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A variant of inverse mean curvature flow for star-shaped hypersurfaces evolving in a cone

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Abstract. Given a smooth convex cone in the Euclidean $(n+1)$-space $(n \geq 2)$, we consider strictly mean convex hypersurfaces with boundary which are star-shaped with respect to the center of the cone and which meet the cone perpendicularly. If those hypersurfaces inside the cone evolve by a variant of inverse mean curvature flow, then, by using the convexity of the cone in the derivation of the gradient and Hölder estimates, we can prove that this evolution exists for all the time and the evolving hypersurfaces converge smoothly to a piece of a round sphere as time tends to infinity.

Tähtimäisten hyperpintojen aikakehitys kartiossa muunnetun käänteisen keskikaarevuusvirtauksen suhteen

Tiivistelmä. Olkoon annettu euklidisen $(n+1)$-avaruuden $(n \geq 2)$ sileä, kupera kartio. Tässä työssä tarkastellaan aidosti keskikuperia reunallisia hyperpintoja, jotka ovat tähtimäisiä kartion keskipisteen suhteen ja kohtaavat kartion kohtisuorasti. Jos kartion sisälle jäävät hyperpinnat kehittyvät muunnetun käänteisen keskikaarevuusvirtauksen mukaisesti, voidaan kartion kuperuutta gradientti- ja Hölderin arvioiden johtamisessa käyttäen todistaa, että tämä kehitys on määritelty kaikilla ajanhetkillä, ja hyperpinnat suppenevat sileästi kohti pyöreän pallopinnan osaa, kun aika lähestyy ääretöntä.

1. Introduction

Recently, Chen, Mao, Tu and Wu [2] considered the evolution of a one-parameter family of closed, star-shaped and strictly mean convex hypersurfaces M_t^n, given by $X(\cdot,t)\colon \mathbb{S}^n \times [0,T) \to \mathbb{R}^{n+1}$ with some $T<\infty$, under the flow

$$\begin{cases} \dfrac{\partial}{\partial t}X = \dfrac{1}{|X|^\alpha H(X)}\nu, \\ X(\cdot,0) = M_0^n, \end{cases} \tag{1.1}$$

where ν is the unit outward normal vector of M_t^n, H is the mean curvature of M_t^n, and $|X|$ is the distance from the point $X(x,t)$ to the origin of $\mathbb{R}^{n+1}$. For $\alpha \geq 0$, they showed the long-time existence and the asymptotical behavior of the flow (1.1). Clearly, when $\alpha = 0$, the flow (1.1) degenerates into the classical inverse mean curvature flow (IMCF for short), and therefore Gerhardt's or Urbas's classical result for the IMCF in $\mathbb{R}^{n+1}$ (see [5, 20]) is covered by the main conclusion of [2] as a special case. There might exist some interesting variants of the classical IMCF by using other constraint terms (not like $|X|^\alpha$) added to the evolution equation of IMCF, and moreover, the asymptotic behavior of those variant flows can be investigated—see, e.g., [17].

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As we know, the classical IMCF is scale invariant. However, generally, the flow (1.1) is non-scale-invariant except the case $\alpha = 0$. The meaning of studying the non-scale-invariant version of inverse curvature flows (ICFs for short) have been revealed clearly by Gerhardt [7], where he investigated the non-scale-invariant version of the classical ICF considered by himself in [5]. This improvement from the scale invariant case to the non-scale-invariant case permits that Gerhardt's main conclusion in [7] covers some interesting conclusions in [12, 18] for the inverse Gauss curvature flow (IGCF for short) and the power of the IGCF. Based on this reason, it also should be interesting to investigate properties of the non-scale-invariant flow (1.1) in different settings—for this purpose, please see the series work [2, 8] of Mao and his collaborators. The flow (1.1) is an initial value problem of second-order parabolic PDEs. *Could we consider the case of boundary value problems?* This motivation forces us to consider the evolution of hypersurfaces with boundary under the ICFs considered in [2].

Marquardt [16] considered the classical IMCF with a Neumann boundary condition (NBC for short), where the embedded flowing hypersurfaces were supposed to be perpendicular to a smooth convex cone in $\mathbb{R}^{n+1}$. He proved that the flow exists for all the time and after rescaling, the evolving hypersurfaces converge smoothly to a piece of a round sphere. Later, Lambert and Scheuer [11] extended this interesting conclusion to the situation that the hypersurfaces are perpendicular to the prescribed sphere. In 2017, Chen, Mao, Xiang and Xu [3] improved Marquardt's main conclusion above to the case that the ambient space is the warped product $I \times_{\lambda(r)} N^n$, where $I \subseteq \mathbb{R}$ is an unbounded interval of $\mathbb{R}$, N^n is an n-dimensional Riemannian manifold with nonnegative Ricci curvature, and the warping function $\lambda(r)$ satisfies some growth assumptions. Inspired by these works, it should be interesting to consider the flow (1.1) with a prescribed NBC. In fact, we can prove the following:

Theorem 1.1. *Let $\alpha > 0$ and $\Sigma^n := \{rx \in \mathbb{R}^{n+1} \mid r > 0,\ x \in \partial M^n\}$ be the boundary of a smooth, convex cone that is centered at the origin and has outward unit normal μ, where $M^n \subset \mathbb{S}^n$ is some piece of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Let $X_0\colon M^n \to \mathbb{R}^{n+1}$ such that $M_0^n := X_0(M^n)$ is a compact, strictly mean convex $C^{2,\gamma}$-hypersurface ($0 < \gamma < 1$) which is a graph over M^n for a positive function $u_0\colon M^n \to \mathbb{R}$, i.e., $M_0^n = \mathrm{graph}_{M^n} u_0$. Assume that*

$$\partial M_0^n \subset \Sigma^n, \quad \langle \mu \circ X_0, \nu_0 \circ X_0 \rangle|_{\partial M^n} = 0,$$

where ν_0 is the outward unit normal to M_0^n. Then

(i) *there exists a family of strictly mean convex hypersurfaces M_t^n given by the unique embedding*

$$X \in C^{2+\gamma,1+\frac{\gamma}{2}}(M^n \times [0,\infty), \mathbb{R}^{n+1}) \cap C^\infty(M^n \times (0,\infty), \mathbb{R}^{n+1})$$

with $X(\partial M^n, t) \subset \Sigma^n$ for $t \geq 0$, satisfying the following system

$$\begin{cases} \dfrac{\partial}{\partial t} X = \dfrac{1}{|X|^\alpha H} \nu & \text{in } M^n \times (0,\infty), \\ \langle \mu \circ X, \nu \circ X \rangle = 0 & \text{on } \partial M^n \times (0,\infty), \\ X(\cdot, 0) = M_0^n & \text{in } M^n, \end{cases} \tag{1.2}$$

where H is the mean curvature of $M_t^n := X(M^n, t) = X_t(M^n)$, ν is the unit outward normal vector of M_t^n, and $|X|$ is the distance from the point $X(x,t)$ to the origin. Moreover, the Hölder norm on the parabolic space $M^n \times (0,\infty)$ is defined in the usual way (see, e.g., [6, Note 2.5.4]*).*

(ii) *the hypersurfaces* M_t^n *are graphs over* M^n*, i.e.,*

$$M_t^n = \mathrm{graph}_{M^n} u(\cdot, t).$$

(iii) *Moreover, the evolving hypersurfaces converge smoothly after rescaling to a piece of a round sphere of radius* r_∞*, where* r_∞ *satisfies*

$$\frac{1}{\sup\limits_{M^n} u_0}\left(\frac{\mathcal{H}^n(M_0^n)}{\mathcal{H}^n(M^n)}\right)^{\frac{1}{n}} \leq r_\infty \leq \frac{1}{\inf\limits_{M^n} u_0}\left(\frac{\mathcal{H}^n(M_0^n)}{\mathcal{H}^n(M^n)}\right)^{\frac{1}{n}},$$

where $\mathcal{H}^n(\cdot)$ *stands for the* n*-dimensional Hausdorff measure of an* n*-manifold.*

Remark 1.1. In order to avoid any potential confusion with the mean curvature H, we use $C^{m+2+\gamma,\frac{m+2+\gamma}{2}}$ not $H^{m+2+\gamma,\frac{m+2+\gamma}{2}}$ used in [7] to represent the parabolic Hölder norm. It is easy to check that all the arguments in the sequel are still valid for the case $\alpha = 0$ except some minor changes should be made. For instance, if $\alpha = 0$, then (3.1) becomes $\phi(x,t) = \frac{1}{n}t + c$. However, in this setting, one can also get the C^0 estimate as well. Clearly, when $\alpha = 0$, the flow (1.2) degenerates into the parabolic system with the vanishing NBC in [16, Theorem 1], and correspondingly, our main conclusion here covers [16, Theorem 1] as a special case.

2. The corresponding scalar equation

2.1. The geometry of graphic hypersurfaces. For an n-dimensional Riemannian manifold (M^n, g), the Riemann curvature (3,1)-tensor Rm is defined by

$$Rm(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z.$$

Pick a local coordinate chart $\{x^i\}_{i=1}^n$ of M^n, and $\frac{\partial}{\partial x^i}, i = 1, 2, \cdots, n$, are the corresponding coordinate vector fields (∂_i for short). The component of the (3,1)-tensor Rm is defined by

$$Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} \doteq R_{ijk}^l \frac{\partial}{\partial x^l}$$

and $R_{ijkl} := g_{lm} R_{ijk}^m$. Then, we have the standard commutation formulas (the Ricci identities):

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\alpha_{k_1\cdots k_r} = \sum_{l=1}^r R_{ijk_l}^m \alpha_{k_1\cdots k_{l-1}mk_{l+1}\cdots k_r}.$$

If furthermore (M^n, g) is an immersed hypersurface in $\mathbb{R}^{n+1}$ with R_{ijkl} the Riemannian curvature of M^n, and let ν be the unit outward normal vector of M^n, then the second fundamental form h_{ij} of the hypersurface M^n with respect to ν can be computed as follows

$$h_{ij} = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle_{\mathbb{R}^{n+1}}.$$

Set $X_{,ij} := \partial_i\partial_j X - \Gamma_{ij}^k X_k$, where Γ_{ij}^k is the Christoffel symbol of the metric g on M^n. We need the following identities

$$X_{,ij} = -h_{ij}\nu, \quad \text{Gauss formula} \tag{2.1}$$

$$\nu_{,i} = h_{ij}X^j, \quad \text{Weingarten formula} \tag{2.2}$$

where $X^j := X_k g^{kj}$,

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \quad \text{Gauss equation} \tag{2.3}$$

$$\nabla_k h_{ij} = \nabla_j h_{ik}, \quad \text{Codazzi equation.} \tag{2.4}$$

We make an agreement that, for simplicity, in the sequel the comma "," in subscripts will be omitted unless necessary. Then, using the Codazzi equation we get

$$\nabla_i\nabla_j h_{kl} = \nabla_i(\nabla_j h_{lk}) = \nabla_i(\nabla_k h_{lj}) = \nabla_i\nabla_k h_{lj}.$$

Using the Ricci identities we have

$$\nabla_i\nabla_j h_{kl} = \nabla_k\nabla_i h_{lj} + R_{iklm}h_j^m + R_{ikjm}h_l^m.$$

Using the Codazzi equation again, it follows that

$$\begin{aligned}\nabla_i\nabla_j h_{kl} &= \nabla_k(\nabla_l h_{ji}) + R_{iklm}h_j^m + R_{ikjm}h_l^m \\ &= \nabla_k\nabla_l h_{ji} + R_{iklm}h_j^m + R_{ikjm}h_l^m.\end{aligned}$$

Using the Gauss equation, we have

$$\nabla_i\nabla_j h_{kl} = \nabla_k\nabla_l h_{ij} + h_j^m(h_{il}h_{km} - h_{im}h_{kl}) + h_l^m(h_{ij}h_{km} - h_{im}h_{kj}). \tag{2.5}$$

2.2. The corresponding scalar equation. In coordinates on the sphere $\mathbb{S}^n$, we equivalently formulate the problem by the corresponding scalar equation. Since the initial $C^{2,\gamma}$-hypersurface is star-sharped (which is a direct consequence of the graphical property of M_0^n), there exists a scalar function $u_0 \in C^{2,\gamma}(M^n)$ such that $X_0\colon M^n \to \mathbb{R}^{n+1}$ has the form $x \mapsto u_0(x)\cdot x$. The hypersurface M_t^n given by the embedding

$$X(\cdot,t)\colon M^n \to \mathbb{R}^{n+1}$$

at time t may be represented as a graph over $M^n \subset \mathbb{S}^n$, and then we can make ansatz

$$X(x,t) = u(x,t)\cdot x$$

for some function $u\colon M^n \times [0,T) \to \mathbb{R}$.

Lemma 2.1. *Define $p := X(x,t)$ and assume that a point on $\mathbb{S}^n$ is described by local coordinates $\xi^1,\dots,\xi^n$, that is, $x = x(\xi^1,\dots,\xi^n)$. Let ∂_i be the corresponding coordinate fields on $\mathbb{S}^n$ and $\sigma_{ij} = g_{\mathbb{S}^n}(\partial_i,\partial_j)$ be the metric on $\mathbb{S}^n$. Let $u_i = D_i u$, $u_{ij} = D_jD_i u$, and $u_{ijk} = D_kD_jD_i u$ denote the covariant derivatives of u with respect to the round metric $g_{\mathbb{S}^n}$ and let ∇ be the Levi–Civita connection of M_t^n with respect to the metric $g := u^2 g_{\mathbb{S}^n} + dr^2$ induced from the standard metric of $\mathbb{R}^{n+1}$. Then, the following formulas hold:*

(i) *The tangential vector on M_t^n is*

$$X_i = \partial_i + u_i\partial_r$$

and the corresponding outward unit normal vector is given by

$$\nu = \frac{1}{v}\left(\partial_r - \frac{1}{u^2}u^j\partial_j\right),$$

where $u^j = \sigma^{ij}u_i$, and $v := \sqrt{1+u^{-2}|Du|^2}$ with Du the gradient of u.

(ii) *The induced metric g on M_t^n has the form*

$$g_{ij} = u^2\sigma_{ij} + u_i u_j$$

and its inverse is given by

$$g^{ij} = \frac{1}{u^2}\left(\sigma^{ij} - \frac{u^i u^j}{u^2 v^2}\right).$$

(iii) *The second fundamental form of M_t^n is given by*

$$h_{ij} = \frac{1}{v}\left(-u_{ij} + u\sigma_{ij} + \frac{2}{u}u_i u_j\right).$$

and

$$h_j^i = g^{ik}h_{jk} = \frac{1}{uv}\delta_j^i - \frac{1}{uv}\widetilde{\sigma}^{ik}\varphi_{jk}, \quad \widetilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2},$$

where $\varphi = \log u$. Naturally, the mean curvature is given by

$$H = \sum_{i=1}^{n} h_i^i = \frac{1}{uv}\left(n - (\sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2})\varphi_{ij}\right).$$

(iv) *Let $p \in \Sigma^n$, $\hat{\mu}(p)$ be the normal to Σ^n at p and $\mu = \mu^i(x)\partial_i$ be the normal to ∂M^n at x. Then*

$$\langle \hat{\mu}(p), \nu(p)\rangle = 0 \iff \mu^i(x)u_i(x,t) = 0.$$

Proof. Let $\overline{\nabla}$ be the covariant connection of $\mathbb{R}^{n+1}$. Since

$$h_{ij} = -\langle \overline{\nabla}_{ij}X, \nu\rangle = -\langle \overline{\nabla}_{\partial_i}\partial_j + u_i\overline{\nabla}_{\partial_j}\partial_r + u_j\overline{\nabla}_{\partial_i}\partial_r + u_i u_j\overline{\nabla}_{\partial_r}\partial_r, \nu\rangle,$$

these formulas can be verified by direct calculation. The details can also be found in [1]. □

Using techniques as in Ecker [4] (see also [5, 6, 16]), the problem (1.2) can be reduced to solving the following scalar equation with the corresponding initial data

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{v}{u^\alpha H} & \text{in } M^n \times (0,\infty), \\ D_\mu u = 0 & \text{on } \partial M^n \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } M^n. \end{cases} \tag{2.6}$$

Define a new function $\varphi(x,t) = \log u(x,t)$ and then the mean curvature can be rewritten as

$$H = \sum_{i=1}^{n} h_i^i = \frac{e^{-\varphi}}{v}\left(n - (\sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2})\varphi_{ij}\right).$$

Hence, the evolution equation in (2.6) can be rewritten as

$$\frac{\partial}{\partial t}\varphi = e^{-\alpha\varphi}(1 + |D\varphi|^2)\frac{1}{[n - (\sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2})\varphi_{ij}]} := Q(\varphi, D\varphi, D^2\varphi).$$

In particular,

$$n - \left(\sigma^{ij} - \frac{\varphi_0^i\varphi_0^j}{v^2}\right)\varphi_{0,ij}$$

is positive on M^n, since M_0^n is strictly mean convex. Thus, the problem (1.2) is again reduced to solving the following scalar equation with the NBC

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = Q(\varphi, D\varphi, D^2\varphi) & \text{in } M^n \times (0,T), \\ D_\mu\varphi = 0 & \text{on } \partial M^n \times (0,T), \\ \varphi(\cdot,0) = \varphi_0 & \text{in } M^n, \end{cases} \tag{2.7}$$

where

$$n - \left(\sigma^{ij} - \frac{\varphi_0^i\varphi_0^j}{v^2}\right)\varphi_{0,ij}$$

is positive on M^n. Clearly, for the initial surface M_0^n,

$$\frac{\partial Q}{\partial \varphi_{ij}}\Big|_{\varphi_0} = \frac{1}{u^{2+\alpha}H^2}\left(\sigma^{ij} - \frac{\varphi_0^i\varphi_0^j}{v^2}\right)$$

is positive on M^n. Based on the above facts, as in [5, 6, 16], we can get the following short-time existence and uniqueness for the parabolic system (1.2).

Lemma 2.2. *Let $X_0(M^n) = M_0^n$ be as in Theorem 1.1. Then there exist some $T > 0$, a unique solution $u \in C^{2+\gamma,1+\frac{\gamma}{2}}(M^n \times [0,T]) \cap C^\infty(M^n \times (0,T])$, where $\varphi(x,t) = \log u(x,t)$, to the parabolic system* (2.7) *with the matrix*

$$n - \left(\sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}\right)\varphi_{ij}$$

positive on M^n. Thus there exists a unique map $\psi\colon M^n \times [0,T] \to M^n$ such that $\psi(\partial M^n, t) = \partial M^n$ and the map $\widehat{X}$ defined by

$$\widehat{X}\colon M^n \times [0,T) \to \mathbb{R}^{n+1}\colon (x,t) \mapsto X(\psi(x,t),t)$$

has the same regularity as stated in Theorem 1.1 and is the unique solution to the parabolic system (1.2).

Let T^* be the maximal time such that there exists some

$$u \in C^{2+\gamma,1+\frac{\gamma}{2}}(M^n \times [0,T^*)) \cap C^\infty(M^n \times (0,T^*))$$

which solves (2.7). In the sequel, we shall prove a priori estimates for those solutions on $[0,T]$ where $T < T^*$.

3. C^0, $\dot{\varphi}$ and gradient estimates

Lemma 3.1. (C^0 estimate) *Let φ be a solution of* (2.7)*. Then for $\alpha > 0$, we have*

$$c_1 \le u(x,t)\Theta^{-1}(t,c) \le c_2, \quad \forall\, x \in M^n,\ t \in [0,T],$$

for some positive constants c_1, c_2, where $\Theta(t,c) := \left\{\frac{\alpha t}{n} + e^{\alpha c}\right\}^{\frac{1}{\alpha}}$ with

$$\inf_{M^n} \varphi(\cdot,0) \le c \le \sup_{M^n} \varphi(\cdot,0).$$

Proof. Let $\varphi(x,t) = \varphi(t)$ (independent of x) be the solution of (2.7) with $\varphi(0) = c$. In this case, the first equation in (2.7) reduces to an ODE

$$\frac{d}{dt}\varphi = e^{-\alpha\varphi}\frac{1}{n}.$$

Therefore,

$$\varphi(t)=\frac{1}{\alpha}\ln\left(\frac{\alpha t}{n}+e^{\alpha c}\right),\quad \text{for }\ \alpha>0. \tag{3.1}$$

Using the maximum principle, we can obtain that

$$\frac{1}{\alpha}\ln\left(\frac{\alpha}{n}t+e^{\alpha\varphi_1}\right)\le\varphi(x,t)\le\frac{1}{\alpha}\ln\left(\frac{\alpha t}{n}+e^{\alpha\varphi_2}\right), \tag{3.2}$$

where $\varphi_1:=\inf_{M^n}\varphi(\cdot,0)$ and $\varphi_2:=\sup_{M^n}\varphi(\cdot,0)$. The estimate is obtained since $\varphi=\log u$. □

Lemma 3.2. ($\dot\varphi$ estimate) *Let φ be a solution of* (2.7) *and Σ^n be a smooth, convex cone, then for $\alpha>0$,*

$$\min\left\{\inf_{M^n}\dot\varphi(\cdot,0)\cdot\Theta(0)^\alpha,\frac{1}{n}\right\}\le\dot\varphi(x,t)\Theta(t)^\alpha\le\max\left\{\sup_{M^n}\dot\varphi(\cdot,0)\cdot\Theta(0)^\alpha,\frac{1}{n}\right\}.$$

Proof. Set

$$\mathcal{M}(x,t)=\dot\varphi(x,t)\Theta(t)^\alpha.$$

Differentiating both sides of the first evolution equation of (2.7), it is easy to get that

$$\begin{cases}\dfrac{\partial\mathcal{M}}{\partial t}=Q^{ij}D_{ij}\mathcal{M}+Q^kD_k\mathcal{M}+\alpha\Theta^{-\alpha}\left(\dfrac{1}{n}-\mathcal{M}\right)\mathcal{M} & \text{in } M^n\times(0,T),\\ D_\mu\mathcal{M}=0 & \text{on } \partial M^n\times(0,T),\\ \mathcal{M}(\cdot,0)=\dot\varphi_0\cdot\Theta(0)^\alpha & \text{on } M^n,\end{cases} \tag{3.3}$$

where $Q^{ij}:=\frac{\partial Q}{\partial\varphi_{ij}}$ and $Q^k:=\frac{\partial Q}{\partial\varphi_k}$. Then the result follows from the maximum principle. □

Lemma 3.3. (Gradient estimate) *Let φ be a solution of* (2.7) *and Σ^n be the boundary of a smooth, convex cone described as in Theorem 1.1. Then we have for $\alpha>0$,*

$$|D\varphi|\le\sup_{M^n}|D\varphi(\cdot,0)|,\quad\forall\ x\in M^n,\ t\in[0,T]. \tag{3.4}$$

Proof. Set $\psi=\frac{|D\varphi|^2}{2}$. By differentiating ψ, we have

$$\frac{\partial\psi}{\partial t}=\frac{\partial}{\partial t}\varphi_m\varphi^m=\dot\varphi_m\varphi^m=Q_m\varphi^m.$$

Then using the evolution equation of φ in (2.7) yields

$$\frac{\partial\psi}{\partial t}=Q^{ij}\varphi_{ijm}\varphi^m+Q^k\varphi_{km}\varphi^m-\alpha Q|D\varphi|^2.$$

Interchanging the covariant derivatives, we have

$$\psi_{ij}=D_j(\varphi_{mi}\varphi^m)=\varphi_{mij}\varphi^m+\varphi_{mi}\varphi^m_j=(\varphi_{ijm}-R^l_{jmi}\varphi_l)\varphi^m+\varphi_{mi}\varphi^m_j.$$

Therefore, we can express $\varphi_{ijm}\varphi^m$ as

$$\varphi_{ijm}\varphi^m=\psi_{ij}+R^l_{jmi}\varphi_l\varphi^m-\varphi_{mi}\varphi^m_j.$$

Then, in view of the fact $R_{jmil}=\sigma_{ji}\sigma_{ml}-\sigma_{lj}\sigma_{im}$ on $\mathbb{S}^n$, we have

$$\frac{\partial\psi}{\partial t}=Q^{ij}\psi_{ij}+Q^k\psi_k-Q^{ij}(\sigma_{ij}|D\varphi|^2-\varphi_i\varphi_j)-Q^{ij}\varphi_{mi}\varphi^m_j-\alpha Q|D\varphi|^2. \tag{3.5}$$

Since the matrix Q^{ij} is positive definite, the third and the fourth terms in the RHS of (3.5) are non-positive. Noticing that the last term in the RHS of (3.5) is also non-positive if $\alpha > 0$. Since Σ^n is convex, using a similar argument to the proof of [16, Lemma 5] (see pp. 1308) implies that

$$D_\mu \psi = -\sum_{i,j=1}^{n-1} h_{ij}^{\partial M^n} D_{e_i}\varphi D_{e_j}\varphi \leq 0 \quad \text{on } \partial M^n \times (0,T),$$

where an orthonormal frame at $x \in \partial M^n$, with $e_1, \ldots, e_{n-1} \in T_x\partial M^n$ and $e_n := \mu$, has been chosen for convenience in the calculation, and $h_{ij}^{\partial M^n}$ is the second fundamental form of the boundary $\partial M^n \subset \Sigma^n$. So, we can get

$$\begin{cases} \dfrac{\partial \psi}{\partial t} \leq Q^{ij}\psi_{ij} + Q^k \psi_k & \text{in } M^n \times (0,T), \\ D_\mu \psi \leq 0 & \text{on } \partial M^n \times (0,T), \\ \psi(\cdot, 0) = \dfrac{|D\varphi(\cdot,0)|^2}{2} & \text{in } M^n. \end{cases}$$

Using the maximum principle, we get the gradient estimate of φ in Lemma 3.3. □

Remark 3.1. It is worth pointing out that the evolving surface M_t^n is always star-shaped under the assumption of Theorem 1.1, since, by Lemma 3.3, we have

$$\left\langle \frac{X}{|X|}, \nu \right\rangle = \frac{1}{v}$$

is bounded from below by a positive constant.

Combining the gradient estimate with $\dot{\varphi}$ estimate, we can obtain

Corollary 3.4. *If φ satisfies (2.7), then we have*

$$0 < c_3 \leq H\Theta \leq c_4 < +\infty, \tag{3.6}$$

where c_3 and c_4 are positive constants independent of φ.

Proof. Since $\varphi = \log u$ satisfies (2.7), so we have

$$H\Theta = \frac{v}{\dot{\varphi} \cdot e^{(\alpha+1)\varphi}} \cdot \Theta = \frac{v}{(\dot{\varphi}\cdot\Theta^\alpha)\cdot e^{(\alpha+1)\varphi}\cdot\Theta^{-(\alpha+1)}} = \frac{\sqrt{1+|D\varphi|^2}}{(\dot{\varphi}\cdot\Theta^\alpha)\cdot(u\Theta^{-1})^{\alpha+1}}.$$

So, combining the C^0 estimate, $\dot{\varphi}$ estimate and the gradient estimate, we have (3.6) and c_3, c_4 are positive constants independent of φ. □

4. Hölder estimates and convergence

Set $\Phi = \frac{1}{|X|^\alpha H}$, $w = \langle X, \nu\rangle = \frac{u}{v}$ and $\Psi = \frac{\Phi}{w}$. We can get the following evolution equations.

Lemma 4.1. *Under the assumptions of Theorem 1.1, we have*

$$\frac{\partial}{\partial t} g_{ij} = 2\Phi h_{ij}, \quad \frac{\partial}{\partial t} g^{ij} = -2\Phi h^{ij}, \quad \frac{\partial}{\partial t}\nu = -\nabla\Phi,$$

$$\begin{aligned}\partial_t h_i^j - \Phi H^{-1}\Delta h_i^j = {} & \Phi H^{-1}|A|^2 h_i^j - \frac{2\Phi}{H^2} H_i H^j - 2\Phi h_{ik}h^{kj} \\ & - \alpha\Phi(\nabla_i \log u \nabla^j \log H + \nabla^j \log u \nabla_i \log H) \\ & + \alpha\Phi u^{-1} u_i^j - \alpha(\alpha+1)\Phi\nabla_i \log u \nabla^j \log u\end{aligned}$$

and

$$\frac{\partial \Psi}{\partial t} = \mathrm{div}_g(u^{-\alpha}H^{-2}\nabla\Psi) - 2H^{-2}u^{-\alpha}\Psi^{-1}|\nabla\Psi|^2 \\ - \alpha\Psi^2 - \alpha\Psi^2 u^{-1}\nabla^i u\langle X, X_i\rangle - \alpha u^{-\alpha-1}H^{-2}\nabla_i u\nabla^i\Psi. \tag{4.1}$$

Proof. The first three evolution equations are easy to obtain and so are omitted. Using the Gauss formula, we have

$$\begin{aligned}\partial_t h_{ij} &= \partial_t\langle\partial_i\partial_j X, -\nu\rangle = \langle\partial_i\partial_j(\Phi\nu), -\nu\rangle - \langle\Gamma_{ij}^k\partial_k X - h_{ij}\nu, \partial_t\nu\rangle \\ &= -\partial_i\partial_j\Phi - \Phi\langle\partial_i\partial_j\nu, \nu\rangle + \Gamma_{ij}^k\Phi_k = -\nabla_{ij}^2\Phi - \Phi\langle\partial_i(h_j^k\partial_k X), \nu\rangle \\ &= -\nabla_{ij}^2\Phi + \Phi h_{ik}h_j^k.\end{aligned}$$

Direct calculation results in

$$\begin{aligned}\nabla_{ij}^2\Phi = \Phi\left(-\frac{1}{H}H_{ij} + \frac{2H_iH_j}{H^2}\right) + \alpha\Phi(\nabla_i\log u\nabla_j\log H + \nabla_j\log u\nabla_i\log H) \\ - \alpha\Phi u^{-1}u_{ij} + \alpha(\alpha+1)\Phi\nabla_i\log u\nabla_j\log u.\end{aligned}$$

Since

$$\Delta h_{ij} = H_{ij} + Hh_{ik}h_j^k - h_{ij}|A|^2,$$

so

$$\begin{aligned}\nabla_{ij}^2\Phi = -\Phi H^{-1}\Delta h_{ij} + \Phi h_{ik}h_j^k - \Phi H^{-1}|A|^2h_{ij} + \frac{2H_iH_j}{H^2}\Phi \\ + \alpha\Phi(\nabla_i\log u\nabla_j\log H + \nabla_j\log u\nabla_i\log H) \\ - \alpha\Phi u^{-1}u_{ij} + \alpha(\alpha+1)\Phi\nabla_i\log u\nabla_j\log u.\end{aligned}$$

Thus,

$$\begin{aligned}\partial_t h_{ij} - \Phi H^{-1}\Delta h_{ij} = \Phi H^{-1}|A|^2h_{ij} - \frac{2\Phi}{H^2}H_iH_j \\ - \alpha\Phi(\nabla_i\log u\nabla_j\log H + \nabla_j\log u\nabla_i\log H) \\ + \alpha\Phi u^{-1}u_{ij} - \alpha(\alpha+1)\Phi\nabla_i\log u\nabla_j\log u.\end{aligned}$$

Obviously, the evolution equation of h_i^j can be directly obtained from the fact $h_i^j = g^{jl}h_{li}$, the evolution equation of the second fundamental form h_{li}, and the evolution equation of the metric. By direct calculation, one furthermore has

$$\begin{aligned}\partial_t H &= \partial_t g^{ij}h_{ij} + g^{ij}\partial_t h_{ij} \\ &= -2\Phi h^{ij}h_{ij} + g^{ij}\left(\Phi H^{-1}\Delta h_{ij} + \Phi H^{-1}|A|^2h_{ij} - \frac{2\Phi}{H^2}\nabla_i H\nabla_j H\right) \\ &\quad + \alpha\Phi g^{ij}\big(- \nabla_i\log u\nabla_j\log H - \nabla_j\log u\nabla_i\log H + u^{-1}u_{ij} \\ &\quad - (\alpha+1)\nabla_i\log u\nabla_j\log u\big) \\ &= \Phi H^{-1}\Delta H - \frac{2\Phi}{H^2}|\nabla H|^2 - \Phi|A|^2 + \alpha\Phi g^{ij}\big(- 2\nabla_i\log u\nabla_j\log H + u^{-1}u_{ij} \\ &\quad - (\alpha+1)\nabla_i\log u\nabla_j\log u\big) \\ &= u^{-\alpha}H^{-2}\Delta H - 2u^{-\alpha}H^{-3}|\nabla H|^2 - u^{-\alpha}H^{-1}|A|^2 - 2\alpha u^{-\alpha-1}H^{-2}\nabla_i u\nabla^i H \\ &\quad + \alpha u^{-\alpha-1}H^{-1}\Delta u - \alpha(\alpha+1)u^{-\alpha-2}H^{-1}|\nabla u|^2.\end{aligned}$$

Clearly,

$$\partial_t w = \Phi + \alpha\Phi u^{-1}\nabla^i u\langle X, X_i\rangle + \Phi H^{-1}\nabla^i H\langle X, X_i\rangle,$$

using the Weingarten equation, we have

$$w_i = h_i^k \langle X, X_k\rangle,$$
$$w_{ij} = h_{i,\ j}^k \langle X, X_k\rangle + h_{ij} - h_i^k h_{kj}\langle X, \nu\rangle = h_{ij,k}\langle X, X^k\rangle + h_{ij} - h_i^k h_{kj}\langle X, \nu\rangle.$$

Thus,

$$\Delta w = H + \nabla^i H \langle X, X_i\rangle - |A|^2\langle X, \nu\rangle$$

and

$$\partial_t w = u^{-\alpha}H^{-2}\Delta w + u^{-\alpha}H^{-2}w|A|^2 + \alpha u^{-\alpha-1}H^{-1}\nabla^i u\langle X, X_i\rangle.$$

Hence

$$\begin{aligned}
\frac{\partial \Psi}{\partial t} &= -\alpha \frac{1}{u^{1+\alpha}}\frac{1}{Hw}\dot{u} - \frac{1}{u^\alpha H^2}\frac{1}{w}\partial_t H - \frac{1}{u^\alpha H}\frac{1}{w^2}\partial_t w \\
&= -\alpha \frac{1}{u^{1+\alpha}}\frac{1}{Hw}\frac{1}{u^{\alpha-1}Hw} - \frac{1}{u^\alpha H^2}\frac{1}{w}\partial_t H - \frac{1}{u^\alpha H}\frac{1}{w^2}\partial_t w \\
&= -\alpha u^{-2\alpha}H^{-2}w^{-2} + \alpha(\alpha+1)u^{-2\alpha-2}H^{-3}w^{-1}|\nabla u|^2 + 2u^{-2\alpha}H^{-5}w^{-1}|\nabla H|^2 \\
&\quad + 2\alpha u^{-2\alpha-1}H^{-4}w^{-1}\nabla_i u\nabla^i H - \alpha u^{-2\alpha-1}H^{-3}w^{-1}\Delta u - u^{-2\alpha}H^{-4}w^{-1}\Delta H \\
&\quad - u^{-2\alpha}H^{-3}w^{-2}\Delta w - \alpha u^{-2\alpha-1}H^{-2}w^{-2}\nabla^i u\langle X, X_i\rangle.
\end{aligned}$$

In order to prove (4.1), we calculate

$$\nabla_i \Psi = -\alpha u^{-\alpha-1}H^{-1}w^{-1}\nabla_i u - u^{-\alpha}H^{-2}w^{-1}\nabla_i H - u^{-\alpha}H^{-1}w^{-2}\nabla_i w$$

and

$$\begin{aligned}
\nabla^2_{ij}\Psi &= \alpha(\alpha+1)u^{-\alpha-2}H^{-1}w^{-1}\nabla_i u\nabla_j u + \alpha u^{-\alpha-1}H^{-2}w^{-1}\nabla_i u\nabla_j H \\
&\quad + \alpha u^{-\alpha-1}H^{-1}w^{-2}\nabla_i u\nabla_j w - \alpha u^{-\alpha-1}H^{-1}w^{-1}\nabla^2_{ij}u \\
&\quad + \alpha u^{-\alpha-1}H^{-2}w^{-1}\nabla_i H\nabla_j u + 2u^{-\alpha}H^{-3}w^{-1}\nabla_i H\nabla_j H \\
&\quad + u^{-\alpha}H^{-2}w^{-2}\nabla_i H\nabla_j w - u^{-\alpha}H^{-2}w^{-1}\nabla^2_{ij}H + \alpha u^{-\alpha-1}H^{-1}w^{-2}\nabla_i w\nabla_j u \\
&\quad + u^{-\alpha}H^{-2}w^{-2}\nabla_i w\nabla_j H + 2u^{-\alpha}H^{-1}w^{-3}\nabla_i w\nabla_j w - u^{-\alpha}H^{-1}w^{-2}\nabla^2_{ij}w.
\end{aligned}$$

Thus

$$\begin{aligned}
u^{-\alpha}H^{-2}\Delta\Psi &= \alpha(\alpha+1)u^{-2\alpha-2}H^{-3}w^{-1}|\nabla u|^2 + 2u^{-2\alpha}H^{-5}w^{-1}|\nabla H|^2 \\
&\quad + 2u^{-2\alpha}H^{-3}w^{-3}|\nabla w|^2 + 2\alpha u^{-2\alpha-1}H^{-4}w^{-1}\nabla_i u\nabla^i H \\
&\quad + 2\alpha u^{-2\alpha-1}H^{-3}w^{-2}\nabla_i u\nabla^i w + 2u^{-2\alpha}H^{-4}w^{-2}\nabla_i H\nabla^i w \\
&\quad - \alpha u^{-2\alpha-1}H^{-3}w^{-1}\Delta u - u^{-2\alpha}H^{-4}w^{-1}\Delta H - u^{-2\alpha}H^{-3}w^{-2}\Delta w.
\end{aligned}$$

So we have

$$\begin{aligned}
\operatorname{div}(u^{-\alpha}H^{-2}\nabla\Psi) &= -\alpha u^{-\alpha-1}H^{-2}\nabla_i\Psi\nabla^i u - 2u^{-\alpha}H^{-3}\nabla_i\Psi\nabla^i H + u^{-\alpha}H^{-2}\Delta\Psi \\
&= (2\alpha^2+\alpha)u^{-2\alpha-2}H^{-3}w^{-1}|\nabla u|^2 + 5\alpha u^{-2\alpha-1}H^{-4}w^{-1}\nabla_i u\nabla^i H \\
&\quad + 3\alpha u^{-2\alpha-1}H^{-3}w^{-2}\nabla_i u\nabla^i w + 4u^{-2\alpha}H^{-5}w^{-1}|\nabla H|^2 \\
&\quad + 4u^{-2\alpha}H^{-4}w^{-2}\nabla_i w\nabla^i H + 2u^{-2\alpha}H^{-3}w^{-3}|\nabla w|^2 \\
&\quad - \alpha u^{-2\alpha-1}H^{-3}w^{-1}\Delta u - u^{-2\alpha}H^{-4}w^{-1}\Delta H - u^{-2\alpha}H^{-3}w^{-2}\Delta w
\end{aligned}$$

and

$$\begin{aligned}
2H^{-1}w|\nabla\Psi|^2 &= 2\alpha^2 u^{-2\alpha-2}H^{-3}w^{-1}|\nabla u|^2 + 2u^{-2\alpha}H^{-5}w^{-1}|\nabla H|^2 \\
&\quad + 2u^{-2\alpha}H^{-3}w^{-3}|\nabla w|^2 + 4\alpha u^{-2\alpha-1}H^{-4}w^{-1}\nabla_i u\nabla^i H \\
&\quad + 4\alpha u^{-2\alpha-1}H^{-3}w^{-2}\nabla_i u\nabla^i w + 4u^{-2\alpha}H^{-4}w^{-2}\nabla_i H\nabla^i w.
\end{aligned}$$

As above, we have

$$\begin{aligned}&\frac{\partial\Psi}{\partial t}-\operatorname{div}(u^{-\alpha}H^{-2}\nabla\Psi)+2H^{-1}w|\nabla\Psi|^2\\&=-\alpha u^{-2\alpha}H^{-2}w^{-2}-\alpha u^{-2\alpha-1}H^{-2}w^{-2}\nabla^i u\langle X,X_i\rangle+\alpha^2u^{-2\alpha-2}H^{-3}w^{-1}|\nabla u|^2\\&\quad+\alpha u^{-2\alpha-1}H^{-4}w^{-1}\nabla_i u\nabla^i H+\alpha u^{-2\alpha-1}H^{-3}w^{-2}\nabla_i u\nabla^i w\\&=-\alpha\Psi^2-\alpha\Psi^2u^{-1}\nabla^i u\langle X,X_i\rangle-\alpha u^{-\alpha-1}H^{-2}\nabla_i u\nabla^i\Psi.\end{aligned}$$

The proof is finished. □

Now, we define the rescaled flow by

$$\widetilde{X}=X\Theta^{-1}.$$

Thus,

$$\widetilde{u}=u\Theta^{-1},$$
$$\widetilde{\varphi}=\varphi-\log\Theta,$$

and the rescaled mean curvature is given by

$$\widetilde{H}=H\Theta.$$

Then, the rescaled scalar curvature equation takes the form

$$\frac{\partial}{\partial t}\widetilde{u}=\frac{v}{\widetilde{u}^{\alpha}\widetilde{H}}\Theta^{-\alpha}-\frac{1}{n}\widetilde{u}\Theta^{-\alpha}.$$

Defining $t=t(s)$ by the relation

$$\frac{dt}{ds}=\Theta^{\alpha}$$

such that $t(0)=0$ and $t(S)=T$. Then $\widetilde{u}$ satisfies

$$\begin{cases}\dfrac{\partial}{\partial s}\widetilde{u}=\dfrac{v}{\widetilde{u}^{\alpha}\widetilde{H}}-\dfrac{\widetilde{u}}{n} & \text{in } M^n\times(0,S),\\ D_\mu\widetilde{u}=0 & \text{on } \partial M^n\times(0,S),\\ \widetilde{u}(\cdot,0)=\widetilde{u}_0 & \text{in } M^n.\end{cases}\tag{4.2}$$

Lemma 4.2. *Let X be a solution of (1.2) and $\widetilde{X}=X\Theta^{-1}$ be the rescaled solution. Then*

$$D\widetilde{u}=Du\Theta^{-1},\quad D\widetilde{\varphi}=D\varphi,\quad \frac{\partial\widetilde{u}}{\partial s}=\frac{\partial u}{\partial t}\Theta^{\alpha-1}-\frac{1}{n}u\Theta^{-1},$$
$$\widetilde{g}_{ij}=\Theta^{-2}g_{ij},\quad \widetilde{g}^{ij}=\Theta^{2}g^{ij},\quad \widetilde{h}_{ij}=h_{ij}\Theta^{-1}.$$

Proof. These relations can be computed directly. □

Lemma 4.3. *Let u be a solution to the parabolic system (2.7), where $\varphi(x,t)=\log u(x,t)$, and Σ^n be a smooth, convex cone described as in Theorem 1.1. Then there exist some $\beta>0$ and some $C>0$ such that the rescaled function $\widetilde{u}(x,s):=u(x,t(s))\Theta^{-1}(t(s))$ satisfies*

$$[D\widetilde{u}]_\beta+\left[\frac{\partial\widetilde{u}}{\partial s}\right]_\beta+[\widetilde{H}]_\beta\le C(\|u_0\|_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)},n,\beta,M^n),\tag{4.3}$$

where $[f]_\beta:=[f]_{x,\beta}+[f]_{s,\frac{\beta}{2}}$ is the sum of the Hölder seminorms of f in $M^n\times[0,S]$ with respect to x and s.

Proof. We divide our proof in three steps[1].

Step 1. We need to prove that

$$[D\widetilde{u}]_{x,\beta}+[D\widetilde{u}]_{s,\frac{\beta}{2}}\leq C(\| u_0 \|_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)},n,\beta,M^n).$$

According to Lemmas 3.1, 3.2 and 3.3, it follows that

$$|D\widetilde{u}|+\left|\frac{\partial\widetilde{u}}{\partial s}\right|\leq C(\| u_0 \|_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)},M^n).$$

Then we can easily obtain the bound of $[\widetilde{u}]_\beta$ for any $0<\beta<1$. Lemma 3.1 in [10, Chap. 2] implies that the bound for $[D\widetilde{u}]_{s,\frac{\beta}{2}}$ follows from a bound for $[\widetilde{u}]_{s,\frac{\beta}{2}}$ and $[D\widetilde{u}]_{x,\beta}$. Hence it remains to bound $[D\varphi]_{x,\beta}$ since $D\widetilde{u}=\widetilde{u}D\widetilde{\varphi}$. Fix s, we know form [9] that the equation (2.7) can be rewritten as an elliptic Neumann problem

$$\mathrm{div}_\sigma\left(\frac{D\widetilde{\varphi}}{\sqrt{1+|D\widetilde{\varphi}|^2}}\right)=\frac{n}{\sqrt{1+|D\widetilde{\varphi}|^2}}-e^{-\alpha\widetilde{\varphi}}\frac{\sqrt{1+|D\widetilde{\varphi}|^2}}{\dot{\widetilde{\varphi}}+\frac{1}{n}}. \tag{4.4}$$

Note that the derivative in the above equation is with respect to s. In fact, the equation (4.4) is of the form $D_i(a^i(p))+a(x)=0$, where the bound of a, the smallest and largest eigenvalues of $a^{ij}(p):=\frac{\partial a^i}{\partial p^j}$ are controlled due to the estimate for $|D\varphi|$ and $|\widetilde{u}|$. The estimate of $[D\widetilde{\varphi}]_{x,\beta}$ for some β follows from a Morrey estimate by calculations similar to the arguments in [10, Chap. 4, §6; Chap. 10, §2] (interior estimate and boundary estimate). For a rigorous proof of this estimate the reader is referred to [15].

Step 2. The next thing to do is to show that

$$\left[\frac{\partial\widetilde{u}}{\partial s}\right]_{x,\beta}+\left[\frac{\partial\widetilde{u}}{\partial s}\right]_{s,\frac{\beta}{2}}\leq C\left(\| u_0 \|_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)},n,\beta,M^n\right).$$

As $\frac{\partial}{\partial s}\widetilde{u}=\widetilde{u}\left(\frac{v}{\widetilde{u}^{1+\alpha}\widetilde{H}}-\frac{1}{n}\right)$, it is enough to bound $\left[\frac{v}{\widetilde{u}^{1+\alpha}\widetilde{H}}\right]_\beta$. Set $w(s):=\frac{v}{\widetilde{u}^{1+\alpha}\widetilde{H}}=\Theta^\alpha\Psi$, and then we have

$$\frac{\partial w}{\partial s}=\frac{\partial}{\partial t}(\Theta^\alpha\Psi)\frac{\partial t}{\partial s}=\frac{\alpha}{n}w+\Theta^{2\alpha}\frac{\partial\Psi}{\partial t}.$$

Let $\widetilde{\nabla}$ be the Levi–Civita connection of $\widetilde{M}_s:=\widetilde{X}(M^n,s)$ with respect to the metric $\widetilde{g}$. Combining the above equation of $\frac{\partial w}{\partial s}$ with (4.1) and Lemma 4.2, we can obtain

$$\begin{aligned}\frac{\partial w}{\partial s}&=\mathrm{div}_{\widetilde{g}}(\widetilde{u}^{-\alpha}\widetilde{H}^{-2}\widetilde{\nabla}w)-2\widetilde{H}^{-2}\widetilde{u}^{-\alpha}w^{-1}|\widetilde{\nabla}w|^2_{\widetilde{g}}\\&\quad+\frac{\alpha}{n}w-\alpha w^2-\alpha w^2P-\alpha\widetilde{u}^{-\alpha-1}\widetilde{H}^{-2}\widetilde{\nabla}_i\widetilde{u}\widetilde{\nabla}^iw,\end{aligned} \tag{4.5}$$

where $P:=u^{-1}\nabla^iu\langle X,X_i\rangle$. Applying Lemma 3.3, we have

$$|P|\leq|\nabla u|_g=\frac{|D\varphi|}{v}\leq C.$$

[1]In the proof of Lemma 4.3, the constant C may differ from each other. However, we abuse the symbol C for the purpose of convenience.

The weak formulation of (4.5) is

$$\begin{aligned}
&\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\frac{\partial w}{\partial s}\eta\,d\mu_s\,ds\\
&=\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\mathrm{div}_{\widetilde{g}}\left(\widetilde{u}^{-\alpha}\widetilde{H}^{-2}\widetilde{\nabla}w\right)\eta-2\widetilde{H}^{-2}\widetilde{u}^{-\alpha}w^{-1}|\widetilde{\nabla}w|^2_{\widetilde{g}}\eta\,d\mu_s\,ds\\
&\quad+\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\left(\frac{\alpha}{n}w-\alpha w^2-\alpha w^2P-\alpha\widetilde{u}^{-\alpha-1}\widetilde{H}^{-2}\widetilde{\nabla}_i\widetilde{u}\widetilde{\nabla}^iw\right)\eta\,d\mu_s\,ds.
\end{aligned}\tag{4.6}$$

Since $\nabla_\mu\widetilde{\varphi}=0$, the interior and boundary estimates are basically the same. We define the test function $\eta:=\xi^2w$, where ξ is a smooth function with values in $[0,1]$ and is supported in a small parabolic neighborhood. Then

$$\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\frac{\partial w}{\partial s}\xi^2w\,d\mu_s\,ds=\frac{1}{2}\parallel w\xi\parallel^2_{2,\widetilde{M}_s}|^{s_1}_{s_0}-\int_{s_0}^{s^1}\int_{\widetilde{M}_s}\xi\dot{\xi}w^2\,d\mu_s\,ds.\tag{4.7}$$

Using integration by parts and Young's inequality, we can obtain

$$\begin{aligned}
&\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\mathrm{div}_{\widetilde{g}}(\widetilde{u}^{-\alpha}\widetilde{H}^{-2}\widetilde{\nabla}w)\xi^2w\,d\mu_s\,ds\\
&=-\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha}\widetilde{H}^{-2}\xi^2\widetilde{\nabla}_iw\widetilde{\nabla}^iw\,d\mu_s\,ds\\
&\quad-2\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha}\widetilde{H}^{-2}\xi w\widetilde{\nabla}_iw\widetilde{\nabla}^i\xi\,d\mu_s\,ds\\
&\le\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha}\widetilde{H}^{-2}|\widetilde{\nabla}\xi|^2_{\widetilde{g}}w^2\,d\mu_s\,ds
\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}
&\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\left(\frac{\alpha}{n}w-\alpha w^2-\alpha w^2P-\alpha\widetilde{u}^{-\alpha-1}\widetilde{H}^{-2}\widetilde{\nabla}_i\widetilde{u}\widetilde{\nabla}^iw\right)\xi^2w\,d\mu_s\,ds\\
&\le C\alpha\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\xi^2(w^2+|w|^3)\,d\mu_s\,ds\\
&\quad+\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\alpha\widetilde{u}^{-\alpha-1}\widetilde{H}^{-2}|\widetilde{\nabla}\widetilde{u}|_{\widetilde{g}}|\widetilde{\nabla}w|_{\widetilde{g}}\xi^2w\,d\mu_s\,ds\\
&\le C\alpha\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\xi^2(w^2+|w|^3)\,d\mu_s\,ds+\frac{\alpha}{2}\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha}\widetilde{H}^{-2}|\widetilde{\nabla}w|^2_{\widetilde{g}}\xi^2\,d\mu_s\,ds\\
&\quad+\frac{\alpha}{2}\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha-2}\widetilde{H}^{-2}|\widetilde{\nabla}\widetilde{u}|^2_{\widetilde{g}}\xi^2w^2\,d\mu_s\,ds.
\end{aligned}\tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we have

$$\begin{aligned}
&\frac{1}{2}\parallel w\xi\parallel^2_{2,\widetilde{M}_s}|^{s_1}_{s_0}+\left(2-\frac{\alpha}{2}\right)\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha}\widetilde{H}^{-2}|\widetilde{\nabla}w|^2_{\widetilde{g}}\xi^2\,d\mu_s\,ds\\
&\le\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\xi|\dot{\xi}|w^2\,d\mu_s\,ds+\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha}\widetilde{H}^{-2}|\widetilde{\nabla}\xi|^2_{\widetilde{g}}w^2\,d\mu_s\,ds\\
&\quad+C\alpha\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\xi^2(w^2+|w|^3)\,d\mu_s\,ds+\frac{\alpha}{2}\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\widetilde{u}^{-\alpha-2}\widetilde{H}^{-2}|\widetilde{\nabla}\widetilde{u}|^2_{\widetilde{g}}\xi^2w^2\,d\mu_s\,ds,
\end{aligned}$$

which implies that

$$\frac{1}{2}\parallel w\xi\parallel^2_{2,\widetilde{M}_s}|^{s_1}_{s_0}+\frac{(2-\frac{\alpha}{2})}{\max(\widetilde{u}^{\alpha}\widetilde{H}^2)}\int_{s_0}^{s_1}\int_{\widetilde{M}_s}|\widetilde{\nabla}w|^2_{\widetilde{g}}\xi^2\,d\mu_s\,ds$$
$$\le\left(1+\frac{1}{\min(\widetilde{u}^{\alpha}\widetilde{H}^2)}\right)\int_{s_0}^{s_1}\int_{\widetilde{M}_s}w^2(\xi|\dot{\xi}|+|\widetilde{\nabla}\xi|^2_{\widetilde{g}})\,d\mu_s\,ds \tag{4.10}$$
$$+\alpha\left(C+\frac{\max(|\widetilde{\nabla}\widetilde{u}|_{\widetilde{g}})^2}{2\min(\widetilde{u}^{2+\alpha}\widetilde{H}^2)}\right)\int_{s_0}^{s_1}\int_{\widetilde{M}_s}\xi^2w^2+\xi^2|w|^3\,d\mu_s\,ds.$$

This means that w belong to the De Giorgi class of functions in $M^n\times[0,S)$. Similar to the arguments in [10, Chap. 5, §1 and §7], there exist constants β and C such that

$$[w]_\beta\le C\parallel w\parallel_{L^\infty(M^n\times[0,S))}\le C(\parallel u_0\parallel_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)},n,\beta,M^n).$$

Step 3. Finally, we have to show that

$$[\widetilde{H}]_{x,\beta}+[\widetilde{H}]_{s,\frac{\beta}{2}}\le C(\parallel u_0\parallel_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)},n,\beta,M^n).$$

This follows from the fact that

$$\widetilde{H}=\frac{\sqrt{1+|D\varphi|^2}}{\widetilde{u}^{1+\alpha}w}$$

together with the estimates for $\widetilde{u}$, w, $D\varphi$. □

Then we can obtain the following higher-order estimates.

Lemma 4.4. *Let u be a solution to the parabolic system (2.7), where $\varphi(x,t)=\log u(x,t)$, and Σ^n be a smooth, convex cone described as in Theorem 1.1. Then for any $s_0\in(0,S)$ there exist some $\beta>0$ and some $C>0$ such that*

$$\parallel\widetilde{u}\parallel_{C^{2+\beta,1+\frac{\beta}{2}}(M^n\times[0,S])}\le C(\parallel u_0\parallel_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)},n,\beta,M^n) \tag{4.11}$$

and for all $k\in\mathbb{N}$,

$$\parallel\widetilde{u}\parallel_{C^{2k+\beta,k+\frac{\beta}{2}}(M^n\times[s_0,S])}\le C(\parallel u_0(\cdot,s_0)\parallel_{C^{2k+\beta,k+\frac{\beta}{2}}(M^n)},n,\beta,M^n). \tag{4.12}$$

Proof. By Lemma 2.1, we have

$$uvH=n-\left(\sigma^{ij}-\frac{\varphi^i\varphi^j}{v^2}\right)\varphi_{ij}=n-u^2\Delta_g\varphi.$$

Since

$$u^2\Delta_g\varphi=\widetilde{u}^2\Delta_{\widetilde{g}}\varphi=-|\widetilde{\nabla}\widetilde{u}|^2+\widetilde{u}\Delta_{\widetilde{g}}\widetilde{u},$$

then

$$\frac{\partial\widetilde{u}}{\partial s}=\frac{\partial u}{\partial t}\Theta^{\alpha-1}-\frac{1}{n}\widetilde{u}=-\frac{uvH}{u^{1+\alpha}H^2}\Theta^{\alpha-1}+\frac{2v}{u^{\alpha}H}\Theta^{\alpha-1}-\frac{1}{n}\widetilde{u}$$
$$=\frac{\Delta_{\widetilde{g}}\widetilde{u}}{\widetilde{u}^{\alpha}\widetilde{H}^2}+\frac{2v}{\widetilde{u}^{\alpha}\widetilde{H}}-\frac{1}{n}\widetilde{u}-\frac{n+|\widetilde{\nabla}\widetilde{u}|^2}{\widetilde{u}^{1+\alpha}\widetilde{H}^2},$$

which is a uniformly parabolic equation with Hölder continuous coefficients. Therefore, the linear theory (see [13, Chap. 4]) yields the inequality (4.11).

Set $\widetilde{\varphi}=\log\widetilde{u}$, and then the rescaled version of the evolution equation in (4.2) takes the form

$$\frac{\partial\widetilde{\varphi}}{\partial s}=e^{-\alpha\widetilde{\varphi}}\frac{v^2}{\left[n-\left(\sigma^{ij}-\frac{\widetilde{\varphi}^i\widetilde{\varphi}^j}{v^2}\right)\widetilde{\varphi}_{ij}\right]}-\frac{1}{n},$$

where $v = \sqrt{1+|D\widetilde{\varphi}|^2}$. According to the $C^{2+\beta,1+\frac{\beta}{2}}$-estimate of $\widetilde{u}$ (see (4.11)), we can treat the equations for $\frac{\partial\widetilde{\varphi}}{\partial s}$ and $D_i\widetilde{\varphi}$ as second-order linear uniformly parabolic PDEs on $M^n \times [s_0, S]$. At the initial time s_0, all compatibility conditions are satisfied and the initial function $u(\cdot, t_0)$ is smooth. We can obtain a $C^{3+\beta,\frac{3+\beta}{2}}$-estimate for $D_i\widetilde{\varphi}$ and a $C^{2+\beta,\frac{2+\beta}{2}}$-estimate for $\frac{\partial\widetilde{\varphi}}{\partial s}$ (the estimates are independent of T) by Theorem 4.3 and Exercise 4.5 in [13, Chap. 4]. Higher regularity can be proven by induction over k. □

Theorem 4.5. *Under the hypothesis of Theorem 1.1, we conclude*

$$T^* = +\infty.$$

Proof. The proof of this result is quite similar to the corresponding argument in [16, Lemma 8] and so is omitted. □

5. Convergence of the rescaled flow

We know that after the long-time existence of the flow has been obtained (see Theorem 4.5), the rescaled version of the system (2.7) satisfies

$$\begin{cases} \frac{\partial}{\partial s}\widetilde{\varphi} = \widetilde{Q}(\widetilde{\varphi}, D\widetilde{\varphi}, D^2\widetilde{\varphi}) & \text{in } M^n \times (0,\infty), \\ D_\mu\widetilde{\varphi} = 0 & \text{on } \partial M^n \times (0,\infty), \\ \widetilde{\varphi}(\cdot, 0) = \widetilde{\varphi}_0 & \text{in } M^n, \end{cases} \tag{5.1}$$

where

$$\widetilde{Q}(\widetilde{\varphi}, D\widetilde{\varphi}, D^2\widetilde{\varphi}) := e^{-\alpha\widetilde{\varphi}} \frac{v^2}{\left[n - \left(\sigma^{ij} - \frac{\widetilde{\varphi}^i\widetilde{\varphi}^j}{v^2}\right)\widetilde{\varphi}_{ij}\right]} - \frac{1}{n}$$

and $\widetilde{\varphi} = \log \widetilde{u}$. Similar to what has been done in the C^1 estimate (see Lemma 3.3), we can deduce a decay estimate of $\widetilde{u}(\cdot, s)$ as follows.

Lemma 5.1. *Let u be a solution of* (2.6), *then for $\alpha > 0$, we have*

$$|D\widetilde{u}(x,s)| \le \sup_{M^n} |D\widetilde{u}(\cdot, 0)| e^{-\lambda s}, \tag{5.2}$$

where λ is a positive constant.

Proof. Set $\psi = \frac{|D\widetilde{\varphi}|^2}{2}$. Similar to that in Lemma 3.3, we can obtain

$$\begin{aligned} \frac{\partial\psi}{\partial s} &= \widetilde{Q}^{ij}\psi_{ij} + \widetilde{Q}^k\psi_k - \widetilde{Q}^{ij}(\sigma_{ij}|D\widetilde{\varphi}|^2 - \widetilde{\varphi}_i\widetilde{\varphi}_j) \\ &\quad - \widetilde{Q}^{ij}\widetilde{\varphi}_{mi}\widetilde{\varphi}^m_j - \alpha\widetilde{Q}|D\widetilde{\varphi}|^2, \end{aligned} \tag{5.3}$$

with the boundary condition

$$D_\mu\psi \le 0.$$

By the C^2 estimate, we can find a positive constant λ such that

$$\begin{cases} \dfrac{\partial\psi}{\partial s} \le \widetilde{Q}^{ij}\psi_{ij} + \widetilde{Q}^k\psi_k - \lambda\psi & \text{in } M^n \times (0,\infty), \\ D_\mu\psi \le 0 & \text{on } \partial M^n \times (0,\infty), \\ \psi(\cdot, 0) = \dfrac{|D\widetilde{\varphi}(\cdot,0)|^2}{2} & \text{in } M^n. \end{cases}$$

Using the maximum principle and Hopf's lemma, we can get the gradient estimates of $\widetilde{\varphi}$, and then the inequality (5.2) holds from the estimate for $D\widetilde{u}$. □

Lemma 5.2. *Let u be a solution of the flow* (2.6)*. Then,*

$$\widetilde{u}(\cdot, s)$$

converges to a real number as $s \to +\infty$.

Proof. We first look at the flow of geodesic spheres. Spheres centered at the origin with radius r are umbilical, their second fundamental forms are given by $h_{ij} = \frac{1}{r} g_{ij}$ and $Dr = 0$. Hence, the flow equation (2.6) can be reduced to

$$\frac{\partial r(t)}{\partial t} = \frac{1}{n r^{\alpha-1}(t)}.$$

Calculating the above ODE, we have

$$r(t) = \left(\frac{\alpha}{n} t + r_0^{\alpha}\right)^{\frac{1}{\alpha}},$$

where $r(0) = r_0$. Taking the same $\Theta(t)$ as in Lemma 3.1, we obtain for any $r_0 > 0$,

$$\lim_{t \to \infty} \frac{r(t)}{\Theta(t)} = 1. \tag{5.4}$$

Let r_1, r_2 be positive constants such that $r_1 \le \inf |u(\cdot, 0)|$, $r_2 \ge \sup |u(\cdot, 0)|$ and

$$r_i(t) = \left(\frac{\alpha}{n} t + r_i^{\alpha}\right)^{\frac{1}{\alpha}}, \quad i = 1, 2.$$

The spheres with radii $r_i(t)$ are the spherical solutions of the flow (2.6) with initial spheres of radius r_i. Due to the maximum principle, the solution $u(\cdot, t)$ of the flow (2.6) satisfies

$$r_1(t) \le u(x, t) \le r_2(t).$$

Thus,

$$r_1(t)\Theta^{-1} \le \tilde{u} \le r_2(t)\Theta^{-1},$$

where $\tilde{u} = u\Theta^{-1}$. Therefore, the convergence of $\tilde{u}$ is determined by (5.4) and interpolation,

$$\lim_{s \to \infty} \tilde{u}(s) = 1 = r_\infty.$$

This completes the proof. □

Remark 5.1. For Lemma 5.2, there exists another proof, which looks a bit more complicated than the one here and which can also establish the convergence of the solution, and moreover using that proof, the third conclusion (iii) of Theorem 1.1 can be derived. For details, we refer readers to our previous manuscript [14, pp. 15–16].

So, we have

Theorem 5.3. *The rescaled flow*

$$\frac{d\widetilde{X}}{ds} = \frac{1}{|\widetilde{X}|^{\alpha} \widetilde{H}} \nu - \frac{\widetilde{X}}{n}$$

exists for all time and the leaves converge in C^∞ to a piece of round sphere of radius r_∞.

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