\varepsilon$ is the union of domains of bi-Lipschitz charts, so it is obviously $(\mu, k, \varepsilon)$-rectifiable. Moreover,

$$\mu([\mathcal{R}_k \cap B_R(p)]\backslash\Omega_\varepsilon) \le \mu(\cup_i B_{r_{x_i}}(x_i)\backslash V_i) = \sum_i \mu(B_{r_{x_i}}(x_i)\backslash V_i)$$
$$\le \varepsilon' \sum_i \mu(B_{r_{x_i}}(x_i)) \le \varepsilon'\mu(\mathcal{R}_k \cap B_R(p)) = \varepsilon. \qquad \square$$

We are now in a position to prove Proposition 4.9.

*Proof of Proposition 4.9.* From the previous lemma, for any $i \in \mathbb{N}\backslash\{0\}$ there exists a Borel set $\Omega_{\varepsilon,i} \subset \mathcal{R}_k \cap B_i(p)$ which is $(\mu, k, 2^{-i}\varepsilon)$-rectifiable and such that $\mu([\mathcal{R}_k \cap B_R(p_i)]\backslash\Omega_{\varepsilon,i}) \le 2^{-i}\varepsilon$. We set $\Omega_\varepsilon := \bigcup_i \Omega_{\varepsilon,i}$. Then

$$\mu(\mathcal{R}_k\backslash\Omega_\varepsilon) \le \lim_{i\to+\infty} \mu([\mathcal{R}_k \cap B_i(p)]\backslash\Omega_\varepsilon) \le \lim_{i\to+\infty} \mu([\mathcal{R}_k \cap B_i(p)]\backslash\Omega_{\varepsilon,i}) = 0.$$

Since for any $i$ there exist countably many $(1+\varepsilon)$-bi-Lipschitz charts $\{(V^\varepsilon_{i,j}, \phi^\varepsilon_{i,j})\}_j$ such that $\mu(\Omega_{\varepsilon,i}\backslash \bigcup_j V^\varepsilon_{i,j}) = 0$, we get that $\Omega_\varepsilon$ (and then $\mathcal{R}_k$) is $(\mu, k, \varepsilon)$-rectifiable. $\square$

Noting that the absolute continuity statement is ensured by v) of Proposition 4.5, we obtain (B) in Theorem 4.4 from the previous proposition.

## 5. Regularity of non-collapsed strong Kato limits

This section is devoted to the structure and regularity of non-collapsed strong Kato limits. We start by recalling some properties of these spaces, then show an almost rigidity result that leads to the Reifenberg regularity stated in Theorem 1.3. In the second part of this section, we prove a Transformation Theorem which, together with Theorem 1.3 and the results of Section 3, implies Theorem 1.2.

**5.1. Non-collapsed strong Kato limits and almost monotone quantity.** Recall that a manifold $(M^n, g) \in \mathcal{K}(n, f)$ satisfies a strong Kato bound if the function $f$ is such that

$$\text{(SK)} \qquad \Lambda := \int_0^T \frac{\sqrt{f(s)}}{s}\,\mathrm{d}s < \infty.$$

Under assumption (SK), the volume bound (3) given by Proposition 2.4 upgrades into the following, as proved in [CMT24].

**Proposition 5.1.** *Let* $(M^n, g) \in \mathcal{K}(n, f)$ *with* $f$ *satisfying* (SK). *Then there exists* $C = C(n, \Lambda) > 0$ *such that for any* $0 < s \le r \le \sqrt{T}$ *we have*

$$\frac{\nu_g(B_r(x))}{\nu_g(B_s(x))} \le C\left(\frac{r}{s}\right)^n.$$

For $v > 0$, $(M^n, g, o)$ belongs to $\mathcal{K}(n, f, v)$ if $f$ satisfies (SK) and moreover $\nu_g(B_{\sqrt{T}}(o)) \ge vT^{\frac{n}{2}}$. Non-collapsed strong Kato limits are elements of the closure $\overline{\mathcal{K}(n, f, v)}$ with respect to Gromov–Hausdorff topology. As proved in [CMT24, Theorem 7.1], volume continuity holds for non-collapsed strong Kato limits.

**Theorem 5.2.** *Let* $\{(M_\alpha, g_\alpha, o_\alpha)\} \subset \mathcal{K}(n, f, v)$ *be a sequence converging in the pointed Gromov–Hausdorff topology to* $(X, \mathsf{d}, o) \in \overline{\mathcal{K}(n, f, v)}$. *Then* $(M_\alpha, g_\alpha, \nu_{g_\alpha}, o_\alpha)$ *converges to* $(X, \mathsf{d}, \mathcal{H}^n, o)$ *in the pointed measured Gromov–Hausdorff topology.*

As a consequence, in this setting the results of Section 3.2 can be revisited. More precisely, if in Theorem 3.8, Propositions 3.9 and 3.10, we replace Kato limits by non-collapsed strong Kato limits, we can assume closedness of balls in the Gromov–Hausdorff topology instead of the measured Gromov–Hausdorff topology. Note that in this case the quantities $\nu$ and $\delta$ also depend on the volume bound $v > 0$.

Now let $(X, \mathsf{d}, o) \in \overline{\mathcal{K}(n, f, v)}$ and let $H\colon \mathbb{R}_+ \times X \times X \to \mathbb{R}_+$ be its heat kernel. For any $t > 0$ and $x \in X$ we consider

$$\uptheta(t, x) = (4\pi t)^{\frac{n}{2}} H(t, x, x).$$

As we recalled in the introduction, in [CMT24] we showed that the map $t \mapsto \uptheta(t, x)$ is almost non-decreasing for all $x \in X$. More precisely, define for any $t \in (0, T]$

$$\upphi(t) := \int_0^t \frac{\sqrt{f(s)}}{s} \, \mathrm{d}s < \infty.$$

Thanks to the Li–Yau inequality given by Proposition 2.9, we get the following (see also [CMT24, Corollaries 5.12 and 5.13]).

**Proposition 5.3.** *Let $(X, \mathsf{d}, o) \in \overline{\mathcal{K}(n, f, v)}$ with $f$ satisfying* (SK). *There is a constant $c_n > 0$ depending only on $n$ such that for any $x \in X$ the function*

$$t \in (0, T) \mapsto e^{c_n \upphi(t)} \uptheta(t, x)$$

*is non-decreasing and such that for any $t \in (0, T)$,*

$$e^{c_n \upphi(t)} \uptheta(t, x) \geq 1.$$

*In particular, the limit $\vartheta(x) = \lim_{t \to 0} \uptheta(t, x)$ is well defined and satisfies $\vartheta(x) \geq 1$.*

**Remark 5.4.** In [CMT24] we also showed that for all $x \in X$, $\vartheta(x)$ is the inverse of the volume density: $\vartheta(x)^{-1} = \lim_{r \to 0}(\mathcal{H}^n(B_r(x))/\omega_n r^n)$, where $\omega_n$ is the volume of the Euclidean unit ball.

One consequence of [CMT24] is that the regular set coincides with the set of points where $\vartheta$ is equal to 1, as we show below.

**Proposition 5.5.** *Let $(X, \mathsf{d}, o) \in \overline{\mathcal{K}(n, f, v)}$ with $f$ satisfying* (SK). *Then*

$$\mathcal{R} = \{x \in X \colon \operatorname{Tan}(X, x) = \{(\mathbb{R}^n, \mathsf{d}_e, 0)\}\} = \{x \in X \colon \vartheta(x) = 1\}.$$

*Proof.* The first equality is a direct consequence of [CMT24, Theorem 6.2(iii)] and of volume continuity as recalled in Theorem 5.2. As for the second one, [CMT24, Theorem 7.2] ensures that if $(\mathbb{R}^n, \mathsf{d}_e, 0)$ is a tangent cone at $x \in X$, then $\vartheta(x) = 1$, so that

$$\mathcal{R} \subset \{x \in X \colon \vartheta(x) = 1\}.$$

To prove the converse inclusion, consider $x \in X$ such that $\vartheta(x) = 1$. The proof of [CMT24, Proposition 6.3] ensures that $\vartheta$ is upper semi-continuous. We have then

$$1 \leq \liminf_{y \to x} \vartheta(y) \leq \limsup_{y \to x} \vartheta(y) \leq \vartheta(x) = 1,$$

so that $\vartheta$ is continuous at $x$. The proof of [CMT24, Theorem 6.2(iii)] then implies that all tangent cones at $x$ are Euclidean, thus $x \in \mathcal{R}$. □

For a manifold $(M^n, g)$ satisfying a strong Kato bound, an upper bound on $\uptheta$ at some point $x$ implies a lower bound on the volume of $B_{\sqrt{T}}(x)$.

**Lemma 5.6.** *Assume that* $(M^n, g)$ *is a closed manifold in* $\mathcal{K}(n, f)$ *with* $f$ *satisfying* (SK). *There is a constant* $v(n) > 0$ *such that if at some* $x \in X$ *and* $t \leq T$ *we have*

$$\theta(t, x) \leq 2,$$

*then* $\nu_g\left(B_{\sqrt{t}}(x)\right) \geq v(n)t^{\frac{n}{2}}$.

*Proof.* Thanks to the heat kernel estimates given by Proposition 2.6, we get

$$\frac{t^{\frac{n}{2}}}{C_n \nu_g(B_{\sqrt{t}}(x))} \leq \theta(x, t) \leq 2,$$

which immediately gives the desired lower bound. □

We are also going to use the following lemma.

**Lemma 5.7.** *Let* $(M^n, g) \in \mathcal{K}(n, f)$ *for* $f$ *satisfying* (SK). *For any* $\delta \in (0, 1)$ *there exists* $\nu > 0$ *depending on* $\delta, f$ *such that if for some* $t \in (0, T]$ *we have* $\mathrm{k}_t(M^n, g) < \nu$, *then for all* $x \in M$ *and* $s \in (0, t]$ *we have* $\theta(s, x) \leq \theta(t, x)(1 + \delta)$.

*Proof.* Assume $\mathrm{k}_t(M, g) < \nu$ and let $c_n$ be the constant appearing in Proposition 5.3. Observe that for any $a \in (0, t)$ we can write

$$\int_0^t \frac{\sqrt{\mathrm{k}_\tau(M^n, g)}}{\tau}\, \mathrm{d}\tau \leq \int_0^a \frac{\sqrt{f(\tau)}}{\tau}\, \mathrm{d}\tau + \sqrt{\nu}\log\left(\frac{T}{a}\right).$$

We can choose $a$ depending on $f$ and $\delta$ such that the first addend in the previous inequality is smaller than $\log(1 + \delta)/2c_n$. Then we can choose $\nu$ depending on $a$ and $\delta$, thus on $f$ and $\delta$, such that the second addend is also smaller than $\log(1 + \delta)/2c_n$. By Proposition 5.3, then we know that for all $x \in M$ and $s \in (0, t]$,

$$\theta(s, x) \leq \theta(t, x)\exp\left(c_n \int_s^t \frac{\sqrt{\mathrm{k}_\tau(M^n, g)}}{\tau}\, \mathrm{d}\tau\right) \leq \theta(t, x)(1 + \delta).$$

**Remark 5.8.** The same argument as in the previous proof implies that for a sequence $\{(M_\ell, g_\ell, o_\ell)\} \subset \mathcal{K}(n, f, v)$ converging to $(X, \mathsf{d}, o) \in \overline{\mathcal{K}(n, f, v)}$ such that $\lim_\ell \mathrm{k}_t(M_\ell, g_\ell) = 0$ for some $t \in (0, T]$, we have that for all $x \in X$ the map $s \mapsto \theta(s, x)$ is monotone non-decreasing and satisfies $\theta(s, x) \geq 1$ for all $s \in (0, t]$.

**5.2. Almost rigidity.** This subsection is devoted to proving the following almost rigidity for $\theta$, which will be the key result to obtain our Reifenberg regularity statement, namely Theorem 1.3.

**Theorem 5.9.** *Let* $f\colon (0, T] \to \mathbb{R}_+$ *be a non decreasing function satisfying* (SK). *For any* $\varepsilon > 0$ *and* $A > 0$ *there exists* $\delta > 0$ *depending only on* $f, n, \varepsilon$ *and* $A$ *such that if* $(M^n, g) \in \mathcal{K}(n, f)$, $x \in M$ *and* $t \leq T$ *satisfy*

$$\mathrm{k}_t(M, g) \leq \delta \quad \text{and} \quad \theta(t, x) \leq 1 + \delta,$$

*then*

$$\mathsf{d}_{\mathrm{GH}}\left(B_{A\sqrt{t}}(x), \mathbb{B}^n_{A\sqrt{t}}\right) < \varepsilon A\sqrt{t}.$$

In order to prove Theorem 5.9, we are going to use a contradiction argument, that we sketch here before giving the detailed proof. We will construct a contradicting sequence for which a ball of radius 1 stays uniformly far from the unit Euclidean ball. Thanks to Lemma 5.6 such sequence is non-collapsing. Then up to extracting a sub-sequence, we obtain a limit $(X, \mathsf{d}, x) \in \overline{\mathcal{K}(n, f, v)}$ such that $B_1(x)$ is at a positive distance from the unit Euclidean ball. We then aim to show that the limit space $(X, \mathsf{d})$

is isometric to the Euclidean space. For that, we use the heat kernel rigidity shown in [CT22]. More precisely, for a non-collapsed strong Kato limit $(X, \mathsf{d}, x) \in \overline{\mathcal{K}(n,f,v)}$ we define

$$\mathbb{P}(t,x,y) = \frac{e^{-\frac{\mathsf{d}^2(x,y)}{4t}}}{(4\pi t)^{\frac{n}{2}}}.$$

If for all $x, y \in X$ and $t > 0$ we have $H(t,x,y) = \mathbb{P}(t,x,y)$, then [CT22, Theorem 1.1] ensures that $(X, \mathsf{d})$ is isometric to the Euclidean space. In order to show that $H$ coincides with $\mathbb{P}$, we will rely on the Li–Yau inequality proven in Proposition 2.9 and on the fact that, thanks to Remark 5.8, $\uptheta$ is monotone non-decreasing.

*Proof.* We assume by contradiction that the statement is false. Then there exist $\varepsilon, A > 0$ such that if we consider the sequence $\delta_\ell = \ell^{-1}$, $\ell \in \mathbb{N}, \ell > 0$, we find $t_\ell \le T$, $(M_\ell, g_\ell) \in \mathcal{K}(n,f)$ and $x_\ell \in M_\ell$ such that

$$\mathrm{k}_{t_\ell}(M_\ell, \tilde{g}_\ell) \le \delta_\ell \quad \text{and} \quad \uptheta(t_\ell, x_\ell) \le 1 + \delta_\ell,$$

but

$$\mathsf{d}_{\mathrm{GH}}(B_{A\sqrt{t_\ell}}(x_\ell), \mathbb{B}^n_{A\sqrt{t_\ell}}) \ge \varepsilon \sqrt{A} t_\ell. \tag{32}$$

Observe that if we define $\tilde{f}(s) = f(sT)$ for all $s \in [0,1]$ and $\tilde{g}_\ell = t_\ell^{-1} g_\ell$ for any $\ell$, then the rescaling properties of $\mathrm{k}_t$ and of the heat kernel imply that each $(M_\ell, \tilde{g}_\ell)$ belongs to $\mathcal{K}(n, \tilde{f})$ and

$$\mathrm{k}_1(M_\ell, \tilde{g}_\ell) = \mathrm{k}_{t_\ell}(M_\ell, g_\ell) \le \delta_\ell, \quad \tilde{\uptheta}(1, x_\ell) = \uptheta(t_\ell, x_\ell) \le 1 + \delta_\ell.$$

Then up to rescaling we can assume that $t_\ell = 1$ for all $\ell \in \mathbb{N}$.

By Lemma 5.6, we also know that there exists $v = v(n) > 0$ such that for any $\ell$,

$$\nu_{g_\ell}(B_1(x_\ell)) \ge v,$$

so that each $(M_\ell, g_\ell, x_\ell)$ belongs to $\mathcal{K}(n,f,v)$. Up to extracting a subsequence, $\{(M_\ell, g_\ell, x_\ell)\}$ converges in the pointed Gromov–Hausdorff topology to $(X, \mathsf{d}, x) \in \overline{\mathcal{K}(n,f,v)}$. Moreover, convergence of the heat kernel given in Proposition 2.8 ensures that

$$\uptheta(1,x) = \lim_\ell \uptheta(1, x_\ell) \le 1.$$

Thanks to Remark 5.8, we also know that $t \mapsto \uptheta(t,x)$ is monotone non-decreasing and larger than one. We then get for all $s \in (0,1]$,

$$\uptheta(s,x) = \uptheta(1,x) = 1. \tag{33}$$

Because of (32), we also have

$$\mathsf{d}_{\mathrm{GH}}(B_A(x), \mathbb{B}^n_A) \ge \varepsilon A. \tag{34}$$

Our setting constructed, we aim to prove that the heat kernel of $X$ satisfies

$$H = \mathbb{P} \tag{35}$$

on $\mathbb{R}_+ \times X \times X$. In order do so, we introduce the function

$$\Phi \colon \mathbb{R}_+ \times X \ni (t,y) \mapsto (4\pi t)^{\frac{n}{2}} H^2(t/2, x, y).$$

*Step 1.* We show that $\Phi$ satisfies

$$4\frac{\partial}{\partial t}\left(\int_X \varphi \Phi \, \mathrm{d}\mathcal{H}^n\right) + \int_X \langle d\varphi, d\Phi\rangle \, \mathrm{d}\mathcal{H}^n \ge 0, \tag{36}$$

for any non-negative $\varphi \in \mathcal{C}_c(X) \cap H^{1,2}(X)$. To this aim, we first observe that

$$4\frac{\partial}{\partial t}\left(\int_X \varphi\Phi \,\mathrm{d}\mathcal{H}^n\right) = \int_X \varphi(y)\left(\frac{2n}{t}\Phi(t,y) + 4(4\pi t)^{\frac{n}{2}} H(t/2,x,y)\frac{\partial H}{\partial t}(t/2,x,y)\right)\mathrm{d}\mathcal{H}^n(y).$$

Then we use the definitions of $L$, $H$ and $\Phi$ to get

$$\int_X \langle d\varphi, d\Phi\rangle \,\mathrm{d}\mathcal{H}^n = \int_X \varphi L\Phi \,\mathrm{d}\mathcal{H}^n$$
$$= 2(4\pi t)^{\frac{n}{2}} \int_X \varphi(y)\left(H(t/2,x,y)L_yH(t/2,x,y) - |d_yH(t/2,x,y)|^2\right)\mathrm{d}\mathcal{H}^n(y)$$
$$= -2(4\pi t)^{\frac{n}{2}} \int_X \varphi(y)\left(H(t/2,x,y)\frac{\partial H}{\partial t}(t/2,x,y) + |d_yH(t/2,x,y)|^2\right)\mathrm{d}\mathcal{H}^n(y).$$

Adding these two identities yields

$$4\frac{\partial}{\partial t}\left(\int_X \varphi\Phi\,\mathrm{d}\mathcal{H}^n\right) + \int_X \langle d\varphi, d\Phi\rangle\,\mathrm{d}\mathcal{H}^n = 2(4\pi t)^{\frac{n}{2}} \int_X \varphi Z\,\mathrm{d}\mathcal{H}^n,$$

where $Z$ is defined by

$$Z(t,y) = \frac{n}{t}H^2(t/2,x,y) + H(t/2,x,y)\frac{\partial H}{\partial t}(t/2,x,y) - |d_yH(t/2,x,y)|^2$$

for any $t \in \mathbb{R}_+$ and $y \in X$. Since $(X,\mathsf{d},o)$ is the limit of manifolds $\{(M_\ell, g_\ell)\}$ such that $\mathrm{k}_1(M_\ell, g_\ell) \to 0$ as $\ell$ goes to infinity, the Li–Yau inequality given by Remark 2.10 holds. Then $Z \geq 0$, this concluding the proof of (36).

*Step 2.* We show that for any $t > 0$ and $y \in X$,

$$H(t,x,y) = \mathbb{P}(t,x,y). \tag{37}$$

The Gaussian estimate given in Proposition 2.6 implies that for any $t > 0$,

$$\lim_{\mathsf{d}(x,y)\to\infty} \Phi(t,y) = 0.$$

Moreover, the fact that $\mathcal{H}^n(B_1(x)) \geq v$ and the volume bound given in Proposition 5.1 imply that for any $y \in X\backslash\{x\}$,

$$\lim_{t\to 0}\Phi(t,y) = 0.$$

By the semi-group law and (33), we know that for any $s \in (0,1]$,

$$\int_X \Phi(s,y)\,\mathrm{d}\mathcal{H}^n(y) = \theta(s,x) = 1.$$

As a consequence we get

$$\lim_{t\to 0}\Phi(t,\cdot) = \delta_x(\cdot).$$

Then for any $t > 0$ and $y \in X$, the function

$$F\colon (0,t) \ni s \mapsto \int_X \Phi(s,z)H((t-s)/4,z,y)\,\mathrm{d}\mathcal{H}^n(z)$$

satisfies

$$\lim_{s\downarrow 0} F(s) = H(t/4,x,y), \quad \lim_{s\uparrow t} F(s) = \Phi(t,y),$$

and a direct computation justified by the Gaussian estimates of Proposition 2.6 yields that for any $s \in (0,t)$,

$$F'(s) = \int_X \Big( \partial_s \Phi(s,z) H((t-s)/4, z, y) + \langle d_z \Phi(s,z), d_z H((t-s)/4, z, y) \rangle \Big) \, \mathrm{d}\mathcal{H}^n(z).$$

As (36) implies that $F' \geq 0$, we obtain that

$$\Phi(t,y) \geq H(t/4, x, y).$$

But we also have, for all $t \in (0,1]$,

$$1 = \int_X \Phi(t,y) \, \mathrm{d}\mathcal{H}^n(y) = \int_X H(t/4, x, y) \, \mathrm{d}\mathcal{H}^n(y),$$

then we obtain, for all $t \in (0,1]$ and $y \in X$,

$$\Phi(t,y) = H(t/4, x, y). \tag{38}$$

We now introduce

$$U(t,x,y) = -4t \log((4\pi t)^{\frac{n}{2}} H(t,x,y)).$$

By Varadhan's formula, we know

$$\lim_{\sigma \to 0} U(\sigma, x, y) = -\mathsf{d}^2(x,y).$$

Because of (38), a simple computation shows that for any $s \in (0,1]$ we have

$$U(s/4, x, y) = U(s/2, x, y).$$

As a consequence, for all $s \in (0,1]$,

$$U(s/2, x, y) = \lim_{\sigma \to 0} U(\sigma, x, y) = -\mathsf{d}^2(x,y).$$

This shows that for all $t \in (0, 1/2]$ and $y \in X$

$$H(t,x,y) = \mathbb{P}(t,x,y).$$

Both expressions in this equality are analytic in $t$, hence we get (37) for any $t > 0$.

*Step 3.* We obtain (35) and conclude. Equality (37) implies in particular that $\theta(t,x) = 1$ for all $t > 0$ and not only for $t \in (0,1]$. By using the estimate on the derivatives of the heat kernel given in the last point of Proposition 2.6, non-collapsing and the volume bound of Proposition 5.1, we get that there exists a constant $C > 0$ such that for any $t > 0$ and $z \in X$,

$$|\theta(t,x) - \theta(t,z)| \leq \frac{C}{\sqrt{t}} \mathsf{d}(x,z).$$

Then for any $z \in X$,

$$\lim_{t \to +\infty} \theta(t,z) = 1.$$

Since by Remark 5.8 the map $t \mapsto \theta(t,z)$ is monotone non-decreasing and larger than one, it must be constantly equal to one. Arguing as in the previous step, the fact that $\theta(t,z) = 1$ for any $z \in X$ and $t > 0$ leads to (35). Then by [CT22, Theorem 1.1], the strong Kato limit $(X, \mathsf{d})$ is isometric to the Euclidean space $(\mathbb{R}^n, \mathsf{d}_e)$, this contradicting inequality (34). □

**Remark 5.10.** Theorem 5.9 can be also proven by using [DPG16, Corollary 1.7], that is the rigidity of the Bishop–Gromov inequality for non-collapsed $\mathrm{RCD}(0,n)$ spaces. We chose to provide a self-contained proof independent of RCD theory.

**5.3. Consequences of almost rigidity.** As an immediate consequence of Theorem 5.9 and of the convergence of heat kernels given by Proposition 2.8 we obtain the following.

**Corollary 5.11.** *Assume that $f$ satisfies* (SK). *For any $\delta > 0$, there is some $\nu > 0$ depending only on $f, n$ and $\delta$ such that if $(M^n, g) \in \mathcal{K}(n, f)$, $x \in M$ and $t \le T$ satisfy*

$$\mathrm{k}_t(M, g) \le \nu \quad \text{and} \quad \theta(t, x) \le 1 + \nu,$$

*then for any $y \in B_{\sqrt{t}}(x)$ we have $\theta(t, y) \le 1 + \delta$.*

By combining Corollary 5.11, the almost monotonicity of $\theta$ (Lemma 5.7) with Theorem 5.9, we get a Reifenberg regularity result for manifolds satisfying a strong Kato bound.

**Corollary 5.12.** *Assume that $f$ satisfies* (SK). *For any $\varepsilon > 0$, there exists $\nu > 0$ depending only on $f, n, \varepsilon$ such that if $(M^n, g) \in \mathcal{K}(n, f)$, $x \in M$ and $t \le T$ satisfy*

$$\mathrm{k}_t(M, g) \le \nu \quad \text{and} \quad \theta(t, x) \le 1 + \nu$$

*then for any $y \in B_{\sqrt{t}}(x)$ and $s \in (0, \sqrt{t})$:*

$$\mathsf{d}_{\mathrm{GH}}\left(B_s(y), \mathbb{B}^n_s\right) \le \varepsilon s.$$

The Reifenberg regularity for non-collapsed strong Kato limits given in Theorem 1.3 is then a direct consequence of Corollary 5.12.

We point out a corollary of the almost rigidity statement Theorem 5.9 and of Proposition 3.9 that we use later to obtain Hölder regularity of the regular set of a non-collapsed strong Kato limit.

**Corollary 5.13.** *Let $v > 0$ and $f$ be a function satisfying* (SK). *For any $\varepsilon > 0$ there exists $\delta > 0$ depending only $f, n, \varepsilon$ such that if $(M^n, g) \in \mathcal{K}(n, f, v)$, $x \in M$ and $t \le T$ satisfy*

$$\mathrm{k}_t(M^n, g) \le \delta \quad \text{and} \quad \theta(t, x) \le 1 + \delta,$$

*then there exists an $(n, \varepsilon)$-splitting $u\colon B_{\sqrt{t}}(x) \to \mathbb{R}^n$.*

**5.4. Transformation theorem.** In order to obtain a quantitative version of Theorem 1.3, we need to prove the following Transformation theorem.

**Theorem 5.14.** (Transformation Theorem) *Let $f$ satisfy* (SK) *and $v > 0$. There exist a constant $\gamma_n > 0$ and $\varepsilon_0 \in (0, 1)$ depending on $n, f$ such that for all $\varepsilon \in (0, \varepsilon_0]$ there exists $\delta > 0$ depending on $\varepsilon, n, f$ and $v$ such that if $(M^n, g) \in \mathcal{K}(n, f)$, $x \in M$ and $r \in (0, \sqrt{T}]$ satisfy*

i) *$\nu_g(B_r(x)) \ge v r^n$;*
ii) *$\mathrm{k}_{r^2}(M^n, g) \le \delta$;*
iii) *for any $s \in (0, r]$, $\mathsf{d}_{\mathrm{GH}}(B_s(x), \mathbb{B}^n_s) \le \delta s$;*

*and if $u\colon B_r(x) \to \mathbb{R}^n$ is an $(n, \delta)$-splitting, then for all $s \in (0, r]$ there exists a $n \times n$ lower triangular matrix $T_s$ such that $\|T_s\| \le (1 + \varepsilon)(r/s)^{\gamma_n \varepsilon}$ and the map $\tilde{u} = T_s \circ u$ is an $(n, \varepsilon)$-splitting on $B_s(x)$.*

**Remark 5.15.** Thanks to Lemma 5.6, we can reformulate the previous theorem replacing the non-collapsing assumption i) by $\theta(r^2, x) \le 2$. In this case the choice of $\delta$ will not depend on $v$.

We obtain Theorem 5.14 as a consequence of the following proposition.

**Proposition 5.16.** *Let $(M,g) \in \mathcal{K}(n,f)$. Then there exist $C_n > 0$ and $\varepsilon_0$, $\lambda \in (0,1)$ depending only on $n$ such that for all $\varepsilon \in (0,\varepsilon_0]$ there exists $\delta > 0$ depending on $n, f, \varepsilon$ such that the following holds. Assume that there exists $r \in (0,\sqrt{T}]$ such that*

$$\mathrm{k}_{r^2}(M^n,g) \leq \delta,$$

*and a ball $B \subset M$ of radius $r$ satisfying*

$$\mathsf{d}_{\mathrm{GH}}(B,\mathbb{B}^n_r) \leq \delta r.$$

*Then for any balanced $(n,\varepsilon)$-splitting $u\colon B \to \mathbb{R}^n$ there exists a $n\times n$ lower triangular matrix $T$ such that $\|T - \mathrm{Id}_n\| \leq C_n\varepsilon$ and the map $\tilde{u} := T \circ u_{|\lambda B}$ is a balanced $(n,\varepsilon)$-splitting of $\lambda B$.*

We postpone the proof of Proposition 5.16 and first give a proof of Theorem 5.14.

*Proof of Theorem 5.14 given Proposition 5.16.* Let $\varepsilon_0, \lambda$ be as in Proposition 5.16, and let $\varepsilon \in (0,\varepsilon_0]$. Consider $\eta \in (0,1]$ to be chosen later depending on $n$ and let $\delta = \delta(n,f,\eta\varepsilon)$ be the quantity given by Proposition 5.16. Assume that

$$\mathrm{k}_{r^2}(M^n,g) \leq \delta, \text{ for all } s \in (0,r] \ \mathsf{d}_{\mathrm{GH}}(B_s(x),\mathbb{B}^n_s) \leq \delta s.$$

Consider a $(n,\eta\varepsilon)$-splitting $u\colon B_r(x) \to \mathbb{R}^n$ and $s \in (0,r]$.

First assume $s \in (\lambda r, r]$. Since $\lambda$ only depends on $n$, then (6) with $\phi = \|G_u - \mathrm{Id}_n\|$ implies

$$\fint_{B_s(x)} \|G_u - \mathrm{Id}_n\| \,\mathrm{d}\nu_g < C(n)\eta\varepsilon.$$

If $C(n)\varepsilon_0 < 1/2$, Remark 3.5 implies the existence of a lower triangular matrix $T_s$ such that $\|T_s\| \leq 1 + C(n)\eta\varepsilon$ and $T_s \circ u\colon B_s(x) \to \mathbb{R}^n$ is a balanced $(n,(1+C(n)\eta\varepsilon)^2\eta\varepsilon)$-splitting. We have no restriction in assuming that $\varepsilon_0$ is lower than $1/4C(n)$, thus we do it. Assume also that

$$\eta \leq \frac{16}{25}.$$

Then $T_s \circ u$ is a balanced $(n,\varepsilon)$-splitting.

Now assume that there exists some positive integer $l$ such that $\lambda^{-l}s \in (\lambda r, r]$. Thanks to assumption iii), we can apply Proposition 5.16 iteratively to get existence of lower triangular matrices $T_0,\ldots,T_l$ such that $\tilde{u} := T_l \circ \ldots \circ T_0 \circ u\colon B_s(x) \to \mathbb{R}^n$ is a balanced $(n,\varepsilon)$-splitting and

$$\|T_j\| \leq (1 + C(n)\eta\varepsilon)$$

for any $j \in \{0,\ldots,l\}$. Set $T{:=}T_l \circ \ldots \circ T_0$. Then

$$\|T\| \leq (1 + C(n)\eta\varepsilon)^{l+1}.$$

Since $\lambda^{-l}s \leq r$ implies $l \leq \frac{\ln(r/s)}{\ln(1/\lambda)}$, we get

$$(1 + C(n)\eta\varepsilon)^l \leq (r/s)^{\frac{\ln(1+C(n)\eta\varepsilon)}{\ln(1/\lambda)}} \leq (r/s)^{\frac{C(n)\varepsilon}{\ln(1/\lambda)}}.$$

Then we set

$$\gamma_n := \frac{C(n)}{\ln(1/\lambda)} \quad \text{and} \quad \eta := \min\left\{\frac{16}{25}, \frac{1}{C(n)}\right\}$$

to get $\|T\| \leq (1+\varepsilon)(r/s)^{\gamma_n\varepsilon}$. This concludes the proof. □

**Remark 5.17.** We point out that, unlike the proof of [CJN21, Proposition 7.7], which relies on a contradiction argument, we provide a direct proof of the Transformation Theorem.

We are left to proving Proposition 5.16. In order to do so, we need the following property of harmonic maps on $\mathbb{B}^n$.

**Proposition 5.18.** *Let $h\colon \mathbb{B}^n \to \mathbb{R}^k$ be a harmonic function and set*

$$\Lambda := \fint_{\mathbb{B}^n} \|G_h - \mathrm{Id}_k\|_1 \,\mathrm{d}x.$$

*Then there exists a constant $C > 0$ depending only on $n$ such that for all $r \in (0, 1/2)$*

$$\fint_{\mathbb{B}^n_r} \|G_h - \textstyle\fint_{\mathbb{B}^n_r} G_h\|_1 \,\mathrm{d}x \leq C\Lambda r. \tag{39}$$

*Proof.* For the sake of brevity, we show an analog statement in the case $k = 1$: consider a harmonic function $h\colon \mathbb{B}^n \to \mathbb{R}$ and a constant $c \in \mathbb{R}$, and set

$$\Lambda_c = \fint_{\mathbb{B}^n} ||dh|^2 - c| \,\mathrm{d}x.$$

Then we show that there exists $C > 0$ only depending on $n$ such that for all $r \in (0, 1/2)$ we have

$$\fint_{\mathbb{B}^n_r} \left| |dh|^2 - \fint_{\mathbb{B}^n_r} |dh|^2 \right| \mathrm{d}x \leq C\Lambda_c r. \tag{40}$$

By arguing as in Lemma 3.11, we obtain the following Hessian bound:

$$\| \operatorname{Hess} h\|_{L^\infty(\frac{5}{8}\mathbb{B}^n)} \leq C_n\sqrt{\Lambda_c}. \tag{41}$$

Now we write

$$h = \ell + \beta,$$

where $\ell$ is the affine part of $h$, namely $\ell(\cdot) = h(0) + dh(0)(\cdot)$, so that $\beta(0) = 0$ and $d\beta(0) = 0$. We also have

$$\operatorname{Hess} h = \operatorname{Hess} \beta,$$

then from (41) we get, for any $x \in \mathbb{B}^n_{\frac{5}{8}}$,

$$|d\beta|(x) \leq C_n\sqrt{\Lambda_c}\,|x|. \tag{42}$$

Using that the coefficients of $dh$ are harmonic and $d\beta(0) = 0$, we obtain

$$\fint_{\mathbb{B}^n} dh = d\ell \quad \text{and} \quad |d\ell| \leq \fint_{\mathbb{B}^n} |dh|.$$

Moreover, for any $r \in (0, 1)$ the mean value of $\langle d\ell, d\beta\rangle$ over $\mathbb{B}^n_r$ is equal to its value at 0, thus it is equal to zero. We then get for any $r \in (0, 1)$

$$\fint_{r\mathbb{B}^n} |dh|^2 = |d\ell|^2 + \fint_{r\mathbb{B}^n} |d\beta|^2$$

so that

$$\fint_{r\mathbb{B}^n} \left| |dh|^2 - \left(\fint_{r\mathbb{B}^n} |dh|^2\right) \right| \leq 2\fint_{r\mathbb{B}^n} |d\beta|^2 + 2\fint_{r\mathbb{B}^n} |\langle d\ell, d\beta\rangle|. \tag{43}$$

By (42), the first term in the right-hand side is smaller than $C_n\Lambda_c r^2$. As for the second term, we use

$$2\langle d\ell, d\beta\rangle = |dh|^2 - |d\ell|^2 - |d\beta|^2$$

to get

$$2\fint_{r\mathbb{B}^n} |\langle d\ell, d\beta\rangle| \leq 2\fint_{r\mathbb{B}^n} |d\beta|^2 + \fint_{r\mathbb{B}^n} \left| |dh|^2 - \left(\fint_{r\mathbb{B}^n} |dh|^2\right) \right|$$

for any $r \in (0,1)$. Choosing $r = 5/8$ gives

$$\fint_{\frac{5}{8}\mathbb{B}^n} |\langle d\ell, d\beta\rangle| \le C_n \Lambda_c.$$

Since $\langle d\ell, d\beta\rangle$ is harmonic, elliptic estimates imply the following gradient estimate

$$\|d\langle d\ell, d\beta\rangle\|_{L^\infty(\frac{1}{2}\mathbb{B}^n)} \le C_n \fint_{\frac{5}{8}\mathbb{B}^n} |\langle d\ell, d\beta\rangle| \le C_n \Lambda_c.$$

Then by using that $\langle d\ell, d\beta\rangle(0)$ vanishes we get for any $x \in \frac{1}{2}\mathbb{B}^n$

$$|\langle d\ell, d\beta\rangle|(x) \le C_n \Lambda_c |x|.$$

As a consequence, for any $r \in (0,1/2)$ the second term in (43) is bounded above by $C_n \Lambda r$. We then get the desired inequality

$$\fint_{\mathbb{B}^n_r} \left| |dh|^2 - \left( \fint_{\mathbb{B}^n_r} |dh|^2 \right) \right| \le C_n \Lambda_c (r^2 + r) \le C_n \Lambda_c r,$$

for any $r \in (0,1/2)$. □

We can now prove Proposition 5.16.

*Proof of Proposition 5.16.* Up to rescaling the distance by a factor $r^{-1}$, we can assume that $r$ is equal to 1. Let $\varepsilon_0, \kappa \in (0,1)$ and $\lambda \in (0,1/4)$ to be chosen later and which will depend only on the dimension $n$. In what follows we note $C(n)$ for a generic constant which depends only on the dimension $n$ and whose value may change from line to line.

Take $\varepsilon \in (0, \varepsilon_0]$ and let $u$ be a balanced $(n,\varepsilon)$-splitting of a ball $B \subset M$ with radius 1. We assume that $(M,g) \in \mathcal{K}(n,f)$ and for some $\delta \in (0, 1/16n)$,

$$\mathrm{k}_1(M^n, g) \le \delta \quad \text{and} \quad \mathsf{d}_{\mathrm{GH}}(B, \mathbb{B}^n_1) \le \delta.$$

By Proposition 3.7, we have

$$\sup_{\frac{3}{4}B} |du| \le (1 + C(n)\varepsilon)\,. \tag{44}$$

If $\delta \le \nu(n, f, v, \kappa\varepsilon, 1/2, \lambda)$, then by Theorem 3.8, there exists a harmonic map $h\colon \frac{1}{2}\mathbb{B}^n \to \mathbb{R}^n$ such that $\|dh\|_{L^\infty(\frac{1}{2}\mathbb{B}^n)} \le 2C(n)$ and

$$\left| \fint_{\frac{1}{2}B} \|G_u - \mathrm{Id}_n\|_1 \,\mathrm{d}\nu_g - \fint_{\frac{1}{2}\mathbb{B}^n} \|G_h - \mathrm{Id}_n\|_1 \,\mathrm{d}x \right| < \kappa\varepsilon, \tag{45}$$

$$\left| \fint_{\lambda B} \|G_u - \overline{G_h}\|_1 \,\mathrm{d}\nu_g - \fint_{\lambda\mathbb{B}^n} \|G_h - \overline{G_h}\|_1 \,\mathrm{d}x \right| < \kappa\varepsilon, \tag{46}$$

where we have noted $\overline{G_h} = \fint_{\lambda\mathbb{B}^n} G_h$, and we introduce similarly $\overline{G_u} = \fint_{\lambda B} G_u \,\mathrm{d}\nu_g$.

We now have that

$$\begin{aligned}
\fint_{\lambda B} \|G_u - \overline{G_u}\| \,\mathrm{d}\nu_g &\leq \fint_{\lambda B} \|G_u - \overline{G_h}\| \,\mathrm{d}\nu_g + \|\overline{G_h} - \overline{G_u}\| \\
&\leq \fint_{\lambda B} \|G_u - \overline{G_h}\| \,\mathrm{d}\nu_g + \left\| \fint_{\lambda B} \left(\overline{G_h} - G_u\right) \mathrm{d}\nu_g \right\| \\
&\leq 2 \fint_{\lambda B} \|G_u - \overline{G_h}\| \,\mathrm{d}\nu_g \\
&\leq 2 \fint_{\lambda B} \|G_u - \overline{G_h}\|_1 \,\mathrm{d}\nu_g \\
&\leq 2 \fint_{\lambda \mathbb{B}^n} \|G_h - \overline{G_h}\|_1 \,\mathrm{d}x + 2\kappa\varepsilon,
\end{aligned}$$

where we have used (46) and $\|\cdot\| \leq \|\cdot\|_1$. But using Proposition 5.18 and then estimate (45), one gets that

$$\begin{aligned}
\fint_{\lambda \mathbb{B}^n} \|G_h - \overline{G_h}\|_1 \,\mathrm{d}x &\leq C(n)\lambda \fint_{\frac{1}{2}\mathbb{B}^n} \|G_h - \mathrm{Id}_n\|_1 \,\mathrm{d}x \\
&\leq C(n)\lambda \left( \kappa\varepsilon + \fint_{\frac{1}{2}B} \|G_u - \mathrm{Id}_n\|_1 \,\mathrm{d}\nu_g \right) \\
&\leq C(n)\lambda \left(\kappa\varepsilon + C(n)\varepsilon\right),
\end{aligned}$$

where in the last inequality, we have used (6) and $\|\cdot\|_1 \leq C(n)\|\cdot\|$. Gathering all the estimates, we get that

$$\fint_{\lambda B} \|G_u - \overline{G_u}\| \,\mathrm{d}\nu_g \leq C(n) \left(\kappa + \lambda\right) \varepsilon.$$

Again (6) implies that

$$\|\overline{G_u} - \mathrm{Id}_n\| \leq \fint_{\lambda B} \|G_u - \mathrm{Id}_n\| \,\mathrm{d}\nu_g \leq C(n,\lambda) \fint_{B} \|G_u - \mathrm{Id}_n\| \,\mathrm{d}\nu_g \leq C(n,\lambda)\varepsilon.$$

If $\varepsilon \leq \frac{1}{4C(n,\lambda)}$, then by Lemma 3.1 there exists a lower triangular matrix $T$ such that

$$T \fint_{\lambda B} G_u \,\mathrm{d}\nu_g {}^tT = \mathrm{Id}_n, \quad \|T\| \leq 1 + C(n)C(n,\lambda)\varepsilon. \tag{47}$$

Then the map $\tilde{u} = Tu \colon \lambda B \to \mathbb{R}^n$ satisfies

$$\fint_{\lambda B} G_{\tilde{u}} \,\mathrm{d}\nu_g = \mathrm{Id}_n,$$

$$\fint_{\lambda B} \|G_{\tilde{u}} - \mathrm{Id}_n\| \,\mathrm{d}\nu_g \leq \|T\|^2 \fint_{\lambda B} \left\| G_u - \fint_{\lambda B} G_u \,\mathrm{d}\nu_g \right\| \mathrm{d}\nu_g \leq \|T\|^2 \, C(n) \left(\kappa + \lambda\right) \varepsilon, \tag{48}$$

and

$$\sup_{\lambda B} |d\tilde{u}| \leq \|T\| \left(1 + C(n)\varepsilon\right). \tag{49}$$

We now make the following choices:

$$\kappa = \lambda = \frac{1}{8C(n)} \quad \text{and} \quad \varepsilon_0 = \min\left\{ \frac{1}{2C(n)}\,;\, \frac{1}{4C(n)C(n,\lambda)} \right\}$$

and assume that

$$\delta = \min\left\{\frac{1}{3C(n)}\,;\,\nu(n,f,v,\kappa\varepsilon,1/2,\lambda)\right\}$$

so that

- $\|T\| \leq 1 + C_n\varepsilon \leq \frac{4}{3} \leq 2$ by (47) and the fact that $\varepsilon \leq \varepsilon_0$,
- $\sup_{\lambda B} |d\tilde{u}| \leq \frac{4}{3}(1 + C(n)\varepsilon) \leq \frac{4}{3}\frac{3}{2} = 2$ by (49),
- $\tilde{u}$ is a balanced $(n,\varepsilon)$-splitting of $\lambda B$ by (48).

This concludes the proof. □

**5.5. Hölder regularity.** We conclude this section by observing that, under suitable assumptions, the results of the previous sections lead to the following Hölder regularity of almost splitting maps.

**Theorem 5.19.** *Assume that $f$ satisfies* (SK). *There exists $\varepsilon_0 \in (0,1)$ depending only on $f, n$ such that for all $\varepsilon \in (0,\varepsilon_0]$ and $\eta \in (0,1)$, there exists $\delta > 0$ depending only on $f, n, \varepsilon, \eta$ such that if $(M^n, g) \in \mathcal{K}(n,f)$, $x \in M$ and $t \in (0,\sqrt{T}]$ satisfy*

$$\mathrm{k}_t(M^n,g) \leq \delta, \quad \theta(t,x) \leq 1 + \delta,$$

*then any $(n,\delta)$-splitting $u\colon B_{\sqrt{t}}(x) \to \mathbb{R}^n$, with $u(x) = 0$, is a diffeomorphism from $B_{(1-\eta)\sqrt{t}}(x)$ onto its image. Moreover, $u$ satisfies for all $y, z \in B_{(1-\eta)\sqrt{t}}(x)$*

$$(1-\varepsilon)\frac{\mathsf{d}_g(y,z)^{1+\varepsilon}}{(\sqrt{t})^{\varepsilon}} \leq |u(y) - u(z)| \leq (1+\varepsilon)\mathsf{d}_g(y,z), \tag{50}$$

*and we have $\mathbb{B}^n_{(1-2\eta)\sqrt{t}} \subset u(B_{(1-\eta)\sqrt{t}}(x)) \subset \mathbb{B}^n_{(1-\eta/2)\sqrt{t}}$.*

As in the proof of [CJN21, Theorem 7.10], Theorem 5.19 follows from the Reifenberg regularity given in Corollary 5.12, Proposition 3.10 and the Transformation Theorem 5.14. We then refer to [CJN21] for the details of the proof.

Theorem 5.19 clearly passes to the limit to give an analog statement on non-collapsed strong Kato limits. Now recall that Corollary 5.13 states that if $\theta(t,x)$ is close enough to 1, then there exists an $(n,\varepsilon)$-splitting on a ball around $x$. As a consequence, we obtain:

**Corollary 5.20.** *Assume that $f$ satisfies* (SK). *Let $(X,\mathsf{d},o) \in \overline{\mathcal{K}(n,f,v)}$. For any $\alpha \in (0,1)$ there exists $\delta$ depending on $\alpha$, $n$ and $f$ such that for any $x \in X$ satisfying $\vartheta(x) < 1 + \delta$ there exist $r \in (0,\sqrt{T})$ and a homeomorphism $u\colon B_r(x) \to u(B_r(x)) \subset \mathbb{R}^n$ such that for all $y, z \in B_r(x)$ we have*

$$\alpha r^{1-\frac{1}{\alpha}}\mathsf{d}(y,z)^{\frac{1}{\alpha}} \leq |u(y) - u(z)| \leq \frac{1}{\alpha}\mathsf{d}(y,z)^{\alpha} r^{1-\alpha}.$$

Theorem 1.3 is then a consequence of this latter result and of a simple covering argument.

## Appendix

**A. Codimension 2.** In this section we prove the following.

**Theorem A.1.** *Assume that* (SK) *holds. Let $(X,\mathsf{d},o) \in \overline{\mathcal{K}(n,f,v)}$. Then the singular set $\mathcal{S} := X \setminus \mathcal{R}$ has Hausdorff dimension at most $n-2$.*

Consider $(X,\mathsf{d},o) \in \overline{\mathcal{K}(n,f,v)}$. From [CMT24, Theorem 6.2], we know that the singular set $\mathcal{S}$ admits a filtration

$$\mathcal{S}^0 \subset \ldots \subset \mathcal{S}^{n-1} = \mathcal{S}$$

where

$$\mathcal{S}^k := \{x \in X\colon \mathbb{R}^\ell \times Z \in \mathrm{Tan}(X,x) \Rightarrow \ell \le k\}$$

for any $k \in \{0, \dots, n-1\}$. Moreover, the Hausdorff dimension of each $\mathcal{S}^k$ is at most $k$. Thus we are left with proving $\mathcal{S}^{n-1} = \mathcal{S}^{n-2}$.

Let us explain why the latter follows from proving that $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ cannot be a tangent cone of $X$ at any $x \in X$. In [CMT24, Theorem A] we proved that any metric measure tangent cone of $X$ is an $\mathrm{RCD}(0,n)$ metric measure cone. As a consequence, if $X_x = Z \times \mathbb{R}^{n-1}$ is a tangent cone of $X$ at $x$, since $X$ has Hausdorff dimension at most $n$, then $Z$ is an $\mathrm{RCD}(0,1)$ metric measure cone over some finite set $F$. If $\#F \ge 2$ then $Z$ has at least two ends and as a consequence splits so that necessarily $Z = \mathbb{R}$. Therefore, we have $\#F = 1$ and then $Z = \mathbb{R}_+$, and this is what we aim to prove impossible.

We prove this by contradiction. With no loss of generality, suppose $T = 1$. Assume that there exists $x \in X$ admitting a metric tangent cone isometric to $\mathbb{R}^+ \times \mathbb{R}^{n-1}$. Then there exist pointed closed Riemannian manifolds $\{(M_\alpha, g_\alpha, o_\alpha)\}$ and positive numbers $\{\varepsilon_\alpha\}$ such that $\varepsilon_\alpha \downarrow 0$,

$$(M_\alpha, \mathsf{d}_{g_\alpha}, o_\alpha) \xrightarrow{\text{pGH}} (\mathbb{R}_+ \times \mathbb{R}^{n-1}, \mathsf{d}_e, 0)$$

and

$$\mathrm{k}_t(M_\alpha, g_\alpha) \le f(\varepsilon_\alpha t)$$

for any $\alpha$ and any $t \in (0, 1/\varepsilon_\alpha]$. Set

$$\tau\mathbb{B}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{B}^n_\tau :\ x_1 \ge 0\}$$

for any $\tau > 0$. By arguing as in the proof of [CMT24, Theorem 7.4], we get harmonic maps

$$\Psi_\alpha = (h_2^\alpha, \dots, h_n^\alpha)\colon B_2(o_\alpha) \to \mathbb{R}^{n-1}$$

which converge uniformly to $(x_2, \dots, x_n)\colon 2\mathbb{B}^n_+ \to \mathbb{R}^{n-1}$ and such that for any $\alpha$,

i) $\|d\Psi_\alpha\|_{L^\infty(B_2(o_\alpha))} \le 1 + \varepsilon_\alpha$,

ii) $\displaystyle\fint_{B_2(o_\alpha)} \|G_{\Psi_\alpha} - \mathrm{Id}_{n-1}\| \,\mathrm{d}\nu_{g_\alpha} \le \varepsilon_\alpha$,

iii) $\displaystyle\fint_{B_2(o_\alpha)} |dG_{\Psi_\alpha}|^2 \,\mathrm{d}\nu_{g_\alpha} \le \varepsilon_\alpha$.

From [CMT24, Proposition A.1], we get existence of uniformly Lipschitz functions $f_1^\alpha \in \mathcal{C}^\infty(B_2(o_\alpha))$ which converge uniformly to $x_1\colon 2\mathbb{B}^n_+ \to \mathbb{R}$. With no loss of generality, we may assume that

$$\Phi^\alpha := (f_1^\alpha, h_2^\alpha, \dots, h_n^\alpha)\colon B_2(o_\alpha) \to 2\mathbb{B}^n_+$$

is an $\varepsilon_\alpha$-GH isometry. We are going to modify each $f_1^\alpha$ into a suitable $h_1^\alpha$. To this aim, we consider a convergent sequence $p_\alpha \in B_1(o_\alpha) \to p = (1/2, 0, \dots, 0)$. Up to working with $\Phi^\alpha$ modified by an additive constant, we can assume that

$$\Phi^\alpha(p_\alpha) = p,$$

and up to considering large enough $\alpha$ only, we can assume that

$$B_{3/8}(p_\alpha) \subset B_1(o_\alpha).$$

For any $\alpha$ let $\tilde{f}_1^\alpha\colon B_2(o_\alpha) \to \mathbb{R}$ be equal to the harmonic replacement of $f_1^\alpha$ on $B_{3/8}(p_\alpha)$ and equal to $f_1^\alpha$ elsewhere. Then the sequence $\{\tilde{f}_1^\alpha\}$ is uniformly bounded in energy and in $L^\infty$, and any of its weak sub-limit in energy is equal to $x_1$ on $2\mathbb{B}^n_+ \setminus B_{3/8}(p)$ and is harmonic on $B_{3/8}(p)$, hence it is equal to $x_1$. Using the energy characterization of harmonic functions and the semicontinuity of the energy, this implies

$$\tilde{f}_1^\alpha \xrightarrow{E} x_1.$$

Moreover, the gradient estimate [CMT24, Lemma 3.6] implies that the convergence is uniform on $B_{5/16}(p_\alpha)$.

For any $\alpha$ let $\chi_\alpha$ be the smooth cut-off function on $M_\alpha$ such that $\chi_\alpha = 1$ on $B_{9/32}(p_\alpha)$ and $\chi_\alpha = 0$ on $M_\alpha \setminus B_{5/16}(p_\alpha)$ with $\operatorname{Lip}\chi_\alpha \leq 64$. Up to extraction of a subsequence, we may assume that $\{\chi_\alpha\}$ converges uniformly to a similar cut-off function on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. For any $\alpha$ set

$$h_1^\alpha := \chi_\alpha \tilde{f}_1^\alpha + (1-\chi_\alpha) f_1^\alpha;$$

then $h_1^\alpha$ is smooth on $B_2(p_\alpha)$ and harmonic on $B_{9/32}(p_\alpha)$. Furthermore, the sequence $\{h_1^\alpha\}$ converges uniformly to $x_1$ on $B_1(p_\alpha)$, and the maps

$$h_\alpha := (h_1^\alpha, h_2^\alpha, \dots, h_n^\alpha)\colon B_1(o_\alpha) \to \mathbb{B}_+^n$$

are $\varepsilon_\alpha$-GH isometries which converge uniformly to the identity function. Moreover,

i) $\|dh_\alpha\|_{L^\infty(B_{17/64}(p_\alpha))} \leq 1 + \varepsilon_\alpha$,

ii) $\displaystyle\fint_{B_{17/64}(p_\alpha)} \|G_{h_\alpha} - \mathrm{Id}_n\| \,\mathrm{d}\nu_{g_\alpha} \leq \varepsilon_\alpha$,

iii) $\displaystyle\fint_{B_{17/64}(p_\alpha)} |dG_{h_\alpha}|^2 \,\mathrm{d}\nu_{g_\alpha} \leq \varepsilon_\alpha$.

Let $\{\tau_\alpha\}, \{\rho_\alpha\} \subset (0,1)$ be such that $\tau_\alpha \uparrow 1$, $\rho_\alpha \uparrow 1/4$, and for any $\alpha$, $\tau_\alpha^2$ is a regular value of $|h_\alpha|^2$ and $\rho_\alpha^2$ is a regular value of $|h_1^\alpha - 1/2|^2 + |\Psi_\alpha|^2$. For a given $\alpha$, set

$$\Omega_\alpha := h_\alpha^{-1}(\mathbb{B}_{\tau_\alpha}^n) \quad \text{and} \quad \mathcal{U}_\alpha := h_\alpha^{-1}(\mathbb{B}_{\rho_\alpha}^n(p)).$$

Since $h_\alpha(\Omega_\alpha) \subset \mathbb{B}_+^n$, we know that $h_\alpha\colon \Omega_\alpha \to \mathbb{B}_{\tau_\alpha}^n$ is not surjective. Moreover, $h_\alpha(\partial\Omega_\alpha) \subset \partial\mathbb{B}_{\tau_\alpha}^n$. Thus for any regular value $x \in \tau_\alpha\mathbb{B}_+^n$ of $h_\alpha$,

$$\#(h_\alpha^{-1}(\{x\}) \cap \Omega_\alpha) \in 2\mathbb{N}. \tag{51}$$

Let us now consider a sequence $q_\alpha \in \mathcal{U}_\alpha \to p$ such that each $h_\alpha(q_\alpha)$ is a regular value of $h_\alpha$. As each $h_\alpha$ is an $\varepsilon_\alpha$-GH isometry, for any $q \in \Omega_\alpha$:

$$h_\alpha(q) = h_\alpha(q_\alpha) \Longrightarrow \mathsf{d}_\alpha(q, q_\alpha) \leq \varepsilon_\alpha.$$

Hence for large enough $\alpha$:

$$\{q \in \Omega_\alpha\colon h_\alpha(q) = h_\alpha(q_\alpha)\} \subset \mathcal{U}_\alpha.$$

But the analysis done in the proof of [CMT24, Theorem 7.4] shows that

- if $\mathcal{U}_\alpha$ is orientable, then the degree of $h_\alpha\colon \mathcal{U}_\alpha \to \mathbb{B}_{\rho_\alpha}(p)$ is $\pm 1$,
- if $\mathcal{U}_\alpha$ is not orientable and if $\pi_\alpha\colon \tilde{\mathcal{U}}_\alpha \to \mathcal{U}_\alpha$ is the 2-fold orientation cover, then the degree of $h_\alpha \circ \pi_\alpha\colon \tilde{\mathcal{U}}_\alpha \to \mathbb{B}_{\rho_\alpha}(p)$ is $\pm 2$.

In any case we get

$$\#\{q \in \Omega_\alpha\colon h_\alpha(q) = h_\alpha(q_\alpha)\} \in 2\mathbb{N} + 1,$$

which contradicts (51).

**B. Proof of Theorem 3.8.** In this section, we obtain Theorem 3.8 as a consequence of a contradiction argument and the following result.

**Theorem B.1.** *Let $\{(M_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, o_\alpha)\} \subset \mathcal{K}_{\mathfrak{m}}(n,f,c)$ be converging to $(X,\mathsf{d},\mu,o)$ in the pointed measured Gromov–Hausdorff topology. For some $r \in (0,\sqrt{T}]$, assume that there exists a harmonic function $h\colon B_r(o) \to \mathbb{R}^k$ such that $h(o)=0$ and $\|dh\|_{L^\infty(B_r(o))} \le L$ for some $L>1$. Let $\eta \in (0,1)$ be given. Then there exist $C(n,\eta) \ge 1$ and $h_\alpha \colon B_{\eta r}(o_\alpha) \to \mathbb{R}^k$ harmonic with $\|dh_\alpha\|_{L^\infty(B_{\eta r}(o_\alpha))} \le LC(n,\eta)$ and $h_\alpha(o_\alpha)=0$ for any $\alpha$, such that $h_\alpha$ converges uniformly to $h$; moreover, the folllowing properties hold:*

(1) *for all $s \in (0,\eta r]$*

$$\fint_{B_s(o_\alpha)} G_{h_\alpha}\,\mathrm{d}\mu_\alpha \to \fint_{B_s(o)} G_h\,\mathrm{d}\mu, \tag{52}$$

(2) *for all $s \in (0,\eta r]$ and $A \in \mathcal{M}_k(\mathbb{R})$*

$$\fint_{B_s(o_\alpha)} \|G_{h_\alpha} - A\|\,\mathrm{d}\mu_\alpha \to \fint_{B_s(o)} \|G_h - A\|\,\mathrm{d}\mu. \tag{53}$$

Before proving it, we need a preliminary lemma.

**Lemma B.2.** *Let $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_{\alpha\in\mathbb{N}\cup\{\infty\}} \subset \overline{\mathcal{K}_{\mathfrak{m}}(n,f,c)}$ be such that*

$$(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha) \to (X_\infty, \mathsf{d}_\infty, \mu_\infty, o_\infty)$$

*in the pointed measured Gromov–Hausdorff topology. Consider $r \in (0,\sqrt{T})$. For any $\alpha$, let $u_\alpha, v_\alpha \in H^{1,2}(B_r(o_\alpha), \mathsf{d}_\alpha, \mu_\alpha)$ be such that*

(1) $u_\alpha \overset{L^2(B_r)}{\to} u_\infty$ and $v_\alpha \overset{L^2(B_r)}{\to} v_\infty$,

(2) $\sup_{\alpha\in\mathbb{N}} \left(\int_{B_r(o_\alpha)} \mathrm{d}\Gamma(u_\alpha)\,\mathrm{d}\mu_\alpha\,,\, \int_{B_r(o_\alpha)} \mathrm{d}\Gamma(v_\alpha)\,\mathrm{d}\mu_\alpha\right) < +\infty.$

*Then for any $s \in (0,r]$,*

$$\fint_{B_s(o_\alpha)} |u_\alpha^2 - v_\alpha^2|\,\mathrm{d}\mu_\alpha \to \fint_{B_s(o_\infty)} |u_\infty^2 - v_\infty^2|\,\mathrm{d}\mu. \tag{54}$$

*Proof.* For any $\gamma > 0$ and $\alpha \in \mathbb{N}\cup\{\infty\}$, set

$$u_{\alpha,\gamma}(\cdot) := \fint_{B_\gamma(\cdot)} u_\alpha\,\mathrm{d}\mu_\alpha, \quad v_{\alpha,\gamma}(\cdot) := \fint_{B_\gamma(\cdot)} v_\alpha\,\mathrm{d}\mu_\alpha.$$

Acting as in the proof of [CMT24, Proposition E.1], it is enough to consider the case $s \in (0,r)$ only.

We first claim that there exists $C_0 > 0$ such that for any $\gamma \in (0, r-s)$,

$$\sup_{\alpha\in\mathbb{N}\cup\{\infty\}} \left| \fint_{B_s(o_\alpha)} |u_\alpha^2 - v_\alpha^2| - |u_{\alpha,\gamma}^2 - v_{\alpha,\gamma}^2|\,\mathrm{d}\mu_\alpha \right| \le C_0\gamma. \tag{55}$$

Indeed,

$$\begin{aligned}
&\left| \fint_{B_s(o_\alpha)} |u_\alpha^2 - v_\alpha^2| - |u_{\alpha,\gamma}^2 - v_{\alpha,\gamma}^2|\,\mathrm{d}\mu_\alpha \right| \\
&\le \fint_{B_s(o_\alpha)} |u_\alpha^2 - u_{\alpha,\gamma}^2|\,\mathrm{d}\mu_\alpha + \fint_{B_s(o_\alpha)} |v_\alpha^2 - v_{\alpha,\gamma}^2|\,\mathrm{d}\mu_\alpha.
\end{aligned}$$

Boundedness in $L^2$ of the averaging operator on doubling spaces (see e.g. [Ald19, Theorem 3.5]) yields the existence of $C_1 > 0$ such that

$$\|u_{\alpha,\gamma}\|_{L^2(B_s(o_\alpha))} \leq C_1 \|u_\alpha\|_{L^2(B_r(o_\alpha))}.$$

Moreover, the $L^2$ strong convergence of $\{u_\alpha\}$ to $u_\infty$ gives $C_2 > 0$ such that

$$\sup_{\alpha \in \mathbb{N}\cup\{\infty\}} \|u_\alpha\|_{L^2(B_r(o_\alpha))} \leq C_2.$$

Finally, the $L^2$ pseudo-Poincaré inequality [CSC93] and assumption (2) yield the existence of $C_3 > 0$ such that

$$\left( \fint_{B_s(o_\alpha)} |u_\alpha - u_{\alpha,\gamma}|^2 \, \mathrm{d}\mu_\alpha \right)^{1/2} \leq C_3 \gamma.$$

Then

$$\begin{aligned} \fint_{B_s(o_\alpha)} |u_\alpha^2 - u_{\alpha,\gamma}^2| \, \mathrm{d}\mu_\alpha &\leq \left( \fint_{B_s(o_\alpha)} |u_\alpha - u_{\alpha,\gamma}|^2 \, \mathrm{d}\mu_\alpha \right)^{1/2} \left( \fint_{B_s(o_\alpha)} |u_\alpha + u_{\alpha,\gamma}|^2 \, \mathrm{d}\mu_\alpha \right)^{1/2} \\ &\leq \frac{(1+C_1)C_2C_3\gamma}{\mu_\alpha(B_s(o_\alpha))^{1/2}} \leq \frac{2(1+C_1)C_2C_3\gamma A(n)}{\mu(B_r(o))^{1/2}} \left(\frac{r}{s}\right)^{C(n)} \end{aligned}$$

where we obtain the last inequality by the doubling condition and by making the assumption, with no loss of generality, that $\inf_\alpha \mu_\alpha(B_r(o_\alpha)) \geq \mu(B_r(o))/2$. This and the symmetry between $u$ and $v$ eventually leads to (55).

We now claim that for any given $\varepsilon > 0$ and $\gamma \in (0, (r-s)/2)$, we can choose $\alpha \in \mathbb{N}$ large enough to ensure

$$\left| \fint_{B_s(o_\alpha)} |u_{\alpha,\gamma}^2 - v_{\alpha,\gamma}^2| \, \mathrm{d}\mu_\alpha - \fint_{B_s(o_\infty)} |u_{\infty,\gamma}^2 - v_{\infty,\gamma}^2| \, \mathrm{d}\mu_\infty \right| \leq \frac{\varepsilon}{3}. \tag{56}$$

The Hölder inequality and a consequence of the doubling condition (see e.g. [CMT24, Proposition 1.2, (v)]) imply that $\{u_{\alpha,\gamma}\}$ and $\{v_{\alpha,\gamma}\}$ are equicontinuous on balls of radius $B_{(s+r)/2}(o_\alpha)$ for any fixed $\gamma \in (0,(r-s)/2)$. Then $u_{\alpha,\gamma} \to u_{\infty,\gamma}$ and $v_{\alpha,\gamma} \to v_{\infty,\gamma}$ uniformly on $B_s$. This yields (56).

To conclude, take $\varepsilon > 0$, choose $\gamma = \varepsilon/(3C_0)$ and then choose $\alpha$ such that (56) holds. Then the triangle inequality, (55) and (56) yield (54). □

**Remark B.3.** The previous proof may be easily adapted to show that for any $a \in \mathbb{R}$ and for all $s \in [0, r]$

$$\fint_{B_s(o_\alpha)} |u_\alpha^2 - v_\alpha^2 - a| \, \mathrm{d}\mu_\alpha \to \fint_{B_s(o)} |u_\infty^2 - v_\infty^2 - a| \, \mathrm{d}\mu.$$

We are now in a position to prove Theorem B.1 and conclude.

*Proof.* We start by treating the case $k = 1$. Consider $\eta' = \eta^{1/2}$ and $\eta'' = \eta^{1/3}$ so that $\eta < \eta' < \eta'' < 1$. Then [CMT24, Proposition E.11] ensures the existence of harmonic functions $h_\alpha \colon B_{\eta'' r}(o_\alpha) \to \mathbb{R}$ uniformly converging to $h|_{B_{\eta'' r}(o)}$ on $B_{\eta'' r}(o)$ and such that for all $s \in (0, \eta'' r]$

$$\fint_{B_s(o_\alpha)} |dh_\alpha|^2 \, \mathrm{d}\mu_\alpha \to \fint_{B_s(o)} |dh|^2 \, \mathrm{d}\mu. \tag{57}$$

By replacing $h_\alpha$ by $h_\alpha - h_\alpha(o_\alpha)$ we can assume that $h_\alpha(o_\alpha) = 0$ for all $\alpha$. Moreover, the convergence of $|dh_\alpha|$ given by (57) and the fact that $\|dh\|_{L^\infty(B_r(o))} \leq L$ imply that for any large enough $\alpha$

$$\fint_{B_s(o_\alpha)} |dh_\alpha|^2 \,\mathrm{d}\mu_\alpha \le 2L^2.$$

We can then apply [CMT24, Lemma 3.6] to get existence of $C(n,\eta) \ge 1$ such that $\|dh_\alpha\|_{L^\infty(B_{\eta' r}(o_\alpha))} \le LC(n,\eta)$. Now consider $s \in (0,\eta r]$. The previous local Lipschitz bound and the Hessian estimate of [CMT24, Proposition 3.5] yield the uniform Hessian bound

$$\sup_\alpha \fint_{B_{\eta r}(o_\alpha)} |\nabla dh_\alpha|^2 \,\mathrm{d}\mu_\alpha \le \frac{C(n,\eta,L)}{r^2}. \tag{58}$$

We are then in a position to apply [CMT24, Proposition E.7] and get $L^2(B_{\eta r})$ strong convergence of $\{|dh_\alpha|\}$ to $|dh|$. Then $\{u_\alpha = |dh_\alpha|\}$ and $\{v_\alpha = 0\}$ satisfy the assumptions of Lemma B.2. We apply it and use Remark B.3 to obtain that for all $a \in \mathbb{R}$ and $s \in (0,\eta r]$

$$\fint_{B_s(o_\alpha)} ||dh_\alpha|^2 - a| \,\mathrm{d}\mu_\alpha \to \fint_{B_s(o)} ||dh|^2 - a| \,\mathrm{d}\mu.$$

We consider now the case $k > 1$. Observe that for all $i,j = 1,\dots,k$ we have

$$(G_{h_\alpha})_{i,j} = \langle d(h_\alpha)_i, d(h_\alpha)_j\rangle = \frac{1}{4}(|d((h_\alpha)_i + (h_\alpha)_j)|^2 - |d((h_\alpha)_i - (h_\alpha)_j)|^2).$$

Set

$$f_\alpha = \frac{1}{2}|d((h_\alpha)_i + (h_\alpha)_j)|, \qquad g_\alpha = \frac{1}{2}|d((h_\alpha)_i - (h_\alpha)_j)|,$$
$$f = \frac{1}{2}|d(h_i + h_j)|, \quad g = \frac{1}{2}|d(h_i - h_j)|.$$

The sequences $\{f_\alpha\}$ and $\{g_\alpha\}$ satisfy the assumptions of Lemma B.2. This immediately yields (52). Moreover, if we consider $A \in \mathcal{M}_k(\mathbb{R})$ with components $a_{i,j}$, by arguing as above we get for all $i,j = 1,\dots,k$,

$$\fint_{B_s(o_\alpha)} |f_\alpha^2 - g_\alpha^2 - a_{i,j}| \,\mathrm{d}\mu_\alpha \to \fint_{B_s(o)} |f^2 - g^2 - a_{i,j}| \,\mathrm{d}\mu,$$

which is equivalent to

$$\fint_{B_s(o_\alpha)} |(G_{h_\alpha})_{i,j} - a_{i,j}| \,\mathrm{d}\mu_\alpha \to \fint_{B_s(o)} |(G_h)_{i,j} - a_{i,j}| \,\mathrm{d}\mu.$$

This shows (53). □

## References

[ABS19] Antonelli, G., E. Brué, and D. Semola: Volume bounds for the quantitative singular strata of non collapsed RCD metric measure spaces. - Anal. Geom. Metr. Spaces 7:1, 2019, 158–178.

[Ald19] Aldaz, J. M.: Boundedness of averaging operators on geometrically doubling metric spaces. - Ann. Acad. Sci. Fenn. Math. 44:1, 2019, 497–503.

[Bam20] Bamler, R.: Structure theory of non-collapsed limits of Ricci flows. - arXiv:2009.03243 [math.DG], 2020.

[BPS21] Brué, E., E. Pasqualetto, and D. Semola: Rectifiability of RCD$(K,N)$ spaces via $\delta$-splitting maps. - Ann. Fenn. Math. 46:1, 2021, 465–482.

[Car19] Carron, G.: Geometric inequalities for manifolds with Ricci curvature in the Kato class. - Ann. Inst. Fourier (Grenoble) 69:7, 2019, 3095–3167.

[CMT24] Carron, G., I. Mondello, and D. Tewodrose: Limits of manifolds with a Kato bound on the Ricci curvature. - Geom. Topol. 28:6, 2024, 2635–2745.

[CMT25] Carron, G., I. Mondello, and D. Tewodrose: Kato meets Bakry–Émery. - Calc. Var. Partial Differential Equations 64:6, 2025, art. no. 193.

[CT22] Carron, G., and D. Tewodrose: A rigidity result for metric measure spaces with Euclidean heat kernel. - J. Éc. Polytech. Math. 9, 2022, 101–154.

[Che99] Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. - Geom. Funct. Anal. 9:3, 1999, 428–517.

[Che01] Cheeger, J.: Degeneration of Riemannian metrics under Ricci curvature bounds. - Lezione Fermiane, Accademia Nazionale dei Lincei, Scuola Normale Superiore, Pisa, 2001.

[CC96] Cheeger, J., and T. H. Colding: Lower bounds on Ricci curvature and the almost rigidity of warped products. - Ann. of Math. (2) 144:1, 1996, 189–237.

[CC97] Cheeger, J., and T. H. Colding: On the structure of spaces with Ricci curvature bounded below. I. - J. Differential Geom. 46:3, 1997, 406–480.

[CC00a] Cheeger, J., and T. H. Colding: On the structure of spaces with Ricci curvature bounded below. II. - J. Differential Geom. 54:1, 2000, 13–35.

[CC00b] Cheeger, J., and T. H. Colding: On the structure of spaces with Ricci curvature bounded below. III. - J. Differential Geom. 54:1, 2000, 37–74.

[CJN21] Cheeger, J., W. Jiang, and A. Naber: Rectifiability of singular sets of noncollapsed limit spaces with Ricci curvature bounded below. - Ann. of Math. (2) 193:2, 2021, 407–538.

[CN15] Cheeger, J., and A. Naber: Regularity of Einstein manifolds and the codimension 4 conjecture. - Ann. of Math. (2) 182:3, 2015, 1093–1165.

[Col97] Colding, T. H.: Ricci curvature and volume convergence. - Ann. of Math. (2) 145:3, 1997, 477–501.

[CSC93] Coulhon, T., and L. Saloff-Coste: Isopérimétrie pour les groupes et les variétés. - Rev. Mat. Iberoamericana 9:2, 1993, 293–314.

[DPG16] De Philippis, G., and N. Gigli: From volume cone to metric cone in the nonsmooth setting. - Geom. Funct. Anal. 26:6, 2016, 1526–1587.

[DPMR17] De Philippis, G., A. Marchese, and F. Rindler: On a conjecture of Cheeger. - In: Measure theory in non-smooth spaces, Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, 2017, 145–155.

[DPR16] De Philippis, G., and F. Rindler: On the structure of $\mathscr{A}$-free measures and applications. - Ann. of Math. (2) 184:3, 2016, 1017–1039.

[Fed14] Federer, H.: Geometric measure theory. - Springer, 2014.

[Gig15] Gigli, N.: On the differential structure of metric measure spaces and applications. - Mem. Amer. Math. Soc. 236:1113, 2015.

[Gig18a] Gigli, N.: Lecture notes on differential calculus on RCD spaces. - Publ. Res. Inst. Math. Sci. 54:4, 2018, no. 4, 855–918.

[Gig18b] Gigli, N.: Nonsmooth differential geometry An approach tailored for spaces with Ricci curvature bounded from below. - Mem. Amer. Math. Soc. 251:1196, 2018.

[Gig] Gigli, N.: The splitting theorem in non-smooth context. - Mem. Amer. Math. Soc. (to appear).

[GMR15] Gigli, N., A. Mondino, and T. Rajala: Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below. - J. Reine Angew. Math. 2015:705, 2015, 233–244.

[GP21] Gigli, N., and E. Pasqualetto: Behaviour of the reference measure on RCD spaces under charts. - Comm. Anal. Geom. 29:6, 2021, 1391–1414.

[Gri95] Grigoryan, A.: Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold. - J. Funct. Anal. 127:2, 1995, 363–389.

[Hei01] Heinonen, J: Lectures on analysis on metric spaces. - Universitext, Springer Science & Business Media, 2001.

[HKST15] Heinonen, J., P. Koskela, N. Shanmugalingam, and J. T. Tyson: Sobolev spaces on metric measure spaces: an approach based on upper gradients. - New Math. Monogr. 27, Cambridge Univ. Press, 2015.

[KM18] Kell, M., and A. Mondino: On the volume measure of non-smooth spaces with Ricci curvature bounded below. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18:2, 2018, 593–610.

[LD11] Le Donne, Enrico: Metric spaces with unique tangents. - Ann. Acad. Sci. Fenn. Math. 36:2, 2011, 683–694.

[MN19] Mondino, A., and A. Naber: Structure theory of metric measure spaces with lower Ricci curvature bounds. - J. Eur. Math. Soc. (JEMS) 21:6, 2019, 1809–1854.

[Pre87] Preiss, D.: Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities. - Ann. of Math. (2) 125, 1987, 537–643.

[RS17] Rose, C., and P. Stollmann: The Kato class on compact manifolds with integral bounds on the negative part of Ricci curvature. - Proc. Amer. Math. Soc. 145:5, 2017, 2199–2210.

[Sim14] Simon, L.: Introduction to geometric measure theory. - Tsinghua Lectures, 2014.

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