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Logarithmic upper bound for weak subsolutions to the fractional Laplace equation

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Abstract. In this note, we present a logarithmic-type upper bound for weak subsolutions to a class of integro-differential problems, whose prototype is the Dirichlet problem for the fractional Laplacian. The bound is slightly smaller than the classical one in this field.

Murtoasteisen Laplacen yhtälön heikkojen aliratkaisujen logaritminen yläraja

Tiivistelmä. Tässä työssä esitetään logaritmistyyppinen yläraja murtoasteisen Laplacen operaattorin Dirichlet'n ongelmaa yleistävän integraali–differentiaaliyhtälöiden luokan aliratkaisuille. Saatu raja parantaa hiukan alan klassista tulosta.

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $N > 2s$ for some $s \in (0,1)$. We consider the following integro-differential problem

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is a non-local operator, whose prototype is the fractional Laplacian, defined in the principle value sense as

$$\mathcal{L}u = P.V. \int_{\mathbb{R}^N} \frac{A(x,y)(u(x) - u(y))}{|x-y|^{N+2s}}\, dy,$$

where A is bounded and measurable, and satisfies $A(x,y) = A(y,x)$, $0 < \lambda \le A(x,y) \le \Lambda$ for constants λ, $\Lambda > 0$. Moreover, we assume $f \in L^q(\Omega)$, where $q > \frac{N}{2s}$.

Under these assumptions, we recall the following well-known definition of weak subsolutions to problem (1.1).

Definition 1.1. We say that $u \in H^s(\mathbb{R}^N)$ with $u \le 0$ on $\mathbb{R}^N \setminus \Omega$, is a weak subsolution to the problem (1.1), if it satisfies

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} A(x,y)(u(x) - u(y))(\phi(x) - \phi(y))\, d\mu \le \int_\Omega f\phi\, dx,$$

for any test functions $\phi \ge 0 \in H^s(\mathbb{R}^N)$ with $\phi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, where $d\mu = \frac{1}{|x-y|^{N+2s}}\, dx\, dy$.

Remark 1.1. The assumptions $q > \frac{N}{2s}$ and $N > 2s$ ensure that the above definition is well-posed. For more details, see Remark 2.2.17 in Chapter 2 of [3].

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For the linear elliptic equations with bounded and measurable coefficients (i.e. when $s \to 1^-$), many monographs established the boundedness result for subsolutions to (1.1), in particular, the following estimate holds

$$\|u_+\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\Omega)}, \quad \text{for } q > \frac{N}{2}, \tag{1.2}$$

where the constant C depends on Ω, N and q. Interested readers may refer to [2, 5, 6, 8, 12] for further details.

In the fractional setting, similar results have also been established, see, for example, [3] and [4]. If u is a subsolution to problem (1.1), then there exists a constant $C = C(\Omega, N, q, s, \lambda)$, such that

$$\|u_+\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\Omega)}, \quad \text{for } q > \frac{N}{2s}. \tag{1.3}$$

The purpose of this note is to provide a logarithmic-type upper bound for subsolutions to problem (1.1), which refines the classical estimate (1.3). Since the well-known result like (1.3) exists, we always assume throughout that u_+ is bounded. Our main result is stated in the following theorem.

Theorem 1.1. *Let u be a weak subsolution to problem* (1.1). *Then*

$$\|u_+\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{\frac{N}{2s}}(\Omega)} \left[\log\left(\frac{\|f\|_{L^q(\Omega)}}{\|f\|_{L^{\frac{N}{2s}}(\Omega)}} + 1\right) + 1\right], \tag{1.4}$$

where $q > \frac{N}{2s}$ and C depends only on λ, N, q, s.

Remark 1.2. The right-hand side of (1.4) is smaller than the corresponding term of (1.3), and the ratio $\frac{\|f\|_{L^q(\Omega)}}{\|f\|_{L^{\frac{N}{2s}}(\Omega)}}$ reflects how much larger the L^q-norm of f is compared to its $L^{\frac{N}{2s}}$-norm.

1.1. Novelty and significance. For uniformly elliptic equations, the index $q > \frac{N}{2}$ in inequality (1.2) is known to be sharp (for $N \geq 3$) in the scale of Lebesgue spaces when deriving the boundedness result. This naturally raises the question of whether an improved upper bound can be obtained that still involves the $L^{\frac{N}{2}}$ norm of the datum f. Xu [13] was the first to provide an improvement by establishing a logarithmic type upper bound for weak subsolutions. Theorem A in [13] states that if u is a subsolution to a uniformly elliptic equation, there exits a constant C, such that

$$\|u_+\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{\frac{N}{2}}(\Omega)} \left[\log\left(\frac{\|f\|_{L^q(\Omega)}}{\|f\|_{L^{\frac{N}{2}}(\Omega)}} + 1\right) + 1\right], \quad \text{for } q > \frac{N}{2}.$$

Subsequently, Cruz-Uribe and Rodney [1] extended this type of result to degenerate elliptic equations, showing an L^∞ bound for the subsolutions when the nonhomogeneous term f belongs to a class of Orlicz spaces with the norm $L^A(\Omega)$, where $A = A(t) = t^{\sigma'} \log(e+t)^q$, with $q > \sigma' > 0$. Moreover, they argued that this condition on q and σ' is almost sharp supported by a counterexample (Example 1.10 in [1]). Then, they also gave an L^∞ upper bound for weak solutions with a logarithmic-type dependence on the Orlicz norm of f.

Motivated by these results, we turn to the case of the fractional Laplace equation. We believe the sharpness of the index $q > \frac{N}{2s}$ for the boundedness result is likely already known in the literature. However, despite our best efforts, we were unable to find any explicit claims or counterexamples that confirm this sharpness in the context of L^∞ bounds for weak subsolutions. What we did find instead was a closely related conclusion of the continuity of solutions, specifically, in Corollary 1.3 of Kuusi, Mingione and Sire of [7], which points to a similar threshold condition.

To address this gap and make our work more complete and accessible to readers, we construct a concrete counterexample showing that the condition $q > \frac{N}{2s}$ is indeed optimal for ensuring boundedness when $N \geq 2$. Interestingly, our example also proves the sharpness of the assumption for the measure used in Corollary 1.3 of [7].

Building on this foundation, we prove Theorem 1.1 using Moser iteration with an exponential-type test function, inspired by the techniques in [1, 13]. The difference lies in the treatment of the nonlocal terms that arise due to the exponential test function. We establish several crucial inequalities by Newton–Leibniz formula in Section 2 to deal with these issues.

1.2. Structure of the paper. In Section 2, we introduce several crucial lemmas which are instrumental in the proof of Theorem 1.1. In Section 3, we give the proof of Theorem 1.1. Finally, we state an example in Section 4 to show the index $q > \frac{N}{2s}$ is sharp, for $N \geq 2$, in order to get the L^∞ result.

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2. Main tools

In this section, we present several Lemmas that are essential for implementing the Moser iteration technique. The first Lemma provides a formula analogous to what one could obtain using the chain rule when applying a local differential operator to the exponential function e^x.

Lemma 2.1. *For any $s,\ t \in \mathbb{R}$, and any fixed $\alpha > 0$, we have the following inequality*

$$(s-t)(e^{\alpha s} - e^{\alpha t}) \geq \frac{1}{\alpha}\left(e^{\frac{\alpha}{2}s} - e^{\frac{\alpha}{2}t}\right)^2.$$

Proof. First, for the case $s > t$, we have

$$e^{\alpha s} - e^{\alpha t} = \alpha \int_t^s e^{\alpha m}\, dm \leq \alpha e^{\alpha s}(s-t),$$

whence, we deduce

$$(s-t)(e^{\alpha s} - e^{\alpha t}) \geq \frac{1}{\alpha e^{\alpha s}}(e^{\alpha s} - e^{\alpha t})^2 = \frac{1}{\alpha}(e^{\frac{\alpha}{2}s} - e^{\frac{\alpha}{2}t + \frac{\alpha}{2}(t-s)})^2 \geq \frac{1}{\alpha}\left(e^{\frac{\alpha}{2}s} - e^{\frac{\alpha}{2}t}\right)^2.$$

For the case $s < t$, similarly, we have

$$e^{\alpha t} - e^{\alpha s} = \alpha \int_s^t e^{\alpha m}\, dm \leq \alpha e^{\alpha t}(t-s).$$

Consequently, we obtain

$$\begin{aligned}(s-t)(e^{\alpha s} - e^{\alpha t}) &= (t-s)(e^{\alpha t} - e^{\alpha s}) \geq \frac{1}{\alpha e^{\alpha t}}(e^{\alpha t} - e^{\alpha s})^2 \\ &= \frac{1}{\alpha}(e^{\frac{\alpha}{2}t} - e^{\frac{\alpha}{2}s + \frac{\alpha}{2}(s-t)})^2 \geq \frac{1}{\alpha}\left(e^{\frac{\alpha}{2}t} - e^{\frac{\alpha}{2}s}\right)^2.\end{aligned}$$

Therefore, the proof of the Lemma 2.1 is complete. □

The following result is similar to the previous lemma and serves as an alternative chain rule for polynomial, when working with the nonlocal differential operator.

Lemma 2.2. *For any $\beta \geq 1$, $s, t \geq 0$, we have*

$$(s-t)(s^\beta - t^\beta) \geq \frac{1}{\beta}\left(s^{\frac{\beta+1}{2}} - t^{\frac{\beta+1}{2}}\right)^2.$$

Proof. The proof is similar to that of Lemma 2.1; for the reader's convenience, we provide the detailed argument below.

First, we consider the case $s \geq t$: we have

$$s^\beta - t^\beta = \beta \int_t^s m^{\beta-1}\, dm \leq \beta s^{\beta-1}(s-t),$$

whence

$$(s-t)(s^\beta - t^\beta) \geq \frac{1}{\beta s^{\beta-1}}(s^\beta - t^\beta)^2 = \frac{1}{\beta}\left(s^{\frac{\beta+1}{2}} - t^{\frac{\beta+1}{2}} \cdot \frac{t^{\frac{\beta-1}{2}}}{s^{\frac{\beta-1}{2}}}\right)^2 \geq \frac{1}{\beta}\left(s^{\frac{\beta+1}{2}} - t^{\frac{\beta+1}{2}}\right)^2.$$

For the case $s < t$, we similarly get

$$t^\beta - s^\beta = \beta \int_s^t m^{\beta-1}\, dm \leq \beta t^{\beta-1}(t-s);$$

hence,

$$\begin{aligned}(s-t)(s^\beta - t^\beta) = (t-s)(t^\beta - s^\beta) &\geq \frac{1}{\beta t^{\beta-1}}(t^\beta - s^\beta)^2 \\ &= \frac{1}{\beta}\left(t^{\frac{\beta+1}{2}} - s^{\frac{\beta+1}{2}} \cdot \frac{s^{\frac{\beta-1}{2}}}{t^{\frac{\beta-1}{2}}}\right)^2 \geq \frac{1}{\beta}\left(t^{\frac{\beta+1}{2}} - s^{\frac{\beta+1}{2}}\right)^2.\end{aligned}$$

Therefore, the proof of the Lemma 2.2 is complete. □

We now present an auxiliary formula that resembles the change of variables technique commonly used in the local differential equations.

Lemma 2.3. *For any s, $t \in \mathbb{R}$ and r_1, r_2, $\alpha > 0$, we have*

$$(s-t)(r_1 e^{\alpha s} - r_2 e^{\alpha t}) \geq \frac{1}{\alpha}(e^{\alpha s} - e^{\alpha t})(r_1 - r_2).$$

Proof. First, when $s \geq t$, we have

$$s - t \geq \frac{1}{e^{\alpha s}} \int_t^s e^{\alpha m}\, dm = \frac{1}{\alpha e^{\alpha s}}(e^{\alpha s} - e^{\alpha t}),$$

and

$$s - t \leq \frac{1}{e^{\alpha t}} \int_t^s e^{\alpha m}\, dm = \frac{1}{\alpha e^{\alpha t}}(e^{\alpha s} - e^{\alpha t}).$$

Thus,

$$\begin{aligned}(s-t)(r_1 e^{\alpha s} - r_2 e^{\alpha t}) = (s-t) r_1 e^{\alpha s} - (s-t) r_2 e^{\alpha t} &\geq \frac{r_1}{\alpha}(e^{\alpha s} - e^{\alpha t}) - \frac{r_2}{\alpha}(e^{\alpha s} - e^{\alpha t}) \\ &= \frac{(e^{\alpha s} - e^{\alpha t})}{\alpha}(r_1 - r_2).\end{aligned}$$

When $s < t$, we similarly have

$$\frac{1}{\alpha e^{\alpha t}}(e^{\alpha t} - e^{\alpha s}) \leq t - s \leq \frac{1}{\alpha e^{\alpha s}}(e^{\alpha t} - e^{\alpha s}).$$

Consequently,

$$
\begin{aligned}
(s-t)(r_1e^{\alpha s}-r_2e^{\alpha t}) &= (t-s)(r_2e^{\alpha t}-r_1e^{\alpha s}) = (t-s)r_2e^{\alpha t}-(t-s)r_1e^{\alpha s}\\
&\geq \frac{r_2}{\alpha}(e^{\alpha t}-e^{\alpha s})-\frac{r_1}{\alpha}(e^{\alpha t}-e^{\alpha s}) = \frac{(e^{\alpha t}-e^{\alpha s})}{\alpha}(r_2-r_1)\\
&= \frac{(e^{\alpha s}-e^{\alpha t})}{\alpha}(r_1-r_2).
\end{aligned}
$$

Therefore, the proof of the Lemma 2.3 is complete. □

The following fractional Sobolev embedding lemma is well known, for the detailed proof, see [Proposition 15.5, 9], [10].

Lemma 2.4. *Let $u \in H^s(\mathbb{R}^N)$ with $s \in (0,1)$, we have*

$$
\|u\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)} \leq C_* \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^{N+2s}}\,dy\,dx\right)^{\frac{1}{2}}.
$$

The following result indicates the exponential integrability of the nonnegative part of the weak subsolution to problem (1.1).

Lemma 2.5. *If u is a subsolution to* (1.1)*, there exists $\alpha \in \left(0, \frac{\lambda}{C_*^2\|f\|_{L^{\frac{N}{2s}}(\Omega)}}\right)$, such that*

$$
\int_\Omega e^{\frac{2\alpha N}{N-2s}u_+(x)}\,dx \leq \left(\frac{2\lambda|\Omega|^{1-\frac{4s}{N+2s}}\|f\|_{L^{\frac{N}{2s}}(\Omega)}}{(\lambda-\alpha C_*^2\|f\|_{L^{\frac{N}{2s}}(\Omega)})}+|\Omega|^{\frac{N-2s}{2N}}\right)^{\frac{2N}{N-2s}},
$$

where C_* is the same constant as in Lemma 2.4.

Proof. Taking $\phi(x) = e^{2\alpha u_+(x)} - 1$ in Definition 1.1 yields,

$$
\frac{\lambda}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(e^{2\alpha u_+(x)}-e^{2\alpha u_+(y)})\,d\mu \leq \int_\Omega f(x)(e^{2\alpha u_+(x)}-1)\,dx. \tag{2.1}
$$

Taking $s = u_+(x)$, $t = u_+(y)$ in Lemma 2.1 and then applying Lemma 2.4, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\left(e^{2\alpha u_+(x)}-e^{2\alpha u_+(y)}\right)d\mu\\
&\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u_+(x)-u_+(y))\left(e^{2\alpha u_+(x)}-e^{2\alpha u_+(y)}\right)d\mu\\
&\geq \frac{2}{\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(e^{\frac{2\alpha}{2}u_+(x)}-e^{\frac{2\alpha}{2}u_+(y)}\right)^2 d\mu\\
&= \frac{2}{\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(e^{\frac{2\alpha}{2}u_+(x)}-1-\left(e^{\frac{2\alpha}{2}u_+(y)}-1\right)\right)^2 d\mu\\
&\geq \frac{2}{\alpha C_*^2}\left(\int_\Omega \left|e^{\frac{2\alpha u_+(x)}{2}}-1\right|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{N}}.
\end{aligned}
$$

Regarding the right side of (2.1), by the Hölder's inequality, we have

$$\begin{aligned}\int_\Omega f(x)\left(e^{2\alpha u_+(x)}-1\right)dx &= \int_\Omega f(x)\left[\left(e^{\frac{2\alpha}{2}u_+(x)}-1\right)^2+2\left(e^{\frac{2\alpha}{2}u_+(x)}-1\right)\right]dx\\ &\le \|f\|_{L^{\frac{N}{2s}}(\Omega)}\left(\int_\Omega\left|e^{\frac{2\alpha u_+(x)}{2}}-1\right|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{N}}\\ &\quad+2\left(\int_\Omega\left|e^{\frac{2\alpha u_+(x)}{2}}-1\right|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{2N}}\|f\|_{L^{\frac{2N}{N+2s}}(\Omega)}.\end{aligned}$$

Combining this with the inequality above, and using the assumption $\alpha\in\left(0,\frac{\lambda}{C_*^2\|f\|_{L^{\frac{N}{2s}}(\Omega)}}\right)$, we have

$$\begin{aligned}&\left(1-\frac{\alpha C_*^2\|f\|_{L^{\frac{N}{2s}}(\Omega)}}{\lambda}\right)\left(\int_\Omega\left|e^{\frac{2\alpha u_+(x)}{2}}-1\right|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{N}}\\ &\le 2\left(\int_\Omega\left|e^{\frac{2\alpha u_+(x)}{2}}-1\right|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{2N}}\|f\|_{L^{\frac{2N}{N+2s}}(\Omega)}.\end{aligned}$$

Without loss of generality, we assume $\left(\int_\Omega\left|e^{\frac{2\alpha u_+(x)}{2}}-1\right|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{2N}}\neq 0$; then, we divide both sides of the above equality by this quantity

$$\left(\int_\Omega\left|e^{\frac{2\alpha u_+(x)}{2}}-1\right|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{2N}}\le\frac{2\lambda\|f\|_{L^{\frac{2N}{N+2s}}(\Omega)}}{\left(\lambda-\alpha C_*^2\|f\|_{L^{\frac{N}{2s}}(\Omega)}\right)}\le\frac{2\lambda|\Omega|^{1-\frac{4s}{N+2s}}\|f\|_{L^{\frac{N}{2s}}(\Omega)}}{\left(\lambda-\alpha C_*^2\|f\|_{L^{\frac{N}{2s}}(\Omega)}\right)}.$$

The proof is complete. □

3. Proof of Theorem 1.1

Due to the homogeneity of the problem (1.1), it is sufficient to assume $\|f\|_{L^{\frac{N}{2s}}(\Omega)}=1$. First, we take the test function $\phi=e^{\alpha u(x)}\eta(x)$, where $\eta(x)=e^{\alpha\beta u_+(x)}-1$, $\beta\ge 1$ and α takes the values in the interval mentioned in Lemma 2.5, then we have

$$\frac{\lambda}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\left(e^{\alpha u(x)}\eta(x)-e^{\alpha u(y)}\eta(y)\right)d\mu\le\int_\Omega f(x)e^{\alpha u(x)}\eta(x)\,dx. \tag{3.1}$$

By applying Lemma 2.3 with $s=u(x)$, $t=u(y)$, $r_1=\eta(x)$, and $r_2=\eta(y)$, we derive

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\left(e^{\alpha u(x)}\eta(x)-e^{\alpha u(y)}\eta(y)\right)d\mu\\ &\ge\frac{1}{\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(e^{\alpha u(x)}-e^{\alpha u(y)}\right)(\eta(x)-\eta(y))\,d\mu.\end{aligned}$$

Then, from Lemma 2.2 with $s = e^{\alpha u_+(x)}$, $t = e^{\alpha u_+(y)}$ and Lemma 2.4, we obtain

$$\begin{aligned}
&\frac{1}{\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(e^{\alpha u(x)} - e^{\alpha u(y)}\right)(\eta(x)-\eta(y))\,d\mu \\
&\geq \frac{1}{\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(e^{\alpha u_+(x)} - e^{\alpha u_+(y)}\right)\left(e^{\alpha\beta u_+(x)} - e^{\alpha\beta u_+(y)}\right)d\mu \\
&\geq \frac{1}{\alpha\beta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(e^{\alpha(\frac{\beta+1}{2})u_+(x)} - e^{\alpha(\frac{\beta+1}{2})u_+(y)}\right)^2 d\mu \\
&\geq \frac{1}{\alpha\beta C_*^2}\left(\int_\Omega \left|e^{\frac{\alpha(\beta+1)u_+(x)}{2}} - 1\right|^{\frac{2N}{N-2s}} dx\right)^{\frac{N-2s}{N}}.
\end{aligned}$$

On the other hand, for the right side of (3.1), Hölder's inequality yields

$$\int_\Omega f(x)e^{\alpha u(x)}\eta(x)\,dx \leq \int_\Omega |f(x)|e^{\alpha(\beta+1)u_+(x)}\,dx \leq \|f\|_{L^q(\Omega)}\left(\int_\Omega \left|e^{\alpha(\frac{\beta+1}{2})u_+(x)}\right|^{2q'} dx\right)^{\frac{1}{q'}},$$

where $q' = \frac{q}{q-1}$. Combining the inequalities above, we obtain the reverse Hölder inequality

$$\begin{aligned}
&\left(\int_\Omega \left|e^{\frac{\alpha(\beta+1)u_+(x)}{2}} - 1\right|^{\frac{2N}{N-2s}} dx\right)^{\frac{N-2s}{2N}} \\
&\leq \left(\frac{2}{\lambda}\right)^{\frac{1}{2}}(\alpha\beta)^{\frac{1}{2}}C_*\|f\|^{\frac{1}{2}}_{L^q(\Omega)}\left(\int_\Omega \left|e^{\alpha(\frac{\beta+1}{2})u_+(x)}\right|^{2q'} dx\right)^{\frac{1}{2q'}},
\end{aligned}$$

then,

$$\begin{aligned}
&\left(\int_\Omega \left|e^{\frac{\alpha(\beta+1)u_+(x)}{2}}\right|^{\frac{2N}{N-2s}} dx\right)^{\frac{N-2s}{2N}} \leq \left(\int_\Omega \left|e^{\frac{\alpha(\beta+1)u_+(x)}{2}} - 1\right|^{\frac{2N}{N-2s}} dx\right)^{\frac{N-2s}{2N}} + |\Omega|^{\frac{N-2s}{2N}} \\
&\leq \left[\left(\frac{2}{\lambda}\right)^{\frac{1}{2}}(\alpha\beta)^{\frac{1}{2}}C_*\|f\|^{\frac{1}{2}}_{L^q(\Omega)} + |\Omega|^{\frac{N-2s}{2N}-\frac{1}{2q'}}\right]\left(\int_\Omega \left|e^{\alpha(\frac{\beta+1}{2})u_+(x)}\right|^{2q'} dx\right)^{\frac{1}{2q'}}.
\end{aligned}$$

Since $2q' < \frac{2N}{N-2s}$, we define $\chi = \frac{N}{(N-2s)q'} > 1$, and set $\frac{\beta+1}{2} = \chi^n$ with $n \in \mathbb{N}_+$, raising both sides of the previous inequality to the power χ^{-n}, we obtain

$$\begin{aligned}
\left(\int_\Omega \left|e^{\frac{2N\alpha u_+(x)}{N-2s}}\right|^{\chi^n} dx\right)^{\frac{N-2s}{2N\chi^n}} &\leq \left[\left(\frac{4\alpha}{\lambda}\right)^{\frac{1}{2\chi^n}}\chi^{\frac{n}{\chi^n}}C_*^{\frac{1}{\chi^n}}\|f\|^{\frac{1}{2\chi^n}}_{L^q(\Omega)} + |\Omega|^{(\frac{N-2s}{2N}-\frac{1}{2q'})\frac{1}{\chi^n}}\right] \\
&\quad \times \left(\int_\Omega \left|e^{\frac{\alpha 2N u_+(x)}{N-2s}}\right|^{\chi^{n-1}} dx\right)^{\frac{N-2s}{2N\chi^{n-1}}}.
\end{aligned}$$

By iteration, we get

$$\begin{aligned}
\left\|e^{\frac{2N\alpha u_+(x)}{N-2s}}\right\|_{L^\infty(\Omega)} &\leq \left(\frac{4\alpha}{\lambda}\right)^{\frac{1}{2(\chi-1)}}\|f\|^{\frac{1}{2(\chi-1)}}_{L^q(\Omega)}C_*^{\frac{1}{\chi-1}}\chi^{\frac{\chi}{(\chi-1)^2}} \\
&\quad \times (|\Omega|^{\frac{N-2s}{2N}-\frac{1}{2q'}}+1)^{\frac{1}{\chi-1}}\left(\int_\Omega \left|e^{\frac{\alpha 2N u_+(x)}{N-2s}}\right| dx\right)^{\frac{N-2s}{2N}},
\end{aligned}$$

i.e.

$$\|u_+\|_{L^\infty(\Omega)} \le \frac{(N-2s)}{2N\alpha}\log\Bigg[\left(\frac{4\alpha}{\lambda}\right)^{\frac{1}{2(\chi-1)}}\|f\|_{L^q(\Omega)}^{\frac{1}{2(\chi-1)}} C_*^{\frac{1}{\chi-1}}\chi^{\frac{\chi}{(\chi-1)^2}}\left(|\Omega|^{\frac{N-2s}{2N}-\frac{1}{2q'}}+1\right)^{\frac{1}{\chi-1}} \times\left(\int_\Omega \left|e^{\frac{2\alpha N u_+(x)}{N-2s}}\right|dx\right)^{\frac{N-2s}{2N}}\Bigg].$$

Consequently, applying Lemma 2.5 yields

$$\begin{aligned}\|u_+\|_{L^\infty(\Omega)} &\le \frac{(N-2s)}{2N\alpha}\log\Bigg[\left(\frac{4\alpha}{\lambda}\right)^{\frac{1}{2(\chi-1)}}\|f\|_{L^q(\Omega)}^{\frac{1}{2(\chi-1)}} C_*^{\frac{1}{\chi-1}}\chi^{\frac{\chi}{(\chi-1)^2}}\left(|\Omega|^{\frac{N-2s}{2N}-\frac{1}{2q'}}+1\right)^{\frac{1}{\chi-1}} \\ &\qquad\times\left(\frac{2\lambda|\Omega|^{1-\frac{4s}{N+2s}}}{(\lambda-\alpha C_*^2)}+|\Omega|^{\frac{N-2s}{2N}}\right)\Bigg] \\ &\le \frac{(N-2s)\Big[\log\Big(\|f\|_{L^q(\Omega)}+1\Big)+1\Big]}{4N(\chi-1)\alpha\log\Big(\big(\frac{4\alpha}{\lambda}\big)^{\frac{1}{2(\chi-1)}}C_*^{\frac{1}{\chi-1}}\chi^{\frac{\chi}{(\chi-1)^2}}\Big(|\Omega|^{\frac{N-2s}{2N}-\frac{1}{2q'}}+1\Big)^{\frac{1}{\chi-1}}\Big)\Big(\frac{2\lambda|\Omega|^{1-\frac{4s}{N+2s}}}{(\lambda-\alpha C_*^2)}+|\Omega|^{\frac{N-2s}{2N}}\Big)},\end{aligned}$$

where $\alpha \in (0, \frac{\lambda}{C_*^2})$. This completes our proof. □

4. Example

Example 4.1. For some $f \in L^{\frac{N}{2s}}(\Omega)$, there exists a weak solution to problem (1.1) that is unbounded.

Proof. We set $\Omega = B_{\frac{1}{e}}(0)$ and define

$$v(x) = \begin{cases} \ln\ln\frac{1}{|x|}, & \text{in } B_{\frac{1}{e}}(0), \\ 0, & \text{in } \mathbb{R}^N \setminus B_{\frac{1}{e}}(0), \end{cases}$$

and then take

$$u(x) = v(x)\eta(x),$$

where $\eta(x) \in C_0^\infty(B_{\frac{1}{e}}(0))$ is a radial cut-off function with $\eta \equiv 1$ in $B_{\frac{1}{2e}}(0)$, $\eta \equiv 0$ in $B_{\frac{1}{e}}(0) \setminus B_{\frac{3}{4e}}(0)$.

It is easy to verify that $u \in H^s(\mathbb{R}^N)$; then we take

$$f(x) = P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \quad \text{for } x \in B_{\frac{1}{e}}(0).$$

Claim 1. $f(x)$ is bounded on $B_{\frac{1}{e}}(0) \setminus B_{c_o}(0)$, where $c_o = \frac{1}{6e}$.

Indeed, for $x \in B_{\frac{1}{e}}(0) \setminus B_{c_o}(0)$

$$\begin{aligned} f(x) &= P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz \\ &= P.V.\int_{B_{\frac{c_o}{2}}(0)}\frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz + P.V.\int_{\mathbb{R}^N\setminus B_{\frac{c_o}{2}}(0)}\frac{u(x)-u(x+z)}{|z|^{N+2s}}\,dz. \end{aligned}$$

For the first term on the right side of the previous equality, since $|z \pm x| \geq |x| - \frac{c_o}{2} \geq \frac{c_o}{2}$, and by applying Taylor's expansion, there exist parameters $\theta_1, \theta_2 \in (0,1)$ such that

$$
\begin{aligned}
&P.V. \int_{B_{\frac{c_o}{2}}(0)} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \\
&= \frac{1}{2} P.V. \int_{B_{\frac{c_o}{2}}(0)} \frac{u(x) - u(x+z) + u(x) - u(x-z)}{|z|^{N+2s}}\, dz \\
&= -\frac{1}{2} \int_{B_{\frac{c_o}{2}}(0)} \frac{\sum_{i,j}^{N,N} u_{x_i,x_j}(x+\theta_1 z) z_i z_j + \sum_{i,j}^{N,N} u_{x_i,x_j}(x-\theta_2 z) z_i z_j}{|z|^{N+2s}}\, dz \\
&\leq \frac{N^2 \max_{|x| \geq \frac{c_o}{2}, i,j \in [1,N]} |D_{ij} u(x)|}{2} \int_{B_{\frac{c_o}{2}}} \frac{1}{|z|^{N+2s-2}}\, dz \\
&= C(N) \max_{|x| \geq \frac{c_o}{2}, i,j \in [1,N]} |D_{ij} u(x)| \int_0^{\frac{c_o}{2}} \frac{1}{r^{2s-1}}\, dr \\
&= \frac{C(N)}{2-2s} \left(\frac{c_o}{2} \right)^{2-2s} \max_{|x| \geq \frac{c_o}{2}, i,j \in [1,N]} |D_{ij} u(x)| < +\infty.
\end{aligned}
$$

As for the second term, we split it into two parts

$$
\begin{aligned}
&P.V. \int_{\mathbb{R}^N \setminus B_{\frac{c_o}{2}}(0)} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \\
&= \int_{\{|z| > \frac{c_o}{2}\} \cap \{|x| \geq |x+z|\}} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz + \int_{\{|z| > \frac{c_o}{2}\} \cap \{|x| \leq |x+z|\}} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz.
\end{aligned}
$$

Regarding the second part, since $u(x) = u(|x|)$ is a decreasing function with respect to $|x|$, and $|x| \geq c_o$, we derive

$$
\begin{aligned}
\int_{\{|z| > \frac{c_o}{2}\} \cap \{|x| \leq |x+z|\}} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz &\leq 2u(|c_o|) \int_{\{|z| \geq \frac{c_o}{2}\}} \frac{1}{|z|^{N+2s}}\, dz \\
&= C(N) u(|c_o|) \int_{\frac{c_o}{2}}^{\infty} \frac{1}{r^{2s+1}}\, dr \\
&= \frac{C(N) v(|c_o|)}{s} \left(\frac{c_o}{2} \right)^{-2s} < +\infty;
\end{aligned}
$$

for the first part, by the Fubini's Theorem, we get

$$
\begin{aligned}
&\int_{\{|z| > \frac{c_o}{2}\} \cap \{|x| \geq |x+z|\}} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz = \int_{\{|z| > \frac{c_o}{2}\} \cap \{|x| \geq |x+z|\}} \frac{\int_{|z+x|}^{|z|} u'(t)\, dt}{|z|^{N+2s}}\, dz \\
&= \int_0^{|x|} u'(t) \int_{\{|z| > \frac{c_o}{2}\} \cap \{|x+z| \leq t\}} \frac{1}{|z|^{N+2s}}\, dz\, dt \\
&= \int_0^{|x| - \frac{c_o}{2}} u'(t) \int_{\{|z| > \frac{c_o}{2}\} \cap \{|x+z| \leq t\}} \frac{1}{|z|^{N+2s}}\, dz\, dt \\
&\quad + \int_{|x| - \frac{c_o}{2}}^{|x|} u'(t) \int_{\{|z| > \frac{c_o}{2}\} \cap \{|x+z| \leq t\}} \frac{1}{|z|^{N+2s}}\, dz\, dt.
\end{aligned}
$$

Notice that when $t \in \left[0, |x| - \frac{c_o}{2}\right]$, we have $\{|z| > \frac{c_o}{2}\} \cap \{|x+z| \le t\} = \{|x+z| \le t\}$. Hence,

$$
\begin{aligned}
&\int_0^{|x|-\frac{c_o}{2}} u'(t) \int_{\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\, dz\, dt \le C(N) \int_0^{|x|-\frac{c_o}{2}} |u'(t)| \int_{|x|-t}^{|x|+t} \frac{r^{N-1}}{r^{N+2s}}\, dr\, dt \\
&= \frac{C(N)}{2s} \int_0^{|x|-\frac{c_o}{2}} |u'(t)| \left(\frac{1}{(|x|-t)^{2s}} - \frac{1}{(|x|+t)^{2s}} \right) dt \\
&\le \frac{C(N)}{2s} \int_0^{|x|-\frac{c_o}{2}} (|v'(t)| + |\eta'(t)||v(t)|) \left(\frac{1}{(|x|-t)^{2s}} - \frac{1}{(|x|+t)^{2s}} \right) dt \\
&\le C(N,s) + C(N,s) \int_0^{|x|-\frac{c_o}{2}} \frac{1}{\ln \frac{1}{t}} \frac{1}{t} \left(\frac{1}{(|x|-t)^{2s}} - \frac{1}{(|x|+t)^{2s}} \right) dt \\
&\le C(N,s).
\end{aligned}
$$

For the remaining part, we have

$$
\begin{aligned}
\int_{|x|-\frac{c_o}{2}}^{|x|} u'(t) \int_{\{|z|>\frac{c_o}{2}\}\cap\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\, dz\, dt &\le \int_{|x|-\frac{c_o}{2}}^{|x|} |u'(t)| \int_{\{|z|>\frac{c_o}{2}\}} \frac{1}{|z|^{N+2s}}\, dz\, dt \\
&\le \frac{C(N)}{2s} \left(\frac{c_o}{2}\right)^{-2s} \int_{|x|-\frac{c_o}{2}}^{|x|} |u'(t)|\, dt \\
&\le \frac{C(N)}{2s} \left(\frac{c_o}{2}\right)^{-2s+1} \max_{t\in[\frac{c_o}{2}, \frac{3}{4e}]} |u'(t)| \\
&< +\infty.
\end{aligned}
$$

Combining all the inequalities above yields

$$
|f(x)| \le C(N,s), \quad \text{for } |x| \in \left[c_o, \tfrac{1}{e}\right].
$$

Claim 2. When $|x| \le c_o$, we have $|f(x)| \le C(N,s)|x|^{-2s} \frac{1}{\ln \frac{1}{|x|}}$.
First, we split f into three parts:

$$
\begin{aligned}
f(x) &= P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \\
&= P.V. \int_{B_{\frac{|x|}{2}}(0)} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz + \int_{B_{2|x|}(0)\setminus B_{\frac{|x|}{2}}(0)} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \\
&\quad + \int_{\mathbb{R}^{\mathbb{N}}\setminus B_{2|x|}(0)} \frac{u(x) - u(x+z)}{|z|^{N+2s}}\, dz \\
&= f_1(x) + f_2(x) + f_3(x).
\end{aligned}
$$

Then, we analyze $f_3(x)$: since $\frac{3}{2}|z| \ge |x+z| \ge |x|$ for $|z| \ge 2|x|$, applying Fubini's Theorem yields

$$
\begin{aligned}
f_3(x) &= \int_{\mathbb{R}^N\setminus B_{2|x|}(0)} \frac{\int_{|x|}^{|x+z|} -u'(t)dt}{|z|^{N+2s}}\,dz = \int_{|x|}^{\frac{3}{4e}} -u'(t)\int_{\{|z+x|\geq t\}\cap\{|z|\geq 2|x|\}} \frac{1}{|z|^{N+2s}}\,dz\,dt \\
&\leq \int_{|x|}^{\frac{3}{4e}} -u'(t)\int_{\{|z|\geq\frac{2}{3}t\}} \frac{1}{|z|^{N+2s}}\,dz\,dt = \frac{C(N)}{2s}\left(\frac{2}{3}\right)^{-2s}\int_{|x|}^{\frac{3}{4e}} -u'(t)t^{-2s}\,dt \\
&\leq C(N,s)\int_{|x|}^{\frac{3}{4e}} |v(t)|t^{-2s}\,dt + C(N,s)\int_{|x|}^{\frac{3}{4e}} |v'(t)|t^{-2s}\,dt.
\end{aligned}
$$

Notice that

$$
\int_{|x|}^{\frac{3}{4e}} |v(t)|t^{-2s}\,dt \leq \ln\ln\frac{1}{|x|}\int_{|x|}^{\frac{3}{4e}} t^{-2s}\,dt = \begin{cases} \frac{\ln\ln\frac{1}{|x|}}{1-2s}\left(\left(\frac{3}{4e}\right)^{1-2s} - |x|^{1-2s}\right), & s\neq\frac{1}{2}, \\ \left(\ln\ln\frac{1}{|x|}\right)\left(\ln\frac{3}{4e|x|}\right), & s=\frac{1}{2}, \end{cases}
$$

and

$$
\int_{|x|}^{\frac{3}{4e}} |v'(t)|t^{-2s}\,dt \leq \int_{|x|}^{\frac{3}{4e}} \frac{1}{\ln\frac{1}{t}}t^{-2s-1}\,dt \leq C(s)\frac{1}{\ln\frac{1}{|x|}}|x|^{-2s}.
$$

Next, we estimate $f_2(x)$; since $|x|\leq\frac{1}{6e}\leq\frac{1}{2e}$ and $|z|\leq 2|x|$, we have

$$
u(x) = v(x) = \ln\ln\frac{1}{|x|},\quad u(x+z) = v(x+z) = \ln\ln\frac{1}{|x+z|};
$$

then, we split it into two parts,

$$
\begin{aligned}
f_2(x) &= \int_{\{\frac{1}{2}|x|\leq|z|\leq 2|x|\}\cap\{|x+z|\geq|x|\}} \frac{\ln\ln\frac{1}{|x|} - \ln\ln\frac{1}{|x+z|}}{|z|^{N+2s}}\,dz \\
&\quad + \int_{\{\frac{1}{2}|x|\leq|z|\leq 2|x|\}\cap\{|x+z|<|x|\}} \frac{\ln\ln\frac{1}{|x|} - \ln\ln\frac{1}{|x+z|}}{|z|^{N+2s}}\,dz \\
&= f_{2,1}(x) + f_{2,2}(x).
\end{aligned}
$$

Regarding $f_{2,1}(x)$, from the Mean Value Theorem we have

$$
\begin{aligned}
f_{2,1}(x) &\leq \int_{\{\frac{1}{2}|x|\leq|z|\leq 2|x|\}\cap\{|x+z|\geq|x|\}} \frac{\ln\ln\frac{1}{|x|} - \ln\ln\frac{1}{|x|+|z|}}{|z|^{N+2s}}\,dz \\
&\leq \int_{\{\frac{1}{2}|x|\leq|z|\leq 2|x|\}} \frac{\ln\ln\frac{1}{|x|} - \ln\ln\frac{1}{|x|+|z|}}{|z|^{N+2s}}\,dz \\
&\leq \int_{\{\frac{1}{2}|x|\leq|z|\leq 2|x|\}} \frac{2}{|x|\ln\frac{2}{|x|}}\frac{1}{|z|^{N+2s-1}}\,dz \\
&\leq \frac{C(N)}{|x|\ln\frac{1}{|x|}}\int_{\frac{1}{2}|x|}^{2|x|} \frac{r^{N-1}}{r^{N+2s-1}}\,dr \\
&\leq \begin{cases} \frac{C(N)}{|x|\ln\frac{1}{|x|}}, & s=\frac{1}{2}, \\ \frac{C(N)}{(1-2s)\ln\frac{1}{|x|}}\left(|2x|^{-2s} - |\frac{1}{2}x|^{-2s}\right), & s\neq\frac{1}{2}. \end{cases}
\end{aligned}
$$

Next, we estimate $f_{2,2}(x)$, from Fubini's Theorem, we get

$$
\begin{aligned}
f_{2,2}(x) &= \int_0^{|x|} \frac{-1}{t\ln\frac{1}{t}} \int_{\{\frac{1}{2}|x|\le|z|\le 2|x|\}\cap\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\,dz\,dt \\
&= \int_0^{\frac{1}{2}|x|} \frac{-1}{t\ln\frac{1}{t}} \int_{\{\frac{1}{2}|x|\le|z|\le 2|x|\}\cap\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\,dz\,dt \\
&\quad + \int_{\frac{1}{2}|x|}^{|x|} \frac{-1}{t\ln\frac{1}{t}} \int_{\{\frac{1}{2}|x|\le|z|\le 2|x|\}\cap\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\,dz\,dt
\end{aligned}
$$

Since $\{|x+z|\le t\}\subset\{\frac{1}{2}|x|\le|z|\le|x|\}$ for $t\in\left[0,\frac{1}{2}|x|\right]$, we obtain

$$
\begin{aligned}
&\int_0^{\frac{1}{2}|x|} \frac{-1}{t\ln\frac{1}{t}} \int_{\{\frac{1}{2}|x|\le|z|\le 2|x|\}\cap\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\,dz\,dt \\
&= \int_0^{\frac{1}{2}|x|} \frac{-1}{t\ln\frac{1}{t}} \int_{\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\,dz\,dt \\
&\le C(N)\int_0^{\frac{1}{2}|x|} \frac{1}{t\ln\frac{1}{t}} \int_{|x|-t}^{|x|+t} \frac{1}{|r|^{1+2s}}\,dr\,dt \\
&= \frac{C(N)}{2s}\int_0^{\frac{1}{2}|x|} \frac{1}{t\ln\frac{1}{t}}\left(\frac{1}{(|x|-t)^{2s}}-\frac{1}{(|x|+t)^{2s}}\right)dt,
\end{aligned}
$$

as

$$
\frac{1}{(|x|-t)^{2s}}-\frac{1}{(|x|+t)^{2s}} \le \frac{4s}{|x|^{2s+1}}t+\frac{C(s)t^2}{|x|^{2s+2}};
$$

then, we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}|x|} \frac{1}{t\ln\frac{1}{t}}\left(\frac{1}{(|x|-t)^{2s}}-\frac{1}{(|x|+t)^{2s}}\right)dt \\
&\le \frac{C(s)}{|x|^{2s+1}}\int_0^{\frac{|x|}{2}}\frac{1}{\ln\frac{1}{t}}\,dt+\frac{C(s)}{|x|^{2s+2}}\int_0^{\frac{|x|}{2}}\frac{t}{\ln\frac{1}{t}}\,dt \le C(s)|x|^{-2s}\frac{1}{\ln\frac{1}{|x|}}.
\end{aligned}
$$

As for $t\in\left[\frac{|x|}{2},|x|\right]$, by direct computation, we have

$$
\begin{aligned}
&\int_{\frac{1}{2}|x|}^{|x|} \frac{-1}{t\ln\frac{1}{t}} \int_{\{\frac{1}{2}|x|\le|z|\le 2|x|\}\cap\{|x+z|\le t\}} \frac{1}{|z|^{N+2s}}\,dz\,dt \\
&\le \frac{C(N)}{|x|\ln\frac{1}{|x|}}\cdot\frac{|x|}{2}\cdot\frac{2^{N+2s}}{|x|^{N+2s}}\cdot|x|^N = C(N,s)|x|^{-2s}\frac{1}{\ln\frac{1}{|x|}}.
\end{aligned}
$$

Finally, we deal with $f_1(x)$: since $|x\pm z|\in\left[\frac{|x|}{2},\frac{3|x|}{2}\right]\subset\left[\frac{|x|}{2},\frac{1}{4e}\right]$, applying Taylor's expansions yields

$$
\begin{aligned}
f_1(x) &= \frac{1}{2}P.V.\int_{B_{\frac{|x|}{2}}(0)} \frac{u(x)-u(x+z)+u(x)-u(x-z)}{|z|^{N+2s}}\,dz \\
&= \frac{1}{2}\int_{B_{\frac{|x|}{2}}(0)} \frac{-\sum_{i=1}^N\sum_{j=1}^N\frac{\partial^2 u(x+\theta_3 z)}{\partial x_i\partial x_j}z_iz_j-\sum_{i=1}^N\sum_{j=1}^N\frac{\partial^2 u(x-\theta_4 z)}{\partial x_i\partial x_j}z_iz_j}{|z|^{N+2s}}\,dz.
\end{aligned}
$$

Here, if $i \neq j$, we have

$$\left|\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right| = \left|\frac{1}{\left(\ln \frac{1}{|x|}\right)^2} \frac{x_i x_j}{|x|^4} + \frac{1}{\ln \frac{1}{|x|}} \frac{x_i x_j}{|x|^4}\right| \leq \frac{1}{|x|^2 \ln \frac{1}{|x|}},$$

and for $i = j$

$$\left|\frac{\partial^2 u(x)}{\partial x_i^2}\right| = \left|\frac{3x_i^2}{|x^4| \ln \frac{1}{|x|}} - \frac{1}{|x|^2 \ln \frac{1}{|x|}}\right| \leq \frac{4}{|x|^2 \ln \frac{1}{|x|}}.$$

Consequently,

$$\left|-\sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 u(x+\theta_3 z)}{\partial x_i \partial x_j} z_i z_j - \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 u(x-\theta_4 z)}{\partial x_i \partial x_j} z_i z_j\right| \leq \frac{4N^2 |z|^2}{|x|^2 \ln \frac{1}{|x|}}.$$

Thus,

$$\begin{aligned}
&\int_{B_{\frac{|x|}{2}}} \frac{-\sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 u(x+\theta_3 z)}{\partial x_i \partial x_j} z_i z_j - \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 u(x-\theta_4 z)}{\partial x_i \partial x_j} z_i z_j}{|z|^{N+2s}}\, dz \\
&\leq \frac{4N^2}{|x|^2 \ln \frac{1}{|x|}} \int_{B_{\frac{|x|}{2}}(0)} \frac{1}{|z|^{N+2s-2}}\, dz = \frac{C(N)}{|x|^2 \ln \frac{1}{|x|}} \int_0^{\frac{|x|}{2}} \frac{1}{r^{2s-1}}\, dr \\
&= C(N) |x|^{-2s} \frac{1}{\ln \frac{1}{|x|}}.
\end{aligned}$$

Adding together the previous inequalities, we get

$$|f(x)| \leq C(N,s) \frac{1}{|x|^{2s} \ln \frac{1}{|x|}} \quad \text{for } x \in B_{\frac{1}{6e}}(0).$$

Finally, combining Claim 1 and Claim 2 immediately yields $f \in L^{\frac{N}{2s}}\big(B_{\frac{1}{e}}(0)\big)$. $\square$

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