Annales Fennici Mathematici
Volumen 50, 2025, 741–759

On $L^p \to L^q$ infinitesimal relative boundedness of Schrödinger operators $(-\Delta)^{\alpha/2} + v$

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Abstract. By analyzing the trace inequality for Bessel potentials, some Morrey-type sufficient conditions are given for which $L^p \to L^q$, $1 < p, q < \infty$, infinitesimal relative boundedness of the Schrödinger operators $(-\Delta)^{\alpha/2} + v$ holds. These results provide new aspects of Morrey spaces and a nice application of weight theory.

Schrödingerin operaattorien $(-\Delta)^{\alpha/2} + v$ suhteellinen $L^p \to L^q$ -rajallisuus häviävän kertoimen kanssa

Tiivistelmä. Tutkimalla Besselin potentiaalien jälkiepäyhtälöä saadaan Morreyn-tyyppisiä riittäviä ehtoja Schrödingerin operaattorien $(-\Delta)^{\alpha/2} + v$ suhteelliselle $L^p \to L^q$ -rajallisuudelle häviävän kertoimen kanssa, kun $1 < p, q < \infty$. Nämä tulokset tarjoavat sekä uusia näkökulmia Morreyn avaruuksiin että näppärän painoteorian sovelluksen.

1. Introduction

The purpose of this paper is to study $L^p \to L^q$, $1 < p, q < \infty$, infinitesimal relative boundedness of the Schrödinger operators $(-\Delta)^{\alpha/2} + v$. Following [1], we clarify the notion of the relative boundedness.

Let A and B be two linear operators in the Banach space X. A basic problem in the perturbation theory of linear operators seeks to extend properties of A to $A + B$ [7]. In many situations, one needs the perturbation B to be relatively A-bounded, i.e. there exist nonnegative constants C_1 and C_2 such that, for any $\varphi \in \mathrm{D}(A) \subseteq \mathrm{D}(B)$,

$$\|B\varphi\|_X \le C_1 \|A\varphi\|_X + C_2 \|\varphi\|_X,$$

where the infimum of such C_1 is called the relative bound of B with respect to A.

In their nice paper [1], Cao, Deng and Jin proved the following.

Theorem. [1, Theorem 1.1] *Let $p \in (1, \infty)$, $\alpha \in (0, n)$ and $a \in (-n/p, \infty)$. Then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\||x|^a \varphi\|_{L^p(\mathrm{d}x)} \lesssim \varepsilon \|(-\Delta)^{\frac{\alpha}{2}} \varphi\|_{L^p(\mathrm{d}x)} + C(\varepsilon) \|\varphi\|_{L^p(\mathrm{d}x)}$$

holds if and only if $a \in (-\alpha, 0]$.

Their proof relies on the following two facts.

- For $0 < \alpha < n$ and $0 < \lambda < \infty$, the identity I can be decomposed as

$$\begin{aligned} I &= (\lambda^2 I - \Delta)^{-\frac{\alpha}{2}} \circ (\lambda^2 I - \Delta)^{\frac{\alpha}{2}} (\lambda^\alpha I + (-\Delta)^{\frac{\alpha}{2}})^{-1} \circ (\lambda^\alpha I + (-\Delta)^{\frac{\alpha}{2}}) \\ &=: J_{\alpha,\lambda} \circ T_m \circ (\lambda^\alpha I + (-\Delta)^{\frac{\alpha}{2}}); \end{aligned}$$

https://doi.org/10.54330/afm.177378
2020 Mathematics Subject Classification: Primary 42B25; Secondary 35J10, 47A55, 42B35.
Key words: Bessel potentials, dyadic cubes, infinitesimal relative bounds, Morrey spaces, Schrödinger operators, trace inequality, Wolff's potential.

- The weighted norm inequality
  $$\||x|^a\varphi\|_{L^p(\mathrm{d}x)} \lesssim \|J_{\alpha,\lambda}\|_{L^p(\mathrm{d}x)\to L^p(|x|^{ap}\,\mathrm{d}x)} \left[\|(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^p(\mathrm{d}x)} + \lambda^\alpha\|\varphi\|_{L^p(\mathrm{d}x)}\right]$$
  holds.

Using these two facts, they verify that if $a \in (\max(-n/p, -\alpha), 0]$, then for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that the operator norm $\|J_{\alpha,\lambda_0}\|_{L^p(\mathrm{d}x)\to L^p(|x|^{ap}\,\mathrm{d}x)} < \varepsilon$, and thereby obtain the theorem. In this paper, under their nice scheme, we establish the following theorems (Theorems 1.1–1.3). For the positive Borel measure μ on $\mathbb{R}^n$, we must, a priori, assume the following condition to hold the theorems.

Condition (A). Let $Q \in \mathcal{Q}(\mathbb{R}^n)$. For all positive number $a \geq 1$ and all sparse family $\mathcal{S} \subset \mathcal{D}(Q)$, we assume that the positive Borel measure μ on $\mathbb{R}^n$ satisfies *the condition* (A):

$$\frac{1}{\mu(Q)^a}\sum_{S\in\mathcal{S}} \mu(S)^a \leq C_{a,\mu,n} < \infty, \tag{1.1}$$

where the finite positive constant $C_{a,\mu,n}$ is independent of the choices of Q.[1]

Theorem 1.1. *Let $0 < \alpha < n$, $1 < p \leq q < \infty$ and μ be a positive Borel measure on $\mathbb{R}^n$ satisfying Condition* (A). *Then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\|\varphi\|_{L^q(\mathrm{d}\mu)} \lesssim \varepsilon\|(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^p(\mathrm{d}x)} + C(\varepsilon)\|\varphi\|_{L^p(\mathrm{d}x)}$$

holds if

$$\max\begin{pmatrix} \displaystyle\sup_{Q\in\mathcal{Q}(\mathbb{R}^n):\,\ell_Q\leq 1} |Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}}, \\ \displaystyle\sup_{Q\in\mathcal{Q}(\mathbb{R}^n):\,\ell_Q> 1} \frac{\mu(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}} \end{pmatrix} < \infty \tag{1.2}$$

and

$$\lim_{\lambda\to\infty}\sup_{Q\in\mathcal{Q}(\mathbb{R}^n):\,\ell_Q\leq 1/\lambda} |Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} = 0. \tag{1.3}$$

Theorem 1.2. *Let $0 < \alpha < n$, $1 < p \leq q < \infty$ and v be a weight (nonnegative locally integrable function in $\mathbb{R}^n$). Then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\|v\varphi\|_{L^q(\mathrm{d}x)} \lesssim \varepsilon\|(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^p(\mathrm{d}x)} + C(\varepsilon)\|\varphi\|_{L^p(\mathrm{d}x)}$$

holds if, for some $r \in (1, \infty)$,

$$\max\begin{pmatrix} \displaystyle\sup_{Q\in\mathcal{Q}(\mathbb{R}^n):\,\ell_Q\leq 1} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(⨍_Q v^{qr}\,\mathrm{d}x\right)^{\frac{1}{qr}}, \\ \displaystyle\sup_{Q\in\mathcal{Q}(\mathbb{R}^n):\,\ell_Q> 1} |Q|^{\frac{1}{q}-\frac{1}{p}}\left(⨍_Q v^{qr}\,\mathrm{d}x\right)^{\frac{1}{qr}} \end{pmatrix} < \infty \tag{1.4}$$

and

$$\lim_{\lambda\to\infty}\sup_{Q\in\mathcal{Q}(\mathbb{R}^n):\,\ell_Q\leq 1/\lambda} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(⨍_Q v^{qr}\,\mathrm{d}x\right)^{\frac{1}{qr}} = 0. \tag{1.5}$$

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[1] If $w \in A_\infty$, the measure $\mu = w(x)\,\mathrm{d}x$ satisfies the Condition (A) (see Remark 5.1).

Remark. That (1.4) was known as Fefferman–Phong condition first due to Fefferman in [3].

Theorem 1.3. *Let $0 < \alpha < n$, $1 < q < p < \infty$ and μ be a positive Borel measure on $\mathbb{R}^n$ satisfying Condition* (A)*. Then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\|\varphi\|_{L^q(\mathrm{d}\mu)} \lesssim \varepsilon \|(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^p(\mathrm{d}x)} + C(\varepsilon)\|\varphi\|_{L^p(\mathrm{d}x)}$$

holds if, for r defined by $1/p + 1/r = 1/q$ and for any sparse family $\mathcal{S} \subset \mathcal{D}(Q)$ for some $Q \in \mathcal{Q}(\mathbb{R}^n)$ large enough (allowing the side length ℓ_Q to tend to infinity),

$$\max\left(\begin{array}{l} \left\| \left[\sum_{S \in \mathcal{S}:\, \ell_S \le 1} \left(\ell_S^\alpha \left(\frac{\mu(S)}{|S|} \right)^{\frac{1}{p}} \right)^{p'} \mathbf{1}_S \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}\mu)}, \\ \left\| \left[\sum_{S \in \mathcal{S}:\, \ell_S > 1} \left(\frac{\mu(S)}{|S|} \right)^{\frac{p'}{p}} \mathbf{1}_S \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}\mu)} \end{array} \right) < \infty \tag{1.6}$$

and

$$\lim_{\lambda \to \infty} \left\| \left[\sum_{S \in \mathcal{S}:\, \ell_S \le 1/\lambda} \left(\ell_S^\alpha \left(\frac{\mu(S)}{|S|} \right)^{\frac{1}{p}} \right)^{p'} \mathbf{1}_S \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}\mu)} = 0. \tag{1.7}$$

In the last section (Section 6), we give results for the power weights (Theorems 6.2 and 6.3). We have used (and will use) the following notation.

(1) Denote by $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^n)$ the family of all cubes in $\mathbb{R}^n$ with sides parallel to the axes. Given a cube $Q \in \mathcal{Q}$, denote by c_Q and ℓ_Q its center and its side length of Q, respectively, and $|Q|$ stands for the volume of Q.

(2) We define the set of all dyadic cubes in $\mathbb{R}^n$ by

$$\mathcal{D} = \mathcal{D}(\mathbb{R}^n) := \{2^{-k}(m + [0,1)^n) \colon k \in \mathbb{Z},\, m \in \mathbb{Z}^n\}.$$

That $\mathcal{D}$ satisfies the following *nested property*:

$$P, Q \in \mathcal{D} \longrightarrow P \cap Q \in \{P, Q, \emptyset\}. \tag{1.8}$$

(3) For a cube $Q \in \mathcal{D}$, denote by $Q^{(1)}$ its dyadic parent, the minimal dyadic cube that strictly contains Q.

(4) Given $Q \in \mathcal{D}$ and $\mathcal{G} \subset \mathcal{D}$, we write

$$\mathcal{G}|_Q := \{Q' \in \mathcal{G} \colon Q' \subseteq Q\},$$

that is, the restriction to Q of $\mathcal{G}$.

(5) For a cube $Q \in \mathcal{Q}(\mathbb{R}^n)$, let $\mathcal{D}(Q)$ be the collection of all dyadic subcubes of Q, that is, all those cubes obtained by dividing Q into 2^n congruent cubes of half its length, dividing each of those into 2^n congruent cubes, and so on. By convention, Q itself belongs to $\mathcal{D}(Q)$.

(6) Given a measurable set $E \subset \mathbb{R}^n$, $\mathbf{1}_E$ denotes the characteristic function of E.

(7) The barred integral $⨍_S f(y)\,\mathrm{d}y$ stands for the usual integral average of f over the set S.

(8) Given $1 < p < \infty$, $p' = p/(p-1)$ denotes the conjugate exponent number of p.

(9) For a rapidly descreasing function f, define its Fourier transform and its inverse Fourier transform as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x \quad \text{and} \quad f^{\vee}(x) := \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi.$$

(10) The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \le A \le cB$ with some positive finite constant c independent of appropriate quantities. We write $X \lesssim Y$, $Y \gtrsim X$ if there is a independent constant c such that $X \le cY$.

2. Preliminaries

In what follows we recall some notions and preliminary facts on Bessel potentials and sparse families. We will also introduce bilinear embedding theorems.

2.1. Bessel potentials. Let us begin with the definition of Bessel potentials. We follow the argument in the book [5].

Definition 2.1. Let $0 < \alpha < n$ and $0 < \lambda < \infty$. The Bessel potential of order α is the operator $J_{\alpha,\lambda} = (\lambda^2 I - \Delta)^{-\alpha/2}$ given by

$$J_{\alpha,\lambda}(f) := \left(\hat{f}\hat{G_{\alpha,\lambda}}\right)^{\vee} = f * G_{\alpha,\lambda},$$

where

$$G_{\alpha,\lambda}(x) = \left((\lambda^2 + 4\pi^2|\xi|^2)^{-\frac{\alpha}{2}}\right)^{\vee}(x).$$

If $\lambda = 1$, we omit the subscript 1 and simply write J_α and G_α, respectively.

By the definition, one sees that

$$G_{\alpha,\lambda}(x) = \lambda^{n-\alpha}G_\alpha(\lambda x), \quad x \in \mathbb{R}^n. \tag{2.1}$$

We show the exponential decay for G_α at infinity.

Lemma 2.2. *Let $0 < \alpha < n$. Then G_α is a smooth function on $\mathbb{R}^n \setminus \{0\}$ that satisfies $G_\alpha(x) > 0$, $x \in \mathbb{R}^n$, and there exist positive finite constants $C_{\alpha,n}$ and $c_{\alpha,n}$ such that*

$$G_\alpha(x) \le \begin{cases} C_{\alpha,n}e^{-\frac{|x|}{2}}, & \text{when } |x| > 1, \\ c_{\alpha,n}|x|^{\alpha-n}, & \text{when } |x| \le 1. \end{cases}$$

Proof. For $A > 0$, we set

$$\Gamma\left(\frac{\alpha}{2}\right) = \int_0^\infty e^{-t}t^{\frac{\alpha}{2}}\,\frac{\mathrm{d}t}{t} = A^{\frac{\alpha}{2}}\int_0^\infty e^{-tA}t^{\frac{\alpha}{2}}\,\frac{\mathrm{d}t}{t}.$$

This implies

$$A^{-\frac{\alpha}{2}} = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty e^{-tA}t^{\frac{\alpha}{2}}\,\frac{\mathrm{d}t}{t}.$$

We use this to obtain

$$\left(1 + 4\pi^2|\xi|^2\right)^{-\frac{\alpha}{2}} = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty e^{-t}e^{-\pi(2\sqrt{\pi t}|\xi|)^2}t^{\frac{\alpha}{2}}\,\frac{\mathrm{d}t}{t}.$$

Take the inverse Fourier transform in ξ and use the fact that the function $e^{-\pi|\xi|^2}$ is equal to its Fourier transform to obtain

$$G_\alpha(x) = \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})}\int_0^\infty e^{-t}e^{-\frac{|x|^2}{4t}}t^{\frac{\alpha-n}{2}}\,\frac{\mathrm{d}t}{t}. \tag{2.2}$$

This proves that $G_\alpha(x) > 0$, $x \in \mathbb{R}^n$, and that G_α is smooth on $\mathbb{R}^n \setminus \{0\}$.

Now suppose $|x| > 1$. Then we have that $t + \frac{|x|^2}{4t} \geq t + \frac{1}{4t}$ and that $t + \frac{|x|^2}{4t} \geq |x|$, and hence,

$$\begin{aligned} t + \frac{|x|^2}{4t} &= \frac{t + \frac{|x|^2}{4t}}{2} + \frac{t + \frac{|x|^2}{4t}}{2} \\ &\geq \frac{t}{2} + \frac{1}{8t} + \frac{|x|}{2}. \end{aligned}$$

It follows from this and (2.2) that

$$\begin{aligned} G_\alpha(x) &\leq \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})} \left(\int_0^\infty e^{-\frac{t}{2}} e^{-\frac{1}{8t}} t^{\frac{\alpha-n}{2}} \frac{\mathrm{d}t}{t} \right) e^{-\frac{|x|}{2}} \\ &= C_{\alpha,n} e^{-\frac{|x|}{2}}. \end{aligned}$$

Now suppose $|x| \leq 1$. It follows from (2.2) that, by letting $t = s|x|^2$,

$$\begin{aligned} G_\alpha(x) &= \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})} \left(\int_0^\infty e^{-s|x|^2} e^{-\frac{1}{4s}} s^{\frac{\alpha-n}{2}} \frac{\mathrm{d}s}{s} \right) |x|^{\alpha-n} \\ &\leq \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})} \left(\int_0^\infty e^{-\frac{1}{4s}} s^{\frac{\alpha-n}{2}} \frac{\mathrm{d}s}{s} \right) |x|^{\alpha-n} \\ &= c_{\alpha,n} |x|^{\alpha-n}. \end{aligned}$$

Combining two estimates we obtain the required conclusion. □

2.2. Sparse families.

Definition 2.3. (See [8]) Let $0 < \eta < 1$. We say that a family $\mathcal{S} \subset \mathcal{D}$ is η-sparse if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \geq \eta |Q|$, and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint.

We now present two technical lemmas of sparse families.

Lemma 2.4. [2, Lemma 4.2] *Let μ be a positive Borel measure on $\mathbb{R}^n$. Suppose that $0 < \alpha_1 < \infty$ and $0 \leq \alpha_2 < \infty$ satisfying $\alpha_1 + \alpha_2 \geq 1$. Then for any sparse family $\mathcal{S} \subset \mathcal{D}$ and any cube $Q \in \mathcal{D}$,*

$$\sum_{Q' \in \mathcal{S}|_Q} |Q'|^{\alpha_1} \mu(Q')^{\alpha_2} \lesssim |Q|^{\alpha_1} \mu(Q)^{\alpha_2}.$$

For each $Q \in \mathcal{S}$, let $\mathrm{ch}_{\mathcal{S}}(Q)$ denote the collection of all maximal $Q' \in \mathcal{S}$ such that $Q' \subseteq Q$. The Pythagoras' theorem for functions adapted to a sparse family is given as follows.

Lemma 2.5. [6, Lemma 4] *Let $1 < p < \infty$ and $\mathcal{S} \subset \mathcal{D}$ be a sparse family. For each $Q \in \mathcal{S}$, if f_Q is a nonnegative function that is supported on Q and is constant on each $Q' \in \mathrm{ch}_{\mathcal{S}}(Q)$, then*

$$\left\| \sum_{Q \in \mathcal{S}} f_Q \right\|_{L^p(\mathrm{d}x)}^p \lesssim \sum_{Q \in \mathcal{S}} \|f_Q\|_{L^p(\mathrm{d}x)}^p.$$

2.3. Bilinear embedding theorems. We first introduce bilinear embedding problem.

Bilinear embedding problem. Let $K\colon \mathcal{D} \to [0,\infty)$ be a map and let σ and ω be positive Borel measures on $\mathbb{R}^n$. We give a necessary and sufficient condition under which the inequality

$$\sum_{Q\in\mathcal{D}} K(Q) \left|\int_Q f\,\mathrm{d}\sigma\right| \left|\int_Q g\,\mathrm{d}\omega\right| \le c_1 \|f\|_{L^p(\mathrm{d}\sigma)} \|g\|_{L^q(\mathrm{d}\omega)} \tag{2.3}$$

holds when $1 < p, q < \infty$.

Bilinear embedding problem can be characterized by two ways. The division line is whether the exponents p and q are in the super-dual range $1/p + 1/q \ge 1$ or in the strictly sub-dual range $1/p + 1/q < 1$.

Lemma 2.6. [12] *Let the exponents p and q be in the super-dual range $1/p + 1/q \ge 1$. Then the necessary and sufficient condition for the inequality* (2.3) *to hold is as follows: For all dyadic cubes $Q \in \mathcal{D}$,*

$$\left(\int_Q \left(\sum_{Q'\in\mathcal{D}|_Q} K(Q')\sigma(Q')\mathbf{1}_{Q'}\right)^{q'} \mathrm{d}\omega\right)^{\frac{1}{q'}} \le c_2 \sigma(Q)^{\frac{1}{p}},$$

$$\left(\int_Q \left(\sum_{Q'\in\mathcal{D}|_Q} K(Q')\omega(Q')\mathbf{1}_{Q'}\right)^{p'} \mathrm{d}\sigma\right)^{\frac{1}{p'}} \le c_2 \omega(Q)^{\frac{1}{q}}.$$

Moreover, the least possible constants c_1 and c_2 are equivalent.

These conditions are called *the Sawyer testing condition*, since this was first introduced by Eric Sawyer.

Lemma 2.7. [11] *Let the exponents p and q be in the strictly sub-dual range $1/p + 1/q < 1$. Then the necessary and sufficient condition for the inequality* (2.3) *to hold is as follows:*

$$\left\|\mathcal{W}^{q'}_{K;\omega}[\sigma]^{\frac{1}{q'}}\right\|_{L^r(\mathrm{d}\sigma)} \le c_2 < \infty,$$

$$\left\|\mathcal{W}^{p'}_{K;\sigma}[\omega]^{\frac{1}{p'}}\right\|_{L^r(\mathrm{d}\omega)} \le c_2 < \infty,$$

for r defined by $1/p + 1/q + 1/r = 1$. Here,

$$\begin{cases} \mathcal{W}^{q'}_{K;\omega}[\sigma](x) := \displaystyle\sum_{Q\in\mathcal{D}} K(Q)\omega(Q) \left(\frac{1}{\omega(Q)} \sum_{Q'\in\mathcal{D}|_Q} K(Q')\sigma(Q')\omega(Q')\right)^{q'-1} \mathbf{1}_Q(x), \\ \mathcal{W}^{p'}_{K;\sigma}[\omega](x) := \displaystyle\sum_{Q\in\mathcal{D}} K(Q)\sigma(Q) \left(\frac{1}{\sigma(Q)} \sum_{Q'\in\mathcal{D}|_Q} K(Q')\sigma(Q')\omega(Q')\right)^{p'-1} \mathbf{1}_Q(x). \end{cases}$$

Moreover, the least possible constants c_1 and c_2 are equivalent.

$\mathcal{W}^{q'}_{K;\omega}[\sigma](x)$ and $\mathcal{W}^{p'}_{K;\sigma}[\omega](x)$ are called *two weight dyadic discrete Wolff potentials.*

3. Dyadic discrete representation of Bessel potentials

In what follows we introduce the so-called Pérez dyadic decomposition of Bessel potentials, which was first due to Pérez in [9] for Riesz potentials.

3.1. Pérez decomposition.

Let, cf. (2.2),

$$g_\alpha(u) := \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-t} e^{-\frac{u^2}{4t}} t^{\frac{\alpha-n}{2}} \frac{\mathrm{d}t}{t}, \quad u > 0.$$

Then one sees that

$$g_\alpha(au) \le a^{\alpha-n} g_\alpha(u) \quad \text{for all } a \ge 1 \text{ and } u > 0. \tag{3.1}$$

Indeed,

$$\begin{aligned} g_\alpha(au) &= \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-t} e^{-\frac{(au)^2}{4t}} t^{\frac{\alpha-n}{2}} \frac{\mathrm{d}t}{t} \\ &= a^{\alpha-n} \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-(a^2-1)s} e^{-s} e^{-\frac{u^2}{4s}} s^{\frac{\alpha-n}{2}} \frac{\mathrm{d}s}{s} \\ &\le a^{\alpha-n} \frac{(2\sqrt{\pi})^{-n}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-s} e^{-\frac{u^2}{4s}} s^{\frac{\alpha-n}{2}} \frac{\mathrm{d}s}{s} \\ &= a^{\alpha-n} g_\alpha(u). \end{aligned}$$

The Bessel potential J_α has the following representation.

Proposition 3.1. *Let $0 < \alpha < n$. Then for the nonnegative function f we have that*

$$J_\alpha f(x) = \int_{\mathbb{R}^n} G_\alpha(x-y)\, f(y)\, \mathrm{d}y \approx \sum_{Q\in\mathcal{D}} g_\alpha(\ell_Q) \int_{3Q} f(y)\, \mathrm{d}y\, \mathbf{1}_Q(x), \quad x \in \mathbb{R}^n. \tag{3.2}$$

Proof. Rewrite by using characteristic functions

$$\sum_{Q\in\mathcal{D}} g_\alpha(\ell_Q) \int_{3Q} f(y)\, \mathrm{d}y\, \mathbf{1}_Q(x) = \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}} g_\alpha(\ell_Q) \mathbf{1}_Q(x) \mathbf{1}_{3Q}(y) \right) f(y)\, \mathrm{d}y.$$

For $x \ne y$, let

$$S(x,y) := \{Q \in \mathcal{D} \colon Q \ni x,\, 3Q \ni y\}.$$

Then by the nested property (1.8) one sees that there exists a minimal dyadic cube $Q(x,y) \in S(x,y)$, and it satisfies

$$\frac{\ell_{Q(x,y)}}{2} < |x-y| < 2\sqrt{n}\ell_{Q(x,y)}.$$

By (3.1) we obtain

$$\sum_{Q\in S(x,y)} g_\alpha(\ell_Q) \approx g_\alpha(\ell_{Q(x,y)}) \le G_\alpha\left(\frac{x-y}{2\sqrt{n}}\right),$$

which yields the equivalence (3.2) by Fubini's theorem. □

3.2. The dyadic grid argument. For $\tau \in \{0, \pm\frac{1}{3}\}^n$, we define the dyadic grid by

$$\mathcal{D}^\tau := \{2^{-k}(m + \tau + [0,1)^n) : k \in \mathbb{Z},\ m \in \mathbb{Z}^n\}.$$

Claim 3.2. *We claim that for any dyadic cube $Q \in \mathcal{D}$, there exist $\tau \in \{0, \pm\frac{1}{3}\}^n$ and τ-shifted dyadic cube $P \in \mathcal{D}^\tau$ such that $3Q \subset P$ and $\ell_P = 8\ell_Q$.*

Proof. We need only verify the one-dimensional case $n = 1$. (The claim for $n > 1$ holds after n steps.) We may assume further $k = 0$.

Let $Q = [m, m+1)$, $m \in \mathbb{Z}$. Then $3Q = [m-1, m+2)$. We cover $3Q$ by disjoint dyadic intervals of $\mathcal{D}(\mathbb{R})$ with the same length 8. If $3Q$ is covered by such an interval P, then we choose $\tau = 0$ and have $P \in \mathcal{D}^\tau(\mathbb{R})$. We assume that $3Q$ is covered by such two intervals as $P_1 \ni (m-1)$ and $P_2 \ni (m+2)$. If $|3Q \cap P_1| \geq \frac{3}{2}$, then we choose $\tau = \frac{1}{3}$ and let $P = \frac{8}{3} + P_1$. If $|3Q \cap P_2| > \frac{3}{2}$, then we choose $\tau = -\frac{1}{3}$ and let $P = -\frac{8}{3} + P_2$. □

This claim implies for $f \geq 0$ and $x \in \mathbb{R}^n$ that

$$\begin{aligned} J_\alpha f(x) &\approx \sum_{Q \in \mathcal{D}} g_\alpha(\ell_Q) \int_{3Q} f(y)\,\mathrm{d}y\,\mathbf{1}_Q(x) \\ &\approx \sum_{\tau \in \{0, \pm\frac{1}{3}\}^n} \sum_{Q \in \mathcal{D}^\tau} g_\alpha(\ell_Q) \int_Q f(y)\,\mathrm{d}y\,\mathbf{1}_Q(x). \end{aligned}$$

In the last step we may have used $g_\alpha(u) \lesssim e^{-u/16}$ instead of $g_\alpha(u) \lesssim e^{-u/2}$ for $u > 1$ (cf. Lemma 2.2). Thus, for the positive cases, we need only estimate the simple linear positive operator

$$T_\alpha f(x) := \sum_{Q \in \mathcal{D}} g_\alpha(\ell_Q) \int_Q f(y)\,\mathrm{d}y\,\mathbf{1}_Q(x), \quad x \in \mathbb{R}^n. \tag{3.3}$$

3.3. The sparse domination argument. We further reduce the simple linear positive operator T_α to the sparse operator S_α.

Proposition 3.3. *Let $0 < \alpha < n$. We have that, for some appropriate sparse family $\mathcal{S} \subset \mathcal{D}$,*

$$T_\alpha f(x) \lesssim S_\alpha f(x), \quad f \geq 0, \quad x \in \mathbb{R}^n.$$

Here,

$$S_\alpha f(x) := \sum_{S \in \mathcal{S}} \frac{\min(\ell_S^\alpha, 1)}{|S|} \int_S f(y)\,\mathrm{d}y\,\mathbf{1}_S(x), \quad f \geq 0, \quad x \in \mathbb{R}^n. \tag{3.4}$$

Proof. For simple notation we let μ be a measure $\mu = f\,\mathrm{d}x$. Let $Q \in \mathcal{D}$ be taken large enough and be fixed. We shall estimate the quantity

$$\sum_{Q' \in \mathcal{D}|_Q} g_\alpha(\ell_{Q'})\mu(Q')\mathbf{1}_{Q'} = \sum_{Q' \in \mathcal{D}|_Q} g_\alpha(\ell_{Q'})|Q'|\frac{\mu(Q')}{|Q'|}\mathbf{1}_{Q'}. \tag{3.5}$$

We define the collection $\mathcal{S}$ of principal cubes to obtain the η-sparse family. Namely,

$$\mathcal{S} := \bigcup_{k=0}^\infty \mathcal{S}_k,$$

where $\mathcal{S}_0 := \{Q\}$,

$$\mathcal{S}_{k+1} := \bigcup_{S \in \mathcal{S}_k} \mathrm{ch}_{\mathcal{S}}(S)$$

and $\mathrm{ch}_{\mathcal{S}}(S)$ is defined by the set of all maximal dyadic cubes $Q' \subset S$ such that

$$\frac{\mu(Q')}{|Q'|} > \frac{\mu(S)}{(1-\eta)|S|}.$$

We write

$$E_{\mathcal{S}}(S) := S \setminus \bigcup_{S' \in \mathrm{ch}_{\mathcal{S}}(S)} S'.$$

Then it is easy to see that the collection $\{E_{\mathcal{S}}(S)\colon S \in \mathcal{S}\}$ is pairwise disjoint. Observe that

$$\sum_{S' \in \mathrm{ch}_{\mathcal{S}}(S)} |S'| \le \frac{(1-\eta)|S|}{\mu(S)} \sum_{S' \in \mathrm{ch}_{\mathcal{S}}(S)} \mu(S') \le (1-\eta)|S|,$$

and, hence,

$$|E_{\mathcal{S}}(S)| = \left| S \setminus \bigcup_{S' \in \mathrm{ch}_{\mathcal{S}}(S)} S' \right| \ge \eta |S|.$$

Thus, $\mathcal{S}$ is a η-sparse family.

For $Q' \in \mathcal{D}|_Q$, we further define the stopping parent $\pi_{\mathcal{S}}(Q')$ by

$$\pi_{\mathcal{S}}(Q') := \min\{S \supset Q'\colon S \in \mathcal{S}\}.$$

Then we can estimate the series in (3.5) as follows:

$$\begin{aligned}
\sum_{Q' \in \mathcal{D}|_Q} g_\alpha(\ell_{Q'})|Q'| \frac{\mu(Q')}{|Q'|} \mathbf{1}_{Q'} &= \sum_{S \in \mathcal{S}} \sum_{Q'\colon \pi_{\mathcal{S}}(Q') = S} g_\alpha(\ell_{Q'})|Q'| \frac{\mu(Q')}{|Q'|} \mathbf{1}_{Q'} \\
&\le \sum_{S \in \mathcal{S}} \frac{\mu(S)}{(1-\eta)|S|} \sum_{Q\colon \pi_{\mathcal{S}}(Q) = S} g_\alpha(\ell_{Q'})|Q'| \mathbf{1}_{Q'} \\
&\lesssim A_0 \sum_{S \in \mathcal{S}} \frac{\min(\ell_S^\alpha,\, 1)}{|S|} \mu(S) \mathbf{1}_S,
\end{aligned}$$

where, letting

$$A_0 := \sum_{k=1}^{\infty} e^{-2^{k-1}} 2^{nk},$$

we have used

$$\sum_{Q\colon \pi_{\mathcal{S}}(Q) = S} g_\alpha(\ell_{Q'})|Q'| \mathbf{1}_{Q'} \lesssim A_0 \min(\ell_S^\alpha,\, 1) \mathbf{1}_S. \tag{3.6}$$

That (3.6) can be verified as follows. Recalling Lemma 2.2, using the nested property (1.8), we have that

$$\begin{aligned}
\sum_{Q\colon \pi_{\mathcal{S}}(Q) = S} g_\alpha(\ell_{Q'})|Q'| \mathbf{1}_{Q'} &\le \mathbf{1}_S \left[\sum_{k=-\infty}^{\min(\log_2 \ell_S,\, 1)} (2^k)^\alpha + \sum_{k=2}^{\log_2 \ell_S} e^{-2^{k-1}} 2^{nk} \right] \\
&\lesssim A_0 \min(\ell_S^\alpha,\, 1) \mathbf{1}_S.
\end{aligned}$$

□

It follows from (2.1) and (3.6) that

$$\sum_{Q\colon \pi_{\mathcal{S}}(Q) = S} g_{\alpha,\lambda}(\ell_{Q'})|Q'| \mathbf{1}_{Q'} = \lambda^{-\alpha} \sum_{Q\colon \pi_{\mathcal{S}}(Q) = S} g_\alpha(\lambda \ell_{Q'})(\lambda \ell_{Q'})^n \mathbf{1}_{Q'} \lesssim \frac{A_0}{\lambda^\alpha} \min((\lambda \ell_S)^\alpha,\, 1) \mathbf{1}_S.$$

From this we define the final target operator $S_{\alpha,\lambda}$, $\lambda > 0$, by

$$S_{\alpha,\lambda} f(x) := \sum_{S\in\mathcal{S}} \frac{\min((\lambda\ell_S)^\alpha,\, 1)}{\lambda^\alpha |S|} \int_S f(y)\,\mathrm{d}y\, \mathbf{1}_S(x), \quad f \geq 0, \quad x \in \mathbb{R}^n, \tag{3.7}$$

which controls $J_{\alpha,\lambda}$.

4. Trace inequalities for S_α

In what follows we analyze the trace inequalities for Bessel potentials in the two cases.

4.1. Trace inequality for measure. It suffices to estimate, for the nonnegative function f and the η-sparse family $\mathcal{S} \subset \mathcal{D}$,

$$S_\alpha f(x) = \sum_{S\in\mathcal{S}} \frac{\min(\ell_S^\alpha,\, 1)}{|S|} \int_S f(y)\,\mathrm{d}y\, \mathbf{1}_S(x), \quad x \in \mathbb{R}^n.$$

Let $0 < \alpha < n$, $1 < p \leq q < \infty$ and μ be a positive Borel measure on $\mathbb{R}^n$. We consider the trace inequality for S_α:

$$\|S_\alpha f\|_{L^q(\mathrm{d}\mu)} \leq c_1 \|f\|_{L^p(\mathrm{d}x)}. \tag{4.1}$$

We use Lemma 2.6, letting a map $K\colon \mathcal{D} \to [0,\infty)$ be

$$K(Q) = \begin{cases} \dfrac{\min(\ell_Q^\alpha,\, 1)}{|Q|}, & Q \in \mathcal{S}, \\ 0, & Q \in \mathcal{D} \setminus \mathcal{S}, \end{cases}$$

and letting $\sigma = \mathrm{d}x$ and $\omega = \mu$. Then the trace inequality (4.1) holds if and only if for all dyadic cubes $S \in \mathcal{S}$

$$I_1(S) := \frac{1}{|S|^{\frac{1}{p}}} \left(\int_S \left(\sum_{S'\in\mathcal{S}|_S} \min(\ell_{S'}^\alpha,\, 1) \mathbf{1}_{S'} \right)^q \mathrm{d}\mu \right)^{\frac{1}{q}} \leq c_2 < \infty,$$

$$I_2(S) := \frac{1}{\mu(S)^{\frac{1}{q'}}} \left(\int_S \left(\sum_{S'\in\mathcal{S}|_S} \frac{\min(\ell_{S'}^\alpha,\, 1)}{|S'|} \mu(S') \mathbf{1}_{S'} \right)^{p'} \mathrm{d}x \right)^{\frac{1}{p'}} \leq c_2 < \infty.$$

Moreover, the least possible constants c_1 and c_2 are equivalent.

Hence, to estimate the operator norm

$$c_1 = \|S_\alpha\|_{L^p(\mathrm{d}x)\to L^q(\mathrm{d}\mu)} \approx c_2,$$

we analyze $I_1(S)$ and $I_2(S)$.

4.2. The estimation of $I_1(S)$. It follows that

$$\begin{aligned} I_1(S) &= \frac{1}{|S|^{\frac{1}{p}}} \left(\int_S \left(\sum_{S'\in\mathcal{S}|_S} \min(\ell_{S'}^\alpha,\, 1) \mathbf{1}_{S'} \right)^q \mathrm{d}\mu \right)^{\frac{1}{q}} \\ &\lesssim \max \begin{pmatrix} |S|^{\frac{\alpha}{n}-\frac{1}{p}} \mu(S)^{\frac{1}{q}}, & \ell_S \leq 1, \\ \dfrac{\mu(S)^{\frac{1}{q}}}{|S|^{\frac{1}{p}}}, & \ell_S > 1 \end{pmatrix}. \end{aligned}$$

4.3. The estimation of $I_2(S)$. Let $\mathcal{S}_1 := \{S \in \mathcal{S} \colon \ell_S \le 1\}$ and $\mathcal{S}_2 := \{S \in \mathcal{S} \colon \ell_S > 1\}$. It follows from Lemma 2.5 that

$$\begin{aligned}
\int_S \left(\sum_{S' \in \mathcal{S}_1|_S} \frac{\ell_{S'}^\alpha}{|S'|} \mu(S') \mathbf{1}_{S'} \right)^{p'} \mathrm{d}x &\lesssim \sum_{S' \in \mathcal{S}_1|_S} \left(\frac{\ell_{S'}^\alpha}{|S'|} \right)^{p'} \mu(S')^{p'} |S'| \\
&= \sum_{S' \in \mathcal{S}_1|_S} (\ell_{S'}^\alpha)^{p'} |S'|^{1-p'} \mu(S')^{p'} \\
&= \sum_{S' \in \mathcal{S}_1|_S} \left[|S'|^{\frac{\alpha}{n} - \frac{1}{p}} \mu(S')^{\frac{1}{q}} \right]^{p'} \mu(S')^{\frac{p'}{q'}} \\
&\le \left[\sup_{S' \in \mathcal{S}_1|_S} |S'|^{\frac{\alpha}{n} - \frac{1}{p}} \mu(S')^{\frac{1}{q}} \right]^{p'} \sum_{S' \in \mathcal{S}_1|_S} \mu(S')^{\frac{p'}{q'}}.
\end{aligned}$$

Deep thanks to Condition (A): (1.1), we have that

$$\sum_{S' \in \mathcal{S}_1|_S} \mu(S')^{\frac{p'}{q'}} \lesssim \mu(S)^{\frac{p'}{q'}},$$

which leads us to conclude that

$$\frac{1}{\mu(S)^{\frac{1}{q'}}} \left(\int_S \left(\sum_{S' \in \mathcal{S}_1|_S} \frac{\ell_{S'}^\alpha}{|S'|} \mu(S') \mathbf{1}_{S'} \right)^{p'} \mathrm{d}x \right)^{\frac{1}{p'}} \lesssim \sup_{Q \in \mathcal{D} \colon \ell_Q \le 1} |Q|^{\frac{\alpha}{n} - \frac{1}{p}} \mu(Q)^{\frac{1}{q}}.$$

similarly, we have that

$$\frac{1}{\mu(S)^{\frac{1}{q'}}} \left(\int_S \left(\sum_{S' \in \mathcal{S}_2|_S} \frac{\ell_{S'}^\alpha}{|S'|} \mu(S') \mathbf{1}_{S'} \right)^{p'} \mathrm{d}x \right)^{\frac{1}{p'}} \lesssim \sup_{Q \in \mathcal{D} \colon \ell_Q > 1} \frac{\mu(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}}.$$

Thus, by (3.7) we have the following proposition.

Proposition 4.1. *Let $0 < \alpha < n$, $1 < p \le q < \infty$ and μ be a positive Borel measure on $\mathbb{R}^n$ satisfying Condition* (A). *Then for $\lambda \in (1, \infty)$, the operator norm $\|S_{\alpha,\lambda}\|_{L^p(\mathrm{d}x) \to L^q(\mathrm{d}\mu)}$ is majorized by*

$$\max \begin{pmatrix} \displaystyle\sup_{Q \in \mathcal{D} \colon \ell_Q \le 1/\lambda} |Q|^{\frac{\alpha}{n} - \frac{1}{p}} \mu(Q)^{\frac{1}{q}}, \\ \displaystyle\lambda^{-\alpha} \sup_{Q \in \mathcal{D} \colon \ell_Q > 1/\lambda} \frac{\mu(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}} \end{pmatrix}.$$

4.4. Trace inequality for weight. Let $0 < \alpha < n$, $1 < p \le q < \infty$ and v be a weight on $\mathbb{R}^n$. We consider the trace inequality for S_α:

$$\|v S_\alpha f\|_{L^q(\mathrm{d}x)} \lesssim \|f\|_{L^p(\mathrm{d}x)}. \tag{4.2}$$

4.5. Fefferman–Phong inequality. We shall estimate (4.2) by way of a duality argument. To this end we take a nonnegative function g with $\|g\|_{L^{q'}(\mathrm{d}x)} = 1$ and evaluate

$$\text{(i)} := \int_{\mathbb{R}^n} g(x) v(x) S_\alpha f(x) \, \mathrm{d}x = \sum_{S \in \mathcal{S}} \frac{\min(\ell_S^\alpha, 1)}{|S|} \int_S f(y) \, \mathrm{d}y \int_S v(x) g(x) \, \mathrm{d}x.$$

It follows that

$$\begin{aligned}\text{(i)} &= \sum_{S\in\mathcal{S}} \min(\ell_S^\alpha,\, 1) \fint_S f(y)\,\mathrm{d}y \fint_S v(x)g(x)\,\mathrm{d}x |S| \\ &\le \eta \sum_{S\in\mathcal{S}} \min(\ell_S^\alpha,\, 1) \fint_S f(y)\,\mathrm{d}y \fint_S v(x)g(x)\,\mathrm{d}x |E_{\mathcal{S}}(S)|.\end{aligned}$$

For some $r\in(1,\infty)$, letting $u=rq$ we have that by Hölder's inequality

$$\fint_S v(x)g(x)\,\mathrm{d}x \le \left(\fint_S v(x)^u\,\mathrm{d}x\right)^{\frac{1}{u}} \left(\fint_S g(x)^{u'}\,\mathrm{d}x\right)^{\frac{1}{u'}},$$

which yields

$$\begin{aligned}\text{(i)} &\lesssim \left[\sup_{S\in\mathcal{S}} \frac{\min(\ell_S^\alpha,\, 1)}{\ell_S^\alpha} \cdot |S|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_S v^u\,\mathrm{d}x\right)^{\frac{1}{u}}\right] \\ &\quad\times \left[\sum_{S\in\mathcal{S}} \fint_S f(y)\,\mathrm{d}y \left(|S|^{\frac{u'}{p}-\frac{u'}{q}} \fint_S g(x)^{u'}\,\mathrm{d}x\right)^{\frac{1}{u'}} |E_{\mathcal{S}}(S)|\right].\end{aligned}$$

Letting $\beta = u'/p - u'/q$,

$$\begin{aligned}\sum_{S\in\mathcal{S}} \fint_S f(y)\,\mathrm{d}y \left(|S|^{\frac{u'}{p}-\frac{u'}{q}} \fint_S g(x)^{u'}\,\mathrm{d}x\right)^{\frac{1}{u'}} |E_{\mathcal{S}}(S)| &\le \int_{\mathbb{R}^n} Mf(x) M_\beta[g^{u'}](x)^{\frac{1}{u'}}\,\mathrm{d}x \\ &\le \|Mf\|_{L^p(\mathrm{d}x)} \|M_\beta[g^{u'}]^{\frac{1}{u'}}\|_{L^{p'}(\mathrm{d}x)} \\ &\lesssim \|f\|_{L^p(\mathrm{d}x)} \|g\|_{L^{q'}(\mathrm{d}x)},\end{aligned}$$

where we have used the Hardy–Littlewood maximal inequality for M and the Hardy–Littlewood–Sobolev maximal inequality for M_β, noticing that $u'/p' = u'/q' - \beta$.

Thus, we obtain

$$\|vS_\alpha f\|_{L^q(\mathrm{d}x)} \lesssim \left[\sup_{Q\in\mathcal{D}} \frac{\min(\ell_Q^\alpha,\, 1)}{\ell_Q^\alpha} \cdot |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_Q v^{qr}\,\mathrm{d}x\right)^{\frac{1}{qr}}\right] \|f\|_{L^p(\mathrm{d}x)}. \tag{4.3}$$

Thus, by (3.7) and (4.3) we have the following proposition.

Proposition 4.2. *Let $0<\alpha<n$, $1<p\le q<\infty$ and v be a weight on $\mathbb{R}^n$. Then for $\lambda\in(1,\infty)$ and for some $r\in(1,\infty)$, the operator norm $\|S_{\alpha,\lambda}\|_{L^p(\mathrm{d}x)\to L^q(v^q)}$ is majorized by*

$$\max\left(\begin{array}{c} \displaystyle\sup_{Q\in\mathcal{D}:\,\ell_Q\le 1/\lambda} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\fint_Q v^{qr}\,\mathrm{d}x\right)^{\frac{1}{qr}}, \\ \displaystyle\lambda^{-\alpha} \sup_{Q\in\mathcal{D}:\,\ell_Q>1/\lambda} |Q|^{\frac{1}{q}-\frac{1}{p}} \left(\fint_Q v^{qr}\,\mathrm{d}x\right)^{\frac{1}{qr}} \end{array}\right).$$

4.6. Trace inequality for measure, revisited. Let $0<\alpha<n$. For the nonnegative function f and the η-sparse family $\mathcal{S}\subset\mathcal{D}$, let

$$S_\alpha f(x) = \sum_{S\in\mathcal{S}} \frac{\min(\ell_S^\alpha,\, 1)}{|S|} \int_S f(y)\,\mathrm{d}y\, \mathbf{1}_S(x), \quad x\in\mathbb{R}^n.$$

In the range $1 < q < p < \infty$, we consider the trace inequality (4.1) for S_α. Letting $\mathcal{S}_1 := \{S \in \mathcal{S} \colon \ell_S \le 1\}$ and $\mathcal{S}_2 := \{S \in \mathcal{S} \colon \ell_S > 1\}$, we first decompose

$$S_\alpha f(x) = \sum_{S \in \mathcal{S}_1} \ell_S^{\alpha-n} \int_S f(y)\,\mathrm{d}y\,\mathbf{1}_S(x) + \sum_{S \in \mathcal{S}_2} \ell_S^{-n} \int_S f(y)\,\mathrm{d}y\,\mathbf{1}_S(x) =: S_{\alpha,1} f(x) + S_{\alpha,2} f(x).$$

For $S_{\alpha,1}$ and $S_{\alpha,2}$ we use Lemma 2.7. Then the necessary and sufficient condition for the trace inequality (4.1) to hold is as follows:

$$\left\| \mathcal{W}^q_{\alpha,i;\mu}[\mathrm{d}x]^{\frac{1}{q}} \right\|_{L^r(\mathrm{d}x)} \le c_2 < \infty, \quad \left\| \mathcal{W}^{p'}_{\alpha,i;\mathrm{d}x}[\mu]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}\mu)} \le c_2 < \infty,$$

for $i = 1, 2$ and for r defined by $1/p + 1/r = 1/q$. Here,

$$\mathcal{W}^q_{\alpha,1;\mu}[\mathrm{d}x](x) := \sum_{S \in \mathcal{S}_1} \ell_S^{\alpha-n} \mu(S) \left(\frac{1}{\mu(S)} \sum_{S' \in \mathcal{S}_1|_S} \ell_{S'}^{\alpha} \mu(S') \right)^{q-1} \mathbf{1}_S(x),$$

$$\mathcal{W}^q_{\alpha,2;\mu}[\mathrm{d}x](x) := \sum_{S \in \mathcal{S}_2} \ell_S^{-n} \mu(S) \left(\frac{1}{\mu(S)} \sum_{S' \in \mathcal{S}_2|_S} \mu(S') \right)^{q-1} \mathbf{1}_S(x),$$

$$\mathcal{W}^{p'}_{\alpha,1;\mathrm{d}x}[\mu](x) := \sum_{S \in \mathcal{S}_1} \ell_S^{\alpha} \left(\frac{1}{|S|} \sum_{S' \in \mathcal{S}_1|_S} \ell_{S'}^{\alpha} \mu(S') \right)^{p'-1} \mathbf{1}_S(x),$$

$$\mathcal{W}^{p'}_{\alpha,2;\mathrm{d}x}[\mu](x) := \sum_{S \in \mathcal{S}_2} \left(\frac{1}{|S|} \sum_{S' \in \mathcal{S}_2|_S} \mu(S') \right)^{p'-1} \mathbf{1}_S(x).$$

Moreover, the least possible constants c_1 and c_2 are equivalent.

Hence, to estimate the operator norm

$$\|S_\alpha\|_{L^p(\mathrm{d}x) \to L^q(\mathrm{d}\mu)} \approx c_2,$$

we analyze the norms of Wolff's potentials.

By Lemma 2.4 and using Condition (A), we can reduce

$$\mathcal{W}^q_{\alpha,1;\mu}[\mathrm{d}x](x) \lesssim \sum_{S \in \mathcal{S}_1} (\ell_S^{\alpha})^q \frac{\mu(S)}{|S|} \mathbf{1}_S(x),$$

$$\mathcal{W}^q_{\alpha,2;\mu}[\mathrm{d}x](x) \lesssim \sum_{S \in \mathcal{S}_2} \frac{\mu(S)}{|S|} \mathbf{1}_S(x),$$

$$\mathcal{W}^{p'}_{\alpha,1;\mathrm{d}x}[\mu](x) \lesssim \sum_{S \in \mathcal{S}_1} (\ell_S^{\alpha})^{p'} \left(\frac{\mu(S)}{|S|} \right)^{p'-1} \mathbf{1}_S(x),$$

$$\mathcal{W}^{p'}_{\alpha,2;\mathrm{d}x}[\mu](x) \lesssim \sum_{S \in \mathcal{S}_2} \left(\frac{\mu(S)}{|S|} \right)^{p'-1} \mathbf{1}_S(x).$$

Claim 4.3. *For $i = 1, 2$, we claim that*

$$\left\| \mathcal{W}^q_{\alpha,i;\mu}[\mathrm{d}x]^{\frac{1}{q}} \right\|_{L^r(\mathrm{d}x)} \lesssim \left\| \mathcal{W}^{p'}_{\alpha,i;\mathrm{d}x}[\mu]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}\mu)}.$$

Proof. Since the same proof is valid, we only treat the case $i = 1$. It follows from Lemma 2.5 that, recalling $1/q = 1/p - 1/r$,

$$\begin{aligned}
\left\|\mathcal{W}^q_{\alpha,1;\mu}[\mathrm{d}x]^{\frac{1}{q}}\right\|^r_{L^r(\mathrm{d}x)} &\lesssim \sum_{S\in\mathcal{S}_1}\left((\ell_S^\alpha)^q\frac{\mu(S)}{|S|}\right)^{\frac{r}{q}}|S| \\
&= \sum_{S\in\mathcal{S}_1}\left((\ell_S^\alpha)^q\frac{\mu(S)}{|S|}\right)^{\frac{r}{q}}\frac{|S|}{\mu(S)}\mu(S) \\
&= \sum_{S\in\mathcal{S}_1}(\ell_S^\alpha)^r\left(\frac{\mu(S)}{|S|}\right)^{\frac{r}{p}}\mu(S) \\
&= \int_{\mathbb{R}^n}\left(\sum_{S\in\mathcal{S}_1}(\ell_S^\alpha)^r\left(\frac{\mu(S)}{|S|}\right)^{\frac{r}{p}}\mathbf{1}_S\right)\mathrm{d}\mu \\
&\le \int_{\mathbb{R}^n}\left(\sum_{S\in\mathcal{S}_1}(\ell_S^\alpha)^{p'}\left(\frac{\mu(S)}{|S|}\right)^{p'-1}\mathbf{1}_S\right)^{\frac{r}{p'}}\mathrm{d}\mu \\
&\le \left\|\left[\sum_{S\in\mathcal{S}_1}\left(\ell_S^\alpha\left(\frac{\mu(S)}{|S|}\right)^{\frac{1}{p}}\right)^{p'}\mathbf{1}_S\right]^{\frac{1}{p'}}\right\|_{L^r(\mathrm{d}\mu)},
\end{aligned}$$

where we have used $p' - 1 = p'/p$. □

Thus, by (3.7) we have the following proposition.

Proposition 4.4. *Let $0 < \alpha < n$, $1 < q < p < \infty$ and μ be a positive Borel measure on $\mathbb{R}^n$ satisfying Condition* (A). *Then for $\lambda \in (1,\infty)$, the operator norm $\|S_{\alpha,\lambda}\|_{L^p(\mathrm{d}x)\to L^q(\mathrm{d}\mu)}$ is majorized by*

$$\max\left(\begin{array}{c}\left\|\left[\sum_{S\in\mathcal{S}:\,\ell_S\le 1/\lambda}\left(\ell_S^\alpha\left(\frac{\mu(S)}{|S|}\right)^{\frac{1}{p}}\right)^{p'}\mathbf{1}_S\right]^{\frac{1}{p'}}\right\|_{L^r(\mathrm{d}\mu)}, \\ \lambda^{-\alpha}\left\|\left[\sum_{S\in\mathcal{S}:\,\ell_S>1/\lambda}\left(\frac{\mu(S)}{|S|}\right)^{\frac{p'}{p}}\mathbf{1}_S\right]^{\frac{1}{p'}}\right\|_{L^r(\mathrm{d}\mu)}\end{array}\right).$$

5. Proof of Theorems 1.1–1.3

In what follows we shall prove Theorems 1.1–1.3. Thanks to Propositions 4.1–4.4, we need only verify Theorem 1.1.

By the argument in Introduction, we shall prove that, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that

$$\|S_{\alpha,\lambda_0}\|_{L^p(\mathrm{d}x)\to L^q(\mathrm{d}\mu)} < \varepsilon, \tag{5.1}$$

when

$$\max\left(\begin{array}{c}\sup_{Q\in\mathcal{D}:\,\ell_Q\le 1}|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}}, \\ \sup_{Q\in\mathcal{D}:\,\ell_Q>1}\frac{\mu(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}}\end{array}\right) = C_0 < \infty \tag{5.2}$$

and

$$\lim_{\lambda\to\infty} \sup_{Q\in\mathcal{D}:\,\ell_Q\le 1/\lambda} |Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} = 0 \tag{5.3}$$

hold.

Take an $\varepsilon>0$. First we choose an integer $N_1\in\mathbb{N}$ so that

$$\frac{C_0}{(2^{N_1})^\alpha} < \varepsilon. \tag{5.4}$$

Next, thanks to (5.3), we choose an integer $N_0>N_1$ so that

$$\sup_{Q\in\mathcal{D}:\,\ell_Q\le 2^{N_1-N_0}} |Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} < \varepsilon. \tag{5.5}$$

Let us use Proposition 4.1 with $\lambda_0=2^{N_0}$. We must verify that

$$\max\begin{pmatrix} \sup_{Q\in\mathcal{D}:\,\ell_Q\le 1/\lambda_0} |Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}}, \\ \sup_{Q\in\mathcal{D}:\,1/\lambda_0<\ell_Q\le 1} (\lambda_0\ell_Q)^{-\alpha}\cdot|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}},{}^{2} \\ (\lambda_0)^{-\alpha}\sup_{Q\in\mathcal{D}:\,\ell_Q>1} \dfrac{\mu(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}} \end{pmatrix} < \varepsilon.$$

Indeed, it follows from (5.5) that

$$\begin{aligned} &\sup\left\{|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} : Q\in\mathcal{D},\, \ell_Q\le 2^{-N_0}\right\} \\ &\le \sup\left\{|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} : Q\in\mathcal{D},\, \ell_Q\le 2^{N_1-N_0}\right\} < \varepsilon, \end{aligned}$$

again from (5.5) that

$$\begin{aligned} &\sup\left\{(2^{N_0}\ell_Q)^{-\alpha}\cdot|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} : Q\in\mathcal{D},\, 2^{-N_0}<\ell_Q\le 2^{N_1-N_0}\right\} \\ &\le \sup\left\{|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} : Q\in\mathcal{D},\, 2^{-N_0}<\ell_Q\le 2^{N_1-N_0}\right\} < \varepsilon, \end{aligned}$$

from (5.2) and (5.4) that

$$\begin{aligned} &\sup\left\{(2^{N_0}\ell_Q)^{-\alpha}\cdot|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} : Q\in\mathcal{D},\, 2^{N_1-N_0}<\ell_Q\le 1\right\} \\ &\le \frac{1}{(2^{N_1})^\alpha}\sup\left\{|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} : Q\in\mathcal{D},\, 2^{N_1-N_0}<\ell_Q\le 1\right\} \le \frac{C_0}{(2^{N_1})^\alpha} < \varepsilon, \end{aligned}$$

and from (5.2), (5.4) and $N_0>N_1$ that

$$\lambda_0^{-\alpha}\sup\left\{\frac{\mu(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}} : Q\in\mathcal{D},\, \ell_Q>1\right\} \le \frac{C_0}{(2^{N_1})^\alpha} < \varepsilon.$$

Thus, the operator norm $\|S_{\alpha,\lambda_0}\|_{L^p(\mathrm{d}x)\to L^q(\mathrm{d}\mu)}<\varepsilon$. This completes the proof.

Remark. We remark that then $C(\varepsilon)\approx(2^{N_0})^\alpha\varepsilon$.

---

[2]We notice that $(\lambda_0\ell_Q)^{-\alpha}\cdot|Q|^{\frac{\alpha}{n}-\frac{1}{p}}\mu(Q)^{\frac{1}{q}} = \lambda^{-\alpha}\dfrac{\mu(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}}$.

Remark 5.1. We remark that, if $w \in A_\infty$, the measure $\mathrm{d}\mu = w(x)\,\mathrm{d}x$ satisfies the Condition (A). This fact can be verified as follows:

Take a sparse family $\mathcal{S} \subset \mathcal{D}(Q)$, $Q \in \mathcal{Q}$. It follows that

$$\begin{aligned}\sum_{S\in\mathcal{S}} \mu(S)^a &= \int_Q \sum_{S\in\mathcal{S}} \mu(S)^{a-1}\mathbf{1}_S\,\mathrm{d}\mu \\ &\lesssim \int_Q \sum_{S\in\mathcal{S}} \mu(S)^{a-1}(M\mathbf{1}_{E_{\mathcal{S}}(S)})^\eta\,\mathrm{d}\mu^3 \\ &\lesssim \int_Q \sum_{S\in\mathcal{S}} \mu(S)^{a-1}\mu(E_{\mathcal{S}}(S)) \\ &\le \mu(Q)^{a-1}\sum_{S\in\mathcal{S}} \mu(E_{\mathcal{S}}(S)) \le \mu(Q)^a.\end{aligned}$$

This is the desired inequality.

6. Applications

In what follows we will consider the particular case $\mathrm{d}\mu(x) = |x|^{aq}\,\mathrm{d}x$ in Theorems 1.1 and 1.3 to obtain a sufficient condition on the power a under which the infinitesimal relative bounds hold. We use the following simple and nice lemma.

Lemma 6.1. [10, Example 113] *Let $Q \in \mathcal{Q}(\mathbb{R}^n)$. If $\beta > -n$, then*

$$\int_Q |x|^\beta\,\mathrm{d}x \approx \max(\ell_Q,\, |c_Q|)^\beta |Q|.$$

Hereafter, we write $w_{aq}(x) = |x|^{aq}$. For its local integrability, one needs $aq+n \ge 0$ and then w_{aq} satisfies the condition (A) because it belongs in A_∞ (see Remark 5.1). Considering $\varphi \in C_c^\infty(\mathbb{R}^n)$ with its support goes to infinity, one needs also $a \le 0$.

6.1. Application of Theorem 1.1. We investigate the conditions (1.2) and (1.3). Thanks to $a \le 0$, it suffices to estimate only the cubes $Q \in \mathcal{Q}$ containing the origin. Then we have

$$|Q|^{\frac{\alpha}{n}-\frac{1}{p}} w_{aq}(Q)^{\frac{1}{q}} \approx \ell_Q^{\alpha-\frac{n}{p}+a+\frac{n}{q}} \to 0, \quad \ell_Q \to 0,$$

whenever $a > -\alpha + n(1/p - 1/q)$, and

$$\frac{w_{aq}(Q)^{\frac{1}{q}}}{|Q|^{\frac{1}{p}}} \approx \ell_Q^{a-\frac{n}{p}+\frac{n}{q}} < \infty,$$

whenever $a \le n(1/p - 1/q)$. Thus, we obtain the following theorem.

Theorem 6.2. *Let $0 < \alpha < n$ and $1 < p \le q < \infty$. For any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\||x|^a\varphi\|_{L^q(\mathrm{d}x)} \lesssim \varepsilon\|(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^p(\mathrm{d}x)} + C(\varepsilon)\|\varphi\|_{L^p(\mathrm{d}x)}$$

holds if

$$\max(-n/q,\, -\alpha + n(1/p - 1/q)) < a \le 0, \quad 1 < p \le q < \infty.$$

[3] Since $w \in A_\infty$, we can take an $\eta > 1$ large enough so that $w \in A_\eta$. Then we apply the vector-valued Hardy–Littlewood maximal theorem (see [4]).

6.2. Application of Theorem 1.3. We investigate the conditions (1.6) and (1.7). We assume that $-n/q < a < -n/r = n(1/p - 1/q)$ and that $\mathcal{S} \subset \mathcal{D}(Q)$ for some sufficiently large $Q \in \mathcal{Q}(\mathbb{R}^n)$ containing the origin.

Letting $\mathcal{S}_{1/\lambda} := \{S \in \mathcal{S} \colon \ell_S \le 1/\lambda\}$, $\lambda \ge 1$, we have that

$$
\begin{aligned}
(\mathrm{I}) &:= \left\| \left[\sum_{S \in \mathcal{S}_{1/\lambda}} \left(\ell_S^{\alpha} \left(\frac{w_{aq}(S)}{|S|} \right)^{\frac{1}{p}} \right)^{p'} \mathbf{1}_S \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}w_{aq})}^r \\
&\lesssim \left\| \left[\sum_{S \in \mathcal{S}_{1/\lambda}} \left(\ell_S^{\alpha} \left(\frac{w_{aq}(S)}{|S|} \right)^{\frac{1}{p}} \right)^{p'} (M \mathbf{1}_{E_{\mathcal{S}}(S)})^{\eta} \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}w_{aq})}^r \quad {}^{4} \\
&\lesssim \left\| \left[\sum_{S \in \mathcal{S}_{1/\lambda}} \left(\ell_S^{\alpha} \left(\frac{w_{aq}(S)}{|S|} \right)^{\frac{1}{p}} \right)^{p'} \mathbf{1}_{E_{\mathcal{S}}(S)} \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}w_{aq})}^r \\
&= \sum_{S \in \mathcal{S}_{1/\lambda}} \left(\ell_S^{\alpha} \left(\frac{w_{aq}(S)}{|S|} \right)^{\frac{1}{p}} \right)^{r} w_{aq}(E_{\mathcal{S}}(S)) \\
&\approx \sum_{S \in \mathcal{S}_{1/\lambda}} \max(\ell_S, |c_S|)^{ar} |S|^{\frac{\alpha r}{n} + 1} \\
&\lesssim \sum_{S \in \mathcal{S}_{1/\lambda}} \max(\ell_S, |c_S|)^{ar} \ell_S^{\alpha r} |E_{\mathcal{S}}(S)|.
\end{aligned}
$$

There hold

$$
\begin{aligned}
&\sum_{S \in \mathcal{S}_{1/\lambda} \colon |c_S| \le 1} \max(\ell_S, |c_S|)^{ar} \ell_S^{\alpha r} |E_{\mathcal{S}}(S)|^{5} \\
&\le (1/\lambda)^{(a+\alpha)r} \sum_{S \in \mathcal{S}_{1/\lambda} \colon |c_S| \le 1} |E_{\mathcal{S}}(S)| \to 0, \quad \lambda \to \infty,
\end{aligned}
$$

whenever $-\alpha < a$, and

$$
\sum_{S \in \mathcal{S}_{1/\lambda} \colon |c_S| > 1} \max(\ell_S, |c_S|)^{ar} \ell_S^{\alpha r} |E_{\mathcal{S}}(S)| \le (1/\lambda)^{\alpha r} \int_{\{|y| > 1\}} |x|^{ar} \, \mathrm{d}x \to 0, \quad \lambda \to \infty,
$$

whenever $a < -n/r$. Thus, if $\max(-n/q, -\alpha) < a < -n/r$, then $(\mathrm{I}) \to 0, \quad \lambda \to \infty$.

In a similar fashion, Letting $\mathcal{S}_2 := \{S \in \mathcal{S} \colon \ell_S > 1\}$, we observe that

$$
\begin{aligned}
\left\| \left[\sum_{S \in \mathcal{S}_2} \left(\frac{w_{aq}(S)}{|S|} \right)^{\frac{p'}{p}} \mathbf{1}_S \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}w_{aq})}^r
&\lesssim \left\| \left[\sum_{S \in \mathcal{S}_2} \left(\frac{w_{aq}(S)}{|S|} \right)^{\frac{p'}{p}} \mathbf{1}_{E_{\mathcal{S}}(S)} \right]^{\frac{1}{p'}} \right\|_{L^r(\mathrm{d}w_{aq})}^r \\
&= \sum_{S \in \mathcal{S}_2} \left(\frac{w_{aq}(S)}{|S|} \right)^{\frac{r}{p}} w_{aq}(E_{\mathcal{S}}(S))
\end{aligned}
$$

---

[4]Since $w_{aq} \in A_\infty$, we can take an $\eta > 1$ large enough so that $w_{aq} \in A_{\eta r/p'}$. Then we apply the vector-valued Hardy–Littlewood maximal theorem.

[5]Since $ar < 0$, one has that $\max(\ell_S, |c_S|)^{ar} \ell_S^{\alpha r} \lesssim (1/\lambda)^{(a+\alpha)r}$.

$$\approx \sum_{S\in\mathcal{S}_2} \max(\ell_S,\, |c_S|)^{ar}|S|$$
$$\lesssim \sum_{S\in\mathcal{S}_2} \max(\ell_S,\, |c_S|)^{ar}|E_{\mathcal{S}}(S)|$$
$$\lesssim 1 + \int_{\{|y|>1\}} |x|^{ar}\,\mathrm{d}x < \infty,$$

whenever $a < -n/r$. Thus, we obtain the following theorem.

Theorem 6.3. *Let $0 < \alpha < n$ and $1 < q < p < \infty$. For any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\||x|^a\varphi\|_{L^q(\mathrm{d}x)} \lesssim \varepsilon\|(-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^p(\mathrm{d}x)} + C(\varepsilon)\|\varphi\|_{L^p(\mathrm{d}x)}$$

holds if

$$\max(-n/q,\, -\alpha) < a < n(1/p - 1/q), \quad 1 < q < p < \infty.$$

Funding. The first named author is supported by Grant-in-Aid for Research Activity Start-up Grant Number 23K19013. The third named author is supported by Grant-in-Aid for Young Scientists (23K03171), the Japan Society for the Promotion of Science. The forth named author is supported by Grant-in-Aid for Scientific Research (C) (15K04918 and 19K03510), the Japan Society for the Promotion of Science.

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Received 3 March 2025 • Accepted 27 October 2025 • Published online 17 November 2025

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