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Anisotropic weighted Levin–Cochran–Lee type inequalities on homogeneous Lie groups

Michael Ruzhansky, Anjali Shriwastawa and Bankteshwar Tiwari

Abstract. In this paper, we first prove the weighted Levin–Cochran–Lee type inequalities on homogeneous Lie groups for arbitrary weights, quasi-norms, and L^p- and L^q-norms. Then, we derive a sharp weighted inequality involving specific weights given in the form of quasi-balls in homogeneous Lie groups. Finally, we also calculate the sharp constants for the aforementioned inequalities.

Tasakoosteisen Lien ryhmän suunnalliset ja painolliset Levinin–Cochranin–Leen-tyyppiset epäyhtälöt

Tiivistelmä. Tässä työssä todistetaan aluksi painolliset Levinin–Cochranin–Leen-tyyppiset epäyhtälöt tasakoosteisen Lien ryhmän mielivaltaisilla painoilla, kvasinormeilla sekä L^p- ja L^q-normeilla. Sitten johdetaan tiettyjä tasakoosteisen Lien ryhmän kvasipallojen muodossa annettuja painoja koskeva tarkka painoepäyhtälö. Lopuksi määritetään näiden epäyhtälöiden tarkat vakiot.

1. History and introduction

In 1984, Cochran and Lee rediscovered an exponential weighted inequality in their paper [5], which was proved earlier in an unnoticed paper of Levin [15] in 1938 written in the Russsian language. We recall the following exponential weighted inequalities proved in the papers of Levin [15] and Cochran and Lee [5].

Theorem 1.1. *Let ϵ and a be two real numbers. Suppose that f is a positive function such that the function $t^{\epsilon-1}\log f(t)$ is locally integrable on $(0,\infty)$. Then the inequality*

$$\int_0^\infty \left[\exp\left(\epsilon\, x^{-\epsilon}\int_0^x t^{\epsilon-1}\log\, f(t)\,dt\right)\right] x^a\,dx \le \left(\exp\frac{a+1}{\epsilon}\right)\int_0^\infty x^a\, f(x)\,dx \tag{1.1}$$

holds for $\epsilon>0$, and

$$\int_0^\infty \left[\exp\left(-\epsilon\, x^{-\epsilon}\int_x^\infty t^{\epsilon-1}\log\, f(t)\,dt\right)\right] x^a\,dx \le \left(\exp\frac{a+1}{\epsilon}\right)\int_0^\infty x^a\, f(x)\,dx \tag{1.2}$$

holds for $\epsilon>0$. Moreover, the constant $\exp\left(\frac{a+1}{\epsilon}\right)$ is the best possible constant.

The inequality (1.1) is called the Levin–Cochran–Lee type inequality and its complementary inequality (1.2) was proved by Love in [16], which was again reproved by Yang and Lin [25]. It is worth noting that inequality (1.1) is a generalization of the famous Knopp inequality [12], which can be obtained by setting $a=0$ and $\epsilon=1$ in (1.1). Thereafter, several works have been devoted to the study of these exponential type inequalities in different forms and in different settings such as higher dimensional

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Euclidean spaces and Euclidean balls, by many authors. It is clearly impossible to give a complete overview of the available literature, therefore we refer to the books, surveys and papers [3, 4, 5, 6, 10, 12, 14] and references therein.

Čižmešija et al. [4] investigated an n-dimensional analogue of (1.1) by replacing the intervals $(0,\infty)$ by $\mathbb{R}^n$ and the means are considered over the balls in $\mathbb{R}^n$ centered as origin. We state this inequality as follows:

Theorem 1.2. *Let f be a positive function on $\mathbb{R}^n$ and let $\mathbb{B}(0,|x|)$ be the ball in $\mathbb{R}^n$ with radius $|x|$, $x \in \mathbb{R}^n$, centered at the origin, with its volume (with respect to the Lebesgue measure on $\mathbb{R}^n$) denoted by $|\mathbb{B}(0,|x|)|$. Then we have the following inequality*

$$\begin{aligned} &\int_{\mathbb{R}^n} \left[\exp\left(\epsilon\, |\mathbb{B}(0,|x|)|^{-\epsilon} \int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|y|)|^{\epsilon-1} \log f(y)\, dy \right) \right] |\mathbb{B}(0,|x|)|^a \, dx \\ &\leq \left(\exp \frac{a+1}{\epsilon} \right) \int_{\mathbb{R}^n} f(x)\, |\mathbb{B}(0,|x|)|^a \, dx, \end{aligned} \tag{1.3}$$

where a and $\epsilon > 0$ are two real numbers. Moreover, the constant $\exp \frac{a+1}{\epsilon}$ appearing in (1.3) is a sharp constant.

The inequality (1.3) was further generalized by Jain et al. [10] to a more general situation involving general weight functions on the Euclidean space.

The main objective of this paper is to prove a Levin–Cochran–Lee type inequality involving general weight functions on homogeneous (Lie) groups equipped with a quasi-norm $|\cdot|$ and a family of dilations compatible with the group law. For a detailed description of analysis on homogeneous groups, we refer to [7, 8, 21]. Particular examples of homogeneous groups are the Euclidean space $\mathbb{R}^n$ (in which case $Q = n$), the Heisenberg group, as well as general stratified groups (homogeneous Carnot groups) and graded groups. Recently, Hardy type inequalities and their best constants have been extensively investigated in non-commutative settings (e.g. Heisenberg groups, graded groups, homogeneous groups); we cite [9, 19, 21, 18, 24] just to a mention a few of them. These Hardy type inequalities have several applications in different branches on mathematics, particularly, in the theory of linear and nonlinear partial-differential equations, we refer to [21] and references therein for more detailed discussion, including the discussion on how these inequalities fit into a broader scale of different inequalities of the mathematical analysis and mathematical physics. The exponential terms in the inequality can make it possible applying them for nonlinear equations with non-polynomial nonlinearities, which is, however, not the subject of the present paper. They also lead to the corresponding uncertainty principles, see [20], or a discussion in [21].

Recently, the first author and Verma [22] obtained several characterizations of weights for two-weight integral Hardy inequalities to hold on general metric measure spaces possessing polar decompositions for the range $1 < p \leq q < \infty$ (see, [23] for the case $0 < q < p$ and $1 < p < \infty$). Using this, one deduced the weighted integral Hardy inequality on homogeneous groups, hyperbolic spaces and Cartan–Hadamard manifolds. In particular, one proved the following theorem [22] which will be useful to establish results of the present paper.

Theorem 1.3. *Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q, and let $1 < p \leq q < \infty$. Suppose that u and v are two weight functions on $\mathbb{G}$.*

Then the inequality

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{B}(0,|x|)} f(y)\,dy\right)^{q} u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{G}} f^{p}(x)\,v(x)\,dx\right)^{\frac{1}{p}} \tag{1.4}$$

holds for all non-negative functions f on $\mathbb{G}$ if and only if

$$\boldsymbol{A}_Q := \sup_{x\in\mathbb{G}}\left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\,dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{B}(0,|x|)} v^{\frac{1}{1-p}}(y)\,dy\right)^{\frac{p-1}{p}} < \infty,$$

and the best constant C in (1.4) *can be estimated in the following way:*

$$\boldsymbol{A}_Q \le C \le \boldsymbol{A}_Q\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} p^{\frac{1}{q}}.$$

Very recently, we have proved a sharp version of Theorem 1.3 in [17]. In fact, we have also calculated the precise value of sharp constants in respective inequalities on homogeneous groups. Using Theorem 1.3, we prove the following result which is one of the main results of this paper.

Theorem 1.4. *Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi norm $|\cdot|$ and let $0 < p \le q < \infty$. Suppose that u and v are two positive weight functions on $\mathbb{G}$. Then, there exists a positive constant C such that, for all positive functions f on $\mathbb{G}$, the following inequality holds*

$$\begin{aligned}&\left(\int_{\mathbb{G}}\left[\exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)}\log f(y)\,dy\right)\right]^{q} u(x)\,dx\right)^{\frac{1}{q}}\\ &\le C\left(\int_{\mathbb{G}} f^{p}(x)\,v(x)\,dx\right)^{\frac{1}{p}},\end{aligned} \tag{1.5}$$

provided that

$$\mathfrak{D}_Q := \sup_{x\in\mathbb{G}} |\mathbb{B}(0,|x|)|^{\frac{1}{q}-\frac{1}{p}} u_1^{\frac{1}{q}}(x)\left[\exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)}\log\frac{1}{v(y)}\,dy\right)\right]^{\frac{1}{p}} < \infty. \tag{1.6}$$

Here u_1 is the spherical average of u, given by

$$u_1(x) := \frac{1}{|\mathfrak{S}|}\int_{\mathfrak{S}} u(|x|\sigma)\,d\sigma, \tag{1.7}$$

where $\mathfrak{S} = \{x \in \mathbb{G}\colon |x| = 1\} \subset \mathbb{G}$ is the unit sphere with respect to the quasi-norm $|\cdot|$ and $|x|\sigma := D_{|x|}(\sigma)$, with $D_{|x|}$ being the dilation on $\mathbb{G}$ by the factor $|x|$. Moreover, the optimal constant C in (1.5) *can be estimated as follows:*

$$0 < C \le \left(\frac{p}{q}\right)^{\frac{1}{q}} e^{\frac{1}{p}}\mathfrak{D}_Q. \tag{1.8}$$

We will also prove a conjugate version (see Theorem 3.2) of Theorem 1.4. Furthermore, we establish some stronger exponential inequalities on the quasi-balls on homogeneous Lie groups (see Theorem 1.5 and Theorem 3.3). In fact, we will prove the following result:

Theorem 1.5. *Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi norm $|\cdot|$. Let $0 < p \le q < \infty$ and $a, b \in \mathbb{R}$. Then for any*

$\epsilon > 0$ and for any arbitrary positive function f on the homogeneous Lie group $\mathbb{G}$, the following inequality

$$\left(\int_{\mathbb{G}}\left[\exp\left(\epsilon|\mathbb{B}(0,|x|)|^{-\epsilon}\int_{\mathbb{B}(0,|x|)}|\mathbb{B}(0,|y|)|^{\epsilon-1}\log f(y)dy\right)\right]^{q}|\mathbb{B}(0,|x|)|^{a}dx\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_{\mathbb{G}}f^{p}(x)|\mathbb{B}(0,|x|)|^{b}\,dx\right)^{\frac{1}{p}} \tag{1.9}$$

holds for a positive finite constant C if and only if

$$p\,(a+1)-q\,(b+1)=0. \tag{1.10}$$

Moreover, the best constant C in (1.9) satisfies

$$\left(\frac{p}{q}\right)^{\frac{1}{q}}\epsilon^{\frac{1}{p}-\frac{1}{q}}\exp\left(\frac{b+1}{\epsilon p}-\frac{1}{p}\right)\leq C\leq\left(\frac{p}{q}\right)^{\frac{1}{q}}\epsilon^{\frac{1}{p}-\frac{1}{q}}\exp\left(\frac{b+1}{\epsilon p}\right). \tag{1.11}$$

For the proof, we follow the method developed in [4, 10] in the (isotropic and abelian) setting of Euclidean spaces. We note that also in the abelian (both isotropic and anisotropic) cases of $\mathbb{R}^n$, our results provide new insights in view of the arbitrariness of the quasi-norm $|\cdot|$ which does not necessarily have to be the Euclidean norm.

Apart from Section 1, this manuscript is divided in two sections. In the next section, we will recall the basics of homogeneous Lie groups and some other useful concepts. The last section is devoted to presenting proofs of the main results of this paper.

Throughout this paper, the symbol $A \asymp B$ means $\exists\, C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$.

2. Preliminaries: Basics on homogeneous Lie groups

In this section, we recall the basics of homogeneous groups. For more details on homogeneous groups as well as several functional inequalities on homogeneous groups, we refer to monographs [7, 8, 21] and references therein.

A Lie group $\mathbb{G}$ (identified with $(\mathbb{R}^N, \circ)$) is called a homogeneous group if it is equipped with the dilation mapping

$$D_\lambda\colon \mathbb{R}^N \to \mathbb{R}^N, \quad \lambda > 0,$$

defined as

$$D_\lambda(x) = (\lambda^{v_1}x_1, \lambda^{v_2}x_2, \ldots, \lambda^{v_N}x_N), \quad v_1, v_2, \ldots, v_N > 0, \tag{2.1}$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$. At times, we will denote the image of $x \in \mathbb{G}$ under D_λ by $\lambda(x)$ or, simply λx. The homogeneous dimension Q of the homogeneous group $\mathbb{G}$ is defined by

$$Q = v_1 + v_2 + \cdots + v_N.$$

It is well known that a homogeneous group is necessarily nilpotent and unimodular. The Haar measure dx on $\mathbb{G}$ is nothing but the Lebesgue measure on $\mathbb{R}^N$.

Let us denote the volume of a measurable set $\omega \subset \mathbb{G}$ by $|\omega|$. Then we have the following consequences: for $\lambda > 0$

$$|D_\lambda(\omega)| = \lambda^Q|\omega| \quad \text{and} \quad \int_{\mathbb{G}} f(\lambda x)\,dx = \lambda^{-Q}\int_{\mathbb{G}} f(x)\,dx. \tag{2.2}$$

A quasi-norm on $\mathbb{G}$ is any continuous non-negative function $|\cdot|\colon \mathbb{G} \to [0,\infty)$ satisfying the following conditions:

(i) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,
(ii) $|\lambda x| = \lambda|x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,
(iii) $|x| = 0 \iff x = 0$.

If $\mathfrak{S} = \{x \in \mathbb{G}\colon |x| = 1\} \subset \mathbb{G}$ is the unit sphere with respect to the quasi-norm $|\cdot|$, then there is a unique Radon measure σ on $\mathfrak{S}$ such that for all $f \in L^1(\mathbb{G})$, we have the following polar decomposition (see [8, Proposition 1.15])

$$\int_{\mathbb{G}} f(x)\,dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1}\,d\sigma(y)\,dr. \tag{2.3}$$

We also note that Balogh and Tyson [2] refine the polar decomposition (2.3) by replacing the curves $\gamma(r,y) := D_r(y) = ry$, $r > 0$, with a family of horizontal curves $\varphi(s,\cdot)\colon (0,\infty) \to \mathbb{G}$ in a certain class of Carnot groups. As mentioned in [2], this refinement has several potential applications, but the decomposition (2.3) will be sufficient for our purposes. We also note that the question of the existence of polar decompositions is interesting in general metric measure spaces, and we can refer to [1] to a recent discussion of this topic.

Here we fix some notation which be used in the sequel. The letters u and v will be always used to denote the weights on homogeneous groups $\mathbb{G}$. A quasi-ball in the homogeneous group $\mathbb{G}$ with radius $|x|$, $x \in \mathbb{G}$, and centred at the origin will be denoted by $\mathbb{B}(0,|x|)$. We denote the (Radon) measure of the unit sphere $\mathfrak{S}$ in $\mathbb{G}$ by $|\mathfrak{S}|$. The Haar measure of the unit quasi-ball $\mathbb{B}(0,|x|)$, denoted by $|\mathbb{B}(0,|x|)|$, can be calculated by using (2.3) as

$$\begin{aligned} |\mathbb{B}(0,|x|)| &= \int_{\mathbb{B}(0,|x|)} dy = \int_0^{|x|} r^{Q-1}\left(\int_{\mathfrak{S}} d\sigma\right) dr \\ &= \int_{\mathfrak{S}} \left(\int_0^{|x|} r^{Q-1}\,dr\right) d\sigma = \frac{|x|^Q|\mathfrak{S}|}{Q}. \end{aligned} \tag{2.4}$$

For a given function u on $\mathbb{G}$, the spherical average u_1 of u is defined by

$$u_1(x) := \frac{1}{|\mathfrak{S}|}\int_{\mathfrak{S}} u(|x|\sigma)\,d\sigma, \tag{2.5}$$

where $\mathfrak{S} = \{x \in \mathbb{G}\colon |x| = 1\} \subset \mathbb{G}$ is the unit sphere with respect to the quasi-norm $|\cdot|$.

3. Main results

In this section, we prove the weighted Levin–Cochran–Lee type inequalities on a homogeneous Lie group equipped with a quasi-norm for arbitrary weights. We will derive sharp weighted inequalities on quasi-balls in homogeneous (Lie) groups involving specific weights and also calculate the sharp constant for these inequalities.

Proof of Theoerem 1.4:. We begin with the proof by rewriting Theorem 1.3 by replacing $\frac{p}{\alpha}$, $\frac{q}{\alpha}$, $u(x)|\mathbb{B}(0,|x|)|^{\frac{-q}{\alpha}}$ and f^α in the places of p, q, $u(x)$ and f, respectively, where $0 < \alpha < p$. Indeed, we get the following inequality

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{B}(0,|x|)} f^\alpha(y)\,dy\right)^{\frac{q}{\alpha}} u(x)|\mathbb{B}(0,|x|)|^{\frac{-q}{\alpha}}\,dx\right)^{\frac{\alpha}{q}} \le C_\alpha \left(\int_{\mathbb{G}} f^{\alpha\frac{p}{\alpha}}(x)\,v(x)\,dx\right)^{\frac{\alpha}{p}},$$

which in turn implies that

$$\left(\int_{\mathbb{G}}\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)}f^{\alpha}(y)\,dy\right)^{\frac{q}{\alpha}}u(x)\,dx\right)^{\frac{1}{q}}\leq C_{\alpha}^{\frac{1}{\alpha}}\left(\int_{\mathbb{G}}f^{p}(x)\,v(x)\,dx\right)^{\frac{1}{p}},\tag{3.1}$$

holds for all non-negative functions $f\in\mathbb{G}$ if

$$\mathbf{A}_{Q,\alpha}=\sup_{x\in\mathbb{G}}\left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)}u(y)\,|\mathbb{B}(0,|y|)|^{-\frac{q}{\alpha}}\,dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{B}(0,|x|)}v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}<\infty,$$

and the constant C_{α} satisfies the following estimate:

$$C_{\alpha}^{\frac{1}{\alpha}}\leq\mathbf{A}_{Q,\alpha}\left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}}.\left(\frac{p}{\alpha}\right)^{\frac{1}{q}}.\tag{3.2}$$

We note that

$$\mathbf{A}_{Q,\alpha}\left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}}.\left(\frac{p}{\alpha}\right)^{\frac{1}{q}}=A_{Q,\alpha}\left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}}\times p^{\frac{1}{q}},$$

with

$$A_{Q,\alpha}=\sup_{x\in\mathbb{G}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)}u(y)|\mathbb{B}(0,|x|)|^{\frac{-q}{\alpha}}\,dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{B}(0,|x|)}v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}.\tag{3.3}$$

Therefore, from (3.2), we have

$$C_{\alpha}^{\frac{1}{\alpha}}\leq A_{Q,\alpha}\left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}}\times p^{\frac{1}{q}}.\tag{3.4}$$

Since

$$\lim_{\alpha\to0^{+}}\left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}}=e^{\frac{1}{p}},\tag{3.5}$$

using (3.5) in (3.4), we get

$$C\leq A_{Q}\,p^{\frac{1}{q}}\,e^{\frac{1}{p}},\tag{3.6}$$

where

$$A_{Q}:=\lim_{\alpha\to0^{+}}A_{Q,\alpha}\quad\text{and}\quad C:=\lim_{\alpha\to0^{+}}C_{\alpha}^{\frac{1}{\alpha}}.$$

Recall that by (2.4) we have,

$$|\mathbb{B}(0,|y|)|:=\int_{\mathbb{B}(0,|y|)}dx=\int_{\mathfrak{S}}\left(\int_{0}^{|y|}r^{Q-1}dr\right)d\sigma=\frac{|y|^{Q}|\mathfrak{S}|}{Q}.$$

Now, let us calculate the first integral from (3.3). We get

$$\left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\,|\mathbb{B}(0,|y|)|^{\frac{-q}{\alpha}}\,dy\right)^{\frac{1}{q}} = \left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|y|^Q}{Q}|\mathfrak{S}|\right)^{\frac{-q}{\alpha}}dy\right)^{\frac{1}{q}}$$

$$= \left(\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{-q}{\alpha}}\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\frac{1}{|y|^{\frac{Qq}{\alpha}}}\,dy\right)^{\frac{1}{q}}$$

$$= \left(\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{-q}{\alpha}}\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|x|}{|y|}\right)^{\frac{Qq}{\alpha}}|x|^{\frac{-Qq}{\alpha}}\,dy\right)^{\frac{1}{q}}$$

$$= \left(\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{-q}{\alpha}}\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|x|}{|y|}\right)^{\frac{Qq}{\alpha}}|x|^{Q}\,|x|^{-Q}\,|x|^{\frac{-Qq}{\alpha}}\,dy\right)^{\frac{1}{q}}$$

$$= \left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{-1}{\alpha}}|x|^{\frac{-Q}{\alpha}}\,|x|^{\frac{Q}{q}}\left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|x|}{|y|}\right)^{\frac{Qq}{\alpha}}|x|^{-Q}\,dy\right)^{\frac{1}{q}}. \tag{3.7}$$

Also, calculating the second integral of (3.3), we get that

$$\left(\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}} = \left(\frac{|\mathbb{B}(0,|x|)|}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}$$

$$= |\mathbb{B}(0,|x|)|^{\frac{p-\alpha}{\alpha p}}\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}$$

$$= \left(\frac{|x|^Q}{Q}|\mathfrak{S}|\right)^{\frac{p-\alpha}{\alpha p}}\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}$$

$$= \left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{p-\alpha}{\alpha p}}|x|^{Q\left(\frac{p-\alpha}{\alpha p}\right)}\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}. \tag{3.8}$$

Next, substituting the values from (3.7) and (3.8) in (3.3), we obtain

$$A_{Q,\alpha} = \sup_{x\in\mathbb{G}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{-1}{\alpha}}\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{p-\alpha}{\alpha p}}|x|^{\frac{-Q}{\alpha}}|x|^{\frac{Q}{q}}|x|^{Q\left(\frac{p-\alpha}{\alpha p}\right)}$$
$$\cdot\left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|x|}{|y|}\right)^{\frac{Qq}{\alpha}}|x|^{-Q}\,dy\right)^{\frac{1}{q}}\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}.$$

Thus we find,

$$A_{Q,\alpha} = \sup_{x\in\mathbb{G}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}|x|^{\frac{Q}{q}-\frac{Q}{p}}\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{-1}{p}}$$
$$\cdot\left(\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|x|}{|y|}\right)^{\frac{Qq}{\alpha}}|x|^{-Q}dy\right)^{\frac{1}{q}}\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}}. \tag{3.9}$$

We set

$$I_\alpha(x) := \frac{1}{\alpha}\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|x|}{|y|}\right)^{\frac{Qq}{\alpha}}|x|^{-Q}\,dy.$$

Now, performing the variable transformation $y = |x|z$ in $I_\alpha(x)$ and then using the polar decomposition by setting $z = r\omega$, we get

$$I_\alpha(x) = \frac{1}{\alpha}\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)} u(y)\left(\frac{|x|}{|y|}\right)^{\frac{Qq}{\alpha}} |x|^{-Q}\,dy = \frac{1}{\alpha}\int_{\mathbb{G}\setminus\mathbb{B}(0,1)} u(|x|z)\,|z|^{\frac{-Qq}{\alpha}}\,dz$$

$$= \frac{1}{\alpha}\int_{\mathfrak{S}}\int_1^\infty u\left(|x|r\omega\right) r^{Q-\frac{Qq}{\alpha}-1}\,dr\,d\omega. \tag{3.10}$$

We observe that

$$\chi_{(1,\infty)}(r)\,Q\left(\frac{q}{\alpha}-1\right) r^{Q-\frac{Qq}{\alpha}-1} \longrightarrow \delta_1(r) \quad \text{as} \quad \alpha \to 0^+,$$

where $\delta_1(r)$ is the Dirac delta function at $r = 1$. Indeed, by choosing a test function $\phi \in C_c^\infty(\mathbb{R})$ we see that

$$\lim_{\alpha\to 0^+}\langle \chi_{(1,\infty)}(r)\,Q\left(\frac{q}{\alpha}-1\right) r^{Q-\frac{Qq}{\alpha}-1}, \phi\rangle = \lim_{\alpha\to 0^+}\int_1^\infty Q\left(\frac{q}{\alpha}-1\right) r^{Q-\frac{Qq}{\alpha}-1}\phi(r)\,dr$$
$$= -\lim_{\alpha\to 0^+}\left(r^{Q-\frac{Qq}{\alpha}}\phi(r)|_{r=1}^{\infty}\right) + \lim_{\alpha\to 0^+}\int_1^\infty r^{Q-\frac{Qq}{\alpha}}\phi'(r)\,dr = \phi(1) = \langle \delta_1, \phi\rangle.$$

This implies that

$$\lim_{\alpha\to 0^+}\chi_{(1,\infty)}(r)\,Q\left(\frac{q}{\alpha}-1\right) r^{Q-\frac{Qq}{\alpha}-1} = \delta_1(r).$$

Indeed, for $r \in (1,\infty)$ and for $\alpha \to 0^+$, we have $\lim_{\alpha\to 0^+} r^{Q-\frac{Qq}{\alpha}} = 0$, which shows that

$$\lim_{\alpha\to 0^+}\chi_{(1,\infty)}(r)\,\frac{1}{\alpha}\,r^{Q-\frac{Qq}{\alpha}-1} = \frac{\delta_1(r)}{Qq}.$$

Thus, from (3.10), we have

$$I_\alpha(x) \to \frac{1}{Qq}\int_{\mathfrak{S}} u\left(|x|\omega\right) d\omega = \frac{1}{Qq}\,|\mathfrak{S}|\,u_1(x) \quad \text{as} \quad \alpha \to 0^+. \tag{3.11}$$

A simple calculation gives, as $\beta \to 0^+$, that

$$\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} f^\beta(y)\,dy\right)^{\frac{1}{\beta}} \to \exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} \log\, f(y)\,dy\right). \tag{3.12}$$

Next, using (3.12) in the second integral of (3.9), we have

$$\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} v^{\frac{\alpha}{\alpha-p}}(y)\,dy\right)^{\frac{p-\alpha}{\alpha p}} \longrightarrow \exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} \log\frac{1}{v(y)}\,dy\right)^{\frac{1}{p}} \tag{3.13}$$

as $\alpha \to 0^+$. Substituting (3.11) and (3.13) in (3.9) as $\alpha \to 0^+$, we have

$$\begin{aligned}
A_Q &= \lim_{\alpha\to 0^+} A_{Q,\alpha} \\
&= \sup_{x\in\mathbb{G}} |x|^{\frac{Q}{q}-\frac{Q}{p}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{-1}{p}} \left(\frac{1}{Qq}|\mathfrak{S}|u_1(x)\right)^{\frac{1}{q}} \\
&\quad \cdot \left(\exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} \log\frac{1}{v(y)}\,dy\right)\right)^{\frac{1}{p}} \\
&= \sup_{x\in\mathbb{G}} q^{\frac{-1}{q}} |x|^{\frac{Q}{q}-\frac{Q}{p}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{1}{q}-\frac{1}{p}} u_1^{\frac{1}{q}}(x) \left(\exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} \log\frac{1}{v(y)}\,dy\right)\right)^{\frac{1}{p}} \\
&= \sup_{x\in\mathbb{G}} q^{-\frac{1}{q}} \left(|x|^Q\frac{|\mathfrak{S}|}{Q}\right)^{\frac{1}{q}-\frac{1}{p}} u_1^{\frac{1}{q}}(x) \left(\exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} \log\frac{1}{v(y)}\,dy\right)\right)^{\frac{1}{p}} \\
&= q^{-\frac{1}{q}}\mathfrak{D}_Q, \qquad (3.14)
\end{aligned}$$

which is finite by the hypothesis of the theorem.

Thus, putting the value of A_Q from (3.14) in (3.6), we get

$$0 < C \le \left(\frac{p}{q}\right)^{\frac{1}{q}} \exp\left(\frac{1}{p}\right)\mathfrak{D}_Q, \tag{3.15}$$

which is same as (1.8).

Finally, using (3.12) in (3.1), we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{G}} \left[\exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)} \log f(y)\,dy\right)\right]^q u(x)\,dx\right)^{\frac{1}{q}} \\
&\le C\left(\int_{\mathbb{G}} f^p(x)\,v(x)\,dx\right)^{\frac{1}{p}},
\end{aligned} \tag{3.16}$$

completing the proof of the theorem. □

Proof of the Theorem 1.5. Assume that equality (1.10) holds. Let f be an arbitrary positive function on the homogeneous Lie group $\mathbb{G}$ equipped with a quasi norm $|\cdot|$. To prove (1.9) we first obtain its equivalent form using polar decomposition on $\mathbb{G}$ and then we use Theorem 1.4 to establish it. Indeed, using the polar decomposition on $\mathbb{G}$, by setting $x = r\sigma$ and $y = t\tau$ in (1.9) on $\mathbb{G}$, we get an equivalent inequality of (1.9):

$$\begin{aligned}
&\left(\int_{\mathfrak{S}}\int_0^\infty \left(\frac{|\mathfrak{S}|}{Q}\right)^a r^{Qa+Q-1}\left[\exp\left(\epsilon r^{-Q\epsilon}\left(\frac{|\mathfrak{S}|}{Q}\right)^{-\epsilon}\right.\right.\right. \\
&\quad \left.\left.\left.\cdot \int_{\mathfrak{S}}\int_0^r \left(\frac{|\mathfrak{S}|}{Q}\right)^{\epsilon-1} t^{Q\epsilon-1}\log f(t\tau)\,dt\,d\tau\right)\right]^q dr\,d\sigma\right)^{\frac{1}{q}} \\
&\le C\left(\int_{\mathfrak{S}}\int_0^\infty r^{Qb+Q-1}\left(\frac{|\mathfrak{S}|}{Q}\right)^b f^p(r\sigma)\,dr\,d\sigma\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.17}$$

Next, we do variable transformation $r = r_1^{\frac{1}{\epsilon}}$ and $t = t_1^{\frac{1}{\epsilon}}$ in (3.17) to obtain

$$\begin{aligned} &\left(\int_{\mathfrak{S}}\int_0^\infty \left(\frac{|\mathfrak{S}|}{Q}\right)^a r_1^{Q\left(\frac{a+1}{\epsilon}-1\right)} r_1^{Q-1}\Bigg[\exp\left(\frac{Qq}{|\mathfrak{S}|r_1^Q}\right.\right. \\ &\quad \left.\left.\cdot \int_{\mathfrak{S}}\int_0^{r_1} t_1^{Q-1}\log f\left(t_1^{\frac{1}{\epsilon}}\tau\right)\, dt_1\, d\tau\right)\Bigg]^q \frac{1}{\epsilon}\, dr_1 d\sigma\right)^{\frac{1}{q}} \\ &\le C\left(\int_{\mathfrak{S}}\int_0^\infty \left(\frac{|\mathfrak{S}|}{Q}\right)^b r_1^{Q\left(\frac{b+1}{\epsilon}-1\right)} r_1^{Q-1} f^p(r_1^{\frac{1}{\epsilon}}\sigma)\frac{1}{\epsilon}\, dr_1\, d\sigma\right)^{\frac{1}{p}}. \end{aligned} \tag{3.18}$$

Recalling the volume of $|\mathbb{B}(0,|z|)|$ in $\mathbb{G}$ from (2.4), that is, $|\mathbb{B}(0,|z|)| = \frac{|z|^Q|\mathfrak{S}|}{Q}$ and using this in (3.18), we have

$$\begin{aligned} &\left(\int_{\mathfrak{S}}\int_0^\infty |\mathbb{B}(0,r_1)|^{\left(\frac{a+1}{\epsilon}-1\right)}\right. \\ &\quad \left.\cdot\left[\exp\left(\frac{q}{|\mathbb{B}(0,r_1)|}\int_{\mathfrak{S}}\int_0^{r_1}\log\, F(t_1\tau)t_1^{Q-1}\, dt_1 d\tau\right)\right]^q r_1^{Q-1}\, dr_1\, d\sigma\right)^{\frac{1}{q}} \\ &\le C\epsilon^{\frac{1}{q}-\frac{1}{p}}\left(\frac{|\mathfrak{S}|}{Q}\right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\epsilon}\right)}\left(\int_{\mathfrak{S}}\int_0^\infty |\mathbb{B}(0,r_1)|^{\left(\frac{b+1}{\epsilon}-1\right)}F^p(r_1\sigma)r_1^{Q-1}\, dr_1\, d\sigma\right)^{\frac{1}{p}}, \end{aligned} \tag{3.19}$$

where we have written $F(r\sigma) = f(r^{\frac{1}{\epsilon}}\sigma)$.

Again using the polar decomposition in $\mathbb{G}$ with $t_1\tau = z$ and $r_1\sigma = w$, the inequality (3.19) yields that

$$\begin{aligned} &\left(\int_{\mathbb{G}} |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{\epsilon}-1\right)}\left[\exp\left(\frac{q}{|\mathbb{B}(0,|w|)|}\int_{\mathbb{B}(0,|w|)}\log\, F(z)\, dz\right)\right]^q dw\right)^{\frac{1}{q}} \\ &\le C\epsilon^{\frac{1}{q}-\frac{1}{p}}\left(\frac{|\mathfrak{S}|}{Q}\right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\epsilon}\right)}\left(\int_{\mathbb{G}} |\mathbb{B}(0,|w|)|^{\left(\frac{b+1}{\epsilon}-1\right)}F^p(w)\, dw\right)^{\frac{1}{p}}. \end{aligned} \tag{3.20}$$

Recalling the assumption from (1.10) that, $\frac{b+1}{p} - \frac{a+1}{q} = 0$, we get

$$\begin{aligned} &\left(\int_{\mathbb{G}} |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{\epsilon}-1\right)}\left[\exp\left(\frac{q}{|\mathbb{B}(0,|w|)|}\int_{\mathbb{B}(0,|w|)}\log\, F(z)\, dz\right)\right]^q dw\right)^{\frac{1}{q}} \\ &\le C\epsilon^{\frac{1}{q}-\frac{1}{p}}\left(\int_{\mathbb{G}} |\mathbb{B}(0,|w|)|^{\left(\frac{b+1}{\epsilon}-1\right)}F^p(w)\, dw\right)^{\frac{1}{p}}. \end{aligned} \tag{3.21}$$

Now, we note that the above inequality (3.21) is equivalent to the inequality (1.9). Therefore, to prove (1.9), it is enough to show that $\mathfrak{D}_Q$ (ref. (1.6)) is finite. Thus, we apply Theorem 1.4 with the corresponding weights $u(w) = |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{\epsilon}-1\right)}$ and $v(w) = |\mathbb{B}(0,|w|)|^{\left(\frac{b+1}{\epsilon}-1\right)}$ on $\mathbb{G}$. For this case,

$$\mathfrak{D}_Q = \sup_{w\in\mathbb{G}} |\mathbb{B}(0,|w|)|^{\frac{1}{q}-\frac{1}{p}}\, u_1^{\frac{1}{q}}(w)\left[\exp\left(\frac{1}{|\mathbb{B}(0,|w|)|}\int_{\mathbb{B}(0,|w|)}\log\frac{1}{v(s)}\, ds\right)\right]^{\frac{1}{p}}, \tag{3.22}$$

where $u_1(w) = \frac{1}{|\mathfrak{S}|}\int_{\mathfrak{S}} u(|w|\sigma)\,d\sigma$. Let us now calculate the value of u_1. In fact, we have

$$u_1^{\frac{1}{q}}(w) = \left(\frac{1}{|\mathfrak{S}|}\int_{\mathfrak{S}} u(|w|\sigma)\,d\sigma\right)^{\frac{1}{q}}$$
$$= \left(\frac{1}{|\mathfrak{S}|}\right)^{\frac{1}{q}} \left(\int_{\mathfrak{S}} |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{\epsilon}-1\right)}\,d\sigma\right)^{\frac{1}{q}}$$
$$= \left(\frac{1}{|\mathfrak{S}|}\right)^{\frac{1}{q}} |\mathbb{B}(0,|w|)|^{\frac{1}{q}\left(\frac{a+1}{\epsilon}-1\right)} \left(\int_{\mathfrak{S}} d\sigma\right)^{\frac{1}{q}} = |\mathbb{B}(0,|w|)|^{\frac{1}{q}\left(\frac{a+1}{\epsilon}-1\right)}. \tag{3.23}$$

Now, we calculate the value of the integral in right hand side of (3.22) by using the polar decomposition $s = (t, w)$ with $|s| = t$ as follows:

$$\int_{\mathbb{B}(0,|w|)} \log\frac{1}{v(s)}\,ds = \int_{\mathbb{B}(0,|w|)} \log|\mathbb{B}(0,|s|)|^{-\left(\frac{b+1}{\epsilon}-1\right)}\,ds$$
$$= \int_{\mathfrak{S}}\int_0^{|w|} \left(1-\frac{b+1}{\epsilon}\right) t^{Q-1}\log\left(\frac{|\mathfrak{S}|}{Q}t^Q\right)dt\,dw$$
$$= \left(1-\frac{b+1}{\epsilon}\right)\int_0^{|w|} |\mathfrak{S}|\,t^{Q-1}\log\left(\frac{|\mathfrak{S}|}{Q}t^Q\right)dt. \tag{3.24}$$

Observe that,

$$\int_0^{\frac{|\mathfrak{S}|}{Q}|w|^Q} \log U\,dU = \log U \times U\Big|_0^{\frac{|\mathfrak{S}|}{Q}|w|^Q} - \int_0^{\frac{|\mathfrak{S}|}{Q}|w|^Q} \frac{d}{dU}(\log U)\times U$$
$$= \left(\log\frac{|\mathfrak{S}|}{Q}|w|^Q\right)\left(\frac{|\mathfrak{S}|}{Q}|w|^Q\right) - \left(\frac{|\mathfrak{S}|}{Q}|w|^Q\right). \tag{3.25}$$

Using (3.25) in (3.24) along with the change of variable $\frac{|\mathfrak{S}|}{Q}t^Q = U$ and using $|\mathbb{B}(0,|w|)| = \frac{|w|^Q|\mathfrak{S}|}{Q}$, we get

$$\int_{\mathbb{B}(0,|w|)} \log\frac{1}{v(s)}\,ds = \left(1-\frac{b+1}{\epsilon}\right)\int_0^{\frac{|\mathfrak{S}|}{Q}|w|^Q} \log U dU$$
$$= \left(1-\frac{b+1}{\epsilon}\right)\left[\left(\log\frac{|\mathfrak{S}|}{Q}|w|^Q\right)\left(\frac{|\mathfrak{S}|}{Q}|w|^Q\right) - \left(\frac{|\mathfrak{S}|}{Q}|w|^Q\right)\right]$$
$$= \left(1-\frac{b+1}{\epsilon}\right)\left(\frac{|\mathfrak{S}|}{Q}|w|^Q\right)\left[\log\left(\frac{|\mathfrak{S}|}{Q}|w|^Q\right) - 1\right]$$
$$= |\mathbb{B}(0,|w|)|\left(\log|\mathbb{B}(0,|w|)|^{\left(1-\frac{b+1}{\epsilon}\right)} + \frac{b+1}{\epsilon} - 1\right). \tag{3.26}$$

Substituting the values from (3.23) and (3.26) in (3.22), we obtain

$$
\begin{aligned}
\mathfrak{D}_Q &= \sup_{w\in\mathbb{G}} |\mathbb{B}(0,|w|)|^{\frac{1}{q}-\frac{1}{p}} |\mathbb{B}(0,|w|)|^{\frac{1}{q}\left(\frac{a+1}{\epsilon}-1\right)} \\
&\quad \cdot \left[\exp\left(\frac{1}{|\mathbb{B}(0,|w|)|} |\mathbb{B}(0,|w|)| \left(\log |\mathbb{B}(0,|w|)|^{\left(1-\frac{b+1}{\epsilon}\right)} + \frac{b+1}{\epsilon} - 1\right)\right)\right]^{\frac{1}{p}} \\
&= \sup_{w\in\mathbb{G}} |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{q\epsilon}-\frac{1}{p}\right)} \left[\exp\left(\log |\mathbb{B}(0,|w|)|^{1-\frac{b+1}{\epsilon}} + \frac{b+1}{\epsilon} - 1\right)\right]^{\frac{1}{p}} \\
&= e^{\frac{1}{p}\left(\frac{b+1}{\epsilon}-1\right)} \sup_{w\in\mathbb{G}} |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{q\epsilon}-\frac{1}{p}\right)} |\mathbb{B}(0,|w|)|^{\frac{1}{p}-\frac{b+1}{\epsilon p}},
\end{aligned} \tag{3.27}
$$

which implies that

$$
\mathfrak{D}_Q = \exp\left(\frac{1}{p}\left(\frac{b+1}{\epsilon}-1\right)\right) \sup_{w\in\mathbb{G}} |\mathbb{B}(0,|w|)|^{\frac{1}{\epsilon}\left(\frac{a+1}{q}-\frac{b+1}{p}\right)}. \tag{3.28}
$$

Thus, using the assumption (1.10) in (3.28), we have

$$
\mathfrak{D}_Q = \exp\left(\frac{1}{p}\left(\frac{b+1}{\epsilon}-1\right)\right), \tag{3.29}
$$

which is finite. Therefore, by Theorem 1.4 inequality (1.9) holds for each positive function f defined on $\mathbb{G}$.

Moreover, by (1.8) and using the explicit value of $\mathfrak{D}_Q$ from (3.29), we get

$$
\epsilon^{\frac{1}{q}-\frac{1}{p}} C \leq \left(\frac{p}{q}\right)^{\frac{1}{q}} \exp\left(\frac{b+1}{\epsilon p}\right) \text{ implies that } C \leq \left(\frac{p}{q}\right)^{\frac{1}{q}} \epsilon^{\frac{1}{p}-\frac{1}{q}} \exp\left(\frac{b+1}{\epsilon p}\right). \tag{3.30}
$$

Conversely, we assume that (1.9) holds for all positive functions f on $\mathbb{G}$. Again we will use the equivalent form (3.20) of (1.9). It is clear from (3.20) that the following inequality is true for any ball $\mathbb{B}(0,|x|)$ of radius $x \in \mathbb{G}$:

$$
\begin{aligned}
&\left(\int_{\mathbb{B}(\nvdash,|\frown|)} |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{\epsilon}-1\right)} \left[\exp \frac{1}{|\mathbb{B}(0,|w|)|} \int_{\mathbb{B}(0,|w|)} \log\, F(z)\, dz\right]^q dw\right)^{\frac{1}{q}} \\
&\leq C\epsilon^{\frac{1}{q}-\frac{1}{p}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\epsilon}\right)} \left(\int_{\mathbb{G}} |\mathbb{B}(0,|w|)|^{\left(\frac{b+1}{\epsilon}-1\right)} F^p(w)\, dw\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.31}
$$

So, we will test the inequality (3.31) (and therefore, equivalently, inequality (1.9)) with the test function, for $x \in \mathbb{G}$ as chosen above,

$$
F_x(z) = |\mathbb{B}(0,|z|)|^{\frac{1}{p}\left(1-\frac{b+1}{\epsilon}\right)} \chi_{\mathbb{B}(0,|x|)}, \quad z \in \mathbb{G},
$$

where $\chi_{\mathbb{B}(0,|x|)}$ is the characteristic function of $\mathbb{B}(0,|x|)$ in $\mathbb{G}$. Indeed, we obtain by noting $|z| \leq |w| \leq |x|$ that

$$
\begin{aligned}
&\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\left(\frac{a+1}{\epsilon}-1\right)} \left[\exp \frac{1}{|\mathbb{B}(0,|w|)|} \int_{\mathbb{B}(0,|w|)} \log |\mathbb{B}(0,|z|)|^{\frac{1}{p}\left(1-\frac{b+1}{\epsilon}\right)} dz\right]^q dw\right)^{\frac{1}{q}} \\
&\leq C\epsilon^{\frac{1}{q}-\frac{1}{p}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\epsilon}\right)} |\mathbb{B}(0,|x|)|^{\frac{1}{p}} = \tilde{C}\, |\mathbb{B}(0,|x|)|^{\frac{1}{p}},
\end{aligned} \tag{3.32}
$$

where

$$\tilde{C} = C\epsilon^{\frac{1}{q}-\frac{1}{p}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\epsilon}\right)}. \tag{3.33}$$

Again, using the polar decomposition on $\mathbb{G}$ and doing calculations similar to (3.24), we get

$$\begin{aligned} &\int_{\mathbb{B}(0,|w|)} \log |\mathbb{B}(0,|z|)|^{\frac{1}{p}\left(1-\frac{b+1}{\epsilon}\right)} dz \\ &= |\mathbb{B}(0,|w|)| \left(\log |\mathbb{B}(0,|w|)|^{\frac{1}{p}\left(1-\frac{b+1}{\epsilon}\right)} + \frac{1}{p}\left(\frac{b+1}{\epsilon}-1\right)\right). \end{aligned} \tag{3.34}$$

Using (3.34) in (3.32) we obtain

$$\begin{aligned} &\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\frac{a+1}{\epsilon}-1}\left(\exp \frac{1}{|\mathbb{B}(0,|w|)|}\left\{|\mathbb{B}(0,|w|)|\right.\right.\right. \\ &\left.\left.\left.\cdot\left(\log |\mathbb{B}(0,|w|)|^{\frac{1}{p}\left(1-\frac{b+1}{\epsilon}\right)} + \frac{1}{p}\left(\frac{b+1}{\epsilon}-1\right)\right)\right\}\right)^q dw\right)^{\frac{1}{q}} \le \tilde{C}|\mathbb{B}(0,|x|)|^{\frac{1}{p}}. \end{aligned}$$

This implies that

$$\begin{aligned} &\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\frac{a+1}{\epsilon}-1}\left(|\mathbb{B}(0,|w|)|^{\frac{1}{p}\left(1-\frac{b+1}{\epsilon}\right)} \exp\left(\frac{1}{p}\left(\frac{b+1}{\epsilon}-1\right)\right)\right)^q dw\right)^{\frac{1}{q}} \\ &\le \tilde{C}|\mathbb{B}(0,|x|)|^{\frac{1}{p}}. \end{aligned} \tag{3.35}$$

Therefore, from (3.35), we can rewrite (3.32) in the following form

$$\exp\left\{\frac{b+1}{\epsilon p}-\frac{1}{p}\right\}\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\frac{a+1}{\epsilon}-1+\frac{q}{p}\left(1-\frac{b+1}{\epsilon}\right)} dw\right)^{\frac{1}{q}} \le \tilde{C}\,|\mathbb{B}(0,|x|)|^{\frac{1}{p}},$$

which further can be rewritten as

$$\begin{aligned} &\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\left(\frac{q}{p}\left(\frac{a+1}{\epsilon}\frac{p}{q}-\frac{b+1}{\epsilon}+1\right)-1\right)} dw\right)^{\frac{1}{q}} \\ &\le \tilde{C}\exp\left\{\frac{1}{p}-\frac{b+1}{\epsilon p}\right\}|\mathbb{B}(0,|x|)|^{\frac{1}{p}}. \end{aligned} \tag{3.36}$$

Therefore, the inequality (3.36) implies that

$$\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\frac{q}{p_0}-1} dw\right)^{\frac{1}{q}} \le \tilde{C}\exp\left\{\frac{1}{p}-\frac{b+1}{\epsilon p}\right\}|\mathbb{B}(0,|x|)|^{\frac{1}{p}}, \tag{3.37}$$

where

$$p_0 = \frac{p}{\frac{a+1}{\epsilon}\frac{p}{q}-\frac{b+1}{\epsilon}+1}. \tag{3.38}$$

Now, again using the polar decomposition on $\mathbb{G}$, from (3.37) we obtain

$$\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\frac{q}{p_0}-1}\, dw\right)^{\frac{1}{q}} = \left(\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{q}{p_0}-1} \int_{\mathbb{B}(0,|x|)} |w|^{Q(\frac{q}{p_0}-1)}\, dw\right)^{\frac{1}{q}}$$

$$= \left(\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{q}{p_0}-1} \int_{\mathfrak{S}}\int_0^{|x|} r^{Q-1}\, r^{Q(\frac{q}{p_0}-1)}\, dr\, d\rho\right)^{\frac{1}{q}}$$

$$= \left(\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{q}{p_0}-1} \int_0^{|x|} |\mathfrak{S}|\, r^{Q\frac{q}{p_0}-1}\, dr\right)^{\frac{1}{q}} = \left(\left(\frac{|\mathfrak{S}|}{Q}\right)^{\frac{q}{p_0}-1} |\mathfrak{S}|\, |x|^{Q\frac{q}{p_0}} \left(\frac{p_0}{Qq}\right)\right)^{\frac{1}{q}}$$

$$= \left(\frac{p_0}{q}\right)^{\frac{1}{q}} \left(\frac{|x|^Q\, |\mathfrak{S}|}{Q}\right)^{\frac{1}{p_0}} = |\mathbb{B}(0,|x|)|^{\frac{1}{p_0}} \left(\frac{p_0}{q}\right)^{\frac{1}{q}}.$$

Thus we have

$$\left(\int_{\mathbb{B}(0,|x|)} |\mathbb{B}(0,|w|)|^{\frac{q}{p_0}-1}\, dw\right)^{\frac{1}{q}} = |\mathbb{B}(0,|x|)|^{\frac{1}{p_0}} \left(\frac{p_0}{q}\right)^{\frac{1}{q}}. \tag{3.39}$$

Therefore, using (3.39) in (3.37), we deduce that

$$|\mathbb{B}(0,|x|)|^{\frac{1}{p_0}} \left(\frac{p_0}{q}\right)^{\frac{1}{q}} \leq \tilde{C} \exp\left(\frac{1}{p} - \frac{b+1}{\epsilon p}\right) |\mathbb{B}(0,|x|)|^{\frac{1}{p}},$$

which implies that

$$|\mathbb{B}(0,|x|)|^{\frac{1}{p_0}-\frac{1}{p}} \leq \tilde{C} \exp\left(\frac{1}{p} - \frac{b+1}{\epsilon p}\right) \left(\frac{q}{p_0}\right)^{\frac{1}{q}}. \tag{3.40}$$

Since (3.40) holds for every $x \in \mathbb{G}$, this implies that $p_0 = p$, that is, from (3.38),

$$\frac{a+1}{\epsilon}\frac{p}{q} - \frac{b+1}{\epsilon} + 1 = 1.$$

Hence

$$\frac{a+1}{q} - \frac{b+1}{p} = 0, \tag{3.41}$$

which is same as (1.10).

Again substituting the value of $\tilde{C}$ from (3.33) in (3.40), we estimate that

$$|\mathbb{B}(0,|x|)|^{\frac{1}{p_0}-\frac{1}{p}} \leq \exp\left(\frac{1}{p} - \frac{b+1}{\epsilon p}\right) \left(\frac{q}{p_0}\right)^{\frac{1}{q}} C \epsilon^{\frac{1}{q}-\frac{1}{p}} \left(\frac{|\mathfrak{S}|}{Q}\right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\epsilon}\right)}$$

which with the fact that $p = p_0$ and (3.41) gives that

$$C \geq \left(\frac{p}{q}\right)^{\frac{1}{q}} \exp\left(\frac{b+1}{\epsilon p} - \frac{1}{p}\right) \epsilon^{\frac{1}{p}-\frac{1}{q}}. \tag{3.42}$$

This completes the proof of the theorem. □

Remark 3.1. We observe that applying Theorem 1.5 with $p = q = 1$ yields a more general result than the Čižmešija–Pečarić–Perić estimate on Euclidean space $\mathbb{R}^n$ in Theorem 1.2 (see [4]).

Next, we also state the following result which is conjugate of Theorem 1.4:

Theorem 3.2. *Let u and v be weight functions on a homogeneous Lie group $\mathbb{G}$ of homogeneous dimension Q, equipped with an arbitrary quasi norm $|\cdot|$, and let $0 < p \le q < \infty$ and $\epsilon > 0$. Then, there exist a positive constant C such that, for all positive functions f on $\mathbb{G}$, the following inequality holds*

$$\left(\int_{\mathbb{G}}\left[\exp\left(\epsilon\,|\mathbb{B}(0,|x|)|^{\epsilon}\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)}|\mathbb{B}(0,|x|)|^{-\epsilon-1}\log f(y)\,dy\right)\right]^{q}u(x)\,dx\right)^{\frac{1}{q}}$$

$$\le C\left(\int_{\mathbb{G}} f^{p}(x)\,v(x)\,dx\right)^{\frac{1}{p}}, \tag{3.43}$$

provided that

$$\tilde{\mathfrak{D}}_Q := \sup_{x\in\mathbb{G}} |\mathbb{B}(0,|x|)|^{\frac{1}{q}-\frac{1}{p}}\,\tilde{u}^{\frac{1}{q}}(x)\left[\exp\left(\frac{1}{|\mathbb{B}(0,|x|)|}\int_{\mathbb{B}(0,|x|)}\log\frac{1}{\tilde{v}(y)}\,dy\right)\right]^{\frac{1}{p}} < \infty, \tag{3.44}$$

where $\tilde{u}$ and $\tilde{v}$ are the spherical average of u and v respectively, given by,

$$\tilde{u}(s) = u(s^{-\frac{1}{\epsilon}})\frac{1}{\epsilon}|s|^{-Q\left(1+\frac{1}{\epsilon}\right)},\quad \tilde{v}(s) = v(s^{-\frac{1}{\epsilon}})\frac{1}{\epsilon}|s|^{-Q\left(1+\frac{1}{\epsilon}\right)}.$$

Moreover, the optimal constant C in (3.43) *can estimated as follows:*

$$0 < C \le \left(\frac{p}{q}\right)^{\frac{1}{q}}\exp\left(\frac{1}{p}\right)\tilde{\mathfrak{D}}_Q. \tag{3.45}$$

Proof. Let f be a positive function on the homogeneous group $\mathbb{G}$ with an arbitrary quasi norm $|\cdot|$. Now, using the polar decomposition of (3.43) on the homogeneous group $\mathbb{G}$ similar to (3.17) in Theorem 1.5 and again, making some variable transformations similar to (3.18) and using (2.4), (3.43) can be written as

$$\left(\int_{\mathbb{G}}\left[\exp\left(\frac{1}{|\mathbb{B}(0,|w|)|}\int_{\mathbb{B}(0,|w|)}\log\, f(z^{-\frac{1}{\epsilon}})\,dz\right)\right]^{q}\tilde{u}(w)\,dw\right)^{\frac{1}{q}}$$

$$\le C\left(\int_{\mathbb{G}} f^{p}(w^{-\frac{1}{\epsilon}})\,\tilde{v}(w)\,dw\right)^{\frac{1}{p}}. \tag{3.46}$$

Again, applying Theorem 1.4, using polar decomposition, (2.4) and some variable transformations we get the required result (3.45). □

Next, we present the conjugate of Theorem 1.5 for the power weights $u(x) = |\mathbb{B}(0,|x|)|^{a}$ and $v(x) = |\mathbb{B}(0,|x|)|^{b}$. We have the following inequality.

Theorem 3.3. *Let $\mathbb{G}$ be a homogeneous group with the homogeneous dimension Q equipped with a quasi norm $|\cdot|$. Let $0 < p \le q < \infty$, and let $a, b \in \mathbb{R}$ and $\epsilon > 0$. Then for all positive functions f on $\mathbb{G}$ the inequality*

$$\left(\int_{\mathbb{G}}\left[\exp\left(\epsilon|\mathbb{B}(0,|x|)|^{\epsilon}\int_{\mathbb{G}\setminus\mathbb{B}(0,|x|)}|\mathbb{B}(0,|y|)|^{(-\epsilon-1)}\log f(y)\,dy\right)\right]^{q}|\mathbb{B}(0,|x|)|^{a}\,dx\right)^{\frac{1}{q}}$$

(3.47)

$$\le C\left(\int_{\mathbb{G}} f^{p}(x)|\mathbb{B}(0,|x|)|^{b}\,dx\right)^{\frac{1}{p}}$$

holds for some finite constant C, if and only if

$$p\,(a+1) - q\,(b+1) = 0.$$

Moreover, the least possible constant C such that (3.47) *holds can be estimated as follows:*

$$\left(\frac{p}{q}\right)^{\frac{1}{q}} \epsilon^{\frac{1}{p}-\frac{1}{q}} \exp\left(-\left(\frac{b+1}{\epsilon p}\right)-\frac{1}{p}\right) \le C \le \left(\frac{p}{q}\right)^{\frac{1}{q}} \epsilon^{\frac{1}{p}-\frac{1}{q}} \exp\left(-\left(\frac{b+1}{\epsilon p}\right)\right).$$

Proof. The proof of this theorem follows exactly same lines as the proof of Theorem 1.5 and therefore, we omit the proof. □

Remark 3.4. If we take $p = q$ and so that $a = b$, then the inequality (3.47) holds with the constant $C = \exp\big(-\frac{b+1}{\epsilon p}\big)$. By using the test function in inequality (3.47)

$$f_\delta(x) = \begin{cases} \exp\big(\frac{b+1}{\epsilon p}\big)|\mathbb{B}(0,1)|^{-(b+1)}|x|^{-\frac{Q}{p}(b+1-\epsilon\delta)}, & x \in \mathbb{B}(0,1), \\ \exp\big(\frac{b+1}{\epsilon p}\big)|\mathbb{B}(0,1)|^{-(b+1)}|x|^{-\frac{Q}{p}(b+1+\epsilon\delta)}, & x \in \mathbb{G}\backslash\mathbb{B}(0,1), \end{cases}$$

and suppose that $\delta \to 0^+$, it can be shown that the constant $C = \exp\big(-\frac{b+1}{\epsilon p}\big)$ is sharp.

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Michael Ruzhansky
Ghent University
Department of Mathematics:
Analysis, Logic and Discrete Mathematics
Krijgslaan 281, B 9000 Gent, Belgium
and Queen Mary University of London
School of Mathematical Sciences
Mile End Road, London E1 4NS, U.K.
michael.ruzhansky@ugent.be

Anjali Shriwastawa
Banaras Hindu University, DST-Centre for
Interdisciplinary Mathematical Sciences
Varanasi-221005, India
anjalisrivastava7077@gmail.com

Bankteshwar Tiwari
Banaras Hindu University, DST-Centre for
Interdisciplinary Mathematical Sciences
Varanasi-221005, India
banktesht@gmail.com