Annales Fennici Mathematici
Volumen 51, 2026, 127–144

On the fiber product of noncompact Riemann surfaces

John A. Arredondo, Saúl Quispe and Camilo Ramírez Maluendas

Abstract. For every pair of non-constant holomorphic maps $\beta_i\colon S_i \to S_0$ between noncompact Riemann surfaces, where $i \in \{1, 2\}$, there exists an associated fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ that has the structure of a singular Riemann surface, endowed with a canonical map β to S_0 satisfying $\beta_i \circ \pi_i = \beta$, where π_i is coordinate projection onto S_i. This paper explores the relationship between the space of ends of this fiber product and the space of ends of its normal fiber product. In addition, we establish conditions on the maps β_1 and β_2 that ensure connectivity in the fiber product. Upon examination of these conditions, we establish a connection between the space of ends of the fiber product and the topological characteristics of the Riemann surfaces S_1 and S_2. Finally, we investigate the fiber product of infinite superelliptic curves by analyzing its connectedness and the space of ends.

Ei-kompaktien Riemannin pintojen säietulosta

Tiivistelmä. Jokaista ei-kompaktien Riemannin pintojen välisten holomorfisten ei-vakiokuvausten $\beta_i\colon S_i \to S_0$ paria ($i \in \{1, 2\}$) kohti on olemassa säietulo $S_1 \times_{(\beta_1,\beta_2)} S_2$, jolla on singulaarisen Riemannin pinnan rakenne, ja sen kanoninen kuvaus β pinnalle S_0, jolla pätee $\beta_i \circ \pi_i = \beta$, missä π_i on koordinaattiprojektio pinnalle S_i. Tässä työssä tutkitaan tämän säietulon ja sen normaalin säietulon päätyavaruuksien välistä suhdetta. Lisäksi esitetään kuvauksille β_1 ja β_2 ehdot, jotka takaavat säietulon yhtenäisyyden. Näitä ehtoja tarkastelemalla saadaan yhteys säietulon päätyavaruuden ja Riemannin pintojen S_1 ja S_2 topologisten ominaisuuksien välille. Lopuksi tutkitaan äärettömien ylielliptisten käyrien säietuloa tarkastelemalla sen yhtenäisyyttä ja päätyavaruutta.

1. Introduction

According to Kerékjártó's classification theorem of noncompact surfaces, the topological type of any surface S is given by the following [7, 16]: (i) its genus $g \in \mathbb{N} \cup \{\infty\}$ and (ii) a couple of nested, compact, metrizable and totally disconnected spaces $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$. The set $\mathrm{Ends}(S)$ (respectively, $\mathrm{Ends}_\infty(S)$) is known as the *space of ends* (respectively, the *space of non-planar ends*) of S. On the other hand, it is well known that in the category of topological spaces, the fiber product is the unique solution to a certain universal problem [10]. In the case of Riemann surfaces (unnecessary compact) S_0, S_1 and S_2 with non-constant holomorphic maps $\beta_1\colon S_1 \to S_0$ and $\beta_2\colon S_2 \to S_0$, one defines the *fiber product* $S_1 \times_{(\beta_1,\beta_2)} S_2$ as a new object with maps to S_1 and S_2 which creates a commutative diagram. At this stage, the fiber product of noncompact surfaces is a singular Riemann surface, and it could be disconnected [9].

https://doi.org/10.54330/afm.179733
2020 Mathematics Subject Classification: Primary 14H37, 14H55, 57N16, 30F10.
Key words: Fiber product, normal fiber product, noncompact surfaces, infinite superelliptic curves, Loch Ness monster.
The second author was partially supported by Project FONDECYT 1220261 and the third author was partially supported by Proyecto Hermes 58898.

In this paper, we study the space of ends of the fiber product of noncompact Riemann surfaces, its connectedness, and the construction of fiber product models from infinite superelliptic curves. Therefore, let us consider the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ of the pairs (S_1, β_1) and (S_2, β_2), where S_0, S_1 and S_2 are noncompact Riemann surfaces and $\beta_1 \colon S_1 \to S_0$ and $\beta_2 \colon S_2 \to S_0$ are non-constant holomorphic maps. It is well known that the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is a singular Riemann surface [9], where the singular locus Sing $\subset S_1 \times_{(\beta_1,\beta_2)} S_2$ consists of those points that have a neighborhood not isomorphic to the unit disc (see Section 2.2). The space $S_1 \times_{(\beta_1,\beta_2)} S_2 \setminus$ Sing, consists of a collection of connected Riemann surfaces, say $\widetilde{R}_j$, where each surface $\widetilde{R}_j$ has a collection of punctures, associated to the points in Sing, and by filling in these points, we obtain a unique Riemann surface R_j, up to biholomorphism, called an *irreducible component* of the surface $S_1 \times_{(\beta_1,\beta_2)} S_2$. We denote by $\widetilde{S_1 \times_{(\beta_1,\beta_2)}} S_2$ the union of all the irreducible components of the surface $S_1 \times_{(\beta_1,\beta_2)} S_2$, which is called the *normal fiber product*.

Based on the elements presented above, in Theorem 3.1, we describe the space of ends of the fiber product in terms of the space of ends of its respective irreducible components when its locus of singular points is finite. More precisely, the result states that the space of ends of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is homeomorphic to the space of ends of the normal fiber product $\widetilde{S_1 \times_{(\beta_1,\beta_2)}} S_2$ if the locus of singular points of the surface $S_1 \times_{(\beta_1,\beta_2)} S_2$ is finite.

By studying when S_0 is the Riemann sphere $\hat{\mathbb{C}}$ and the other two Riemann surfaces S_1 and S_2 are compact, Fulton and Hansen proved that the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is connected [4, p. 160]. However, if S_0 is a compact Riemann surface of genus at least one, then the fiber product could be disconnected: see examples 2 and 3 [9]. Therefore, one can be asked about the connectedness of the fiber product of noncompact Riemann surfaces. For this, if we consider noncompact Riemann surfaces S_1 and S_2, as well as branched covering maps $\beta_i \colon S_i \to \mathbb{C}$ with finite degree p_i for each $i \in \{1, 2\}$, we prove the connectedness of $S_1 \times_{(\beta_1,\beta_2)} S_2$ in two cases: In Theorem 3.3 by assuming that the fiber product possesses a singular point in each of its irreducible components, and in Theorem 3.4 when the degree of the branched covering maps β_1 and β_2 satisfying either p_1, p_2 are coprime, or $p_1 = p_2 = 2$. Connectedness also occurs if the Riemann surfaces S_1, S_2 are not biholormorphically equivalent. Also, in Theorem 3.8 we observe that the space of ends of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is homeomorphic to the fiber product $\mathrm{Ends}(S_1) \times_{(\mathrm{Ends}(\beta_1),\mathrm{Ends}(\beta_2))} \mathrm{Ends}(S_2)$.

Finally, in Section 4, we restrict ourselves to the case of infinite superelliptic curves (see [1]). It is well known that if $(w_l)_{l\in\mathbb{N}}$ is a sequence of different complex numbers such that its norm sequence $(|w_l|)_{l\in\mathbb{N}}$ diverges, then there is a Weierstrass theorem ensuring the existence of an entire map $f \colon \mathbb{C} \to \mathbb{C}$ whose only zeros are the points of this sequence and, each one of them is simple. Therefore, for $n \geq 2$ the affine plane curve

$$S(f) = \{(z_1, z_2) \colon z_2^n = f(z_1)\}$$

is called *infinite superelliptic curve*. If $n = 2$, the affine curve $S(f)$ is known as *infinite hyperelliptic curve*.

Now if we consider an infinite superelliptic curve $S(f)$ and a suitable finite or infinite superelliptic curve $S(g)$; and the projection maps onto the first coordinate $\beta_1 \colon S(f) \to \mathbb{C}$ and $\beta_2 \colon S(g) \to \mathbb{C}$, we take the (singular Riemann surface) fiber product $\mathcal{S}(f, g) := S(f) \times_{(\beta_1,\beta_2)} S(g)$. In Theorem 4.6, we describe the connectedness and the space of ends of the surface $\mathcal{S}(f, g)$. In addition, in Theorem 4.8 we determine the necessary and sufficient conditions that guarantee that these two singular Riemann surfaces $\mathcal{S}(f, g)$ and $\mathcal{S}(f, h)$ are isomorphic.

The paper is organized as follows. In Section 2, we recall the definition of ends and the space of ends of topological spaces, singular Riemann surfaces, and the fiber product. Sections 3 and 4 are devoted to proving our main results.

2. Preliminaries

2.1. Space of ends of topological spaces. The objects that we now call ends were introduced by Freudenthal in [5]. Geometrically, an end of a suitable topological space is a point at infinity. We will briefly describe the concept of ends for certain topological spaces and surfaces.

Definition 2.1. [5, 1. Kapitel] Let X be a locally compact, locally connected, connected Hausdorff space, and let $(U_n)_{n\in\mathbb{N}}$ be an infinite nested sequence $U_1 \supset U_2 \supset \ldots$ of non-empty connected open subsets of X such that the following holds.

(1) For each $n \in \mathbb{N}$, the boundary ∂U_n of U_n is compact.
(2) The intersection $\bigcap_{n\in\mathbb{N}} \overline{U_n} = \emptyset$.
(3) For each compact $K \subset X$ there is $m \in \mathbb{N}$ such that $K \cap U_m = \emptyset$.

Two nested sequences $(U_n)_{n\in\mathbb{N}}$ and $(U'_n)_{n\in\mathbb{N}}$ are *equivalent* if for each $n \in \mathbb{N}$ there exist $j, k \in \mathbb{N}$ such that $U_n \supset U'_j$, and $U'_n \supset U_k$. The corresponding equivalence classes $[U_n]_{n\in\mathbb{N}}$ of these sequences are called the *ends* of X, and the set of all ends of X is denoted by $\mathrm{Ends}(X)$.

Remark 2.2. If the ends $[U_n]_{n\in\mathbb{N}}$ and $[V_n]_{n\in\mathbb{N}}$ of X are different, then there are $l, m \in \mathbb{N}$ such that $U_l \cap V_m = \emptyset$. We can suppose without loss of generality that $U_1 \cap V_1 = \emptyset$.

The *space of ends* of X is the topological space with the ends of X as elements and endowed with the following topology: for every non-empty connected open subset U of X with a compact boundary, we define

$$U^* := \{[U_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(X) \mid U_j \subset U \text{ for some } j \in \mathbb{N}\}.$$

Then we consider the set of all such U^*, with U open with a compact boundary in X, as a basis for the topology of $\mathrm{Ends}(X)$. Afterwards, we check that if $U \subset V$ are open subsets of X with a compact boundary, then $U^* \subset V^*$.

Theorem 2.3. [15, Theorem 1.5] *The space* $\mathrm{Ends}(X)$ *is Hausdorff, totally disconnected, and compact. In other words, the space of ends* $\mathrm{Ends}(X)$ *is a closed subset of the Cantor set.*

Lemma 2.4. [17, §5.1, p. 320] *The space X has exactly $n \in \mathbb{N}$ ends if and only if for all compact subsets $K \subset X$ there is a compact subset $K' \subset X$ such that $K \subset K'$ and $X \setminus K'$ are n components connected.*

2.1.1. Space of ends of surfaces. Topologically orientable and connected surfaces S are classified up to homeomorphisms by their genus $g(S) \in \mathbb{N}_0 \cup \{\infty\}$, the space of ends $\mathrm{Ends}(S)$, and the subspace $\mathrm{Ends}_\infty(S) \subseteq \mathrm{Ends}(S)$ for all *non-planar ends* of S (or ends accumulated by genus). In addition, any pair of nested closed subsets of the Cantor set can be performed as the space of ends of a connected orientable topological surface. For more details, we refer the reader to [16].

Theorem 2.5. (Classification of topological surfaces, [7, §7], [16, Theorem 1]) *Two orientable surfaces S_1 and S_2 with the same genus are topologically equivalent* *if and only if there exists a homeomorphism* $f\colon \mathrm{Ends}(S_1) \to \mathrm{Ends}(S_2)$ *such that* $f(\mathrm{Ends}_\infty(S_1)) = \mathrm{Ends}_\infty(S_2)$.

2.2. Singular Riemann surfaces. In this subsection, we will explore some elements of the theory of singular Riemann surfaces as the locus of singular points, irreducible components, isomorphism, and the group of automorphisms of a singular Riemann surface. The Poincaré disk will be denoted by Δ.

Definition 2.6. [9, Subsection 2.2] A *singular Riemann surface*[1] is a complex analytic surface of one dimension S such that for each point p of S there exists a neighborhood holomorphically equivalent to a subspace of the form

$$V_{n,m} = \{(z,w) \in \Delta \times \Delta\colon z^n = w^m\} \subset \Delta \times \Delta,$$

for some integers $n, m \geq 1$ where p corresponds to the point $(0,0) \in V_{n,m}$.

If $n = 1$ or $m = 1$, then $V_{n,m}$ is holomorphically equivalent to Δ. Now if $n, m \geq 2$ and $d \geq 1$ is the greatest common divisor of n and m, then we write $n = d\widehat{n}$ and $m = d\widehat{m}$, where $\widehat{n}, \widehat{m} \geq 1$ are relatively prime integers. Thus, the space $V_{n,m}$ can be written as

$$V_{n,m} = \left\{ (z,w) \in \Delta \times \Delta\colon \prod_{k=0}^{d-1} (z^{\widehat{n}} - \omega^k w^{\widehat{m}}) = 0 \right\},$$

and it is homeomorphic to a collection of d cones with a common vertex at $(0,0)$ (as seen in Figure 1), where ω is a d^{th} primitive root of unity. In particular, if $d = 1$, then the space $V_{n,m}$ is holomorphically equivalent to Δ.

If $d \geq 2$, then the point $p \in S$ is called *singular*. The *locus of singular points of* S, denoted by $\mathrm{Sing}(S)$, is a discrete subset of S. It follows that each connected component $\widetilde{R}$ of $S \setminus \mathrm{Sing}(S)$ has a Riemann surface structure, and the points in $\mathrm{Sing}(S)$ define the punctures in $\widetilde{R}$. By adding these punctures, we obtain another Riemann surface R, that contains $\widetilde{R}$, which is called an *irreducible component* of S. If S has only one irreducible component, then it is called *irreducible*; otherwise, it is called *reducible*.

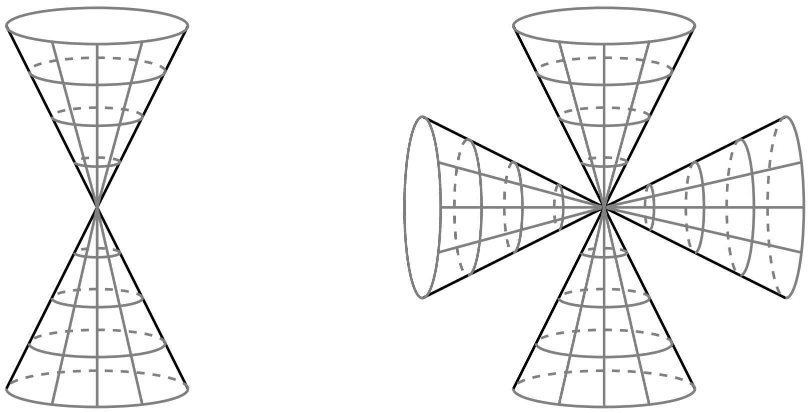
Remark 2.7. If S is a compact singular Riemann surface, then it follows the properties below:

(1) The locus of singular points $\mathrm{Sing}(S)$ of S is a finite set.
(2) $S \setminus \mathrm{Sing}(S)$ has finitely many irreducible components.
(3) Each irreducible component R of $S \setminus \mathrm{Sing}(S)$ is a compact Riemann surface, and $\widetilde{R}$ is also a Riemann surface with a finite number of punctures.

Definition 2.8. [9, Subsection 2.2] Consider two singular Riemann surfaces S_1 and S_2. A homeomorphism $f\colon S_1 \to S_2$ is called *isomorphism* if it satisfies the properties below:

(1) The homeomorphism f sends the locus singular points of S_1 onto the locus singular points of S_2: $f(\mathrm{Sing(S_1)}) = \mathrm{Sing}(S_2)$.
(2) The restriction $f\colon S_1 \setminus \mathrm{Sing}(S_1) \to S_2 \setminus \mathrm{Sing}(S_2)$ is holomorphic.

[1]It means a Riemann surface that allows singularities.



Case $d = 2$. Case $d = 4$.

Figure 1. Neighborhood of a singular Riemann surface.

Two singular Riemann surfaces S_1 and S_2 are *isomorphic* if there exists an isomorphism $f\colon S_1 \to S_2$. An *automorphism* of a singular Riemann surface S is an isomorphism of S to itself. The automorphism set of a singular Riemann surface S, which will be denoted by $\mathrm{Aut}(S)$, has a group structure with the composition operation.

2.3. Fiber product of Riemann surfaces. Now we introduce the concept of the fiber product of Riemann surfaces. Furthermore, we explore the irreducible components associated with the fiber product and define the normal fiber product.

Definition 2.9. [9, Subsection 2.3] Fix three Riemann surfaces (not necessarily compact) S_0, S_1 and S_2, as well as the surjective holomorphic maps $\beta_1\colon S_1 \to S_0$ and $\beta_2\colon S_2 \to S_0$. The *fiber product* associated with the pair (S_1, β_1) and (S_2, β_2) is defined as

$$S_1 \times_{(\beta_1,\beta_2)} S_2 := \{(z_1, z_2) \in S_1 \times S_2 \colon \beta_1(z_1) = \beta_2(z_2)\}$$

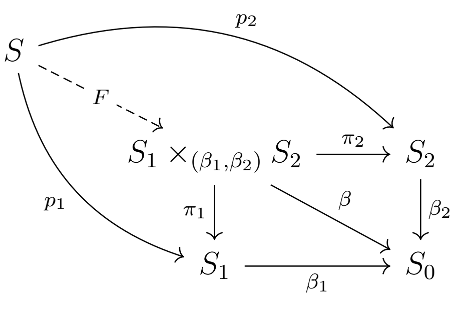
and is endowed with the topology induced by the product topology of $S_1 \times S_2$. There is a natural continuous map $\beta\colon S_1 \times_{(\beta_1,\beta_2)} S_2 \to S_0$ such that

$$\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2, \tag{1}$$

where $\pi_j\colon S_1 \times_{(\beta_1,\beta_2)} S_2 \to S_j$ is the projection map $\pi_j(z_1, z_2) = z_j$, for $j \in \{1, 2\}$.

The fiber product of the pairs (S_1, β_1) and (S_2, β_2) enjoys the following universal property. If S is a topological space and there are continuous maps $p_1\colon S \to S_1$ and $p_2\colon S \to S_2$ such that $\beta_1 \circ p_1 = \beta_2 \circ p_2$, then there exists a unique continuous map $F\colon S \to S_1 \times_{(\beta_1,\beta_2)} S_2$ as in the diagram (2), that satisfies $p_j = \pi_j \circ F$, for $j \in \{1, 2\}$.

(2)



The fiber product is, up to homeomorphisms, the unique topological space that satisfies the above property. For more details on the fiber product of topological spaces, we refer the reader to [6, p. 30].

The following result ensures that the fiber product is a singular Riemann surface.

Proposition 2.10. [9, Proposition 2.3] *The fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is a singular Riemann surface. Moreover, if $p = (z_1, z_2) \in S_1 \times_{(\beta_1,\beta_2)} S_2$, n_j is the local degree of β_j at z_j (for $j = 1, 2$), and d is the greatest common divisor of n_1 and n_2, then the point p is a singular point if and only if $d \geq 2$. Thus, it has a neighborhood of the form V_{n_1,n_2}.*

The locus of singular points $\operatorname{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ of the fiber product corresponds to the pairs (z_1, z_2), in which z_i is a ramification point of S_i, with $i \in \{1, 2\}$. On the other hand, the space

$$S_1 \times_{(\beta_1,\beta_2)} S_2 \setminus \operatorname{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2),$$

consists of a collection $\{\widetilde{R}_\alpha \colon \alpha \in \mathcal{A}\}$ of connected Riemann surfaces for a suitable set of indices $\mathcal{A}$. These surfaces satisfy the properties below.

(1) If the locus $\operatorname{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ is a finite set, then the collection of Riemann surfaces $\left\{\widetilde{R}_\alpha \colon \alpha \in \mathcal{A}\right\}$ is finite.

(2) The map β described in (1) restricted to $\widetilde{R}_\alpha$ is not necessarily a surjective holomorphic map.

(3) The Riemann surface $\widetilde{R}_\alpha$ has a collection of punctures associated with the points in $\operatorname{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2)$. Furthermore, by filling in these points, we obtain a unique irreducible component R_α, up to biholomorphism, of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ and a surjective holomorphic map $\beta \colon R_\alpha \to S_0$, which extends the one given in item **(2)**.

Definition 2.11. The union of all the irreducible components of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is called the *normal fiber product*, which is denoted by $S_1 \widetilde{\times_{(\beta_1,\beta_2)}} S_2$. This is the normalization of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ when it is considered a complex algebraic variety.

From the previous definition, there is a one-to-one correspondence between the components of the normal fiber product and the irreducible components of the fiber product.

Note that if $B \subset S_0$ is the discrete subset consisting of the union of the branch values of β_1 and β_2, as well as $S_0^* = S_0 \setminus B$, $S_1^* = S_1 \setminus \beta_1^{-1}(B)$ and $S_2^* = S_2 \setminus \beta_2^{-1}(B)$, then

$$S_1^* \times_{(\beta_1,\beta_2)} S_2^* = \bigcup_{\alpha\in\mathcal{A}} R_\alpha^* \subset \left(S_1 \times_{(\beta_1,\beta_2)} S_2 \setminus \operatorname{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2)\right) = \bigcup_{\alpha\in\mathcal{A}} \widetilde{R}_\alpha \subset \bigcup_{\alpha\in\mathcal{A}} R_\alpha,$$

where R_α^* is a connected Riemann surface.

The following fact is a consequence of the universal property of the fiber product and the above description.

Lemma 2.12. [9, Lemma 2.5] *If S is a connected Riemann surface and $p_j \colon S \to S_j^*$ are surjective holomorphic maps such that $\beta_1 \circ p_1 = \beta_2 \circ p_2$, then there exist an irreducible component R_α^* and a holomorphic map $h \colon S \to R_\alpha^*$ so that $p_j = \pi_j \circ h$, where $\pi_j \colon S_1 \times_{(\beta_1,\beta_2)} S_2 \to S_j$ is the projection map $\pi_j(z_1, z_2) = z_j$, for $j \in \{1, 2\}$.*

For the case in which both holomorphic maps β_j are of finite degree, the following result provides an upper bound on the number of irreducible components of the fiber product.

Proposition 2.13. [9, Proposition 2.6] *If β_1 and β_2 both have a finite degree, then the number of irreducible components of the fiber product of the pairs (S_1, β_1) and (S_2, β_2) is at most the greatest common divisor of the degrees of β_1 and β_2.*

Note that in Proposition 2.13 if the degrees of β_1 and β_2 are coprime, then the fiber product has exactly one irreducible component.

3. Space of ends of the fiber product of Riemann surfaces

Let S_0, S_1, and S_2 be Riemann surfaces (not necessarily compact), and let $\beta_1\colon S_1 \to S_0$ and $\beta_2\colon S_2 \to S_0$ be surjective holomorphic maps. The fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is a subspace of the product space $S_1 \times S_2$ inheriting the following properties: Hausdorff, second countable, locally compact and locally connected. So we can associate the respective end spaces with each connected component of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$. Recall that if the locus of singular points of $S_1 \times_{(\beta_1,\beta_2)} S_2$ is finite, it has finite irreducible components $R_1, \ldots, R_k$ for some $k \in \mathbb{N}$. Given that such irreducible components are Riemann surfaces, the space of ends of the normalization of the fiber product is

$$\mathrm{Ends}(S_1 \widetilde{\times_{(\beta_1,\beta_2)}} S_2) = \bigcup_{i=1}^{k} \mathrm{Ends}(R_i).$$

The following result describes the space of ends of a connected fiber product of Riemann surfaces $S_1 \times_{(\beta_1,\beta_2)} S_2$ when its locus of singular points is finite.

Theorem 3.1. *Suppose that the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is connected and its locus of singular points is finite; the space* $\mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ *is then homeomorphic to the space* $\mathrm{Ends}(S_1 \widetilde{\times_{(\beta_1,\beta_2)}} S_2)$.

The following remark is necessary for the proof of the result above.

Remark 3.2. Assume that the locus $\mathrm{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is finite. The following properties hold.

(1) The subset
$$\mathrm{Sing}(R_i) := R_i \setminus \widetilde{R}_i \subset R_i,$$
is a finite compact space for each $i \in \{1, \ldots, k\}$.

(2) If we consider the end $[U_n]_{n\in\mathbb{N}}$ in the irreducible component R_i and the compact space $\mathrm{Sing}(R_i)$, by using Definition 2.1, there exists $s \in \mathbb{N}$ such that $\mathrm{Sing}(R_i) \cap U_s = \emptyset$. We can suppose without loss of generality that $s = 1$. This implies that $\mathrm{Sing}(R_i) \cap U_n = \emptyset$ or, equivalently, $U_n \subset \widetilde{R}_i = R_i \setminus \mathrm{Sing}(R_i)$ for all $n \in \mathbb{N}$.

(3) Given that the Riemann surface $\widetilde{R}_i$ is a subspace of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$, the infinite nested sequence $(U_n)_{n\in\mathbb{N}}$ of $\widetilde{R}_i$ is then also an infinite nested sequence of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$, which defines the *common end*
$$[\widetilde{U}_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2),$$
so the sequence $(U_n)_{n\in\mathbb{N}}$ belongs to the class $[\widetilde{U}_n]_{n\in\mathbb{N}}$. The nomenclature for the common end appears in [14] Remark 3.6.

(4) If the infinite nested sequences $(V_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$ of R_i are equivalent, then, by the construction above, these infinite sequences define their respective common ends $[\widetilde{V}_n]_{n\in\mathbb{N}}$ and $[\widetilde{U}_n]_{n\in\mathbb{N}}$ in the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ such that $[\widetilde{V}_n]_{n\in\mathbb{N}} = [\widetilde{U}_n]_{n\in\mathbb{N}}$.

3.1. Proof of Theorem 3.1. Let us start defining the map

$$F\colon \bigcup_{i=1}^{k} \mathrm{Ends}(R_i) \to \mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$$

such that it sends the end $[U_n]_{n\in\mathbb{N}} \in \bigcup_{i=1}^{k} \mathrm{Ends}(R_i)$ to the common end $[\widetilde{U}_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$; see (3) in Remark 3.2. The map F is a well-defined map by (4) in Remark 3.2.

Injectivity. Let us consider the different ends $[U_n]_{n\in\mathbb{N}}$ and $[V_n]_{n\in\mathbb{N}}$ in $\bigcup_{i=1}^{k}\mathrm{Ends}(R_i)$. We must prove that $[\widetilde{U}_n]_{n\in\mathbb{N}} = F([U_n]_{n\in\mathbb{N}})$ is different from $F([V_n]_{n\in\mathbb{N}}) = [\widetilde{V}_n]_{n\in\mathbb{N}}$. Therefore, we study the cases below.

Case 1. The ends are in different irreducible components: there are $k_1 \neq k_2 \in \{1,\dots,k\}$ such that $[U_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(R_{k_1})$ and $[V_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(R_{k_2})$. From (1) in Remark 3.2, we hold that $\mathrm{Sing}(R_{k_i})$ is a finite compact subset of R_{k_i}, for each $i \in \{1,2\}$, and, from (2) in Remark 3.2, we obtain that for all $n \in \mathbb{N}$

$$\mathrm{Sing}(R_{k_1}) \cap U_n = \emptyset \quad \text{and} \quad \mathrm{Sing}(R_{k_2}) \cap V_n = \emptyset.$$

By hypothesis, the classes $[U_n]_{n\in\mathbb{N}}$ and $[V_n]_{n\in\mathbb{N}}$ are different, and by using Remark 2.2 we obtain that $U_1 \cap V_1 = \emptyset$. As infinite nested sequences $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ belong to the class $[\widetilde{U}_n]_{n\in\mathbb{N}}$ and $[\widetilde{V}_n]_{n\in\mathbb{N}}$ of $\mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$, respectively, there are then $m, l \in \mathbb{N}$ such that $\widetilde{U}_l \cap \widetilde{V}_m = \emptyset$.

Case 2. The ends are in the same irreducible component: there is $k' \in \{1,\dots,k\}$ such that the ends $[U_n]_{n\in\mathbb{N}}, [V_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(R_{k'})$. Since these ends are different, from Remark 2.2 it follows that

$$U_1 \cap V_1 = \emptyset. \tag{3}$$

From (1) in Remark 3.2, we state that $\mathrm{Sing}(R_{k'})$ is a finite compact subset of $R_{k'}$. By using **(2)** in the preceding remark, we obtain that for all $n \in \mathbb{N}$

$$\mathrm{Sing}(R_{k'}) \cap U_n = \emptyset \quad \text{and} \quad \mathrm{Sing}(R_{k'}) \cap V_n = \emptyset. \tag{4}$$

Because infinite nested sequences $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ belong to the class $[\widetilde{U}_n]_{n\in\mathbb{N}}$ and $[\widetilde{V}_n]_{n\in\mathbb{N}}$ of $\mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$, respectively, from (3) and (4) it follows that there are $l, m \in \mathbb{N}$ such that $\widetilde{U}_l \cap \widetilde{V}_m = \emptyset$. This shows that the map F is injective.

Surjectivity. We consider an end $[\widetilde{U}_n]_{n\in\mathbb{N}}$ of $S_1 \times_{(\beta_1,\beta_2)} S_2$, then we must prove that there exists at least one end $[U_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(R_{k'})$ for any $k' \in \{1,\dots,k\}$ such that $F([U_n]_{n\in\mathbb{N}}) = [\widetilde{U}_n]_{n\in\mathbb{N}}$. From Definition 2.1, we can assume that $\mathrm{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2) \cap \widetilde{U}_n = \emptyset$ for all $n \in \mathbb{N}$. Because the open subset $\widetilde{U}_n \subset S_1 \times_{(\beta_1,\beta_2)} S_2$ is connected, the infinite nested sequence $(\widetilde{U}_n)_{n\in\mathbb{N}}$ then belongs to any connected component of

$$S_1 \times_{(\beta_1,\beta_2)} S_2 / \mathrm{Sing}(S_1 \times_{(\beta_1,\beta_2)} S_2) = \bigcup_{i=1}^{k} \widetilde{R}_i.$$

In other words, there is $k' \in \{1, \ldots, k\}$ such that $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is an infinite nested sequence of $\widetilde{R}_{k'}$, which defines the end $[U_n]$ of $R_{k'}$ such that $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is in the class $[U_n]_{n\in\mathbb{N}}$; see **(3)** in Remark 3.2. Finally, by the definition of the map F, we obtain $F([U_n]_{n\in\mathbb{N}}) = [\widetilde{U}_n]_{n\in\mathbb{N}}$.

Continuity. We consider an end $[U_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(R_{k'})$, for any $k' \in \{1, \ldots, k\}$, and the open subset W^* of $\mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ such that $F([U_n]_{n\in\mathbb{N}}) = [\widetilde{U}_n]_{n\in\mathbb{N}}$ for any non-empty connected open subset W of $S_1 \times_{(\beta_1,\beta_2)} S_2$ with compact boundary. Thus, we must prove that there is an open subset $V^* \subset \mathrm{Ends}(R_{k'})$ with the end $[U_n]_{n\in\mathbb{N}}$ such that $F(V^*) \subset W^*$.

From the surjectivity of the map F, it follows that $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is an infinite nested sequence of $R_{k'}$, which belongs to the class $[U_n]_{n\in\mathbb{N}} \in R_{k'}$, for any $k' \in \{1, \ldots, k\}$. By hypothesis, the class $[\widetilde{U}_n]_{n\in\mathbb{N}}$ is in the open W^*: $\widetilde{U}_m \subset W$ for any $m \in \mathbb{N}$. Because the infinite nested sequence $(\widetilde{U}_n)_{n\in\mathbb{N}}$ is in the class $[U_n]_{n\in\mathbb{N}}$, then there exists a natural number $p(m) \in \mathbb{N}$ such that $W \supset \widetilde{U}_m \supset U_{p(m)}$. Hence, the connected open subset $U^*_{p(m)} \subset \mathrm{Ends}(R_{k'})$ contains the end $[U_n]_{n\in\mathbb{N}}$ and satisfies that $F(U^*_{p(m)}) \subset W^*$.

As the continuous map F is defined from a compact space to a Hausdorff space, from [3, Theorem 2.1, p. 226], it follows that the function map F is closed. Thereby, we conclude that F is a homeomorphism. □

3.2. Branched covering maps. We will consider the fiber product coming from covering branched maps, and we will establish conditions regarding such maps to guarantee the connectedness of the fiber product. Recall that a topological surface is of *infinite-type* if it has an infinitely generated group. Let S_1 and S_2 be Riemann surfaces such that S_1 is of infinite-type and S_2 has $p \geq 2$ ends and genus $g \in \mathbb{N}_0 \cup \{\infty\}$. Let $\beta_i\colon S_i \to \mathbb{C}$ be a branched covering map of degree $p_i \geq 2$ for each $i \in \{1, 2\}$. Within this setting for the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$, we have the following.

Theorem 3.3. *Assume that there is a singular point of the fiber product that belongs to each of its irreducible components. Then, the fiber product is connected.*

Proof. Take a point $q \in \mathrm{Sing}\big(S_1 \times_{(\beta_1,\beta_2)} S_2\big)$, which belongs to each connected component of the fiber product. As each component is connected, their union is connected. □

The following result is a consequence of [9, Proposition 2.6].

Theorem 3.4. *Let us assume that the degree p_1 of $\beta_1\colon S_1 \to S_0$ and the degree p_2 of $\beta_2\colon S_2 \to S_0$ are either*

(i) *coprime or*
(ii) *$p_1 = p_2 = 2$, and S_1 is not biholomorphically equivalent to S_2.*

Then, the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ is connected.

Proof. (i) From [9, Proposition 2.6], it follows that there is exactly one irreducible component, so it is connected.

(ii) From [9, Proposition 2.6], we obtain that the number of irreducible components is at most two. If there are two, then necessarily each of them is isomorphic for both S_1 and S_2, which is a contradiction. □

Now we consider the general case when $\beta_i\colon S_i \to S_0$ is a branched covering map for each $i \in \{1, 2\}$ such that S_0 is a noncompact Riemann surface. Therefore, such a branched covering map can be extended to a surjective map $\mathrm{Ends}(\beta_i)$ from the space of ends $\mathrm{Ends}(S_i)$ of S_i to the space of ends $\mathrm{Ends}(S_0)$ of S_0.

Theorem 3.5. *Each one of the branched covers* $\beta_i\colon S_i \to S_0$ *induces a surjective continuous map* $\mathrm{Ends}(\beta_i)\colon \mathrm{Ends}(S_i) \to \mathrm{Ends}(S_0)$.

The following remark is necessary for proving the result above.

Remark 3.6. Recall that a Riemann surface is a σ-compact space. Thus, for each $i \in \{1, 2\}$ the surface S_i can be represented as the union

$$S_i = \bigcup_{n\in\mathbb{N}} K_n^i,$$

where each K_n^i is a compact subset of S_i and $K_n^i \subset K_{k+1}^i$ for each $n \in \mathbb{N}$. The complements of such compact subset define the space of ends of the surface S_i. More precisely, we can write

$$S_i \setminus K_n^i = U_1^n \sqcup \ldots \sqcup U_{k(n)}^n \sqcup \ldots \sqcup U_{r_n}^n$$

so that the following hold.

(1) For each $k(n) \in \{1, \ldots, r_n\}$, the set $U_{k(n)}^n$ is a connected component with a closure in S_i that is noncompact but has the compact boundary $\partial U_{k(n)}^n$.
(2) For every $k(n+1) \in \{1, \ldots, r_{n+1}\}$, there exists $k(n) \in \{1, \ldots, r_n\}$ such that $U_{k(n)}^n \supset U_{k(n+1)}^{n+1}$.

Thus, the space $\mathrm{Ends}(S_i)$ is the set of all the classes defined by the nested sequences $\left(U_{k(n)}^n\right)_{n\in\mathbb{N}}$.

3.3. Proof of Theorem 3.5. We shall define the surjective continuous map $\mathrm{Ends}(\beta_i)\colon \mathrm{Ends}(S_i) \to \mathrm{Ends}(S_0)$ from the map $\beta_i\colon S_i \to S_0$. We take the nested sequence $\left(U_{k(n)}^n\right)_{n\in\mathbb{N}}$ of S_i (as in Remark 3.6); therefore,

$$\mathrm{Ends}(\beta_i)\left(U_{k(n)}^n\right)_{n\in\mathbb{N}} = \left(V_{s(n)}^n\right)_{n\in\mathbb{N}},$$

where $\left(V_{s(n)}^n\right)_{n\in\mathbb{N}}$ is a nested sequence of S_0, as detailed below.

As β_i is a surjective continuous map and S_0 is a σ-compact space, the surface S_0 can then be represented as the union

$$S_0 = \bigcup_{n\in\mathbb{N}} L_n^i,$$

where each $\beta_i(K_n^i) = L_n^i$ is a compact subset of S_0, and $L_n^i \subset L_{n+1}^i$ for each $n \in \mathbb{N}$ and where each K_n^i is a compact subset of S_i. In addition, the connected component of the complement

$$S_0 \setminus L_n^i = V_1^n \sqcup \ldots \sqcup V_{s(n)}^n \sqcup \ldots \sqcup V_{t_n}^n$$

satisfies both items described in Remark 3.6. We note the following.

(1) For each $k(n) \in \{1, \ldots, r_n\}$, there exists a unique connected component $V_{s(n)}^n$ of $S_0 \setminus L_n^i$ containing the connected $\beta_i\big(U_{k(n)}^n\big)$. In other words, $\beta_i\big(U_{k(n)}^n\big) \subset V_{s(n)}^n$.
(2) Given that $U_{k(n)}^n \supset U_{k(n+1)}^{n+1}$, the connected components $V_{s(n)}^n$ and $V_{s(n+1)}^{n+1}$ satisfy that $V_{s(n)}^n \supset V_{s(n+1)}^{n+1}$.

By preceding both conditions, each nested sequence $\left(U^n_{k(n)}\right)_{n\in\mathbb{N}}$ of S_i, has an associated unique nested sequence $\left(V^n_{s(n)}\right)_{n\in\mathbb{N}}$ of S_0. Thus, we define

$$\mathrm{Ends}(\beta_i)\left(U^n_{k(n)}\right)_{n\in\mathbb{N}} = \left(V^n_{s(n)}\right)_{n\in\mathbb{N}}.$$

The surjectivity and continuity are immediate, from the definition of $\mathrm{Ends}(\beta_i)$. $\square$

As the singular Riemann surfaces are also σ-compact spaces, the ideas of the previous proof can then be extended, and one can prove that a surjective continuous map between singular Riemann surfaces induces a surjective continuous map between its respective space of ends.

Corollary 3.7. *The surjective continuous map* $f\colon S_1 \to S_2$ *between singular Riemann surface induces a surjective continuous map* $\mathrm{Ends}(f)\colon \mathrm{Ends}(S_1) \to \mathrm{Ends}(S_2)$.

As a consequence of these last results, the following result guarantees that the end space of the fiber product $\mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ is homeomorphic to the fiber product $\mathrm{Ends}(S_1) \times_{(\mathrm{Ends}(\beta_1),\mathrm{Ends}(\beta_2))} \mathrm{Ends}(S_2)$.

Theorem 3.8. *The spaces* $\mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ *and* $\mathrm{Ends}(S_1) \times_{(\mathrm{Ends}(\beta_1),\mathrm{Ends}(\beta_2))} \mathrm{Ends}(S_2)$ *are homeomorphic.*

Proof. By Theorem 3.5, we attain the surjective continuous maps $\mathrm{Ends}(\beta_i)\colon \mathrm{Ends}(S_i) \to \mathrm{Ends}(S_0)$ for $i = 1, 2$. We then consider the diagram

(5)
$$\begin{array}{ccc} S_1 \times_{(\beta_1,\beta_2)} S_2 & \xrightarrow{\pi_2} & S_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \beta_2} \\ S_1 & \xrightarrow[\beta_1]{} & S_0 \end{array}$$

and we have the commutative diagram

(6)
$$\begin{array}{ccc} \mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2) & \xrightarrow{\mathrm{Ends}(\pi_2)} & \mathrm{Ends}(S_2) \\ {\scriptstyle \mathrm{Ends}(\pi_1)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ends}(\beta_2)} \\ \mathrm{Ends}(S_1) & \xrightarrow[\mathrm{Ends}(\beta_1)]{} & \mathrm{Ends}(S_0) \end{array}$$

On the other hand, by considering the surjective continuous maps $\mathrm{Ends}(\beta_i)\colon \mathrm{Ends}(S_i) \to \mathrm{Ends}(S_0)$, we have the following commutative diagram:

(7)
$$\begin{array}{ccc} \mathrm{Ends}(S_1) \times_{(\mathrm{Ends}(\beta_1),\mathrm{Ends}(\beta_2))} \mathrm{Ends}(S_2) & \xrightarrow{\widetilde{\pi}_2} & \mathrm{Ends}(S_2) \\ {\scriptstyle \widetilde{\pi}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Ends}(\beta_2)} \\ \mathrm{Ends}(S_1) & \xrightarrow[\mathrm{Ends}(\beta_1)]{} & \mathrm{Ends}(S_0) \end{array}$$

The universal property of the fiber product (as seen in Section 2.3) implies that the space $\mathrm{Ends}(S_1 \times_{(\beta_1,\beta_2)} S_2)$ is heomeomorphic to the space $\mathrm{Ends}(S_1) \times_{(\mathrm{Ends}(\beta_1),\mathrm{Ends}(\beta_2))} \mathrm{Ends}(S_2)$. $\square$

4. Fiber product of infinite superelliptic curves

Let $(w_l)_{l\in\mathbb{N}}$ be a sequence of different complex numbers such that $\lim_{l\to\infty} |w_l| = \infty$. By the Weierstrass theorem in [12, p. 498], there then exists a meromorphic function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ (called the *Weierstrass map*) with simple zeroes that are given by the points $w_1, w_2, \ldots$. Moreover, f is uniquely determined (by multiplication) by a zero-free entire map (for example the map e^z). Such functions as f admit the representation

$$f(z) = h(z)z^m \prod_{l=1, w_l \neq 0}^{\infty} \left(1 - \frac{z}{w_l}\right) E_l(z),$$

where h is a zero-free entire function ($m = 0$ if $w_l \neq 0$ for every $l \in \mathbb{N}$. In the other case, $m = 1$ if $w_l = 0$ for some l), and $E_l(z)$ is a function of the form

$$E_l(z) = \exp\left[\sum_{s=1}^{d(l)} \frac{1}{s}\left(\frac{z}{w_l}\right)^s\right]$$

for a suitably large no-negative integer $d(l)$. We observe that by choosing the function $f(z)$ one may take $h(z) = 1$ without loss of generality.

Now if we consider the holomorphic function $F\colon \mathbb{C}^2 \to \mathbb{C}$ given by $F(z_1, z_2) = z_2^n - f(z_1)$, for $n \in \mathbb{N}$ such that $n \geq 2$, we then obtain the affine plane curve

$$S(f) := \left\{(z_1, z_2) \in \mathbb{C}^2\colon z_2^n = f(z_1)\right\}.$$

Definition 4.1. [1, Subsection 6.3] The affine curve $S(f)$ is a Riemann surface called an *infinite superelliptic curve*. Furthermore, if $n = 2$, the affine curve $S(f)$ is known as an *infinite hyperelliptic curve.*

Let us observe that the curves $S(f)$ admits a conformal automorphism φ_f of order n given by

$$\varphi_f(z_1, z_2) = (z_1, e^{2\pi i/n} z_2).$$

Moreover, if $G_f := \langle \varphi_f \rangle$, then the quotient space $S(f)/G_f$ is equal to the complex plane $\mathbb{C}$, and that G_f is the Galois group of the cyclic branched covering $\pi_{z_1}\colon S(f) \to \mathbb{C}$ given by $\pi_{z_1}(z_1, z_2) = z_1$. Additionally, if $\delta \subset \mathbb{C} \setminus \{w_l\colon l \in \mathbb{N}\}$ is a simple loop, then it lifts under π_{z_1} to exactly n simple loops when such a loop surrounds a multiple of n punctures.

Below, the following result describes the topology type of an infinite superelliptic curve.

Theorem 4.2. [1, Theorem 6.12] *The infinite superelliptic curve $S(f)$ is a connected Riemann surface homeomorphic to the Loch Ness monster.*

The following result details the conditions when two infinite hyperelliptic curves are biholomorphically equivalent.

Theorem 4.3. [1, Theorem 6.13] *If $n = 2$, i.e., for the hyperelliptic case, the pairs $(S(f), G_f)$ and $(S(g), G_g)$ are biholomorphically equivalent if and only if there exists a holomorphic automorphism of the complex plane $\mathbb{C}$ carrying the zeros of f onto the zeros of g.*

4.1. Fiber product of infinite superelliptic curves. We consider a sequence of different complex numbers $(w_l)_{l \in \mathbb{N}}$ such that $\lim_{l \to \infty} |w_l| = \infty$. Let f denote the Weierstrass map with simple zeros at the points of the sequence $(w_l)_{l \in \mathbb{N}}$. Now we fix a subset A of the set of points $\{w_l\colon l \in \mathbb{N}\}$. Such a set might be finite or finite.

(1) If A is a finite set, then we assume that its cardinality is a multiple of $q \geq 2$, and we define the map

$$g(z_1) = \prod_{a \in A} (z_1 - a).$$

(2) If A is an infinite set, then we let g be a Weierstrass map associated with the set A.

Associated with the holomorphic functions above, we have the following affine plane curves:

$$S(f) = \{(z_1, z_2) \in \mathbb{C}^2 \colon z_2^p = f(z_1)\}, \tag{8}$$

$$S(g) = \{(z_1, z_3) \in \mathbb{C}^2 \colon z_3^q = g(z_1)\}, \tag{9}$$

such that $p, q \geq 2$. The following observation describes the topological type of $S(f)$ and $S(g)$, respectively.

Remark 4.4. From Theorem 4.2, it follows that the infinite superelliptic curve $S(f)$ described in (8) is a Riemann surface topologically equivalent to the Loch Ness monster. Moreover, as a consequence of the implicit function theorem [8], [11, p. 10, Theorem 2.1], the affine plane curve $S(g)$ defined in (9) admits a Riemann surface structure with the topological types described below.

(a) If the subset $A \subset \mathbb{C}$ is finite and $|A| = qk$, with $k \geq 1$ because $S(g)$ is defined, every simple loop in $\mathbb{C} \setminus A$ that lifts to exactly q loops is when it surrounds a multiple of q points of A. By the Riemann–Hurwitz's theorem [11, Theorem 4.16], the affine plane curve $S(g)$ has q non-planar ends and a genus equal to $(q-1)(kq-2)/2$.

(b) If A is a subsequence of $(w_l)_{l \in \mathbb{N}}$, then, by Theorem 4.2, it follows that the affine plane curve $S(g)$ is an infinite superelliptic curve topologically equivalent to the Loch Ness monster.

Now we consider $\beta_1 \colon S(f) \to \mathbb{C}$ and $\beta_2 \colon S(g) \to \mathbb{C}$ the projection maps onto the first coordinate and obtain the fiber product

$$S(f) \times_{(\beta_1, \beta_2)} S(g) = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon z_2^p = f(z_1),\ z_3^q = g(z_1)\right\}, \tag{10}$$

such that $p, q \geq 2$. For convenience in this section, we will denote the above fiber product by

$$\mathcal{S}(f, g) := S(f) \times_{(\beta_1, \beta_2)} S(g). \tag{11}$$

Thus, we consider the projection map $\beta \colon \mathcal{S}(f, g) \to \mathbb{C}$ as defined in (1), where

$$\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2, \tag{12}$$

and the maps $\pi_1 \colon \mathcal{S}(f, g) \to S(f)$ and $\pi_2 \colon \mathcal{S}(f, g) \to S(g)$ are given by $\pi_1(z_1, z_2, z_3) = (z_1, z_2)$ and $\pi_2(z_1, z_2, z_3) = (z_1, z_3)$, respectively.

Remark 4.5. The projection maps β_1 and β_2 are proper branched covering maps. Moreover, for each $z \in \mathbb{C} \setminus \{w_l \colon l \in \mathbb{N}\}$, the fiber $\beta_1^{-1}(z)$ consists of $p \geq 2$ points; moreover, for each $z \in \mathbb{C} \setminus A$, the fiber $\beta_2^{-1}(z)$ consists of $q \geq 2$ points. As a consequence, the map β is proper because $\beta^{-1}(K)$ is a closed subset of the compact $K \times h_p^{-1}[f(K)] \times h_q^{-1}[g(K)]$, where $h_k(w) = e^{2\pi i/k} w$.

On the fiber product $\mathcal{S}(f, g)$, the implicit function theorem does not apply everywhere: on the points (z_1, z_2, z_3) such that $z_1 \in A$. By Proposition 2.10, we conclude that the fiber product $\mathcal{S}(f,g)$ is a singular Riemann surface with a locus of singular points that is the discrete set

$$\mathrm{Sing}(\mathcal{S}(f,g)) = \{(z_1,z_2,z_3) \in \mathcal{S}(f,g) \colon z_1 \in A\}\,, \tag{13}$$

and the subspace

$$\mathcal{S}(f,g) \setminus \mathrm{Sing}(\mathcal{S}(f,g)) \tag{14}$$

of the fiber product $\mathcal{S}(f,g)$ is a Riemann surface.

From the Remark 4.5 and Theorems 3.3, 3.8 the following results, which describe the connectedness and the space of ends of the fiber product $\mathcal{S}(f,g)$, immediately hold.

Theorem 4.6. *The singular Riemann surface $\mathcal{S}(f,g)$ is connected. Moreover, the space of ends of $\mathcal{S}(f,g)$ has*

(i) *one end, if A is infinite,*
(ii) *and q ends, if A is finite.*

Corollary 4.7. *For each $k \in \{1,\ldots,q\}$, the irreducible component R_k of the fiber product $\mathcal{S}(f,g)$ is biholomorphically equivalent to the Loch Ness monster $S(f)$.*

Now we give necessary and sufficient conditions for the isomorphisms of arbitrary singular Riemann surfaces coming from infinite hyperelliptic curves.

We consider an arbitrary subset B of $\{w_l \colon l \in \mathbb{N}\}$, which can be finite or infinite by satisfying the same conditions of the set A, such a subset also defines a Weierstrass function or a polynomial map h, depending on the cardinality of B, with simple zeros that are the complex numbers in B. Hence, as in (9) we define the affine plane curve

$$S(h) = \left\{(z_1,z_3) \in \mathbb{C}^2 \colon z_3^2 = h(z_1)\right\}. \tag{15}$$

Now we consider the hyperelliptic curve $S(f)$ defined in (8) and the projection maps onto the first coordinate $\beta_1 \colon S(f) \to \mathbb{C}$ and $\beta_3 \colon S(h) \to \mathbb{C}$. Thus, we hold another singular Riemann surface as the fiber product

$$\mathcal{S}(f,h) = \left\{(z_1,z_2,z_3) \in \mathbb{C}^3 \colon z_2^2 = f(z_1),\ z_3^2 = h(z_1)\right\}. \tag{16}$$

Let $G_{f,g}$ (respectively, $G_{f,h}$) be the group isomorphic to the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which is generated by the involutions $\alpha_i \colon \mathcal{S}(f,g) \to \mathcal{S}(f,g)$ (respectively, $\alpha_i \colon \mathcal{S}(f,h) \to \mathcal{S}(f,h)$) given by

$$\alpha_1(z_1,z_2,z_3) = (z_1,z_2,-z_3) \quad \text{and} \quad \alpha_2(z_1,z_2,z_3) = (z_1,-z_2,z_3).$$

With all the above, the next result holds.

Theorem 4.8. *There is an isomorphism between the singular Riemann surfaces $\mathcal{S}(f,g)$ and $\mathcal{S}(f,h)$ that conjugates $G_{f,g}$ in $G_{f,h}$ if and only if the sets A and B have the same cardinality and there exists a biholomorphism of the complex plane $T : \mathbb{C} \to \mathbb{C}$ such that it renders the points of the sequence $(w_l)_{l\in\mathbb{N}}$ and $T(A) = B$ invariant.*

Proof. One direction is clear if there exists an isomorphism

$$\tilde{T} \colon \mathcal{S}(f,g) \to \mathcal{S}(f,h),$$

by conjugating $G_{f,g}$ in $G_{f,h}$, then it descends to an isomorphism T of the quotient spaces $\mathcal{S}(f,g)/G_{f,g}$ and $\mathcal{S}(f,h)/G_{f,h}$. These quotient spaces are the complex plane with cone points that are the points in the sequence $(w_l)_{l\in\mathbb{N}}$. The points in A (respectively, in B) are exactly those cone points of $S(f,g)/G_{f,g}$ (respectively, $\mathcal{S}(f,h)/G_{f,h}$)

that are the projections of those points with a stabilizer that is the full group $G_{f,g}$ (respectively, $G_{f,h}$).

Let us now assume there is the map $T\colon \mathbb{C} \to \mathbb{C}$ as in the theorem statement. As $f \circ T$ (respectively, $h \circ T$) is a Weierstrass function for $(w_l)_{l\in\mathbb{N}}$ (respectively, A), there are then nonzero holomorphic maps $e_1, e_2\colon \mathbb{C} \to \mathbb{C}$ such that

$$f \circ T(z) = e_1(z) f(z) \quad \text{and} \quad h \circ T(z) = e_2(z) g(z).$$

Now because, as e_j has no zeroes in $\mathbb{C}$, there exists a holomorphic map $b_j\colon \mathbb{C} \to \mathbb{C}$ such that $b_j^2 = e_j$. The map $\tilde{T}_e\colon \mathcal{S}(f,g) \to \mathcal{S}(f,h)$, given by

$$(z_1, z_2, z_3) \mapsto (T(z_1), b_1(z_1) z_2, b_2(z_1) z_3),$$

is an isomorphism as desired. □

4.2. Double covering of infinite hyperelliptic curves. We consider the fiber product of surfaces (as seen in (10)):

$$\mathcal{S}(f,g) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \colon z_2^2 = f(z_1),\ z_3^2 = g(z_1) \right\},$$

which admits two automorphisms α_1 and α_2 of order 2 that are given by

$$\alpha_1(z_1, z_2, z_3) = (z_1, z_2, -z_3) \quad \text{and} \quad \alpha_2(z_1, z_2, z_3) = (z_1, -z_2, z_3). \tag{17}$$

The group $G_{f,g} = \langle \alpha_1, \alpha_2 \rangle$ is a subgroup of $\mathrm{Aut}(\mathcal{S}(f,g))$ that is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Therefore, the cyclic subgroup $\langle \alpha_i \rangle$ acts on the singular Riemann surface $\mathcal{S}(f,g)$ for each $i \in \{1, 2\}$.

On the other hand, if S_1 is a surface (singular or not) and G is a finite group, a *Galois cover* of S_1 with group G, shortly a G*-cover* of S_1, is a finite morphism $\pi : S_2 \to S_1$, where S_2 is a surface (singular or not) with an effective action by G such that π is G-invariant and induces an isomorphism $S_2/G \cong S_1$.

With all the above, the following result ensures that the quotient surface $\mathcal{S}(f,g)/\langle \alpha_i \rangle$ is a Riemann surface and describes its topological type.

Lemma 4.9. *The quotient surfaces $\mathcal{S}(f,g)/\langle \alpha_1 \rangle$ and $\mathcal{S}(f,g)/\langle \alpha_2 \rangle$ are Riemann surfaces biholomophic to $S(f)$ and $S(g)$, respectively.*

Proof. We consider the $\langle \alpha_1 \rangle$-cover $p_1\colon \mathcal{S}(f,g) \to \mathcal{S}(f,g)/\langle \alpha_1 \rangle$. If we consider the projection function $\pi_1\colon \mathcal{S}(f,g) \to S(f)$ as in (12), we then have the following diagram

$$\begin{array}{ccc} \mathcal{S}(f,g) & \xrightarrow{\ p_1\ } & \mathcal{S}(f,g)/\langle \alpha_1 \rangle \\ {\scriptstyle \pi_1}\big\downarrow & \swarrow {\scriptstyle \pi_1 \circ p_1^{-1}} & \\ S(f) & & \end{array} \tag{18}$$

where $\pi_1 \circ p_1^{-1}$ is constant in the fibers. Furthermore, by the transgression theorem described in [3, p. 123], it follows that $\pi_1 \circ p_1^{-1}$ is a biholomorphism. Therefore, the surface $\mathcal{S}(f,g)/\langle \alpha_1 \rangle$ is a Riemann surface biholomorphic to the Loch Ness monster $S(f)$.

Analogously, by considering the $\langle \alpha_2 \rangle$-cover $p_2\colon \mathcal{S}(f,g) \to \mathcal{S}(f,g)/\langle \alpha_2 \rangle$, and the projection function $\pi_2\colon \mathcal{S}(f,g) \to S(g)$ as in (12), we observe that the map $\pi_2 \circ p_2^{-1}\colon \mathcal{S}(f,g)/\langle \alpha_2 \rangle \to S(g)$ is constant in the fibers. Thus, by the transgression theorem, it follows that $\pi_2 \circ p_2^{-1}$ is a biholomorphism. So, the surface $\mathcal{S}(f,g)/\langle \alpha_2 \rangle$ is a Riemann surface biholomorphic to the Riemann surface $S(g)$. □

Let S be a singular Riemann surfaces that is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-covering of the complex plane with $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \langle \alpha_1, \alpha_2 \rangle = G_{f,g}$. Moreover, from Lemma 4.9 and the universal property of the fiber product, it holds that S is isomorphic to the surface $\mathcal{S}(f,g)$.

Theorem 4.10. *Let S be a singular Riemann surfaces that is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-covering of the complex plane $\mathbb{C}$, with $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \langle \alpha_1, \alpha_2 \rangle = G_{f,g}$. If $\pi_{\alpha_i} \colon S \to S_i = S/\langle \alpha_i \rangle$, $\beta_1 \colon S_1 \to S/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\beta_2 \colon S_2 \to S/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are the obvious projection maps, then the singular Riemann surface S is isomorphic to the fiber product $\mathcal{S}(f,g)$ defined by the following commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{\pi_{\alpha_2}} & S_2 \\ {\scriptstyle \pi_{\alpha_1}}\downarrow & & \downarrow{\scriptstyle \beta_2} \\ S_1 & \xrightarrow[\beta_1]{} & S/\mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array} \tag{19}$$

From the result above, we hold the following observation.

Remark 4.11. The singular Riemann surface $\mathcal{S}(f,g)$ is an unramified double cover of the Loch Ness monster $S(f)$ and the surface $S(g)$.

5. Conclusions

The fiber product is a fundamental concept in studying problems in algebraic geometry. In general, it is obtained or constructed by properly gluing two spaces (geometric-topological) through specific functions. This process allows capturing and analyzing the geometric and topological properties from the perspective of the objects involved in its construction.

In this work, the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ of interest to us is derived from the (not necessarily compact) Riemann surface S_j, with $j \in \{0,1,2\}$, and the non-constant holomorphic function $\beta_i \colon S_i \to S_0$, with $i \in \{1,2\}$. In this setting, the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ turns out to be a topological space admitting a singular Riemann surface structure. Note that the topological type of a Riemann surface is determined by its genus, its end spaces, and its ends of infinite genus. From this perspective, we have been able to determine the end spaces of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$ as a finite union of end spaces of Riemann surfaces, which arises by considering the normal fiber product of $S_1 \times_{(\beta_1,\beta_2)} S_2$.

In the case that the mapping $\beta_i \colon S_i \to S_0 = \mathbb{C}$ is a branched covering function where S_i is a noncompact Riemann surface for each $i \in \{1,2\}$, we have established suitable conditions on the β_i's functions to guarantee the connectedness of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$. As a possible line of research, one could consider S_0 to be an arbitrary infinite-type Riemann surface and explore the conditions necessary for the connectedness of the fiber product $S_1 \times_{(\beta_1,\beta_2)} S_2$.

There are very few explicit models of noncompact Riemann surfaces described as zero loci of holomorphic (transcendental) functions. Using the fiber product, we aimed to provide new models of (singular) noncompact Riemann surfaces starting from certain previously known models (infinite superelliptic curves). Also, Bishop and Rempe [2] proved that there exist Belyi functions mapping non-compact Riemann surfaces onto the Riemann sphere. This raises the question of whether explicit Belyi functions can be found from infinite superelliptic curves to the Riemann sphere.

References

[1] Atarihuana, Y., J. García, R. A. Hidalgo, S. Quispe, S., and C. Ramírez Malundas: Dessins d'enfants and some holomorphic structures on the Loch Ness monster. - Q. J. Math. 73:1, 2021, 349–369.

[2] Bishop, C. J., and L. Rempe: Non-compact Riemann surfaces are equilaterally triangulable. - Preprint, arXiv:2103.16702 [math.CV], 2021.

[3] Dugundji, J.: Topology. - Allyn and Bacon Series in Advanced Mathematics, Allyn and Bacon, Inc., Boston, Mass.–London–Sydney, 1978.

[4] Fulton, W., and J. Hansen: A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings. - Ann. of Math. (2) 110:1, 1979, 159–166.

[5] Freudenthal, H.: Neuaufbau Der Endentheorie. - Ann. of Math. (2) 43:2, 1942, 261–279.

[6] Harris, J.: Algebraic geometry: a first course. - Grad. Texts in Math. 133, Springer Science & Business Media, 2013.

[7] Kerékjártó, B.: Vorlesungen über Topologie I. - Mathematics: Theory & Applications, Springer, Berlín, 1923.

[8] Krantz, S. G., and H. R. Parks: The implicit function theorem: History, theory, and applications. - Birkhäuser Boston, Inc., Boston, MA, 2002.

[9] Hidalgo, R. A., S. Reyes-Carocca, and A. A. Vega, Angélica: Fiber product of Riemann surfaces. - In: Automorphisms of Riemann surfaces, subgroups of mapping class groups and related topics, Contemp. Math. 776, Amer. Math. Soc., Providence, RI, 2022, 161–175.

[10] Iitaka, S.: Algebraic geometry: An introduction to birational geometry of algebraic varieties. - North-Holland Mathematical Library 24, Springer-Verlag, New York-Berlin, 1982.

[11] Miranda, R.: Algebraic curves and Riemann surfaces. - Grad. Stud. in Math. 5, Amer. Math. Soc., Providence, RI, 1995.

[12] Palka, B. P.: An introduction to complex function theory. - Undergrad. Texts Math., Springer Science & Business Media, 1991.

[13] Phillips, A., and D. Sullivan: Geometry of leaves. - Topology 20:2, 1981, 209–218.

[14] Ramírez Maluendas, C., and F. Valdez: Veech groups of infinite-genus surfaces. - Algebr. Geom. Topol. 17:1, 2017, 529–560.

[15] Raymond, F.: The end point compactification of manifolds. - Pacific J. Math. 10, 1960, 947–963.

[16] Richards, I.: On the classification of noncompact surfaces. - Trans. Amer. Math. Soc. 106, 1963, 259–269.

[17] Specker, E.: Die erste Cohomologiegruppe von Überlagerungen und Homotopie-eigenschaften dreidimensionaler Mannigfaltigkeiten. - Comment. Math. Helv. 23:1, 1949, 303–333.

Received 22 July 2025 • Revision received 22 July 2025 • Accepted 10 January 2026
Published online 10 February 2026

John A. Arredondo
Fundación Universitaria Konrad Lorenz
Facultad de Matemáticas e Ingenierías
Bogotá 110231, Colombia
alexander.arredondo@konradlorenz.edu.co

Saúl Quispe
Universidad de La Frontera
Departamento de Matemática y Estadística
Temuco 4780000, Chile
saul.quispe@ufrontera.cl

Camilo Ramírez Maluendas
Universidad Nacional de Colombia, Sede Manizales
Departamento de Matemáticas y Estadística
Manizales 170004, Colombia
camramirezma@unal.edu.co