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Hyperbolicity of the sub-Riemannian affine-additive group

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Dedicated to Hans Martin Reimann

Abstract. We consider the affine-additive group as a metric measure space with a canonical left-invariant measure and a left-invariant sub-Riemannian metric. We prove that this metric measure space is locally 4-Ahlfors regular and it is hyperbolic, meaning that it has a non-vanishing 4-capacity at infinity. This implies that the affine-additive group is not quasiconformally equivalent to the Heisenberg group or to the roto-translation group in contrast to the fact that both of these groups are globally contactomorphic to the affine-additive group. Moreover, each quasiregular map, from the Heisenberg group to the affine-additive group must be constant.

Esi-Riemannin affiinin yhteenlaskuryhmän hyperbolisuus

Tiivistelmä. Tässä työssä tarkastellaan affiinia yhteenlaskuryhmää vasemmalta vakaalla mitalla ja esi-Riemannin metriikalla varustettuna metrisenä mitta-avaruutena. Tämä avaruus osoitetaan Ahlforsin 4-säännölliseksi ja hyperboliseksi, mikä tarkoittaa, että sillä on häviämätön 4-kapasiteetti äärettömyydessä. Tästä seuraa, että affiini yhteenlaskuryhmä ei ole kvasikonformisesti yhtäpitävä Heisenbergin ryhmän eikä kierto–siirtoryhmän kanssa – vastakohtana sille, että nämä molemmat ryhmät ovat kontaktigeometrisessa mielessä yhdenmuotoisia affiinin yhteenlaskuryhmän kanssa. Lisäksi ainoat kvasisäännölliset kuvaukset Heisenbergin ryhmästä affiiniin yhteenlaskuryhmään ovat vakioita.

1. Introduction and statement of the main result

Due to work of Heinonen and Koskela [12], the theory of quasiconformal mappings has been developed in the setting of general metric measure spaces satisfying some mild regularity properties. For the related analytic machinery including upper gradients, capacities and Sobolev spaces we refer to the book of Heinonen [11], or the book of Heinonen, Koskela, Shanmungalingam and Tyson [13].

An important class of examples where these results apply is the geometric setting of sub-Riemannian spaces, including Heisenberg groups. Motivated by Mostow rigidity [21], the theory of quasiconformal mappings in the Heisenberg group has been developed by Pansu [22] and Korányi and Reimann in [15] and [16]. This theory is rather advanced, examples of non-trivial quasiconformal maps acting between Heisenberg groups have been constructed as flows of contact vector fields by Korányi and Reimann [15, 16] and by lifting of planar symplectic maps by Capogna and Tang

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[6]. Extremal quasiconformal maps that are similar to the planar stretch map, acting between Heisenberg groups were found by Balogh, Fässler and Platis [3]. Using the flow method of Korányi and Reimann, Balogh established in [1] the existence of quasiconformal maps between Heisenberg groups distorting the Hausdorff dimension of Cantor sets in a rather arbitrary fashion.

By a theorem of Darboux (see Theorem 18.19 in Lee [18]), every contact manifold is locally bi-Lipschitz to the Heisenberg group, one would expect that the results of quasiconformal maps could be transposed from the Heisenberg setting to general contact manifolds endowed with a sub-Riemannian metric. However, this turns out not to be the case as not all contact manifolds are *globally* quasiconformal to the Heisenberg group. A remarkable example of this has been found by Fässler, Koskela and Le Donne [8] who proved that the sub-Riemannian roto-translation group is not globally quasiconformal to the Heisenberg group, in contrast to the fact, that there exists a global contactomorphism between these spaces.

In the present paper we consider another natural three dimensional Lie group: the affine-additive group endowed with a sub-Riemannian metric. We prove that it is also globally contactomorphic to both, the Heisenberg group and (by [8]) also to the roto-translation group. However, the affine-additive group is not globally quasiconformal to neither the Heisenberg, nor to the roto-translation group. The reason for the non-existence of a global quasiconformal map between these groups is their behaviour at infinity as formulated by Zorich in [25] (see also [14, 10]). We prove that the affine-additive group has a non-vanishing 4-capacity at infinity, thus it is hyperbolic in the terminology of [25], while both the Heisenberg and the roto-translation groups are parabolic, having a vanishing 4-capacity at infinity.

To be more precise we define the affine-additive group $(\mathcal{AA}, \star)$ as the Cartesian product of $\mathbb{R}$ and the hyperbolic right half plane:

$$\mathcal{AA} = \mathbb{R} \times \mathbf{H}^1_{\mathbb{C}} \quad \text{where} \quad \mathbf{H}^1_{\mathbb{C}} := \{(\lambda, t) \colon \lambda > 0,\ t \in \mathbb{R}\}$$

together with the group law

$$(a', \lambda', t') \star (a, \lambda, t) = (a' + a, \lambda'\lambda, \lambda' t + t')$$

and the contact 1-form

$$\vartheta = \frac{dt}{2\lambda} - da.$$

For a detailed presentation of the geometric structure of the affine-additive group $\mathcal{AA}$ we refer to Section 3 of this paper. At this point, we can say that the Carnot–Carathéodory distance $d_{\mathcal{AA}}$ will be defined as the sub-Riemannian distance on $\mathcal{AA}$ generated by the horizontal vector fields

$$U = \partial_a + 2\lambda\partial_t, \quad V = 2\lambda\partial_\lambda,$$

and a sub-Riemannian metric making $\{U, V\}$ an orthonormal frame. The left-invariant Haar measure on the group $\mathcal{AA}$ is given by $d\mu_{\mathcal{AA}} = \frac{da\, d\lambda\, dt}{\lambda^2}$. The main result of the paper is the following:

Theorem 1.1. *The metric measure space $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$ is a locally 4-Ahlfors regular space. It is globally contactomorphic to the first Heisenberg group $\mathbb{H}$. The sub-Riemannian manifold, $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$ is 4 hyperbolic, in particular there is no non-trivial quasiregular map $f \colon \mathbb{H} \to \mathcal{AA}$.*

The paper is organized as follows: in the Section 2 we fix notation, recall preliminaries on metric measure spaces and give a sufficient condition on the parabolicity of a metric measure space. In Section 3 we consider the sub-Riemannian metric of the affine-additive group in greater detail. Here we prove that the affine-additive group is globally contactomorphic to the Heisenberg group. In Section 4 we prove the main result of this paper about the hyperbolicity of the affine-additive group and discuss its consequences.

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2. Preliminaries on metric measure spaces

We start by recalling some concepts and results on the theory quasiconformal (QC) maps in the setting of general metric measure spaces. For more details we refer to the paper of Heinonen and Koskela [12], the book of Heinonen [11] and the book of Heinonen, Koskela, Shanmugalingam and Tyson [13].

Let us recall that a homeomorphism $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called *quasiconformal* if there exists $K \geq 1$ such that

$$\limsup_{r\to 0} \frac{\sup_{d_X(p,q)\leq r} d_Y(f(p), f(q))}{\inf_{d_X(p,q)\geq r} d_Y(f(p), f(q))} := H_f(p) \leq K, \tag{1}$$

for all p in X.

A metric measure space is a triple (X, d_X, μ_X) comprising a non empty set X, a distance function d_X and a regular Borel measure μ_X such that (X, d_X) is a complete, and separable metric space and every metric ball has positive and finite measure. This setting will be our standing assumption throughout this paper.

Given a point $p \in X$ and a radius $r > 0$, we employ the following notation for balls:

$$B_{d_X}(p, r) = \{q \in X\colon d_X(p, q) < r\} \text{ and } \overline{B}_{d_X}(p, r) = \{q \in X\colon d_X(p, q) \leq r\}.$$

Where it will not cause confusion, we will replace $B_{d_X}(p, r)$ by $B(p, r)$.

A metric measure space (X, d_X, μ_X) is called *Ahlfors Q-regular*, $Q > 1$, if there exists a constant $C \geq 1$ such that for all $p \in X$ and $0 < r \leq \operatorname{diam} X$, we have

$$C^{-1}r^Q \leq \mu_X(\overline{B}_{d_X}(p, r)) \leq Cr^Q. \tag{2}$$

Further, we say that (X, d_X, μ_X) is *locally Ahlfors Q-regular*, if for every compact subset $V \subset X$, there is a constant $C \geq 1$ and a radius $r_0 > 0$ such that for each point $p \in V$ and each radius $0 < r \leq r_0$ we have

$$C^{-1}r^Q \leq \mu_X(\overline{B}_{d_X}(p, r)) \leq Cr^Q. \tag{3}$$

An important geometric quantity in the theory of quasiconformal mappings is the Q-modulus of a curve family. Let us recall, that Γ is a family of curves in the metric measure space (X, d_X, μ_X), a Borel function $\rho\colon X \to [0, \infty]$ is said to be *admissible* for Γ if for every rectifiable $\gamma \in \Gamma$, we have

$$1 \leq \int_\gamma \rho \, d\ell_X.$$

Such a ρ shall be also called a density and the set of all densities shall be denoted by $\mathrm{Adm}(\Gamma)$. If $Q > 1$ then the *Q-modulus* of Γ is

$$\mathrm{Mod}_Q(\Gamma) = \inf_{\rho\in \mathrm{Adm}(\Gamma)} \int_X \rho^Q \, d\mu_X.$$

It follows immediately from this definition that if Γ_0 and Γ are two curve families such that each curve $\gamma \in \Gamma$ has a sub-curve $\gamma_0 \in \Gamma_0$, then

$$\operatorname{Mod}_Q(\Gamma) \leq \operatorname{Mod}_Q(\Gamma_0). \tag{4}$$

Let us recall that by Theorem 3.8 in [17], if (X, d_X, μ_X) and (Y, d_Y, μ_Y) are separable metric measure spaces of locally finite measure that are both locally Ahlfors Q-regular for some given $Q > 1$ and $f\colon X \to Y$ is a quasiconformal map then there exists $H \geq 1$ such that

$$\operatorname{Mod}_Q(\Gamma) \leq H \operatorname{Mod}_Q(f(\Gamma)), \tag{5}$$

for every curve family Γ in X, i.e., the Mod_Q is quasi-preserved by quasiconformal maps.

For two disjoint compact sets $E, F \subset X$ we consider the number $\operatorname{Mod}_Q(E, F) = \operatorname{Mod}_Q(\Gamma)$ where Γ is the set of all rectifiable curves connecting E and F. If $x_0 \in X$ is a fixed point and $0 < r < R < \operatorname{diam} X$, $E = \partial B(x_0, r)$ and $F = \partial B(x_0, R)$ then the quantity $\operatorname{Mod}_Q(E, F) = \operatorname{Mod}_Q(\mathcal{D}(r, R))$ is the so called modulus of the ring domain

$$\mathcal{D}(r, R) = \{x \in X\colon r < d(x, x_0) < R\}.$$

The following definition is a reformulation in the setting of metric spaces of the corresponding concept by Zorich [25]. For related results we refer also to Holopainen and Rickman [14], Coulhon, Holopainen and Saloff-Coste [7], Fässler, Lukyanenko and Tyson [10].

Definition 2.1. The metric measure spaces (X, d_X, μ_X) is Q-parabolic if and only if for some $x_0 \in X$ and $R_0 > 0$ we have

$$\lim_{R \to \infty} \operatorname{Mod}_Q(\mathcal{D}(R_0, R)) = 0. \tag{6}$$

Otherwise we call (X, d_X, μ_X) Q-hyperbolic.

Let us note that this property does not depend on the choice of $x_0 \in X$ and $R_0 > 0$, in particular when x_0, R_0 satisfying (6) exist then (6) holds true for any other choices of $x_0' \in X$, $R_0' > 0$. This means that parabolicity of a metric measure space is a property about the behaviour of the space at infinity.
We also remark that Q-parabolicity of a metric measure space can be defined equivalently by capacity of condensers (see Section 7 in [25] and Definition 4.5.4 in [10]).

The following sufficient condition seems to be known to experts, however we could not locate a precise reference and we include it for the sake of completeness.

Proposition 2.2. *Let $Q > 1$ and (X, d_X, μ_X) be a metric measure space such that there exists $x_0 \in X$, $R_0 > 0$ and $K > 0$ such that for all $R > R_0$ we have*

$$\mu_X(B(x_0, R)) \leq K R^Q. \tag{7}$$

Then (X, d_X, μ_X) is Q'-parabolic for any $Q' \geq Q$.

Proof. We shall consider the ring domain $\mathcal{D}(R_0, R) = \{x \in X\colon R_0 < d_X(x, x_0) < R\}$ for $R > R_0$. Our purpose is to show that

$$\lim_{R \to \infty} \operatorname{Mod}_{Q'}(\mathcal{D}(R_0, R)) = 0.$$

To do this, we consider the integer $N \in \mathbb{N}$ defined by the property that $2^N R_0 \geq R > 2^{N-1} R_0$. Note, that if $R \to \infty$, then $N \to \infty$. Consider the density

$$\rho_N(x) = \begin{cases} \frac{3}{N} \cdot \frac{1}{d_X(x_0, x)} & \text{if } x \in \mathcal{D}(R_0, R), \\ 0 & \text{otherwise.} \end{cases}$$

Let us check that the ρ_N is an admissible density for the curve family Γ connecting $\partial B(x_0, R_0)$ and $\partial B(x_0, R)$. To do so we consider the integers $1 < k < N$ and denote by $B_k = B(x_0, 2^k R_0)$ and $D_k = B_k \setminus B_{k-1}$. For $\gamma \in \Gamma$ denote by $\gamma_k = D_k \cap \gamma$. By this notation, we observe that the length of γ_k, $\ell_X(\gamma_k) \geq 2^{k-1} R_0$ and if $x \in \gamma_k$, then $\rho_N(x) \geq \frac{3}{N} \cdot \frac{1}{2^k R_0}$. Using this information we can write

$$\int_\gamma \rho_N \, d\ell_X \geq \sum_{k=2}^{N-1} \int_{\gamma_k} \rho_N \, d\ell_X \geq \sum_{k=2}^{N-1} \frac{3}{N} \cdot \frac{1}{2^k R_0} \ell(\gamma_k) \geq \frac{3(N-2)}{2N} \geq 1,$$

if $N \geq 6$. Note, that by our assumption on the upper of the measure (7) we have that $\mu_X(B_k) \leq K 2^{kQ} R_0^Q$. Using this upper estimate on the measure of B_k, the assumption $Q' \geq Q$ and the fact that for $x \in B_k$ we have $\rho(x) \leq \frac{3}{N} \frac{1}{2^{k-1} R_0}$, we can estimate

$$\begin{aligned}
\operatorname{Mod}_{Q'} \mathcal{D}(R_0, R) &\leq \int_{\mathcal{D}(R_0,R)} \rho_N^{Q'} \, d\mu_X \leq \sum_{k=1}^{N} \int_{D_k} \rho_N^{Q'} \, d\mu_X \\
&\leq \sum_{k=1}^{N} \int_{B_k} \left(\frac{3}{N} \frac{1}{2^{k-1} R_0} \right)^{Q'} d\mu_X = \sum_{k=1}^{N} \left(\frac{3}{N} \frac{1}{2^{k-1} R_0} \right)^{Q'} \mu_X(B_k) \\
&\leq K \left(\frac{6}{N} \right)^{Q'} R_0^{Q-Q'} \sum_{k=1}^{N} 2^{k(Q-Q')} \to 0 \quad \text{as} \quad N \to \infty.
\end{aligned}$$

Since $R \to \infty$ implies that $N \to \infty$ we obtain the statement. □

As expected, our next statement is a formulation of the fact that a parabolic metric measure space cannot be quasiconformally equivalent to a hyperbolic one. In order to formulate the statement we recall that a metric space is proper, if its closed metric balls are compact.

Theorem 2.3. *Let $Q > 1$ and let (X, d_X, μ_X), $(X', d_{X'}, \mu_{X'})$ be two locally Ahlfors Q-regular metric measure spaces. Assume that both spaces are proper and (X, d_X, μ_X) is hyperbolic and $(X', d_{X'}, \mu_{X'})$ is a parabolic space. Then there is no QC map $f \colon X \to X'$.*

Proof. Assume by contradiction that there is a QC map $f \colon X \to X'$. Since (X, d_X, μ_X) is assumed to be hyperbolic, there exist a point $x_0 \in X$, $R_0 > 0$, a sequence $R_n \to \infty$, and a number $M > 0$ such that

$$\operatorname{Mod}_Q(\Gamma_n) \geq M > 0, \quad n \geq n_0,$$

where Γ_n is the set of curves connecting $\partial B_X(x_0, R_0)$ and $\partial B_X(x_0, R_n)$. By the relation (5) there exists $H \geq 1$ such that

$$\operatorname{Mod}_Q(f(\Gamma_n)) \geq \frac{\operatorname{Mod}_Q(\Gamma_n)}{H} \geq \frac{M}{H} > 0.$$

Let us denote by $y_0 = f(x_0) \in X'$. Since X is proper, $\bar{B}_X(x_0, R_0)$ is compact and thus $f(B_X(x_0, R_0))$ is bounded in X'. We conclude that there exists a number $R_0' > 0$ such that $f(B_X(x_0, R_0)) \subseteq B_{X'}(y_0, R_0')$. Let us denote by

$$R_n' := \min\{d_{X'}(f(x_0), f(x)) \colon x \in \partial B_X(x_0, R_n)\}.$$

We claim that $R_n' \to \infty$. For otherwise, we find a sequence $x_n \in X$ with $d_X(x_0, x_n) = R_n$ such that $d_X'(f(x_0), f(x_n)) \leq M'$ for some fixed constant $M' > 0$. Since the space X' is a proper metric space, we obtain that (up to a subsequence) $f(x_n) \to y$ for some $y \in X'$. Let us denote by $x_1 = f^{-1}(y) \in X$ the preimage of y. Since f is a homeomorphism we have that $f(B_X(x_1, r))$ is a neighborhood of $y \in X'$ for any fixed $r > 0$. Since $f(x_n) \to y$ we must have that for n large enough $f(x_n) \in f(B_X(x_1, r))$, which is a contradiction to the injectivity of f.

Let us note that any curve in $f(\Gamma_n)$ has a sub-curve connecting $\partial B_{X'}(y_0, R_0')$ and $\partial B_{X'}(y_0, R_n')$. This implies by (4) that

$$\mathrm{Mod}_Q(D(R_0', R_n')) \geq \mathrm{Mod}_Q(f(\Gamma_n)) \geq \frac{M}{H},$$

which is a contradiction to the parabolicity of $(X', d_{X'}, \mu_{X'})$, concluding the proof. □

The metric spaces considered in this paper are 3-dimensional Lie groups $\mathbb{G}$ with group multiplication $\star$. We shall assume that $\mathbb{G}$ is equipped with a left-invariant contact form $\vartheta_{\mathbb{G}}$. Using this contact form we define a distribution of planes in the tangent bundle $\mathrm{T}_{\mathbb{G}}$ of $\mathbb{G}$ as $\mathcal{H}_{\mathbb{G}} = \ker \vartheta_{\mathbb{G}}$. Next, a left-invariant sub-Riemannian metric is constructed on $\mathbb{G}$ as follows. If X and Y are left-invariant vector fields such that $\mathcal{H}_{\mathbb{G}} = \mathrm{span}\{X, Y\}$, then a left-invariant sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathbb{G}}$ is considered in $\mathcal{H}_{\mathbb{G}}$, making $\{X, Y\}$ an orthonormal basis of $\mathcal{H}_{\mathbb{G}}$.

An absolutely continuous curve $\gamma\colon [a, b] \to \mathbb{G}$, $\gamma = \gamma(s)$ shall be called horizontal if $\dot\gamma(s) \in \ker(\vartheta_{\mathbb{G}})_{\gamma(s)}$ for almost every $s \in [a, b]$. Then, the horizontal velocity of γ is

$$|\dot\gamma(s)|_{\mathbb{G}} = \sqrt{\langle \dot\gamma(s), X_{\gamma(s)} \rangle_{\mathbb{G}}^2 + \langle \dot\gamma(s), Y_{\gamma(s)} \rangle_{\mathbb{G}}^2}.$$

The horizontal length of γ is

$$\ell_{\mathbb{G}}(\gamma) = \int_a^b |\dot\gamma(s)|_{\mathbb{G}}\, ds.$$

The corresponding sub-Riemannian or Carnot–Carathéodory distance $d_{\mathbb{G}}$ associated to the sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathbb{G}}$ is defined in $\mathbb{G}$ as follows: let $p, q \in \mathbb{G}$ and consider the family $\Gamma_{\mathbb{G}}(p, q)$ of horizontal curves $\gamma\colon [a, b] \to \mathbb{G}$ such that $\gamma(a) = p$ and $\gamma(b) = q$. Then

$$d_{\mathbb{G}}(p, q) = \inf_{\gamma \in \Gamma_{\mathbb{G}}(p,q)} \{\ell_{\mathbb{G}}(\gamma)\}. \tag{8}$$

We remark that the above definition only depends on the values of $\langle \cdot, \cdot \rangle_{\mathbb{G}}$ on $\mathcal{H}_{\mathbb{G}}$. Moreover, since $\mathcal{H}_{\mathbb{G}}$ is completely non integrable, the distance $d_{\mathbb{G}}$ is finite, geodesic, and induces the manifold topology (see e.g. [19, 20]). This will make the space $(X, d_X) = (\mathbb{G}, d_{\mathbb{G}})$ a metric space. We consider the measure $\mu_X = \mu_{\mathbb{G}}$ induced by the contact form $\vartheta_{\mathbb{G}}$ by $\mu_{\mathbb{G}} = \vartheta_{\mathbb{G}} \wedge d\vartheta_{\mathbb{G}}$ (up to a multiplicative constant different from 0) that is also left-invariant and gives our metric measure space $(\mathbb{G}, d_{\mathbb{G}}, \mu_{\mathbb{G}})$.

A well-known example of such a structure is the *first Heisenberg group* $\mathbb{H}$. Its underlying manifold is $\mathbb{C} \times \mathbb{R}$ with coordinates $(z = x + iy, t)$ and the group multiplication $\star$ is given by

$$p' \star p = (z' + z, t' + t + 2\Im(\overline{z}' z))$$

for every $p = (z, t)$ and $p' = (z', t')$ in $\mathbb{C} \times \mathbb{R}$.

The *contact form* of $\mathbb{H}$ is given by:

$$\vartheta_{\mathbb{H}} = dt + 2\Im(\overline{z}\, dz) = dt + 2(x\, dy - y\, dx).$$

The horizontal bundle $\mathcal{H}_{\mathbb{H}}$ of the tangent bundle is spanned by the vector fields

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t.$$

Denote the sub-Riemannian metric in $\mathbb{H}$ by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ making $\{X, Y\}$ an orthonormal frame. The horizontal length of a curve $\gamma = \gamma(s)$, $s \in [a, b]$, $\gamma(s) = (z(s), t(s))$ is

$$\ell_{\mathbb{H}}(\gamma) = \int_a^b |\dot{z}(s)|\, ds.$$

Denote also the corresponding Carnot–Carathéodory distance by $d_{\mathbb{H}}$. The measure $\mu_{\mathbb{H}}$ is a bi-invariant Haar measure for $\mathbb{H}$ and it coincides with the 3-dimensional Lebesgue measure in $\mathbb{C} \times \mathbb{R}$ denoted with $\mathcal{L}^3$. It turns out that $(\mathbb{H}, d_{\mathbb{H}}, \mu_{\mathbb{H}})$ is a parabolic, 4-Ahlfors regular metric measure space. It follows from Proposition 2.2 that the metric measure space $(\mathbb{H}, d_{\mathbb{H}}, \mu_{\mathbb{H}})$ is 4-parabolic. We note, that there is an elaborate theory of QC maps on the Heisenberg group (see e.g. [22, 15, 16, 6, 3, 24]). It is therefore of interest to identify those sub-Riemannian Lie groups that are QC equivalent to the Heisenberg group.

The second example is the *roto-translation group* $\mathcal{RT}$ (see Chapter 3 in [5] and [8]). Its underlying manifold is $\mathbb{C} \times \mathbb{R}$ with coordinates $p = (z = x + iy, t)$ and the group multiplication $\star$ is given by

$$p' \star p = \left(e^{it'} z + z', t' + t\right) \in \mathbb{C} \times \mathbb{R}$$

for every $p = (z, t)$ and $p' = (z', t')$ in $\mathbb{C} \times \mathbb{R}$.

The *contact form* of $\mathcal{RT}$ is given by

$$\vartheta_{\mathcal{RT}} = \sin t\, dx - \cos t\, dy.$$

The horizontal bundle $\mathcal{H}_{\mathcal{RT}}$ of the tangent bundle is spanned by the vector fields

$$X = \cos t\, \partial_x + \sin t\, \partial_y, \quad Y = \partial_t.$$

Denote the sub-Riemannian metric in $\mathcal{RT}$ by $\langle \cdot, \cdot \rangle_{\mathcal{RT}}$ making $\{X, Y\}$ an orthonormal frame. The horizontal length of a curve $\gamma = \gamma(s)$, $s \in [a, b]$, $\gamma(s) = (z(s), t(s))$ is

$$\ell_{\mathcal{RT}}(\gamma) = \int_a^b |\dot{\gamma}(s)|_{\mathcal{RT}}\, ds,$$

where

$$|\dot{\gamma}(s)|_{\mathcal{RT}} = \sqrt{(\dot{x}(s) \cos t(s) + \dot{y}(s) \sin t(s))^2 + \dot{t}(s)^2}.$$

Denote also the corresponding Carnot–Carathéodory distance by $d_{\mathcal{RT}}$. The measure $\mu_{\mathcal{RT}}$ is again a bi-invariant Haar measure of $\mathcal{RT}$ and it is the 3-dimensional Lebesgue measure in $\mathbb{C} \times \mathbb{R}$.

Later on we will need the following (Lemma 5.5 in [8]):

Proposition 2.4. *The manifolds $(\mathcal{RT}, \vartheta_{\mathcal{RT}})$ and $(\mathbb{H}, \vartheta_{\mathbb{H}})$ are globally contactomorphic: i.e., there is a diffeomorphism $f\colon \mathcal{RT} \to \mathbb{H}$ such that*

$$f^* \vartheta_{\mathbb{H}} = \sigma\, \vartheta_{\mathcal{RT}},$$

where $\sigma\colon \mathcal{RT} \to \mathbb{R}$ is a nowhere vanishing smooth function.

We will also need (Corollary 5.9 in [8]):

Proposition 2.5. *There exists $R_0 > 0$, and $C_0 > 0$ such that if $B_{\mathcal{RT}}(e_{\mathcal{RT}}, r)$ is the open CC-ball of centre $e_{\mathcal{RT}}$ and radius r then:*

$$\mathcal{L}^3(B_{\mathcal{RT}}(e_{\mathcal{RT}}, r)) \le C_0 r^3, \quad \text{for all } r \ge R_0. \tag{9}$$

The remarkable result of Fässler, Koskela and Le Donne states that in contrast to the fact that both spaces $(\mathbb{H}, d_{\mathbb{H}}, \mu_{\mathbb{H}})$ and $(\mathcal{RT}, d_{\mathcal{RT}}, \mu_{\mathcal{RT}})$ are 4-parabolic and by Proposition 2.4 locally bi-Lipschitz equivalent, they are still not QC equivalent (see Corollary 1.2 in [8]).

3. The affine-additive group as a metric measure space

The main subject of this paper is the affine-additive group, which we describe below. In particular, after introducing the group, we discuss its sub-Riemannian structure. For more details about the affine-additive group, we refer to [4].

Our starting point is the hyperbolic plane, defined as

$$\mathbf{H}^1_{\mathbb{C}} := \{\zeta = \xi + i\eta \in \mathbb{C} \colon \xi > 0\} \text{ with the Riemannian metric } g = \frac{|d\zeta|^2}{4\xi^2} = \frac{d\xi^2 + d\eta^2}{4\xi^2}.$$

We consider affine transformations on $\mathbf{H}^1_{\mathbb{C}}$, composed by dilations D_λ, $\lambda > 0$, defined by $D_\lambda(\zeta) = \lambda\zeta$, and translations T_t, $t \in \mathbb{R}$, defined by $T_t(\zeta) = \zeta + it$, for $\zeta \in \mathbf{H}^1_{\mathbb{C}}$ resulting in maps of the form

$$M(\lambda, t)(\zeta) = (T_t \circ D_\lambda)(\zeta) = \lambda\zeta + it.$$

It is clear that $\mathbf{H}^1_{\mathbb{C}}$ is in bijection with the set of transformations of the above form: to each point $\xi + i\eta$ we uniquely assign the transformation $M(\xi, \eta)$. Therefore we define a group structure on $\mathbf{H}^1_{\mathbb{C}}$ by considering the composition of any two transformations $M(\lambda', t')$ and $M(\lambda, t)$:

$$\begin{aligned}(M(\lambda', t') \circ M(\lambda, t))(\zeta) &= M(\lambda', t')(\lambda\zeta + it) = \lambda'\lambda\zeta + i(\lambda' t + t') \\ &= M(\lambda'\lambda, \lambda' t + t')(\zeta).\end{aligned}$$

To sum up, (compare to Section 4.4.2 in [23]) the group operation on $\mathbf{H}^1_{\mathbb{C}}$ is given by

$$(\lambda', t') \cdot (\lambda, t) = (\lambda'\lambda, \lambda' t + t'). \tag{10}$$

We wish to extend the previous construction over the space $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$. We define the group operation as follows: if $p' = (a', \lambda', t')$ and $p = (a, \lambda, t)$ are points of $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$, then

$$p' \star p = (a' + a, \lambda'\lambda, \lambda' t + t'), \tag{11}$$

which is again a point in $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$. This group operation is the group operation of the Cartesian product of the additive group $(\mathbb{R}, +)$ and the group $(\mathbf{H}^1_{\mathbb{C}}, \cdot)$, where $\cdot$ is as in (10).

Definition 3.1. The pair $\mathcal{AA} = (\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}, \star)$ shall be called the affine-additive group.

We define a 1-form on $\mathcal{AA}$ as follows:

$$\vartheta = \frac{dt}{2\lambda} - da. \tag{12}$$

Since $d\vartheta = \frac{1}{2\lambda^2}\, dt \wedge d\lambda$ we obtain $\vartheta \wedge d\vartheta = \frac{dt \wedge da \wedge d\lambda}{2\lambda^2}$ and thus $(\mathcal{AA}, \vartheta)$ is a contact manifold. In what follows we identify the left invariant vector fields and define a left invariant sub-Riemannian metric on the group $\mathcal{AA}$.

Proposition 3.2. *The vector fields*

$$U = \partial_a + 2\lambda\partial_t, \quad V = 2\lambda\partial_\lambda, \quad W = -\partial_a$$

are left-invariant and form a basis for the tangent bundle $T(\mathcal{AA})$ *of* $\mathcal{AA}$*. They satisfy the following Lie bracket relations:*

$$[U,W] = [V,W] = 0 \quad \textit{and} \quad [U,V] = -2(U+W); \tag{13}$$

Moreover, a left-invariant Haar measure for $\mathcal{AA}$ *is* $d\mu_{\mathcal{AA}} = \frac{da\,d\lambda\,dt}{\lambda^2}$.

Proof. By the definition of U, V and W we have the relations:

$$\partial_a = -W, \quad \partial_\lambda = \frac{U}{2\lambda}, \quad \partial_t = \frac{U+W}{2\lambda},$$

and thus $\{U,V,W\}$ is a basis for $T(\mathcal{AA})$. Now we are going to verify that U, V and W are left-invariant. We set $e = e_{\mathcal{AA}} = (0,1,0)$ and we define the following three tangent vectors spanning a basis for $T_e(\mathcal{AA})$:

$$U_e = (\partial_a + 2\partial_t)_{|e}, \quad V_e = (2\partial_\lambda)_{|e}, \quad W_e = (-\partial_a)_{|e}.$$

If we fix a point $p' = (a', \lambda', t') \in \mathcal{AA}$ we can consider the left translation on $\mathcal{AA}$ given by

$$L_{p'}(p) = p' \star p = (a' + a, \lambda'\lambda, \lambda' t + t'),$$

where the Jacobian matrix of the derivative $(L_{p'})_{*,p}$ of $L_{p'}$ evaluated at p is

$$(DL_{p'})_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda' \end{bmatrix}.$$

We construct U, V and W by using $(L_p)_{*,e}\colon T_e(\mathcal{AA}) \to T_p(\mathcal{AA})$ and verifying that

$$(L_p)_{*,e}\,(\partial_a + 2\partial_t)_{|e} = U_p, \quad (L_p)_{*,e}\,(2\partial_\lambda)_{|e} = V_p, \quad (L_p)_{*,e}\,(-\partial_a)_{|e} = W_p.$$

This proves the first claim. The verification of the Lie bracket relations and the left-invariance of $\mu_{\mathcal{AA}}$ are left to the reader. □

Note, that $\vartheta(U) = \vartheta(V) = 0$ and thus, the horizontal bundle of $\mathcal{AA}$ is $\mathcal{H}_{\mathcal{AA}} = \operatorname{Span}\{U,V\}$. The sub-Riemannian structure in $\mathcal{AA}$ is defined by a sub-Riemannian metric on $\mathcal{H}_{\mathcal{AA}}$ making $\{U,V\}$ an orthonormal basis. In order to define the sub-Riemannian or Carnot–Carathéodory distance on $\mathcal{AA}$ let $\gamma\colon [0,1] \to \mathcal{AA}$, $\gamma(s) = (a(s), \lambda(s), t(s))$ be an absolutely continuous curve. Its tangent vector at $\gamma(s)$ is

$$\dot\gamma(s) = \frac{\dot t(s)}{2\lambda(s)} U_{\gamma(s)} + \frac{\dot\lambda(s)}{2\lambda(s)} V_{\gamma(s)} + \left(\frac{\dot t(s)}{2\lambda(s)} - \dot a(s)\right) W_{\gamma(s)}.$$

The curve γ is a *horizontal curve* if and only if $\dot\gamma(s) \in \ker \vartheta_{\gamma(s)}$ for almost every $s \in [0,1]$. This is equivalent to the ODE

$$\frac{\dot t(s)}{2\lambda(s)} - \dot a(s) = 0, \quad \text{a.e. } s \in [0,1]. \tag{14}$$

It follows that for a horizontal curve

$$\dot\gamma(s) = \frac{\dot t(s)}{2\lambda(s)} U_{\gamma(s)} + \frac{\dot\lambda(s)}{2\lambda(s)} V_{\gamma(s)} \in (\mathcal{H}_{\mathcal{AA}})_{\gamma(s)}.$$

The horizontal velocity $|\dot\gamma|_H$ of γ is now defined by the relation

$$|\dot\gamma|_H = \left(\langle\dot\gamma, U\rangle_{\mathcal{AA}}^2 + \langle\dot\gamma, V\rangle_{\mathcal{AA}}^2\right)^{1/2} = \frac{\sqrt{\dot\lambda^2 + \dot t^2}}{2\lambda}. \tag{15}$$

Here, $\langle\cdot,\cdot\rangle_{\mathcal{AA}}$ is the sub-Riemannian metric on $\mathcal{H}_{\mathcal{AA}}$. Let $\pi\colon \mathcal{AA}\to \mathbf{H}^1_{\mathbb{C}}$ denote the canonical projection given by $\pi(a,\lambda,t)=(\lambda,t)$, $(a,\lambda,t)\in\mathcal{AA}$, the horizontal length of γ is then given by

$$\ell(\gamma)=\int_0^1 \frac{\sqrt{\dot\lambda^2+\dot t^2}}{2\lambda}\,ds=\ell_h(\pi\circ\gamma), \tag{16}$$

where $\ell_h(\pi\circ\gamma)$ is the hyperbolic length of the projected curve $\pi\circ\gamma$ in $\mathbf{H}^1_{\mathbb{C}}$. It is straightforward to prove that the horizontal length is invariant under left-translations.

Conversely, if $\tilde\gamma$ is a C^1 curve in $\mathbf{H}^1_{\mathbb{C}}$, $\tilde\gamma(s)=(\xi(s),\eta(s))$, $s\in[0,1]$, passing from a point $q_0=\gamma(s_0)$, then the curve $\gamma\colon[0,1]\to\mathcal{AA}$ given by $\gamma(s)=(a(s),\lambda(s),t(s))$, where

$$a(s)=\int_{s_0}^s \frac{t(u)}{2\lambda(u)}\,du+a_0,\quad \lambda(s)=\xi(s),\quad t(s)=\eta(s),$$

is a horizontal curve passing from a point $p_0=(a_0,q)$ in the fibre of q.

The corresponding Carnot–Carathéodory distance $d_{\mathcal{AA}}$ associated to the sub-Riemannian metric $\langle\cdot,\cdot\rangle$ is defined for all $p,q\in\mathcal{AA}$ as follows:

$$d_{\mathcal{AA}}(p,q)=\inf_{\gamma\in\Gamma_{\mathcal{AA}}}\{\ell(\gamma)\}, \tag{17}$$

where $\Gamma_{\mathcal{AA}}$ is the following family of horizontal curves:

$$\Gamma_{\mathcal{AA}}=\{\gamma,\gamma\colon[0,1]\to\mathcal{AA}\colon \gamma \text{ horizontal},\ \gamma(0)=p,\ \gamma(1)=q\}.$$

We recall that the distance $d_{\mathcal{AA}}$ is finite, geodesic and induces the manifold topology. Our main object of study is the metric measure space $(\mathcal{AA},d_{\mathcal{AA}},\mu_{\mathcal{AA}})$.

It is well known, that by Darboux theorem each three dimensional contact manifold is locally contactomorphic to the Heisenberg group. Our next statement is a stronger, global version of this fact.

Proposition 3.3. *The manifolds $(\mathcal{AA},\vartheta)$ and $(\mathbb{H},\vartheta_{\mathbb{H}})$ are globally contactomorphic. That is: there exists a smooth bijective map $g\colon\mathbb{H}\to\mathcal{AA}$ such that $g^*\vartheta=\nu\vartheta_{\mathbb{H}}$ for some non-vanishing smooth function $\nu\colon\mathbb{H}\to\mathbb{R}$. In particular the metric spaces $(\mathbb{H},d_{\mathbb{H}})$ and $(\mathcal{AA},d_{\mathcal{AA}})$ are locally bi-Lipschitz equivalent.*

Proof. We define the smooth contactomorphism $g\colon(\mathbb{H},\vartheta_{\mathbb{H}})\to(\mathcal{AA},\vartheta)$ explicitly by the formula

$$g(x,y,t)=\left(xe^{-y},e^y,\frac{1}{2}(t-2xy+4x)\right)\quad\text{for } (x,y,t)\in\mathbb{H}. \tag{18}$$

Clearly, g is a smooth diffeomorphism between $\mathbb{H}$ and $\mathcal{AA}$. Its inverse map $g^{-1}\colon\mathcal{AA}\to\mathbb{H}$ is given by

$$g^{-1}(a,\lambda,t)=(a\lambda,\ln\lambda,2t+2a\lambda(\ln\lambda-2))\quad\text{for } (a,\lambda,t)\in\mathcal{AA}.$$

To check the contact property of g we compute directly:

$$\begin{aligned} g^*\vartheta&=\frac{(1/2)\,dt-x\,dy-y\,dx+2\,dx}{2e^y}-e^{-y}\,dx+xe^{-y}\,dy\\ &=\frac{dt+2x\,dy-2y\,dx}{4e^y}=\frac{1}{4e^y}\vartheta_{\mathbb{H}}. \end{aligned}$$

□

Combining Proposition 2.4 and Proposition 3.3 we deduce

Proposition 3.4. *The manifolds $(\mathcal{RT},\vartheta_{\mathcal{RT}})$, $(\mathbb{H},\vartheta_{\mathbb{H}})$ and $(\mathcal{AA},\vartheta)$ are all globally contactomorphic to each other.*

Another consequence of Proposition 3.3 is the following:

Proposition 3.5. *The metric measure space* $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$ *is locally Ahlfors 4-regular.*

Proof. We are going to prove a stronger property for $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$; actually, that there exist a $C \geq 1$ and an $r_0 > 0$ such that

$$C^{-1} r^4 \leq \mu_{\mathcal{AA}}(\overline{B}_{d_{\mathcal{AA}}}(p, r)) \leq C r^4, \tag{19}$$

for all $0 < r \leq r_0$ and for all $p \in \mathcal{AA}$. Due to the left-invariance of both the sub-Riemannian distance $d_{\mathcal{AA}}$ and the measure $\mu_{\mathcal{AA}}$, it suffices to prove (19) for balls $B_{\mathcal{AA}}(e, r)$ centered at the neutral element $e = e_{\mathcal{AA}}$. We have that $(\mathbb{H}, \vartheta_{\mathbb{H}})$ and $(\mathcal{AA}, \vartheta)$ are globally contactomorphic thanks to Proposition 3.3, so let us consider the map

$$g\colon (\mathbb{H}, d_{\mathbb{H}}, \mathcal{L}^3) \to (\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}}), \quad g^*\vartheta = \nu \vartheta_{\mathbb{H}},$$

given in (18). Since $g(e_{\mathbb{H}}) = e$, where $e_{\mathbb{H}} = (0, 0, 0)$ is the neutral element of $\mathbb{H}$ and $g\colon \mathbb{H} \to \mathcal{AA}$ is locally bi-Lipschitz, we have the inclusions

$$g(B_{\mathbb{H}}(e_{\mathbb{H}}, L^{-1} r)) \subseteq B_{\mathcal{AA}}(e, r) \subseteq g(B_{\mathbb{H}}(e_{\mathbb{H}}, Lr))$$

for some fixed number $L \geq 1$ and any $0 \leq r \leq 1$. Since $g^*\mu_{\mathcal{AA}} = \nu^2 \mu_{\mathbb{H}} = \nu^2 \mathcal{L}^3$ (up to multiplicative constants different from 0) and $\mathcal{L}^3(B_{\mathbb{H}}(e_{\mathbb{H}}, r)) = Cr^4$ for some fixed constant $C > 0$, the claim follows. □

Applying Propositions 2.2 and 2.5 the following statement follows:

Proposition 3.6. *The metric measure spaces* $(\mathbb{H}, d_{\mathbb{H}}, \mu_{\mathbb{H}})$ *and* $(\mathcal{RT}, d_{\mathcal{RT}}, \mu_{\mathcal{RT}})$ *are locally* 4*-Ahlfors regular and* 4*-parabolic.*

Another consequence of Proposition 3.3 is the following result about the existence of non-smooth QC maps of $\mathcal{AA}$ that distort the Hausdorff dimension $\dim_H$ of certain Cantor sets in $\mathcal{AA}$ in an arbitrary fashion:

Proposition 3.7. *For any* s, t, $0 < s < t < 4$ *there exist Cantor sets* $C_s \subset \mathcal{AA}$ *and* $C_t \subset \mathcal{AA}$ *such that* $\dim_H(C_s) = s$ *and* $\dim_H(C_t) = t$ *and a QC map* $F\colon \mathcal{AA} \to \mathcal{AA}$ *such that* $F(C_s) = C_t$.

Proof. The proof is based on the corresponding result in [1] for the case of the Heisenberg group. In fact Theorem 1.1 in [1] states that if $0 < s < t < 4$ there exist Cantor sets $K_s \subset \mathbb{H}$ and $K_t \subset \mathbb{H}$ and a QC map $G\colon \mathbb{H} \to \mathbb{H}$ such that $\dim_H(K_s) = s$, $\dim_H K_t = t$, $K_s \subset B_{\mathbb{H}}(e, 1)$, $K_t \subset B_{\mathbb{H}}(e, 1)$, $G(K_s) = K_t$ and $G = id_{\mathbb{H}}$ outside of $B_{\mathbb{H}}(e, 1)$. We note that the map $F\colon \mathcal{AA} \to \mathcal{AA}$ defined by $F = g \circ G \circ g^{-1}$ is a QC map. To see this observe that $G\colon B_{\mathbb{H}}(e, 1) \to B_{\mathbb{H}}(e, 1)$ is quasiconformal and the map $g\colon B_{\mathbb{H}}(e, 1) \to g(B_{\mathbb{H}}(e, 1))$ is bi-Lipschitz. This shows that $F\colon g(B_{\mathbb{H}}(e, 1)) \to g(B_{\mathbb{H}}(e, 1))$ is a QC map. On the other hand since $G = \mathrm{id}_{\mathbb{H} \setminus B_{\mathbb{H}}(e,1)}$, this implies that the map $F_{\mathbb{H} \setminus g(B_{\mathbb{H}}(e,1))} = \mathrm{id}_{\mathbb{H} \setminus g(B_{\mathbb{H}}(e,1))}$. This implies that F is a global QC map and it does satisfy the properties in the statement for the sets $C_s = g(K_s)$ and $C_t = g(K_t)$. □

More results on quasiconformal maps defined on the affine-additive group $\mathcal{AA}$, including methods of constructing such maps and study of their extremality will be contained in the forthcoming paper [2] of the authors as well as in the dissertation of the second author [4].

4. Proof of the main result

Let us observe, first, that hyperbolicity of a metric measure space (X, d_X, μ_X) holds, if there exists a compact set $E \subset X$ and sequence of compact sets $F_n \subseteq X$

such that

$$\operatorname{dist}(E, F_n) := \inf\{d_X(x,y) \colon x \in E,\ y \in F_n\} \to \infty$$

and

$$\liminf_{n\to\infty} \operatorname{Mod}_Q(E, F_n) > 0.$$

To do this, let us pick $x_0 \in E$.

We shall consider $R_n = \inf\{d_X(x_0, y) \colon y \in F_n\}$. Note, that $R_n \to \infty$ and any curve connecting E and F_n must have a sub-curve connecting $\partial B(x_0, R_0)$ and $\partial B(x_0, R_n)$. Thus, by (4) we have the inequality

$$\operatorname{Mod}_Q(\mathcal{D}(R_0, R_n)) \geq \operatorname{Mod}_Q(E, F_n).$$

Since $\liminf_{n\to\infty} \operatorname{Mod}_Q(E, F_n) > 0$ we obtain that (X, d_X, μ_X) is hyperbolic.

The idea of the proof of Theorem 1.1 is to construct compact sets E and F_n in $\mathcal{AA}$ with the above properties. This is explicitly done as follows:

Let $n \in \mathbb{N}$, $n \geq 2$. We define

$$E = \{(a, 1, t) \in \mathcal{AA} \colon a \in [-1,1] \text{ and } t \in [-1,1]\},$$
$$F_n = \{(a, \frac{1}{n}, t) \in \mathcal{AA} \colon a \in [-1,1] \text{ and } t \in [-1,1]\}.$$

Next, for each such n we define the following curve families of piecewise smooth horizontal curves:

$$\Gamma_n = \{\gamma, \gamma \colon [0,1] \to \mathcal{AA} \text{ such that } \gamma(0) \in E \text{ and } \gamma(1) \in F_n\}. \tag{20}$$

The following estimate holds.

Proposition 4.1. *There exists some $M > 0$ such that $\operatorname{Mod}_4(\Gamma_n) > M$ for all $n \in \mathbb{N}$, $n \geq 2$.*

Proof. We consider the sub-family $\Gamma_n^0 \subset \Gamma_n$ which comprises curves $\gamma \colon [0,1] \to \mathcal{AA}$ given by

$$\gamma(s) = \left(a, 1 - \left(1 - \frac{1}{n}\right) s, t\right), \quad a \in [-1,1],\ t \in [-1,1].$$

It is straightforward to check that the curves in Γ_n^0 are horizontal with $\gamma(0) \in E$ and $\gamma(1) \in F_n$ for all $n \in \mathbb{N}$, $n \geq 2$. Further, from (15) we obtain that

$$|\dot\gamma(s)|_H = \frac{1 - \frac{1}{n}}{2\left(1 - \left(1 - \frac{1}{n}\right) s\right)}.$$

If now $\rho \in \operatorname{Adm}(\Gamma_n^0)$, then we have

$$\int_0^1 \rho\left(a, 1 - \left(1 - \frac{1}{n}\right) s, t\right) \frac{1 - \frac{1}{n}}{2\left(1 - \left(1 - \frac{1}{n}\right) s\right)}\, ds \geq 1,$$

which, under integration by substitution with $\lambda(s) = 1 - \left(1 - \frac{1}{n}\right) s$, gives

$$\int_{\frac{1}{n}}^1 \frac{\rho(a, \lambda, t)}{2\lambda}\, d\lambda \geq 1, \quad \forall n \geq 2. \tag{21}$$

Next, by integrating (21) with respect to $a \in [-1,1]$ and $t \in [-1,1]$, we obtain

$$\int_{-1}^1 \int_{-1}^1 \int_{\frac{1}{n}}^1 \frac{\rho(a, \lambda, t)}{\lambda}\, da\, d\lambda\, dt \geq 8, \quad \forall n \geq 2. \tag{22}$$

At this point, for $n \geq 2$ we define the sets

$$P_n = \left\{(a,\lambda,t) \in \mathcal{AA}\colon a \in [-1,1],\ \lambda \in \left[\tfrac{1}{n},1\right],\ t \in [-1,1]\right\}$$

and we apply Hölder's inequality in (22) with respect to $\frac{\rho(a,\lambda,t)}{\sqrt{\lambda}} \cdot \frac{\mathcal{X}_{P_n}(a,\lambda,t)}{\sqrt{\lambda}}$ and with conjugated exponents 4 and $\frac{4}{3}$, to obtain

$$\left(\int_{\mathcal{AA}} \rho^4(a,\lambda,t)\frac{da\,d\lambda\,dt}{\lambda^2}\right)^{\frac{1}{4}} \left(\int_{\mathcal{AA}} \mathcal{X}_{P_n}(a,\lambda,t)\frac{1}{\lambda^{\frac{2}{3}}}\,da\,d\lambda\,dt\right)^{\frac{3}{4}} \geq 8, \quad \forall n \geq 2. \tag{23}$$

Now we observe that

$$\int_{\mathcal{AA}} \mathcal{X}_{P_n}(a,\lambda,t)\frac{1}{\lambda^{\frac{2}{3}}}\,da\,d\lambda\,dt \leq 4\int_0^1 \frac{1}{\lambda^{\frac{2}{3}}}\,d\lambda = 12, \quad \forall n \geq 2.$$

The latter inequality combined with (23) gives

$$\int_{\mathcal{AA}} \rho^4(a,\lambda,t)\frac{da\,d\lambda\,dt}{\lambda^2} \geq \frac{2^4}{3^3}.$$

Finally, the proof is concluded by taking the infimum over all $\rho \in \operatorname{Adm}(\Gamma_n^0)$. □

The first consequence of Theorem 1.1 is the following:

Corollary 4.2. *There is no QC map between $\mathcal{AA}$ and $\mathbb{H}$ or $\mathcal{AA}$ and $\mathcal{RT}$.*

For a map f between metric spaces, let's recall the quantity $H_f(\cdot)$ from (1) and define B_f as the branch set (i.e., the set of points where f does not define a local homeomorphism). We use the following definition of quasiregular (QR) maps from [9].

Definition 4.3. Let M and N be any sub-Riemannian manifolds among $\mathbb{H}$, $\mathcal{RT}$ and $\mathcal{AA}$. We call a mapping $f\colon M \to N$ K-quasiregular if it is constant, or if:

(1) f is a branched cover onto its image (i.e., continuous, discrete, open and sense-preserving),
(2) $H_f(\cdot)$ is locally bounded on M,
(3) $H_f(p) \leq K$ for almost every $p \in M$,
(4) the branch set B_f and its image have measure zero.

A mapping is said to be *quasiregular* if it is K-quasiregular for some $1 \leq K < \infty$.

From the definition it is clear, every QC map is QR. On the other hand, the class of QR maps can be substantially larger than the class of QC maps.

Let us recall, that by Theorem 4.8.1 from [10] if $f\colon \mathbb{H} \to N$ is a QR map where N is 4-hyperbolic then f must be constant. Applying this statement to our situation, we obtain the following result:

Theorem 4.4. *If $f\colon \mathbb{H} \to \mathcal{AA}$ is a quasiregular map, then f is constant.*

In contrast to the previous statement, we note that there can be plenty of examples of QR maps $f\colon \mathcal{AA} \to \mathbb{H}$. One such map is the following:

Example 4.5. Let $f\colon \mathcal{AA} \to \mathbb{H}$ be the map defined by

$$f(a,\lambda,t) = (-\sqrt{\lambda}\cos a, \sqrt{\lambda}\sin a, t), \quad (a,\lambda,t) \in \mathcal{AA}.$$

By a direct calculation one can verify the contact property of f, namely $f^*\vartheta_{\mathbb{H}} = 2\lambda\vartheta$. Moreover, denoting $f(a,\lambda,t) = (x,y,t) \in \mathbb{H}$, one can check that $f_*U = yX - xY$ and $f_*V = xX + yY$. Using this, we have that

$$f_*(\alpha U + \beta V) = (\alpha y + \beta x)X + (\beta y - \alpha x)Y,$$

for any $\alpha, \beta \in \mathbb{R}$.

Since $\{U, V\}$, resp. $\{X, Y\}$, is the orthonormal basis in the sub-Riemannian metric of $\mathcal{AA}$, resp. $\mathbb{H}$, we obtain that

$$|f_*(\alpha U + \beta V)|_{\mathbb{H}} = \sqrt{(\alpha^2 + \beta^2)(x^2 + y^2)}$$

and therefore

$$H_f(a, \lambda, t) = \frac{\max\{|f_*(\alpha U + \beta V)|_{\mathbb{H}} \colon \alpha^2 + \beta^2 = 1\}}{\min\{|f_*(\alpha U + \beta V)|_{\mathbb{H}} \colon \alpha^2 + \beta^2 = 1\}} = 1,$$

for every point $(a, \lambda, t) \in \mathcal{AA}$. See also Proposition 2.4 in [9] for a different way to compute the value of $H_f(\cdot)$.

Furthermore, note, that a direct computation, gives $\det f_* = \frac{1}{2}$; and thus f is a local diffeomorphism at every point. This means that the branch set B_f of f is empty, and thus f is an immersion of $\mathcal{AA}$ into $\mathbb{H}$. Consequently we conclude that f is 1-quasiregular.

Further examples of QR maps $g \colon \mathcal{AA} \to \mathbb{H}$ can be obtained as compositions $g = h \circ f$ where $h \colon \mathbb{H} \to \mathbb{H}$ is a QC map of the Heisenberg group.

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