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Maximal operators and differentiation associated to collections of shifted convex bodies

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Abstract. Maximal operators and differentiation of integrals associated to collections of shifted balls in $\mathbb{R}^n$ (i.e., balls that may not contain the origin) have been studied by various authors. One of the motivations has been the intimate connection of these concepts with the boundary behaviour of Poisson integrals along regions more general than cones. Generalizing the corresponding results of Nagel and Stein, and Hagelstein and Parissis (established for the case of collections of balls) we give characterizations of the two classes of monotone collections Ω of shifted convex bodies in $\mathbb{R}^n$ that are defined by the following properties respectively: 1) the maximal operator associated to Ω (i.e., to the means $(1/|B|)\int_{B+x}|f|$ $(B\in\Omega)$) satisfies the weak type $(1,1)$ inequality; 2) the means over the sets $B+x$ $(B\in\Omega)$ are a.e. convergent for the characteristic function of an arbitrary measurable subset of $\mathbb{R}^n$.

Siirrettyihin kuperiin kappaleisiin liittyvät enimmäismuunnokset ja derivointi

Tiivistelmä. Useat kirjoittajat ovat tutkineet avaruuden $\mathbb{R}^n$ siirrettyihin (ts. ei välttämättä origon sisältäviin) kuuliin liittyvien integraalien enimmäismuunnoksia ja derivoituvuutta. Yksi virike tälle on ollut näiden läheinen yhteys kartioita yleisemmillä alueilla laskettujen Poissonin integraalien reunakäytökseen. Yleistämällä Nagelin ja Steinin sekä Hagelsteinin ja Parissiksen vastaavia (kuulakokoelmia koskevia) tuloksia määritetään tässä työssä ne kaksi avaruuden $\mathbb{R}^n$ siirrettyjen kuperien kappaleiden monotonista kokoelmaa, joilla on toinen seuraavista ominaisuuksista: 1) kokoelmaan Ω (ts. keskiarvoihin $(1/|B|)\int_{B+x}|f|$, missä $B\in\Omega$) liittyvä enimmäismuunnos toteuttaa heikon $(1,1)$-epäyhtälön; 2) joukoilla $B+x$, missä $B\in\Omega$, lasketut keskiarvot suppenevat melkein kaikkialla avaruuden $\mathbb{R}^n$ mielivaltaisen mitallisen osajoukon ilmaisimeen.

1. Introduction

Let Ω be a non-empty collection of bounded measurable subsets of $\mathbb{R}^n$ with positive measure. We define the *maximal operator* associated to Ω by

$$M_\Omega(f)(x)=\sup_{B\in\Omega}\frac{1}{|B|}\int_{B+x}|f|\quad (f\in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ x\in\mathbb{R}^n).$$

Let us call Ω a *regular collection in* $\mathbb{R}^n$ if it consists of bounded measurable subsets of $\mathbb{R}^n$ with positive measure and satisfies the condition $\inf_{B\in\Omega}\operatorname{diam}(B\cup\{0\})=0$.

We will say that a regular collection Ω *differentiates integrals of functions from a class* $F\subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (briefly, *differentiates a class* $F\subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$) if for every $f\in F$,

$$\lim_{B\in\Omega,\ \operatorname{diam}(B\cup\{0\})\to 0}\frac{1}{|B|}\int_{B+x}f=f(x)$$

for a.e. $x\in\mathbb{R}^n$. If Ω differentiates the class of all characteristic functions of measurable subsets of $\mathbb{R}^n$, then Ω will be said to have the *density property*.

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Below everywhere balls will be assumed to be open.

The classical theorem of Lebesgue (see, e.g., [3, Section I.3]) asserts that if Ω consists of all balls centered at the origin, or more generally, if Ω is a regular collection of balls weakly shifted in the sense that $\sup_{B(x,r)\in\Omega}\|x\|/r<\infty$, then Ω differentiates $L^1(\mathbb{R}^n)$ (and consequently, Ω has the density property). Here and below $B(x,r)$ denotes the ball centered at x with radius r.

Motivated by the study of boundary behaviour of Poisson integrals along regions more general than cones, Nagel and Stein [8] investigated maximal operators and differentiation associated to collections Ω of balls that do not satisfy the condition $\sup_{B(x,r)\in\Omega}\|x\|/r<\infty$. In [8] (see also [11, Section II.3]), it was proved that the maximal operator M_Ω associated with a collection Ω of balls satisfies the weak type $(1,1)$ inequality if and only if Ω has the following covering property (referred to below as the *S-covering property*): there exists a fixed N such that, for any given $r>0$, the balls in Ω with radius at most r are contained in a union of at most N balls of radius r. Consequently, the validity of the S-covering property for a regular collection Ω of balls implies that Ω differentiates $L^1(\mathbb{R}^n)$.

In [8] (see also [11, Section II.3]), it is also shown that the S-covering property is equivalent to each of the following conditions: 1) M_Ω satisfies the strong type (p,p) inequality for every $p\in(1,\infty)$; 2) M_Ω satisfies the strong type (p,p) inequality for some $p\in(1,\infty)$.

Let $\Omega=\{B(x_k,r_k)\colon k\in\mathbb{N}\}$ be a regular collection, where (r_k) is a decreasing sequence tending to zero. One of the interesting corollaries in [8] shows that, even when $\sup_{k\in\mathbb{N}}\|x_k\|/r_k=\infty$, the collection Ω still differentiates $L^1(\mathbb{R}^n)$, provided that $\sup_{k\in\mathbb{N}}\|x_{k+1}\|/r_k<\infty$. This result was also established by Aversa and Preiss [1].

Hagelstein and Parissis [5] have shown that for a regular collection Ω of balls the following three conditions are equivalent: 1) Ω has the density property; 2) Ω differentiates $L^1(\mathbb{R}^n)$; 3) There exists $r>0$ for which the collection of all balls $B\in\Omega$ with $\operatorname{diam}(B\cup\{0\})<r$ has the S-covering property. This result for the case of a regular collection Ω consisting of pairwise disjoint balls was established earlier by Csörnyei [2].

The papers of Moonens and Rosenblatt [7], and Laba and Pramanik [6], should also be mentioned. In [7], there are studied maximal operators and differentiation associated to collections Ω of the type $\{I_k+x_k\colon k\in\mathbb{N}\}$, where (I_k) is a sequence of two-dimensional intervals centered at the origin. The paper [6] deals with the topic for dilation-invariant collections Ω of sparse one-dimensional sets.

We will refer to any bounded, open, non-empty convex set in $\mathbb{R}^n$ as a convex body.

The aim of the paper is to study the maximal operators and differentiation associated to collections Ω of convex bodies that are *monotone* in the sense that for any two sets B and B' from Ω there exists a shift mapping one of them into a subset of the other. Note that if sets from Ω are not "shifted", i.e., all sets from Ω contain the origin, then Ω differentiates $L^1(\mathbb{R}^n)$ (see, e.g., [3, Section I.3]).

Let Ω be a monotone collection consisting of convex bodies in $\mathbb{R}^n$. For a set $B\in\Omega$ let us denote by $\Omega(B)$ the collection of all sets $R\in\Omega$ for which there exists a shift mapping R into a subset of B. Let us say that the collection Ω has the *S-covering property* if there exists $N\in\mathbb{N}$ such that for every set B from Ω the sets from the collection $\Omega(B)$ lie in a union of N shifts of B. Note that for collections of balls this definition is equivalent to the one given above (see Remark 2.1 below).

The theorems below extend the characterizations of Nagel and Stein, and Hagelstein and Parissis to the monotone collections of convex bodies.

Theorem 1.1. *Let Ω be a monotone collection consisting of convex bodies in $\mathbb{R}^n$. Then the following four statements are equivalent:*

1) *M_Ω satisfies the weak type $(1,1)$ inequality;*
2) *M_Ω satisfies the strong type (p,p) inequality for every $p \in (1,\infty)$;*
3) *M_Ω satisfies the strong type (p,p) inequality for some $p \in (1,\infty)$;*
4) *Ω has the S-covering property.*

Theorem 1.2. *Let Ω be a monotone regular collection consisting of convex bodies in $\mathbb{R}^n$. Then the following three statements are equivalent:*

1) *Ω has the density property;*
2) *Ω differentiates $L^1(\mathbb{R}^n)$;*
3) *There exists $r > 0$ for which the collection of all sets $B \in \Omega$ with $\operatorname{diam}(B \cup\{0\}) < r$ has the S-covering property.*

2. Auxiliary statements

Remark 2.1. Let Ω be a collection of balls. Then the following two statements are equivalent: 1) There exists $N \in \mathbb{N}$ such that for every $r > 0$ the balls from Ω having radii at most r lie in a union of N balls of radius r; 2) There exists $M \in \mathbb{N}$ such that for every ball B from Ω the balls from $\Omega(B)$ lie in a union of M shifts of B.

The proof of this statement follows from Theorem 1.1 and the result of Nagel and Stein given in the introduction. Below we provide its direct proof as well.

For a ball B denote by $r(B)$ and $c(B)$ its radius and center, respectively.

The implication 1) $\Rightarrow$ 2) is obvious.

Suppose that the statement 2) is true. Take any $t > 0$. If the collection $\{B \in \Omega\colon r(B) \leqslant t\}$ is empty or there is a ball $B \in \Omega$ with $r(B) = t^* = \sup\{r(B)\colon B \in \Omega, r(B) \leqslant t\}$, then clearly we can find balls $B(x_1,t),\ldots,\ B(x_M,t)$ which cover the union $\bigcup_{B\in\Omega, r(B)\leqslant t} B$. Otherwise we can find a sequence of balls $B_k \in \Omega$ with $r(B_1) \leqslant r(B_2) \leqslant \cdots < t^*$ and $\lim_{k\to\infty} r(B_k) = t^*$. For each k denote by Ω_k the collection of all balls from Ω with radius at most $r(B_k)$. Then for every k we can find balls $B_{k,1},\ldots B_{k,M}$ which are shifts of B_k, $\bigcup_{B\in\Omega_k} B \subset \bigcup_{m=1}^M B_{k,m}$ and $B_{k,m} \cap \big(\bigcup_{B\in\Omega_k} B\big) \neq \varnothing$ $(m = 1,\ldots,M)$. Then the sequences of centers $(c(B_{k,1})),\ldots,(c(B_{k,M}))$ are bounded (otherwise it is easy to see that the statement 2) would not be true). Consequently, we can find sub-sequences $(c(B_{k_j,1})),\ldots,(c(B_{k_j,M}))$ which are convergent to some points $x_1,\ldots,x_M$. Then it is easy to see that $\bigcup_{B\in\Omega,r(B)\leqslant t} B \subset \bigcup_{m=1}^M B(x_m, 2t)$. This completes the proof of the implication 2) $\Rightarrow$ 1).

For a bounded set B with center of symmetry x and for a number $\alpha > 0$, we will denote by $\alpha \times B$ the dilation of B by the coefficient α, i.e., the set $\{x+\alpha(y-x)\colon y \in B\}$.

Recall that by virtue of the lemma of John (see, e.g., [3, Section VI.2]), for an arbitrary convex body $B \subset \mathbb{R}^n$ there exists an open ellipsoid T such that $T \subset B \subset n \times T$. This lemma easily implies the following statement.

Lemma 2.2. *Let B be a convex body in $\mathbb{R}^n$. Then there exists an open n-dimensional rectangle R such that $R \subset B \subset n^2 \times R$.*

Lemma 2.3. *Let B_1 and B_2 be convex bodies in $\mathbb{R}^n$, and let there exists a shift mapping B_1 into a subset of B_2. Then the set $B_1 + B_2$ can be covered by $4^n n^{2n}$ shifts of B_2. The same is true for the set $B_1 - B_2$. Consequently, $|B_1 \pm B_2| \leqslant 4^n n^{2n} |B_2|$.*

Proof. By Lemma 2.2 we can choose an open rectangle such that $R \subset B_2 \subset n^2 \times R$. Then denoting by τ the shift mapping B_1 into a subset of B_2 we have that $B_1 + B_2 \subset \tau^{-1}(B_2) + B_2 \subset \tau^{-1}(n^2 \times R) + n^2 \times R$. The last set is a shift of a rectangle $2n^2 \times R$ which clearly can be covered by $4^n n^{2n}$ shifts of R, and consequently, by the same number of shifts of B_2. The proof is analogous for the case of the set $B_1 - B_2$. □

Below we will switch to a more general terminology related with maximal operators and differentiation of integrals.

A mapping $\mathbf{B}$ defined on $\mathbb{R}^n$ is called a *differentiation basis* (briefly: a *basis*) if for each $x \in \mathbb{R}^n$ the value $\mathbf{B}(x)$ is a collection of bounded measurable sets of positive measure such that there exists a sequence (B_k) of sets from $\mathbf{B}(x)$ with $\lim_{k\to\infty} \operatorname{diam}(B_k \cup \{x\}) = 0$.

Let $\mathbf{B}$ be a basis. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ it is said that $\mathbf{B}$ *differentiates* $\int f$ (or $\int f$ *is differentiable with respect to* $\mathbf{B}$) if for almost every $x \in \mathbb{R}^n$, $\frac{1}{|B|}\int_B f \to f(x)$ as $B \in \mathbf{B}(x)$ and $\operatorname{diam}(B) \to 0$. If this is true for each f in the class of functions F, we say that $\mathbf{B}$ *differentiates* F. If $\mathbf{B}$ differentiates the class of all characteristic functions of measurable subsets of $\mathbb{R}^n$, then $\mathbf{B}$ is called a *density basis.*

Let us call a mapping $\mathbf{B}$ defined on $\mathbb{R}^n$ a *semi-basis* if for each $x \in \mathbb{R}^n$ the value $\mathbf{B}(x)$ is a collection of bounded measurable sets of positive measure.

The *maximal operator* associated with a semi-basis $\mathbf{B}$ is defined as follows

$$M_{\mathbf{B}}(f)(x) = \sup_{B \in \mathbf{B}(x)} \frac{1}{|B|} \int_B |f| \quad (f \in L^1_{\text{loc}}(\mathbb{R}^n),\ x \in \mathbb{R}^n).$$

A semi-basis $\mathbf{B}$ is called *shift-invariant* if for every $x \in \mathbb{R}^n$, $\mathbf{B}(x) = \{B + x \colon B \in \mathbf{B}(0)\}$.

Let Ω be a collection of bounded measurable subsets of $\mathbb{R}^n$ with positive measure. Denote by $\mathbf{B}_\Omega$ the shift-invariant semi-basis for which $\mathbf{B}(0)$ coincides with Ω. Obviously, $M_\Omega = M_{\mathbf{B}_\Omega}$. Note also that, if Ω is a regular collection in $\mathbb{R}^n$, then the process of differentiation of integrals associated to Ω (defined in the introduction) coincides with the one with respect to the basis $\mathbf{B}_\Omega$.

Let $\mathbf{B}$ be a semi-basis. For a point $x_0 \in \mathbb{R}^n$ and a set $B_0 \in \mathbf{B}(x_0)$ by $H_{\mathbf{B}}(x_0, B_0)$ denote the union of all sets $B \cup \{x\}$, where $x \in \mathbb{R}^n$, $B \in \mathbf{B}(x)$, $(B \cup \{x\}) \cap (B_0 \cup \{x_0\}) \neq \varnothing$ and $|B| \leqslant 2|B_0|$.

By $|E|_*$ we will denote the outer measure of a set $E \subset \mathbb{R}^n$.

Lemma 2.4. *Let Ω be a monotone collection of convex bodies in $\mathbb{R}^n$ which has the S-covering property. Then for every $x_0 \in \mathbb{R}^n$ and $B_0 \in \mathbf{B}_\Omega(x_0)$ we have that $|H_{\mathbf{B}_\Omega}(x_0, B_0)|_* \leqslant C|B_0|$, where the constant C depends only on the dimension n and the constant N from the definition of the S-covering property.*

Proof. Taking into account monotonicity of Ω and continuity of the outer measure from below, it is sufficient to show that for every set $R \in \Omega$ with $|B_0| \leqslant |R| \leqslant 2|B_0|$ the estimate $|H_{\mathbf{B}_{\Omega(R)}}(x_0, B_0)|_* \leqslant C|B_0|$ holds, where C depends only on the dimension n and the constant N associated to the S-covering property.

Let us take any $R \in \Omega$ with $|B_0| \leqslant |R| \leqslant 2|B_0|$. By the S-covering property of Ω we can find shifts $R_1, \dots, R_N$ of R for which

$$\bigcup_{B \in \Omega(R)} B \subset R_1 \cup \dots \cup R_N.$$

Denote by Λ the set of all pairs (x, B) such that $x \in \mathbb{R}^n$, $B \in \mathbf{B}_{\Omega(R)}(x)$ and $(B \cup \{x\}) \cap (B_0 \cup \{x_0\}) \neq \varnothing$. We have that

$$H_{\mathbf{B}_{\Omega(R)}}(x_0, B_0) = \bigcup_{(x,B)\in\Lambda} (B \cup \{x\}).$$

Note that the condition $(B \cup \{x\}) \cap (B_0 \cup \{x_0\}) \neq \varnothing$ is valid if and only if at least one among the four following conditions is true: 1) $x = x_0$; 2) $B \ni x_0$; 3) $x \in B_0$; and 4) $B \cap B_0 \neq \varnothing$. Hence, setting

$$\begin{aligned}
&\Lambda_1 = \{(x,B) \in \Lambda \colon x = x_0\}, \quad \Lambda_2 = \{(x,B) \in \Lambda \colon B \ni x_0\},\\
&\Lambda_3 = \{(x,B) \in \Lambda \colon x \in B_0\}, \quad \Lambda_4 = \{(x,B) \in \Lambda \colon B \cap B_0 \neq \varnothing\},\\
&H^{(k)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0) = \bigcup_{(x,B)\in\Lambda_k} (B \cup \{x\}) \qquad (k = 1,2,3,4),
\end{aligned}$$

we have that

$$H_{\mathbf{B}_{\Omega(R)}}(x_0, B_0) = \bigcup_{k=1}^{4} H^{(k)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0).$$

Let us estimate the outer measures of the sets $H^{(k)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0)$. We will discuss in detail the case $k = 4$. The other three cases are analogous.

Let J_0 be the set from Ω for which $B_0 = J_0 + x_0$.

Suppose $(x, B) \in \Lambda_4$. Let J be a set from $\Omega(R)$ for which $B = J + x$. Then taking into account that $B \cap B_0 = (J + x) \cap (J_0 + x_0) \neq \varnothing$ and $J \subset R_1 \cup \dots \cup R_N$ we have

$$\begin{aligned}
B \cup \{x\} \subset {} & [(x_0 + J_0 - R_1 \cup \dots \cup R_N) + R_1 \cup \dots \cup R_N]\\
& \cup [x_0 + J_0 - R_1 \cup \dots \cup R_N].
\end{aligned}$$

Since (x, B) is any pair from Λ_4, we obtain that

$$H^{(4)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0) \subset \bigcup_{i,j=1}^{N} (x_0 + J_0 - R_i + R_j) \ \cup \ \bigcup_{i=1}^{N} (x_0 + J_0 - R_i).$$

Let $i, j \in \{1, \dots, N\}$. By Lemma 2.3,

$$|x_0 + J_0 - R_i| \leqslant c|R_i| = c|R| \leqslant 2c|B_0|,$$

where c depends only on the dimension n. Now noticing that $x_0 + J_0 - R_i$ is a convex body which contains a shift of R_j, by Lemma 2.3 again we obtain

$$|x_0 + J_0 - R_i + R_j| \leqslant c|x_0 + J_0 - R_i| \leqslant 2c^2|B_0|.$$

From the last two estimates we conclude that

$$|H^{(4)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0)|_* \leqslant 2N^2c^2|B_0| + 2Nc|B_0|.$$

By similar reasoning it can be shown that $|H^{(1)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0)|_* \leqslant 2N|B_0|$, $|H^{(2)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0)|_* \leqslant 2N^2c|B_0| + 2N|B_0|$ and $|H^{(3)}_{\mathbf{B}_{\Omega(R)}}(x_0, B_0)|_* \leqslant 2Nc|B_0| + 2N|B_0|$. □

Lemma 2.5. *Let* $\mathbf{B}$ *be a semi-basis in* $\mathbb{R}^n$*. If there exists a constant* $C > 0$ *for which* $|H_{\mathbf{B}}(x, B)|_* \leqslant C|B|$ *for every* $x \in \mathbb{R}^n$ *and* $B \in \mathbf{B}(x)$*, then*

$$|\{M_{\mathbf{B}}(f) > \lambda\}|_* \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} |f|$$

for every $f \in L^1(\mathbb{R}^n)$ *and* $\lambda > 0$*.*

Proof. Suppose $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. For every $x \in \{M_{\mathbf{B}}(f) > \lambda\}$ let us choose a set $B_x \in \mathbf{B}(x)$ such that $\frac{1}{|B_x|}\int_{B_x} |f| > \lambda$. Using the greedy algorithm of choice (see, e.g., [10, Section I.1.7]) for the collection $\{B_x \cup \{x\} \colon x \in \{M_{\mathbf{B}}(f) > \lambda\}\}$ with respect to the value of the measure of a set, and taking into account the estimate $|H_{\mathbf{B}}(x, B)|_* \leqslant C|B|$ $(x \in \mathbb{R}^n,\ B \in \mathbf{B}(x))$ we find a sequence (x_k) of points from the set $\{M_{\mathbf{B}}(f) > \lambda\}$ for which the sets $B_{x_k} \cup \{x_k\}$ $(k \in \mathbb{N})$ are pairwise disjoint and

$$\Bigg|\bigcup_{x\in\{M_{\mathbf{B}}(f)>\lambda\}} (B_x \cup \{x\})\Bigg|_* \leqslant C\sum_k |B_{x_k} \cup \{x_k\}| = C\sum_k |B_{x_k}|.$$

Hence, we obtain that

$$\begin{aligned}|\{M_{\mathbf{B}}(f) > \lambda\}|_* &\leqslant \Bigg|\bigcup_{x\in\{M_{\mathbf{B}}(f)>\lambda\}} (B_x \cup \{x\})\Bigg|_* \leqslant C\sum_k |B_{x_k}| \\ &< C\sum_k \frac{1}{\lambda}\int_{B_{x_k}} |f| \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} |f|.\end{aligned}$$

The lemma is proved. □

Remark 2.6. Let Ω be a monotone collection of convex bodies in $\mathbb{R}^n$ containing the origin. Then Ω has the S-covering property and, consequently, by Lemmas 2.4 and 2.5, the maximal operator $M_\Omega = M_{\mathbf{B}(\Omega)}$ satisfies the weak type $(1, 1)$ inequality. The S-covering property is verified as follows. Fix an arbitrary $B \in \Omega$. Then every set in the collection $\Omega(B)$ is contained in $B + B - B$. By Lemma 2.3, the latter set can be covered by $4^{2n}n^{4n}$ shifts of the set B.

Let Ω be a monotone collection consisting of convex bodies in $\mathbb{R}^n$. Let us define the *completion* of Ω as the collection of all sets R such that R is a shift of some set from Ω and R contains some set from Ω. We will use the notation $\overline{\Omega}$ for the completion of Ω.

Lemma 2.7. *Let* Ω *be a monotone collection consisting of convex bodies in* $\mathbb{R}^n$*. Then the following two statements are equivalent:*

1) Ω *has the* S*-covering property;*
2) *there exists* $C > 0$ *such that* $\big|\bigcup_{R\in\overline{\Omega(B)}} R\big| \leqslant C|B|$ *for every* $B \in \Omega$*.*

Proof. Assume that 1) is true. Let $B \in \Omega$. Then we can find sets $B_1, \dots, B_N$ which are shifts of B and cover the union of sets from $\Omega(B)$. Take any $R \in \overline{\Omega(B)}$. Note that R contains some set from $\Omega(B)$ and, consequently, $R \cap B_k \neq \varnothing$ for some $k \in \{1, \dots, N\}$. Hence, taking into account that R can be mapped into a subset of B_k by some shift, we have

$$R \subset B_k - B_k + B_k.$$

From the last inclusion we get

$$\bigcup_{R\in\overline{\Omega(B)}} R \subset \bigcup_{k=1}^{N} (B_k - B_k + B_k).$$

This by Lemma 2.3 implies that the statement 2) is true.

Now suppose that the statement 2) is true. Let $B \in \Omega$. Note that $\bigcup_{R\in\overline{\Omega(B)}} R$ can be represented as a union of shifts of B. Indeed, for every $R \in \overline{\Omega(B)}$ there is a set B_R which is a shift of B such that $R \subset B_R$. Then $\bigcup_{R\in\overline{\Omega(B)}} R = \bigcup_{R\in\overline{\Omega(B)}} B_R$. Denote by Λ the collection $\{B_R\colon R \in \overline{\Omega(B)}\}$.

Let us choose sets $E_1, \dots, E_K \in \Lambda$ in the following way: Take $E_1 \in \Lambda$ arbitrarily. If $E_1, \dots, E_k$ have already been chosen and the collection $\Lambda_k = \{E \in \Lambda\colon E \cap (E_1 \cup \dots \cup E_k) \neq \varnothing\}$ is non-empty then choose E_{k+1} from Λ_k arbitrarily.

The sets $E_1, \dots, E_K \in \Lambda$ chosen in this way are pairwise disjoint and $E \cap (E_1 \cup \dots \cup E_K) \neq \varnothing$ for every $E \in \Lambda$. From the disjointness we have the estimate

$$K \leqslant \Big|\bigcup_{R\in\overline{\Omega(B)}} R\Big| / |B| \leqslant C|B|/|B| = C. \tag{2.1}$$

On the other hand, by the condition $E \cap (E_1 \cup \dots \cup E_K) \neq \varnothing$ $(E \in \Lambda)$ we have

$$\bigcup_{R\in\Omega(B)} R \subset \bigcup_{R\in\overline{\Omega(B)}} R \subset \bigcup_{k=1}^{K} (E_k - E_k + E_k). \tag{2.2}$$

From Lemma 2.2 it follows that each set $E_k - E_k + E_k$ $(k = 1, \dots, K)$ can be covered by M shifts of B, where M depends only on the dimension n. This together with the relations (2.1) and (2.2) implies the validity of the S-covering property for Ω with constant N equal to CM. □

Remark 2.8. The proof of the implication 2) ⇒ 1) of Lemma 2.7 actually shows that the collection $\overline{\Omega}$ has the S-covering property. Consequently, taking into account the implication 1) ⇒ 2) of Lemma 2.7 we see that a monotone collection of convex bodies Ω has the S-covering property if and only if its completion $\overline{\Omega}$ has the same property.

Remark 2.9. The validity of the estimate $\big|\bigcup_{R\in\Omega(B)} R\big| \leqslant C|B|$ $(B \in \Omega)$ does not imply the S-covering property for a monotone collection Ω of convex bodies. Indeed, let (I_k) be a sequence of one-dimensional open intervals such that $I_k \subset (0,\infty)$ and $\sup I_{k+1} < |I_k|/(k2^k)$ for every $k \in \mathbb{N}$. For each k, let $I_{k,1}, \dots, I_{k,k}$ be pairwise disjoint open sub-intervals of I_k with lengths equal to $|I_k|/k$, and let $B_{k,1}, \dots, B_{k,k}$ be open intervals which are concentric to $I_{k,1}, \dots, I_{k,k}$ respectively, and $|B_{k,j}| = |I_k|/(k2^k)$ $(j = 1, \dots, k)$. Then for the collection $\Omega = \{B_{k,j}\colon k \in \mathbb{N},\ j = 1, \dots, k\}$ the estimate $\big|\bigcup_{R\in\Omega(B)} R\big| \leqslant 3|B|$ $(B \in \Omega)$ holds although Ω does not have the S-covering property.

For a collection Ω of subsets of $\mathbb{R}^n$ by Ω^* we will denote the collection $\{B-B\colon B \in \Omega\}$.

Lemma 2.10. *Let Ω be a monotone collection consisting of convex bodies in $\mathbb{R}^n$. Then for every $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have that*

$$M_{\mathbf{B}_{\overline{\Omega}}}(f)(x) \leqslant C M_{\mathbf{B}_{\Omega^*}}(M_{\mathbf{B}_\Omega}(f))(x),$$

where C is a constant depending only on the dimension n.

Proof. Taking into account the shift-invariance of $\mathbf{B}_\Omega, \mathbf{B}_{\overline{\Omega}}$, and $\mathbf{B}_{\Omega^*}$ it suffices to show that

$$M_{\mathbf{B}_{\overline{\Omega}}}(f)(0) \leqslant C M_{\mathbf{B}_{\Omega^*}}(M_{\mathbf{B}_\Omega}(f))(0) \tag{2.3}$$

for every $f \in L^1(\mathbb{R}^n)$.

Take an arbitrary $f \in L^1(\mathbb{R}^n)$. Let R be an arbitrary set from $\mathbf{B}_{\overline{\Omega}}(0) = \overline{\Omega}$ and B be its subset belonging to Ω. Then we have

$$M_{\mathbf{B}_\Omega}(f) \geqslant M_{\mathbf{B}_\Omega}(f\chi_R) \geqslant (|f|\chi_R) * (\chi_{-B}/|B|), \tag{2.4}$$

$$R - R \in \mathbf{B}_{\Omega^*}(0), \tag{2.5}$$

$$\operatorname{supp}\,(|f|\chi_R) * (\chi_{-B}/|B|) \subset R - B \subset R - R, \tag{2.6}$$

$$\begin{aligned} \int_{R-R} (|f|\chi_R) * (\chi_{-B}/|B|) &= \int_{\mathbb{R}^n} (|f|\chi_R) * (\chi_{-B}/|B|) \\ &= \int_{\mathbb{R}^n} |f|\chi_R \int_{\mathbb{R}^n} \chi_{-B}/|B| = \int_R |f|. \end{aligned} \tag{2.7}$$

Note also that by Lemma 2.3,

$$|R - R| \leqslant C|R|, \tag{2.8}$$

where C depends only on the dimension n.

By (2.4)–(2.8) we have

$$\begin{aligned} \frac{1}{|R|}\int_R |f| \leqslant C\frac{1}{|R-R|}\int_R |f| &= C\,\frac{1}{|R-R|}\int_{R-R} (|f|\chi_R) * (\chi_{-B}/|B|) \\ &\leqslant C\,\frac{1}{|R-R|}\int_{R-R} M_{\mathbf{B}_\Omega}(f) \leqslant C M_{\mathbf{B}_{\Omega^*}}(M_{\mathbf{B}_\Omega}(f))(0). \end{aligned}$$

Hence, by the arbitrariness of $R \in \mathbf{B}_{\overline{\Omega}}(0)$ we conclude the needed estimate (2.3). □

Denote by $\mathbf{B}^r$ ($r \in (0, \infty]$) the *truncation* of a semi-basis $\mathbf{B}$ at the level r defined by $\mathbf{B}^r(x) = \{B \in \mathbf{B}(x) \colon \operatorname{diam}(B \cup \{x\}) < r\}$ ($x \in \mathbb{R}^n$).

We will need the following characterization of shift-invariant density bases.

Theorem 2.11. *Let* $\mathbf{B}$ *be a shift-invariant basis in* $\mathbb{R}^n$*. Then the following two properties are equivalent:* (a) $\mathbf{B}$ *is a density basis;* (b) *For each* $\lambda \in (0,1)$*, there exist positive constants* $r(\mathbf{B},\lambda)$ *and* $c(\mathbf{B},\lambda)$ *such that* $\left|\left\{M_{\mathbf{B}^{r(\mathbf{B},\lambda)}}(\chi_E) \geqslant \lambda\right\}\right| \leqslant c(\mathbf{B},\lambda)|E|$ *for every measurable set* $E \subset \mathbb{R}^n$*.*

This characterization for the case of shift-invariant bases $\mathbf{B}$ for which $B \ni 0$ for every $B \in \mathbf{B}(0)$ was given in the works of Oniani [9], and Hagelstein and Parissis [4]. The general case can be reduced to the above mentioned one by associating to a given shift-invariant basis $\mathbf{B}$ the basis $\widehat{\mathbf{B}}$ defined by $\widehat{\mathbf{B}}(x) = \{B \cup \{x\} \colon B \in \mathbf{B}(x)\}$ ($x \in \mathbb{R}^n$).

3. Proofs

Proof of Theorem 1.2. The implication 3) ⇒ 2) follows from Lemmas 2.4 and 2.5. The implication 2) ⇒ 1) is clear. So let us deal with 1) ⇒ 3).

Let C be the constant from Lemma 2.10. By Theorem 2.11 there exist constants r and c such that

$$|\{M_{(\mathbf{B}_\Omega)^r}(\chi_E) \geqslant 1/2C\}| \leqslant c|E|, \tag{3.1}$$

for every measurable set $E \subset \mathbb{R}^n$.

Let us show that the collection $\Omega(r) = \{B \in \Omega\colon \operatorname{diam}(B \cup \{0\}) < r\}$ has the S-covering property.

Let B be a set from $\Omega(r)$. Note that

$$\bigcup_{R\in\overline{\Omega(r)(B)}} (-R) \subset \{M_{\mathbf{B}_{\overline{\Omega(r)(B)}}}(\chi_{B-B}) \geqslant 1\}. \tag{3.2}$$

Indeed, take an arbitrary set R from the collection $\overline{\Omega(r)(B)} = \mathbf{B}_{\overline{\Omega(r)(B)}}(0)$. Then for every $x \in R$ we have that $R - x \in \mathbf{B}_{\overline{\Omega(r)(B)}}(-x)$ and $R - x \subset R - R$. Since there is a shift of B containing R, we have the inclusion $R - R \subset B - B$. Hence,

$$M_{\mathbf{B}_{\overline{\Omega(r)(B)}}}(\chi_{B-B})(-x) \geqslant \frac{1}{|R-x|}\int_{R-x} \chi_{B-B} = 1.$$

Thus, $-R \subset \{M_{\mathbf{B}_{\overline{\Omega(r)(B)}}}(\chi_{B-B}) \geqslant 1\}$.

By Lemma 2.10 we have

$$\begin{aligned} |\{M_{\mathbf{B}_{\overline{\Omega(r)(B)}}}(\chi_{B-B}) \geqslant 1\}| &\leqslant |\{M_{\mathbf{B}_{\Omega(r)(B)^*}}(M_{\mathbf{B}_{\Omega(r)(B)}}(\chi_{B-B})) \geqslant 1/C\}| \\ &\leqslant |\{M_{\mathbf{B}_{\Omega^*}}(M_{\mathbf{B}_{\Omega(r)(B)}}(\chi_{B-B})) \geqslant 1/C\}|. \end{aligned} \tag{3.3}$$

Note that $\mathbf{B}_{\Omega^*}$ is a shift-invariant basis for which $\mathbf{B}_{\Omega^*}(0)$ is a monotone collection of convex bodies containing the origin. Hence, by Remark 2.6, the maximal operator $M_{\mathbf{B}_{\Omega^*}}$ satisfies the weak type $(1, 1)$ inequality. Consequently, $M_{\mathbf{B}_{\Omega^*}}$ also satisfies the estimate

$$|\{M_{\mathbf{B}_{\Omega^*}}(f) \geqslant \lambda\}| \leqslant \frac{A}{\lambda}\int_{\{|f|\geqslant\lambda/2\}} |f|, \tag{3.4}$$

where $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, and A is a constant not depending on f and λ.

By (3.3) and (3.4), we have

$$\begin{aligned} |\{M_{\mathbf{B}_{\overline{\Omega(r)(B)}}}(\chi_{B-B}) \geqslant 1\}| &\leqslant AC \int_{\{M_{\mathbf{B}_{\Omega(r)(B)}}(\chi_{B-B})\geqslant 1/2C\}} M_{\mathbf{B}_{\Omega(r)(B)}}(\chi_{B-B}) \\ &\leqslant AC|\{M_{\mathbf{B}_{\Omega(r)(B)}}(\chi_{B-B}) \geqslant 1/2C\}|. \end{aligned}$$

Now taking into account that $\Omega(r)(B) \subset \Omega(r)$ and $\mathbf{B}_{\Omega(r)} = (\mathbf{B}_\Omega)^r$, by (3.1) and Lemma 2.3 we obtain that

$$|\{M_{\mathbf{B}_{\overline{\Omega(r)(B)}}}(\chi_{B-B}) \geqslant 1\}| \leqslant ACc|B - B| \leqslant C'|B|,$$

where C' is a constant not depending on B. Hence by (3.2) and Lemma 2.7 we obtain the validity of the S-covering property for $\Omega(r)$. □

Proof of Theorem 1.1. The implication 4) ⇒ 1) follows from Lemmas 2.4 and 2.5. The implication 1) ⇒ 2) follows from the Marcinkiewicz interpolation theorem, and 2) ⇒ 3) is obvious. The proof of the implication 3) ⇒ 4) follows the scheme as used for the implication 1) ⇒ 3) from Theorem 1.2 with a single distinction related to the estimation of the measure of the set $\{M_{\mathbf{B}_{\Omega(B)}}(\chi_{B-B}) \geqslant 1/2C\}$ $(B \in \Omega)$. Here we should use the weak type (p, p) inequality for the operator $M_{\mathbf{B}_\Omega}$ (instead of Theorem 2.11). □

Remark 3.1. The conditions in Theorem 1.1 are equivalent to the following condition: 3*) M_Ω satisfies the weak type (p, p) inequality for some $p \in (1, \infty)$. This follows from the proof of Theorem 1.1 as well as from the Marcinkiewicz interpolation theorem (used for showing the equivalence of conditions 3) and 3*)).

Remark 3.2. The scheme of the proofs given above allows us to generalize Theorems 1.1 and 1.2 to monotone collections Ω of bounded open sets containing the origin which have the following two properties:

1) There exists $N \in \mathbb{N}$ such that if $B_1, B_2, B_3 \in \Omega$ and $k \in \{1, 2, 3\}$ is a number for which $|B_k| = \max(|B_1|, |B_2|, |B_3|)$, then the set $B_1 - B_2 + B_3$ can be covered by N shifts of B_k;
2) The maximal operator $M_{\mathbf{B}_{\Omega^*}}$ satisfies the weak type $(1, 1)$ inequality. Here Ω^* is the collection $\{B - B \colon B \in \Omega\}$.

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