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Geometric characterizations of freely quasiconformal mappings in real Banach spaces

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Abstract. In this paper, we establish a characterization of freely quasiconformal mappings in real Banach spaces. This characterization is in terms of the geometric moduli of rings which was introduced by Tukia and Väisälä in 2021. As an application, we obtain a generalization of geometric characterizations of quasiconformal mappings in $\mathbb{R}^n$ obtained by Gehring and Väisälä.

Reaalisen Banachin avaruuden vapaasti kvasikonformisten kuvausten yhtäpitäviä geometrisia määritelmiä

Tiivistelmä. Tässä työssä löydetään reaalisen Banachin avaruuden vapaasti kvasikonformisille kuvauksille yhtäpitävä määritelmä Tukian ja Väisälän vuonna 2021 esittelemän geometrisen rengasvakion avulla. Tämän sovelluksena yleistetään Gehringin ja Väisälän todistamia avaruuden $\mathbb{R}^n$ kvasikonformisten kuvausten yhtäpitäviä geometrisia määritelmiä.

1. Introduction

A domain (nonempty, open and connected set) A in a normed space X with dimension at least 2 is called a *ring* or a *ring domain* if its complement $X \setminus A$ has exactly two components, a bounded C_0 and an unbounded C_1. Also, we say that A is a ring with components C_0 and C_1. For a domain G in X, a ring A is called a *ring* in G if its closure is contained in G, where the closure of A is taken in X.

For a ring A in $\mathbb{R}^n$, its modulus is defined as follows:

$$M(A) = \left(\frac{\omega_{n-1}}{\operatorname{mod}(\Gamma_{\mathrm{A}})}\right)^{1/(n-1)},$$

where ω_{n-1} is the $(n-1)$-measure of the unit sphere in $\mathbb{R}^n$, Γ_A denotes the family of all curves in A joining the two components of its complement, and $\operatorname{mod}(\Gamma_A)$ stands for the n-modulus of Γ_A.

It is known that the modulus $M(A)$ can be estimated by Euclidean quantities (cf. [11, Lemma 2.1]), and every self-homeomorphism of $\mathbb{R}^n$ is strictly ring-preserving (see Appendix for the definition). Based on these facts, in 1997, Rohde obtained a geometric characterization for bilipschitz self-homeomorphisms of $\mathbb{R}^n$. That is, a self-homeomorphism f of $\mathbb{R}^n$ is bilipschitz if and only if there is a constant L such that $|M(f(A)) - M(A)| \le L$ for all rings A in $\mathbb{R}^n$ (see [11, Theorem 1.1]). A natural problem is whether there is an analogue of [11, Theorem 1.1] in the setting of more general metric spaces. To study this problem, there are at least two difficulties. First, the moduli of curve families is not well defined even in the setting of Banach spaces (cf. [22, 1.4]). Second, by [13, Example 2.12], there is a self-homeomorphism

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f of $E = \ell_2$ which satisfies that for each $i > 1$ and $t \geq 2$, $f(t\vec{e}_i) = it\vec{e}_i$. By letting $A = \mathbb{B}(0,3)\setminus\overline{\mathbb{B}}(0,2)$, we see that A is a ring in E, but $f(A)$ is not a ring in E since the closures of both $f(\overline{\mathbb{B}}(0,2))$ and $f(E\setminus\mathbb{B}(0,3))$ are unbounded, and therefore, $E\setminus f(A)$ consists of two unbounded component. Thus f is not strictly ring-preserving, even not ring-preserving (i.e., maps a ring to a ring). To overcome these deficiencies, in [13], Tukia and Väisälä raised the concept of the so-called geometric modulus of a ring (see (1.1) below). The obvious advantage is that this concept does not depend on the families of curves, and no measure is required either. Also, they focused their investigation on a special class of self-homeomorphisms of normed spaces, namely set-bounded self-homeomorphisms (see Appendix for the definition). Based on these preparations, they established an analogue of [11, Theorem 1.1] in the setting of normed spaces (see [13, Theorems 2.4 and 2.5]).

From early 1990s onwards, with the help of the quasihyperbolic metric, Väisälä developed the theory of (dimension) freely quasiconformal (briefly, FQC in the following) mappings in real Banach spaces (see [18, 19, 20, 21, 22]) (see Section 2 for the definition of FQC mappings). It is known that, in $\mathbb{R}^n$, FQC mappings coincide with quasiconformal (briefly, QC) mappings (cf. [18, Remark 3.5]). See Section 2 for the definition of QC mappings, and [1, 10, 17] for the basic properties of QC mappings.

By comparing with the theory of QC mappings, one will realize that still there is no characterization for FQC mappings like the geometric definition of QC mappings in $\mathbb{R}^n$. In this paper, we consider this problem. Our main purpose is to establish a characterization of FQC mappings in terms of geometric moduli of rings. The motivation of this study mainly comes from new ideas of Tukia and Väisälä in [13] and the discussions on the conceptually different but equivalent definitions of quasiconformal mappings (see, e.g., [3, 7, 15, 16, 17] for the case of $\mathbb{R}^n$ and [9, 14, 23] for the case of metric spaces).

For convenience, in the following, we use E and E' to denote real Banach spaces with dimension at least 2, and $G \subset E$ and $G' \subset E'$ are domains. It is possible that $G = E$ and $G' = E'$. When we require that G or G' must be a proper subdomain of E or E', we shall use the notation $G \subsetneq E$ or $G' \subsetneq E'$. Also, we use $\mathbb{B}(x,r)$ to denote the open ball with center $x \in E$ and radius $r > 0$, that is, $\mathbb{B}(x,r) = \{y \in E\colon |y-x| < r\}$, and $\overline{\mathbb{B}}(x,r)$ means its closure, where $|\cdot|$ denotes the norm on Banach spaces. Unless stated otherwise, the notions and notations appeared in the rest of the section will be defined in Section 2.

As in [13], the *geometric modulus* of a non-degenerate ring A with components C_0 and C_1 is defined by the number

$$GM(A) = \log\left(1 + \frac{2d(C_1,C_0)}{\min\{d(C_0),d(C_1)\}}\right), \tag{1.1}$$

where $d(C_1,C_0)$ denotes the distance between C_1 and C_0, that is, $d(C_1,C_0) = \inf\{|x-y|\colon x \in C_1 \text{ and } y \in C_0\}$, and $d(C_i)$ stands for the diameter of C_i for $i \in \{0,1\}$, that is, $d(C_i) = \sup\{|x-y|\colon x,y \in C_i\}$. Here a ring with components C_0 and C_1 being *non-degenerate* means that $\min\{\mathrm{card}\{C_0\},\mathrm{card}\{C_1\}\} > 1$, where $\mathrm{card}\{C_i\}$ denotes the cardinality of C_i for $i \in \{0,1\}$. For a non-degenerate ring A with components C_0 and C_1, obviously, one of the two quantities $d(C_0)$ and $d(C_1)$ is finite and the other is infinite.

Note that FQC mappings are not always strictly ring-preserving (see Example 4.1 in the appendix). This fact indicates that, in order to characterize FQC mappings in terms of geometric moduli of rings, we need to consider a class of mappings which

is more general than strictly ring-preserving mappings, that is, strictly Whitney spherical ring-preserving mappings. To state the definition of this class of mappings, let us introduce the following class of rings. For any point $x \in G$ and any reals r and R with $0 < r < R < \delta_G(x)$, we easily see that $\mathbb{B}(x, R) \setminus \overline{\mathbb{B}}(x, r)$ is a ring in G, which is called an (R, r)-*Whitney spherical ring with centre* x, where $\delta_G(x)$ stands for the distance from x to ∂G, the boundary of G (for the case when $G = E$, we assume that $\delta_G(x) = \infty$ for any $x \in G$). In the rest of this paper, we use the notation $S_x(R, r)$ to denote $\mathbb{B}(x, R) \setminus \overline{\mathbb{B}}(x, r)$. When the center is not important for the discussion, we use briefly $S(R, r)$ to denote an (R, r)-Whitney spherical ring.

Definition 1.2. A homeomorphism $f\colon G \to G'$ between two domains G and G' in normed spaces is called *strictly semi-Whitney spherical ring-preserving* if for any Whitney spherical ring $S(R, r)$ in G, its image $f(S(R, r))$ under f is also a ring in G' and the bounded component C_0' of $f(A)$ satisfies $C_0' = \overline{f(\mathbb{B}(x, r))}$. If both f and its inverse f^{-1} are strictly semi-Whitney spherical ring-preserving, then we say that f is *strictly Whitney spherical ring-preserving*.

The relationship between strictly Whitney spherical ring-preserving mappings and freely quasiconformal mappings will be discussed in the appendix, and the main result is Theorem 4.2. Also, from Theorem 4.2, we see that for any Whitney spherical ring $S(R, r)$, the geometric modulus $GM(f(S(R, r)))$ of the image of $S(R, r)$ under an FQC mapping f is meaningful. The following geometric characterization of FQC mappings is the main result of this paper.

Theorem 1.3. *Suppose that $f\colon G \to G'$ is a homeomorphism. Then the following statements are quantitatively equivalent:*

(1) *f is φ-FQC.*
(2) *f is (K, L)-FQC.*

Here, a strictly Whitney spherical ring-preserving mapping $f\colon G \to G'$ is called

(i) *freely (K, L)-semi-quasiconformal* (briefly, *(K, L)-FSQC*) if there are constants $K \geq 1$ and $L \geq 0$ such that for any (R, r)-Whitney spherical ring $S(R, r)$ in G,

$$\frac{1}{K} GM(S(R, r)) - L \leq GM(f(S(R, r))) \leq K \cdot GM(S(R, r)) + L.$$

(ii) *freely (K, L)-quasiconformal* (briefly, *(K, L)-FQC*) if both f and its inverse f^{-1} are (K, L)-FSQC.

For two conditions Φ and Ψ, we say that Condition Φ quantitatively implies Condition Ψ if Condition Φ implies Condition Ψ and the data of Condition Ψ depends only on that of Condition Φ. If Condition Ψ also quantitatively implies Condition Φ, then we say that Condition Φ is equivalent to Condition Ψ, quantitatively.

As an application of Theorem 1.3, we have the following equivalence relations.

Theorem 1.4. *Suppose that $G, G' \subset \mathbb{R}^n$ with $n \geq 2$ are domains and $f\colon G \to G'$ is a homeomorphism. Then the following are quantitatively equivalent:*

(1) *f is K_1-QC.*
(2) *There is a constant $K_2 \geq 1$ such that for any (R, r)-Whitney spherical ring $S(R, r)$ in G,*

$$\frac{1}{K_2} M(S(R, r)) \leq M(f(S(R, r))) \leq K_2 \cdot M(S(R, r)). \tag{1.5}$$

(3) *There is a constant* $K_3 \geq 1$ *such that for any* (R, r)*-Whitney spherical ring* $S(R, r)$ *in* G,

$$M(f(S(R,r))) \geq \frac{1}{K_3} M(S(R,r)). \tag{1.6}$$

(4) f *is* (K_4, L)*-FQC.*
(5) f *is* φ*-FQC.*

Theorems 1.3 and 1.4 will be shown in Section 3. In Section 2, some necessary preliminaries will be introduced, and several auxiliary lemmas will be proved.

2. Preliminaries

This section consists of two subsections. In the first subsection, some necessary terminology will be introduced, some known results will be recalled, and in the second subsection, several auxiliary lemmas will be proved.

2.1. Freely quasiconformal mappings, quasisymmetric mappings and weakly geometrically freely quasiconformal mappings. Suppose that $G \subsetneq E$ is a domain and γ is a rectifiable curve in G. Then the *quasihyperbolic length* of γ is defined as

$$l_{k,G}(\gamma) = \int_\gamma \frac{|dx|}{\delta_G(x)}.$$

The *quasihyperbolic distance* between the points $x, y \in G$ is

$$k_G(x,y) = \inf_\gamma \{l_{k,G}(\gamma)\},$$

where the infimum is taken over all rectifiable curves γ in G connecting x and y. See [5, 6] for basic properties of the quasihyperbolic metric k_G.

Next, we introduce several classes of homeomorphisms.

Definition 2.1. Let $0 < t_0 \leq 1$, and let $G \subsetneq E$ and $G' \subsetneq E'$ be domains. Suppose that $f\colon G \to G'$ is a homeomorphism, φ is a self-homeomorphism of $[0, \infty)$, and $\theta\colon [0, t_0) \to [0, \infty)$ is an embedding with $\theta(0) = 0$. Then f is called

(1) φ*-semisolid* if for all x, $y \in G$,

$$k_{G'}(f(x), f(y)) \leq \varphi(k_G(x,y)),$$

(2) φ*-solid* if both f and its inverse f^{-1} are φ-semisolid,
(3) (θ, t_0)*-relative* if for all x, $y \in G$ with $|x - y| < t_0 \delta_G(x)$,

$$\frac{|f(x) - f(y)|}{\delta_{G'}(f(x))} \leq \theta\Big(\frac{|x-y|}{\delta_G(x)}\Big).$$

In particular, if $t_0 = 1$, then we say that f is θ*-relative.*

Here and in the following, we say that a homeomorphism $f\colon G \to G'$ has *fully a given property* if for each subdomain $D \subset G$, the restriction $f|_D$ has this property. Especially, if $G = E$ or $G' = E'$, then we say that a homeomorphism $f\colon G \to G'$ has *fully a given property* if for each proper subdomain $D \subsetneq G$, the restriction $f|_D$ has this property. We thus obtain the classes of *fully* φ*-semisolid, fully* φ*-solid, fully* (θ, t_0)*-relative* and *fully* θ*-relative* mappings, respectively.

Suppose that $f\colon G \to G'$ is a homeomorphism, φ and η are self-homeomorphisms of $[0, \infty)$, and $K \geq 1$ and $q \in (0, 1)$ are constants. Then f is called

(1) *freely* φ*-quasiconformal* (*briefly,* φ*-FQC*) if f is fully φ-solid,

(2) *K-quasiconformal* (*briefly, K-QC*) if for every $x \in G$,

$$H(x,f) = \limsup_{r\to 0} \frac{\sup\{|f(x)-f(y)|\colon |x-y| \le r\}}{\inf\{|f(x)-f(y)|\colon |x-y| \ge r\}} \le K,$$

(3) *η-quasisymmetric* (*briefly, η-QS*) if for all distinct points $a, b, x \in G$,

$$\frac{|f(a)-f(x)|}{|f(b)-f(x)|} \le \eta\left(\frac{|a-x|}{|b-x|}\right),$$

(4) *q-locally η-QS* (briefly, *(q,η)-LQS*) if for each $x \in G$, the restriction $f|_{\mathbb{B}(x,qr)}$ is η-QS whenever $\mathbb{B}(x,r) \subset G$.

We remark that, when $G = E$, f is (q,η)-LQS for a fixed $q \in (0,1)$ if and only if it is η-QS.

The following known results are useful for us.

Theorem A. [18, Theorem 5.10] For a homeomorphism $f\colon G \to G'$, the following statements are quantitatively equivalent:

(1) f is φ-FQC.
(2) Both f and f^{-1} are (q,η)-LQS for some $q \in (0,1)$.
(3) For every $p \in (0,1)$, there is a self-homeomorphism η_p of $[0,\infty)$ such that both f and f^{-1} are (p,η_p)-LQS.

Theorem B. [18, Remark 3.9.1] For a homeomorphism $f\colon G \to G'$, the following statements are quantitatively equivalent:

(1) f is fully φ-semisolid.
(2) f is fully θ-relative.
(3) f is fully (θ, t_0)-relative.

In particular, when $t_0 = 1$, θ can be chosen as follows:

$$\theta(t) = e^{\varphi(t/(1-t))} - 1$$

(cf. the proof of [18, Theorem 3.7]).

Theorem C. [18, Theorem 5.20] If $f\colon G \to G'$ is φ-FQC, then f is φ_1-FQC, where $\varphi_1(t) = C\max\{t^\alpha, t\}$, and the constants $C \ge 1$ and $0 < \alpha \le 1$ depend only on φ.

Theorem D. [4, Theorem 4] and [18, Remark 3.5] Suppose that G, $G' \subset \mathbb{R}^n$ are domains and $f\colon G \to G'$ is a homeomorphism. Then the following statements are quantitatively equivalent:

(1) f is φ-FQC.
(2) f is fully θ-relative.
(3) f is K-QC.

We end this subsection with the following definitions.

Suppose that G and G' are domains in normed spaces and $f\colon G \to G'$ is a strictly Whitney spherical ring-preserving mapping. It is called

(1) *weakly (ϕ_1,ϕ_2)-semi-geometrically freely quasiconformal* (briefly, *(ϕ_1,ϕ_2)-WSGFQC*) if there are two embeddings $\phi_1, \phi_2\colon [0,\infty) \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \phi_1(t) = \lim_{t\to\infty} \phi_2(t) = \infty,$$

and for any (R,r)-Whitney spherical ring $S(R,r)$ in G,

$$\phi_1(GM(S(R,r))) \le GM(f(S(R,r))) \le \phi_2(GM(S(R,r))).$$

Also, the embeddings ϕ_1 and ϕ_2 are called *control functions* of f.

(2) *weakly* (ϕ_1,ϕ_2)*-geometrically freely quasiconformal* (briefly, (ϕ_1,ϕ_2)*-WGFQC*) if both f and its inverse f^{-1} are (ϕ_1,ϕ_2)-WSGFQC.

Obviously, every (K,L)-FQC mapping is (ϕ_1,ϕ_2)-WGFQC with $\phi_1(t)=t/K-L$ and $\phi_2(t)=Kt+L$. It follows from Theorem 1.3 and Lemma 3.3 below that the converse is true as well.

2.2. Auxiliary results. The following elementary properties of geometric moduli of rings directly follows from the definition of geometric moduli.

Proposition 2.2. (i) *For any point $x\in G$ and any reals r and R with $0<r<R<\delta_G(x)$, we have*

$$d\big(\mathbb{B}(x,r)\big)=2r \quad \text{and} \quad d\big(\overline{\mathbb{B}}(x,r),E\setminus\mathbb{B}(x,R)\big)=R-r.$$

(ii) *For any Whitney spherical ring $S(R,r)$, we have*

$$GM(S(R,r))=\log\frac{R}{r}.$$

Lemma 2.3. *Suppose that $D\subset E$ is a bounded set and $x\in D$. Then for any $M>2$, there is $y\in D$ such that*

$$|x-y|\geq\frac{1}{M}d(D).$$

Proof. Let M be a constant such that $M>2$. Suppose that for any y, $z\in D$,

$$|x-y|<\frac{1}{M}d(D) \quad \text{and} \quad |x-z|<\frac{1}{M}d(D).$$

Then we get

$$d(D)=\sup_{y,z\in D}\{|y-z|\}\leq\frac{2}{M}d(D)<d(D).$$

This is the desired contradiction. □

Lemma 2.4. *Suppose that $G\subset E$ is a domain, $y\in G$ and $R\in(0,\delta_G(y))$. Then for any $s\in(0,\infty)$, there is an (R,r)-Whitney spherical ring $S_y(R,r)$ in G such that*

$$GM(S_y(R,r))=s.$$

Proof. For $s\in(0,\infty)$, let $r=Re^{-s}$. Then $0<r<R<\delta_G(y)$. Obviously, $S_y(R,r)\subset G$ and

$$GM(S_y(R,r))=s.$$

This proves the lemma. □

Lemma 2.5. *Suppose that $G\subset\mathbb{R}^n$ is a domain with $n\geq 2$. Then there is a constant $C>0$ such that for any ring A in G,*

$$|GM(A)-M(A)|\leq C. \tag{2.6}$$

Proof. By [11, Lemma 2.1], we see that there is a constant $\tau>0$ such that for any ring A in G with components C_0 and C_1,

$$\left|M(A)-\log\left(1+\frac{d(C_0,C_1)}{\min\{d(C_0),d(C_1)\}}\right)\right|\leq\tau.$$

Then it follows from the fact

$$0\leq GM(A)-\log\left(1+\frac{d(C_0,C_1)}{\min\{d(C_0),d(C_1)\}}\right)\leq\log 2$$

that

$$|GM(A) - M(A)| \le \tau + \log 2.$$

By taking $C = \tau + \log 2$, we see that the inequality in (2.6) is true. □

We remark that the constant C in Lemma 2.5 may depend on the dimension n.

3. Geometric characterizations of freely quasiconformal mappings

In this section, we shall prove Theorems 1.3 and 1.4. We start the proofs with several auxiliary lemmas.

Lemma 3.1. *Suppose that $f\colon G \to G'$ is strictly Whitney spherical ring-preserving. Let $x \in G$, $R \in (0, \delta_G(x))$ and $\{r_n\}$ be a sequence such that $0 < r_n < R < \delta_G(x)$ for each n and $\lim_{n\to\infty} r_n = R$. Then*

$$\lim_{n\to\infty} GM(f(S_x(R, r_n))) = 0.$$

Proof. Since f is strictly Whitney spherical ring-preserving, we see that $f(S_x(R, r_n))$ is a ring in G'. Choose $b \in E$ be such that $|b - x| = R$. For $n \ge 1$, let $x_n = x + (r_n/R)(b - x)$. Then $|x_n - x| = r_n$. Thus b, $x_n \in G$. On the other hand, let $0 < \varepsilon < \delta_G(x) - R$ and $y \in \partial\mathbb{B}(x, R + \varepsilon)$ be such that y, x, and b are collinear. For $n \ge 1$, take $y_n = y - (1 - 1/n)(y - b)$, and then, $y_n \in (E \setminus \mathbb{B}(x, R)) \cap G$ as $\overline{\mathbb{B}}(x, R + \varepsilon) \subset G$. Therefore

$$\lim_{n\to\infty} x_n = b = \lim_{n\to\infty} y_n.$$

It follows from the fact

$$\lim_{n\to\infty} f(x_n) = f(b) = \lim_{n\to\infty} f(y_n)$$

that

$$\lim_{n\to\infty} GM(f(S_x(R, r_n))) = 0,$$

as required. □

Lemma 3.2. *Suppose that $f\colon G \to G'$ is (ϕ_1, ϕ_2)-WSGFQC. Then $\phi_1(0) \le 0$ and $\phi_2(0) \ge 0$.*

Proof. Let $x \in G$, $R \in (0, \delta_G(x))$ and $\{r_n\}$ be a sequence such that $0 < r_n < R < \delta_G(x)$ for each n and $\lim_{n\to\infty} r_n = R$. Obviously,

$$\lim_{n\to\infty} GM(S(R, r_n)) = 0.$$

Since f is strictly Whitney spherical ring-preserving, we know from Lemma 3.1 that

$$\lim_{n\to\infty} GM(f(S(R, r_n))) = 0.$$

Then the assumption that f is (ϕ_1, ϕ_2)-WSGFQC ensures that

$$\phi_1(GM(S(R, r_n))) \le GM(f(S(R, r_n))) \le \phi_2(GM(S(R, r_n))),$$

which implies that $\phi_1(0) \le 0$ and $\phi_2(0) \ge 0$. □

Lemma 3.3. *Let $f\colon G \to G'$ be a strictly Whitney spherical ring-preserving mapping. Suppose that there is an embedding $\phi_1\colon [0, \infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} \phi_1(t) = \infty$, and such that for any (R, r)-Whitney spherical ring $S(R, r)$ in G,*

$$GM(f(S(R, r))) \ge \phi_1(GM(S(R, r))). \tag{3.4}$$

Then there is a self-homeomorphism φ of $[0,\infty)$ depending only on ϕ_1 such that f is fully φ-semisolid.

Proof. From the proof of Lemma 3.2, we see that $\phi_1^{-1}(0)$ exists. Let $\kappa = e^{-\phi_1^{-1}(0)}/2$. Then $\kappa \le 1/2$. By Theorem B, to prove the lemma, we only need to show that f is fully (θ,κ)-relative. Let D be a subdomain such that $D \ne G$ if $G = E$ or $G' = E'$, and let $D' = f(D)$. For $x \in D$, let $y \in \mathbb{B}(x, \kappa\delta_D(x))$, $R = \frac{1}{2}\delta_D(x)$ and $r = |x-y|$. Then $S_x(R,r)$ is a (R,r)-Whitney spherical ring in G with the complementary components $C_0 = \overline{\mathbb{B}}(x,r)$ and $C_1 = E \setminus \mathbb{B}(x,R)$.

Since f is strictly Whitney spherical ring-preserving, we know that $f(S_x(R,r))$ is a ring with the bounded component $C_0' = f(C_0)$ and the unbounded component $C_1' = E' \setminus f(\mathbb{B}(x,R))$. This implies that

$$\min\{d(C_0'), d(C_1')\} = d(C_0').$$

Moreover, since the fact $y \in C_0$ implies

$$|f(x) - f(y)| \le d(C_0'),$$

and the fact $\partial\mathbb{B}(x,R) \subset C_1 \cap D$ implies $f(\partial\mathbb{B}(x,R)) \subset C_1' \cap D'$, it follows from Lemmas 4.4 and 4.8 in the appendix that

$$\partial f(\mathbb{B}(x,R)) \subset C_1' \cap D'.$$

Then the fact $f(x) \in C_0'$ ensures that

$$\delta_{D'}(f(x)) \ge d(C_1', C_0').$$

These lead to

$$\frac{\delta_{D'}(f(x))}{|f(x)-f(y)|} \ge \frac{d(C_1', C_0')}{d(C_0')} = \frac{1}{2}\big(e^{GM(f(S_x(R,r)))} - 1\big),$$

and thus, by (3.4), we have

$$\frac{\delta_{D'}(f(x))}{|f(x)-f(y)|} \ge \frac{d(C_1', C_0')}{d(C_0')} = \frac{1}{2}\big(e^{\phi_1(GM(S_x(R,r)))} - 1\big).$$

Then the fact

$$GM(S_x(R,r)) = \log\frac{\delta_D(x)}{2|x-y|}$$

gives

$$\frac{\delta_{D'}(f(x))}{|f(x)-f(y)|} \ge \frac{1}{2}\left(e^{\phi_1\left(\log\frac{\delta_D(x)}{2|x-y|}\right)} - 1\right).$$

As the choice of y implies that

$$e^{\phi_1\left(-\log\frac{2|x-y|}{\delta_D(x)}\right)} > 1,$$

we get

$$\frac{|f(x)-f(y)|}{\delta_{D'}(f(x))} \le 2\left(e^{\phi_1\left(-\log\frac{2|x-y|}{\delta_D(x)}\right)} - 1\right)^{-1} =: \theta_1\left(\frac{|x-y|}{\delta_D(x)}\right).$$

Since $\lim_{t\to\infty}\phi_1(t) = \infty$, we know that $\lim_{t\to 0}\theta_1(t) = 0$. Let

$$\theta(t) = \begin{cases} 0 & \text{for } t = 0, \\ \theta_1(t) & \text{for } t \in (0,\kappa). \end{cases}$$

Then $\theta\colon [0,\kappa) \to [0,\infty)$ is an embedding with $\theta(0) = 0$. This proves that f is (θ,κ)-relative in D. Then we conclude from the arbitrariness of D in G that f is fully (θ,κ)-relative, and hence, the proof of the lemma is complete. □

Lemma 3.5. *Suppose that $f\colon G \to G'$ is φ-FQC. Then for any (R,r)-Whitney spherical ring $S(R,r)$ in G,*

$$GM(f(S(R,r))) > 0.$$

Proof. Let $0 < r < R < \delta_G(x)$, where $x \in G$. Since $GM(f(S(R,r))) > 0$ if and only if $d(\partial f(\mathbb{B}(x,r)), \partial f(\mathbb{B}(x,R))) > 0$, to prove the lemma, it suffices to show that

$$d(\partial f(\mathbb{B}(x,r)), \partial f(\mathbb{B}(x,R))) > 0. \tag{3.6}$$

For this, take $y \in \partial\mathbb{B}(x,r)$ and $z \in \partial\mathbb{B}(x,R)$. We may assume that $G \subsetneq E$ since the proof for the case of $G = E$ is similar. By Theorem A, we see that f is (q,η_q)-LQS, where $R/\delta_G(x) < q < 1$. Let $A = \{y,z\}$ and $B = \mathbb{B}(x,R)$. Then [12, Theorem 2.5] ensures that

$$\frac{|f(y)-f(z)|}{d(f(\mathbb{B}(x,R)))} \geq \frac{1}{2}\eta_q\left(\frac{d(\mathbb{B}(x,R))}{|y-z|}\right)^{-1} \geq \frac{1}{2}\eta_q\left(\frac{2R}{R-r}\right)^{-1}.$$

This leads to

$$|f(y)-f(z)| \geq \frac{1}{2}\eta_q\left(\frac{2R}{R-r}\right)^{-1} d(f(\mathbb{B}(x,R))),$$

which implies that

$$d(\partial f(\mathbb{B}(x,r)), \partial f(\mathbb{B}(x,R))) \geq \frac{1}{2}\eta_q\left(\frac{2R}{R-r}\right)^{-1} d(f(\mathbb{B}(x,R))) > 0.$$

This proves (3.6), and hence, the lemma is true. □

Lemma 3.7. *Suppose that both $f\colon G \to G'$ and $f^{-1}\colon G' \to G$ are fully θ-relative. Then f is (ϕ_1,ϕ_2)-WGFQC, where*

$$\phi_1(t) = \log\left(2 + \frac{1}{\theta(e^{-t})}\right) - \log 3 \quad \textit{and} \quad \phi_2(t) = \log\left(\frac{4}{\theta^{-1}(e^{-t})} - 3\right).$$

Proof. Under the assumption of the lemma, we see that there is an embedding $\theta\colon [0,1) \to [0,\infty)$ with $\theta(0) = 0$ such that for any $w \in G$ and $r_0 \in (0,\delta_G(w))$, f and f^{-1} are θ-relative in $\mathbb{B}(w,r_0)$ and $f(\mathbb{B}(w,r_0))$, respectively.

Let $x \in G$ and $0 < r < R < \delta_G(x)$. Then Theorem C, together with Theorem 4.2 in the appendix, ensures that $f(S_x(R,r))$ is a ring in G' with components $C_0' = f(\overline{\mathbb{B}}(x,r))$ and $C_1' = E' \setminus f(\mathbb{B}(x,R))$. It follows that

$$\min\{d(C_0'), d(C_1')\} = d(C_0').$$

Let $U = f(\mathbb{B}(x,R))$ and $\varepsilon_0 = d(C_0',C_1')/2$. Then by Lemma 3.5, $\varepsilon_0 > 0$, and so, there is $b \in \partial\mathbb{B}(x,R)$ such that

$$|f(x)-f(b)| \leq \delta_U(f(x)) + \varepsilon_0.$$

Obviously, the intersection of the segment $[f(x),f(b)]$ and the boundary of $f(\mathbb{B}(x,r))$ is nonempty. Let $a' \in [f(x),f(b)] \cap \partial f(\mathbb{B}(x,r))$. Then there must exist $a \in \partial\mathbb{B}(x,r)$ such that $a' = f(a)$. Since

$$|f(a)-f(b)| \geq d(C_0',C_1'),$$

it follows from the fact

$$|f(x)-f(b)| = |f(x)-f(a)| + |f(a)-f(b)|$$

that

$$\delta_U(f(x)) \geq |f(x)-f(a)| + \frac{1}{2}d(C_0', C_1').$$

Then the fact $|f(x)-f(a)| \leq d(C_0')$ leads to

$$\frac{d(C_1', C_0')}{d(C_0')} \leq \frac{d(C_1', C_0')}{|f(x)-f(a)|} \leq \frac{2\delta_U(f(x))}{|f(x)-f(a)|} - 2. \tag{3.8}$$

Moreover, the assumption that f^{-1} is θ-relative in U gives

$$\frac{\delta_U(f(x))}{|f(x)-f(a)|} \leq \left(\theta^{-1}\left(\frac{|x-a|}{\delta_{\mathbb{B}(x,R)}(x)}\right)\right)^{-1},$$

and thus, we obtain from (3.8), together with the facts $\delta_{\mathbb{B}(x,R)}(x) = R$ and $|x-a| = r$, that

$$\frac{d(C_1', C_0')}{d(C_0')} \leq 2\left(\theta^{-1}\left(\left(\frac{\delta_{\mathbb{B}(x,R)}(x)}{|x-a|}\right)^{-1}\right)\right)^{-1} - 2 = 2\left(\theta^{-1}\left(\left(\frac{R}{r}\right)^{-1}\right)\right)^{-1} - 2.$$

Further, it follows from Proposition 2.2(i) that

$$\begin{aligned}\frac{d(C_0', C_1')}{d(C_0')} &\leq 2\left(\theta^{-1}\left(\left(1+\frac{2d(\overline{\mathbb{B}}(x,r), E\setminus\mathbb{B}(x,R))}{d(\overline{\mathbb{B}}(x,r))}\right)^{-1}\right)\right)^{-1} - 2\\ &\leq \frac{2}{\theta^{-1}\left(e^{-GM(S_x(R,r))}\right)} - 2,\end{aligned}$$

and thus, we get

$$GM(f(S_x(R,r))) \leq \phi_2(GM(S_x(R,r))),$$

where

$$\phi_2(t) = \log\left(\frac{4}{\theta^{-1}(e^{-t})} - 3\right).$$

To obtain a lower bound for the quantity $GM(f(S_x(R,r)))$, we still need to find three points in G. First, by Lemma 2.3, there must be a point $y \in \partial\mathbb{B}(x,r)$ such that

$$|f(x)-f(y)| \geq \frac{1}{3}d(C_0'). \tag{3.9}$$

Second, by Theorem B, we know that f is φ-FQC, where φ can be chosen to depend only on θ. Then the proof of Lemma 3.5 shows that $d(C_1', C_0') > 0$, and thus, there are two points $y^* \in \partial\mathbb{B}(x,r)$ and $z \in \partial\mathbb{B}(x,R)$ such that

$$|f(y^*)-f(z)| \leq 2d(C_1', C_0'). \tag{3.10}$$

To continue the discussions, we consider two cases.

Case 3.1. Suppose that $|f(y^*)-f(x)| \geq |f(y)-f(x)|$. Under this assumption, we get from (3.9) that

$$|f(y^*)-f(x)| \geq \frac{1}{3}d(C_0').$$

Then we have

$$\begin{aligned}\frac{d(C_0',C_1')}{d(C_0')} &\geq \frac{|f(y^*)-f(z)|}{6|f(y^*)-f(x)|} && \text{(by (3.10))}\\ &\geq \frac{|f(x)-f(z)|}{6|f(y^*)-f(x)|}-\frac{1}{6} && \text{(by the triangle inequality)}\\ &\geq \frac{\delta_U(f(x))}{6|f(y^*)-f(x)|}-\frac{1}{6} && \text{(since } f(z)\in\partial U).\end{aligned}$$

Since f is θ-relative in $\mathbb{B}(x,R)$, we know that

$$\frac{\delta_U(f(x))}{|f(y^*)-f(x)|}\geq\Big(\theta\Big(\big(\frac{\delta_{\mathbb{B}(x,R)}(x)}{|y^*-x|}\big)^{-1}\Big)\Big)^{-1},$$

and thus, we obtain

$$\frac{d(C_0',C_1')}{d(C_0')}\geq\frac{1}{6}\Big(\theta\Big(\big(\frac{\delta_{\mathbb{B}(x,R)}(x)}{|y^*-x|}\big)^{-1}\Big)\Big)^{-1}-\frac{1}{6}.$$

Hence it follows from Proposition 2.2(i) that

$$\frac{d(C_1',C_0')}{d(C_0')}\geq\frac{1}{6}\Big(\theta\Big(\Big(1+\frac{2d(\overline{\mathbb{B}}(x,r),E\setminus\mathbb{B}(x,R))}{d(\overline{\mathbb{B}}(x,r))}\Big)^{-1}\Big)\Big)^{-1}-\frac{1}{6},$$

which gives

$$GM(f(S_x(R,r)))\geq\phi_1(GM(S_x(R,r))),$$

where $\phi_1(t)=\log\big(2+\frac{1}{\theta(e^{-t})}\big)-\log 3$.

Case 3.2. Suppose that $|f(y^*)-f(x)|<|f(y)-f(x)|$. We first have

$$\begin{aligned}\frac{d(C_0',C_1')}{d(C_0')} &\geq \frac{|f(y^*)-f(z)|}{6|f(y)-f(x)|} && \text{(by (3.9) and (3.10))}\\ &\geq \frac{|f(x)-f(z)|}{6|f(y)-f(x)|}-\frac{1}{6} && \text{(since } |f(y^*)-f(x)|<|f(y)-f(x)|)\\ &\geq \frac{\delta_U(f(x))}{6|f(y)-f(x)|}-\frac{1}{6} && \text{(since } f(z)\in\partial U).\end{aligned}$$

Since f is θ-relative in $\mathbb{B}(x,R)$, the similar reasoning as in the discussion in Case 3.1 guarantees that

$$GM(f(S_x(R,r)))\geq\phi_1(GM(S_x(R,r))).$$

Now, we have proved that f is (ϕ_1,ϕ_2)-WSGFQC. Similarly, we can demonstrate that f^{-1} is (ϕ_1,ϕ_2)-WSGFQC as well. These prove that f is (ϕ_1,ϕ_2)-WGFQC. □

Lemma 3.11. *If $f\colon G\to G'$ is φ-FQC, then f is (ψ_1,ψ_2)-WGFQC, where*

$$\psi_1(t)=\log\left(\left(e^{C\max\left\{(e^t-1)^{-1},\,(e^t-1)^{-\alpha}\right\}}-1\right)^{-1}+2\right)-\log 3,$$

$$\psi_2(t)=\log\left(C^{\frac{1}{\alpha}}\big(\log(e^{-t}+1)\big)^{-\frac{1}{\alpha}}+\frac{1}{4}\right)+\log 4,$$

and the constants $C\geq 1$ and $0<\alpha\leq 1$ depend only on ϕ_1 and ϕ_2.

Proof. By Theorem C, f is φ_1-FQC with

$$\varphi_1(t) = C\max\{t, t^\alpha\},$$

where we can choose $C \geq 1$ and $0 < \alpha \leq 1$ to depend only on φ.

Further, by Theorem B, we know that both f and f^{-1} are fully θ-relative with $\theta\colon [0,1) \to [0,\infty)$ being the following homeomorphism:

$$\theta(t) = e^{\varphi_1\left(\frac{t}{1-t}\right)} - 1 = e^{C\max\left\{\frac{t}{1-t}, \left(\frac{t}{1-t}\right)^\alpha\right\}} - 1, \tag{3.12}$$

and so, it follows from Lemma 3.7 that f is (ϕ_1', ϕ_2')-WGFQC with

$$\phi_1'(t) = \log\left(\frac{1}{\theta(e^{-t})} + 2\right) - \log 3 \quad \text{and} \quad \phi_2'(t) = \log\left(\frac{4}{\theta^{-1}(e^{-t})} - 3\right). \tag{3.13}$$

Let $\psi_1 = \phi_1'$. Then (3.12) and (3.13) ensure that for every (R,r)-Whitney spherical ring $S(R,r)$ in G,

$$GM\big(f(S(R,r))\big) \geq \psi_1\big(GM(S(R,r))\big). \tag{3.14}$$

Next, let us find the explicit expression for ϕ_2'. Obviously, it suffices to get the explicit expression for $\theta^{-1}(e^{-t})$ with $t > 0$. By letting $s = e^{-t}$, we see that we only need to find the explicit expression for $\theta^{-1}(s)$ with $0 < s < 1$. For this, let

$$u = (\log 2)^{1/\alpha}/(C^{1/\alpha} + (\log 2)^{1/\alpha}).$$

Since $C \geq 1 > \log 2$, we know that $u < 1/2$. As $u/(1-u) = (\log 2/C)^{1/\alpha}$, we get that $\theta(u) = 1$. Hence $\theta^{-1}(1) = u < 1/2$, and then, we only need to consider the case of $t \leq 1/2$.

When $t \leq 1/2$, we see that

$$\theta(t) = e^{C\left(\frac{t}{1-t}\right)^\alpha} - 1,$$

which implies

$$\theta^{-1}(s) = \frac{(\log(s+1))^{\frac{1}{\alpha}}}{C^{\frac{1}{\alpha}} + (\log(s+1))^{\frac{1}{\alpha}}} \tag{3.15}$$

for $s \in [0,1)$. Let $\lambda = (\log(s+1))^{\frac{1}{\alpha}}/\big(C^{\frac{1}{\alpha}} + (\log(s+1))^{\frac{1}{\alpha}}\big)$. Then $\theta^{-1}\colon [0,1) \to [0,\lambda)$ is increasing and continuous, and thus, θ^{-1} is an embedding.

Let $\psi_2 = \phi_2'$. Then we infer from (3.13) and (3.15) that for every (R,r)-Whitney spherical ring $S(R,r)$ in G,

$$GM\big(f(S(R,r))\big) \leq \psi_2\big(GM(S(R,r))\big). \tag{3.16}$$

Now, we conclude from (3.14) and (3.16) that the lemma is true. □

Lemma 3.17. *Let ψ_1 and $\psi_2\colon [0,\infty) \to \mathbb{R}$ be the functions defined in Lemma 3.11. Then*

$$\lim_{t\to\infty} \frac{\psi_1(t)}{t} = \alpha \quad \textit{and} \quad \lim_{t\to\infty} \frac{\psi_2(t)}{t} = \frac{1}{\alpha}.$$

Proof. First, we find the limit $\lim_{t\to\infty} \psi_1(t)/t$. For this, let $\lambda_\alpha(t) = e^{C(e^t-1)^{-\alpha}} - 1$. Since for sufficiently large t,

$$\log\big((\lambda_\alpha(t))^{-1}\big) - \log 3 \leq \psi_1(t) = \log\big((\lambda_\alpha(t))^{-1} + 2\big) - \log 3 \leq \log\big(2(\lambda_\alpha(t))^{-1}\big),$$

we see from the fact

$$\lim_{x\to+0} \frac{\log(e^x - 1)}{\log x} = 1$$

that

$$\lim_{t\to\infty}\frac{\psi_1(t)}{t}=\lim_{t\to\infty}\frac{\log(\lambda_\alpha(t))^{-1}}{t}=\alpha.$$

Next, we find the limit $\lim_{t\to\infty}\psi_2(t)/t$. For this, let $\mu_\alpha(t)=\big(\log(e^{-t}+1)\big)^{-\frac{1}{\alpha}}$. Since for sufficiently large t,

$$\log\big(C^{\frac{1}{\alpha}}\mu_\alpha(t)\big)\le\psi_2(t)\le\log\big(5C^{\frac{1}{\alpha}}\mu_\alpha(t)\big),$$

we obtain that

$$\lim_{t\to\infty}\frac{\psi_2(t)}{t}=\lim_{t\to\infty}\frac{\log\big(C^{\frac{1}{\alpha}}\mu_\alpha(t)\big)}{t}=\frac{1}{\alpha}.$$

The lemma is proved. □

3.1. Proof of Theorem 1.3. (1) ⇒ (2). Assume that f is φ-FQC. By Theorem B and Lemma 3.7, f is (ϕ_1,ϕ_2)-WGFQC. Then it follows from Lemma 3.11 that f is (ψ_1,ψ_2)-WGFQC with ψ_1 and ψ_2 defined in Lemma 3.11. Further, Lemma 3.17 shows that

$$\lim_{t\to\infty}\frac{\psi_1(t)}{t}=\alpha\quad\text{and}\quad\lim_{t\to\infty}\frac{\psi_2(t)}{t}=\frac{1}{\alpha}.$$

We see that there is a constant $M>0$ such that for all $t\in(M,\infty)$,

$$\psi_1(t)\ge\frac{\alpha}{2}t\quad\text{and}\quad\psi_2(t)\le\frac{2}{\alpha}t.$$

Since both ψ_1 and ψ_2 are increasing, we know that for all $t\in[0,M]$,

$$\psi_1(t)\ge\psi_1(0)\ge\log\frac{2}{3}\quad\text{and}\quad\psi_2(t)\le\psi_2(M).$$

These show that for all $t\in[0,\infty)$,

$$\psi_1(t)\ge\frac{\alpha}{2}t+\log\frac{2}{3}-M\quad\text{and}\quad\psi_2(t)\le\frac{2}{\alpha}t+\psi_2(M).\tag{3.18}$$

Let $K=2/\alpha$ and $L=\max\{\log(3/2)+M,\psi_2(M)\}$. Then (3.18) guarantees that f is (K,L)-FQC.

(2) ⇒ (1). Assume that f is (K,L)-FQC. Obviously, both f and f^{-1} are (ϕ_1,ϕ_2)-WSGFQC with $\phi_1(t)=t/K-L$ and $\phi_2(t)=Kt+L$. By Lemma 3.3, both f and f^{-1} are fully φ-semisolid. This implies that f is φ-FQC. □

Remark 3.19. (1) In terms of moduli of curve families, in [9], Heinonen, Koskela, Shanmugalingam and Tyson introduced the metric definition, the analytic definition, and the geometric definition of quasiconformality in the setting of metric measure spaces with locally Q-bounded geometry, and then, they proved that these three definitions are equivalent to each other (see [9, Theorem 9.8]). In terms of the moduli of rings, in [23], Williams obtained a geometric characterization of quasiconformality in the setting of doubling and locally linearly locally connected metric measure spaces (see [23, Theorem 1.2 and Remark 4.3]). It is known that every metric measure space with locally Q-bounded geometry is of finite dimension (cf. [8, §8.7]), and each doubling metric measure space is of finite dimension as well (cf. [2, page 66]). However, our main result, Theorem 1.3, is independent of the dimensions of the underlying spaces. Also, in Theorem 1.3, no measure is required.

(2) The mappings studied in [11] and [13] are self-homeomorphisms of the underlying spaces. But the domains G of the mappings involved in Theorem 1.3 include the possibilities of $G=E$ and $G\subsetneq E$.

(3) In [20] or [21], Väisälä proved that FQC mappings can be characterized by a geometric condition involving spherical rings, that is, (M,α,β)-ring condition (see [20, Corollary 2.21] or [21, Corollary 5.5]). But this condition is much different from the characterization obtained in Theorem 1.3.

3.2. Proof of Theorem 1.4. (1) $\Rightarrow$ (2). Assume that f is K_1-QC. Then by [17, Theorem 13.2(3) and Remark 34.2], the converse f^{-1} of f is K_1'-QC with K_1' depending only on K_1 and n, and thus, the implication follows from [17, Corollary 36.2].

(2) $\Rightarrow$ (3). This implication holds for $K_3 = K_2$.

(3) $\Rightarrow$ (4). Assume that there is $K_3 \geq 1$ such that for every (R,r)-Whitney spherical ring $S(R,r)$ in G,

$$M(f(S(R,r))) \geq \frac{1}{K_3} M(S(R,r)). \tag{3.20}$$

Then we have

$$\begin{aligned} GM(f(S(R,r))) &\geq M(f(S(R,r))) - C && \text{(by Lemma 2.5)}\\ &\geq \frac{1}{K_3} M(S(R,r)) - C && \text{(by (3.20))}\\ &\geq \frac{1}{K_3} GM(S(R,r)) - C\Big(\frac{1}{K_3}+1\Big) && \text{(by Lemma 2.5).} \end{aligned}$$

Let $\phi_1(t) = \frac{1}{K_3}t - C(\frac{1}{K_3}+1)$. By Lemma 3.3, f is fully φ-semisolid, and hence, f is φ_1-FQC by Theorems C and E. Then it follows from Theorem 1.3 that the implication is true.

Obviously, the implication (4) $\Rightarrow$ (5) follows from Theorem 1.3, and the implication (5) $\Rightarrow$ (1) follows from Theorem D. Hence the proof of the theorem is complete. □

Remark 3.21. (1) In [7], Gehring and Väisälä proved that a homeomorphism of a domain in $\mathbb{C}$ is K-QC if and only if the modulus condition (1.6) is satisfied by all rings in G (cf. [7, Theorem 3]). Notice that a ring in $\mathbb{C}$ is conformally equivalent to a Whitney spherical ring in $\mathbb{C}$ by the Riemann mapping theorem, and hence, the equivalence between (1) and (3) in Theorem 1.4 coincides with [7, Theorem 3] in $\mathbb{C}$.

(2) As one of the main purposes, in [3], Gehring proved that a homeomorphism of a domain G in $\mathbb{R}^3$ is K-QC if and only if the modulus condition (1.5) is satisfied by all bounded rings in G (see [3, Corollary 4]). See [17, Corollary 36.2] for the analogue of [3, Corollary 4] in $\mathbb{R}^n$ for all $n \geq 3$. Since in Theorem 1.4(2), only a part of spherical rings in G, i.e., Whitney spherical rings, is involved, we see that the equivalence between (1) and (2) in Theorem 1.4 is a generalization of [3, Corollary 4] and [17, Corollary 36.2].

(3) The proofs of [7, Theorem 3] and [17, Corollary 36.2] rely on the dimension and the measure of the underlying space. However, our proof of Theorem 1.4 is based on the geometric modulus, which is independent of the dimension and the measure of the underlying space.

(4) It is well known that the moduli of rings in $\mathbb{R}^n$ are conformally invariant (cf. [17, Theorem 8.1]). However, the geometric moduli of rings do not possess this property. Let $u\colon G \to G$ with $u(x) = x/|x|^2$, where $G = \mathbb{R}^2 \setminus \{0\}$, and $A = (\mathbb{B}(0,3) \setminus [2,3]) \setminus \overline{\mathbb{B}}(0,1)$. Then both A and $u(A) = \mathbb{B}(0,1) \setminus (\overline{\mathbb{B}}(0,1/3) \cup [1/3,1/2])$ are rings in G since $\overline{A} = \overline{\mathbb{B}}(0,3) \setminus \mathbb{B}(0,1) \subset G$ and $\overline{u(A)} = \overline{\mathbb{B}}(0,1) \setminus \mathbb{B}(0,1/3) \subset G$. However, $GM(u(A)) \neq GM(A)$ as $GM(u(A)) = \log(11/5)$ and $GM(A) = \log 2$.

4. Appendix

We start the appendix with the definitions of several classes of mappings.

Suppose that G and G' are domains in normed spaces and f is a homeomorphism from G to G'. It is called

(1) *strictly semi-ring-preserving* if for any ring A in G with the bounded component C_0, its image $f(A)$ under f is also a ring in G' and the bounded component C_0' of $f(A)$ satisfies $f(C_0 \cap G) = C_0' \cap G'$. If both f and its inverse f^{-1} are strictly semi-ring-preserving, then we say that f is *strictly ring-preserving*.

(2) *set-bounded* if the image of a bounded set and the pre-image of a bounded set are still bounded. Obviously a self-homeomorphism f of a normed space X is set-bounded if and only if f extends to a self-homeomorphism $\widehat{f}$ of $\widehat{X} = X \cup \{\infty\}$ with $\widehat{f}(\infty) = \infty$. From this, we see that a set-bounded self-homeomorphism of a normed space is strictly ring-preserving.

(3) *ball-bounded* if for any $x \in G$, $x' \in G'$, $0 < r < \delta_G(x)$ and $0 < r' < \delta_{G'}(x')$, both the image $f(\mathbb{B}(x, r))$ and the pre-image $f^{-1}(\mathbb{B}(x', r'))$ are bounded.

The following example shows that there is no implication from FQC mapping to strictly ring-preserving mapping or set-bounded mapping.

Example 4.1. For $n \geq 2$, let $h\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be the mapping defined by $h(x) = x/|x|^2$. Then

(1) h is φ-FQC.

(2) h is neither set-bounded nor strictly ring-preserving.

But the following result illustrates that the implication from FQC mapping to strictly Whitney spherical ring-preserving mapping exists.

Theorem 4.2. *Every φ-FQC mapping is strictly Whitney spherical ring-preserving, and the converse is invalid.*

To prove Theorem 4.2, we introduce the following definition.

Definition 4.3. For a homeomorphism $f\colon G \to G'$, we say that f has the *semi-closure-property* (briefly, S-C-*Property*) if for any point $x \in G$ and any number R with $0 < R < \delta_G(x)$,

$$\overline{f(\mathbb{B}(x, R))} = f(\overline{\mathbb{B}}(x, R)).$$

If both f and its inverse f^{-1} have the S-C-Property, then f is called the *closure-property* (briefly, C-Property).

The following lemma shows that all FQC mappings and all strictly Whitney spherical ring-preserving mappings have the C-Property.

Lemma 4.4. *Suppose that $f\colon G \to G'$ is an FQC mapping or a strictly Whitney spherical ring-preserving mapping. Then f has the* C-*Property.*

Proof. Obviously, it suffices to show that f has the S-C-Property since similarly we can prove that f^{-1} has the S-C-Property as well. For this, let $x \in G$ and R be a number with $0 < R < \delta_G(x)$. Then the continuity of f guarantees that

$$f(\overline{\mathbb{B}}(x, R)) \subset \overline{f(\mathbb{B}(x, R))}. \tag{4.5}$$

In the following, we check the converse inclusion relation in (4.5). We first consider the case when f is an FQC mapping. Let q be a number with $q > R/\delta_G(x)$. Then Theorem A guarantees that the restriction $f|_{\overline{\mathbb{B}}(x,R)}$ is η_q-QS. Since $\overline{\mathbb{B}}(x, R)$ is

complete, [12, Theorem 2.24] implies that the image $f(\overline{\mathbb{B}}(x,R))$ is complete as well, and thus, $f(\overline{\mathbb{B}}(x,R))$ is closed. Then

$$\overline{f(\mathbb{B}(x,R))} \subset \overline{f(\overline{\mathbb{B}}(x,R))} = f(\overline{\mathbb{B}}(x,R)). \tag{4.6}$$

For the case when f is strictly Whitney spherical ring-preserving, let $r>0$ be a number with $r<R<q\delta_G(x)$ and $\overline{f(\overline{\mathbb{B}}(x,r))} \subset G'$. Since

$$\overline{f(\mathbb{B}(x,R))} = \overline{f(\mathbb{B}(x,R)\setminus\overline{\mathbb{B}}(x,r))} \cup \overline{f(\overline{\mathbb{B}}(x,r))},$$

and since the assumption of f being strictly Whitney spherical ring-preserving implies that $\overline{f(\mathbb{B}(x,R)\setminus\overline{\mathbb{B}}(x,r))} \subset G'$, we see that

$$\overline{f(\mathbb{B}(x,R))} \subset G'.$$

Let $y' \in \overline{f(\mathbb{B}(x,R))}$, $y=f^{-1}(y')$ and $y_n=(y-x)(1-1/n)+x$. Then $y_n \in \mathbb{B}(x,R)$ and $y_n \to y$ as $n\to\infty$. This ensures that $y\in\overline{\mathbb{B}}(x,R)$, and thus, $y'\in f(\overline{\mathbb{B}}(x,R))$. Hence

$$\overline{f(\mathbb{B}(x,R))} \subset f(\overline{\mathbb{B}}(x,R)). \tag{4.7}$$

By combining (4.5), (4.6) and (4.7), we see from the arbitrariness of x in G and R in $(0,\delta_G(x))$ that f has the S-C-Property, and hence, the proposition is proved. □

Lemma 4.8. *Suppose that $f\colon G\to G'$ is a homeomorphism. Then f has the* S-C-*Property if and only if for any point x in G and any number R with $0<R<\delta_G(x)$,*

$$f(\partial\mathbb{B}(x,R)) = \partial f(\mathbb{B}(x,R)).$$

Proof. For any point x in G, let R be a number such that $0<R<\delta_G(x)$. Then the following facts are obvious:

$$f(\partial\mathbb{B}(x,R)) = f(\overline{\mathbb{B}}(x,R))\setminus f(\mathbb{B}(x,R)) \quad\text{and}\quad f(\overline{\mathbb{B}}(x,R)) = f(\mathbb{B}(x,R))\cup f(\partial\mathbb{B}(x,R)).$$

These ensure that the proposition is true. □

Still, the proof of Theorem 4.2 needs the following example.

Example 4.9. For any integer $n\geq 1$, let $I_n=[2^{-n},2^{1-n})\times(-1,1)$ and $I_n'=[2^{-n^2},2^{-(n-1)^2})\times(-1,1)$ be two subsets in $\mathbb{R}^2$, and let $g_n\colon I_n\to I_n'$ be the homeomorphism defined by

$$g_n(x)=(a_n+k_n(x_1-b_n),x_2)$$

for $x=(x_1,x_2)\in I$, $a_n=2^{-(n-1)^2}$, $b_n=2^{1-n}$ and $k_n=(2^{2n-1}-1)2^{-n(n-1)}$.

Let $I=(0,1)\times(-1,1)$, and define a homeomorphism $g\colon I\to I$ by setting $g|_{I_n}=g_n$ for each $n\geq 1$. Then

(1) g is strictly Whitney spherical ring-preserving.
(2) g is not φ-FQC for any self-homeomorphism φ of $[0,\infty)$.

Proof. The boundedness of I and the construction of g guarantee that g is strictly Whitney spherical ring-preserving. For the proof of the statement (2), let $y_n=(2^{-n},0)$ and $z_n=(2^{n^2},0)$ with $n\geq 2$. Since for any $x\in[2^{-n},2^{1-n}]\times\{0\}$, the distance $\delta_I(x)$ from x to the boundary of I is x_1, we get

$$k_I(y_n,y_{n-1}) \leq \int_{2^{-n}}^{2^{1-n}} \frac{|dx_1|}{x_1} = \log 2. \tag{4.10}$$

Moreover, we know from [18, Lemma 2.2(1)] that

$$k_I(z_n, z_{n-1}) \geq \log \frac{2^{n^2}}{2^{(n-1)^2}} = (2n-1)\log 2. \tag{4.11}$$

Suppose on the contrary that there is a self-homeomorphism φ of $[0,\infty)$ such that g is φ-FQC. Then we know from the assumption

$$k_I\left(g(y_n), g(y_{n-1})\right) \leq \varphi\left(k_I(y_n, y_{n-1})\right),$$

together with (4.10) and (4.11), that for all $n \geq 2$,

$$(2n-1)\log 2 \leq \varphi(\log 2).$$

This is the desired contradiction. □

Now, we are ready to prove Theorem 4.2.

4.1. Proof of Theorem 4.2. Suppose first that f is φ-FQC. We know from Theorem A that both f and f^{-1} are (p, η_p)-LQS for every $0 < p < 1$. Then it follows from [12, Corollary 2.6] that f is ball-bounded. This, together with Lemma 4.4, shows that the first part is true. The second part follows from Example 4.9. □

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References

[1] Ahlfors, L. V.: Lectures on quasiconformal mappings. - Van Nostrand, Princeton, N.J., Toronto-New York-London, 1966.

[2] Björn, A., and J. Björn: Nonlinear potential theory on metric spaces. - EMS Tracts Math. 17, Eur. Math. Soc. (EMS), Zürich, 2011.

[3] Gehring, F. W.: Rings and quasiconformal mappings in space. - Trans. Amer. Math. Soc. 103, 1962, 353–393.

[4] Gehring, F. W.: The Carathéodory convergence theorem for quasiconformal mappings in space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 336/11, 1963, 21 pp.

[5] Gehring, F. W., and B. G. Osgood: Uniform domains and the quasi-hyperbolic metric. - J. Analyse Math. 36, 1979, 50–74.

[6] Gehring, F. W., and B. P. Palka: Quasiconformally homogeneous domains. - J. Analyse Math. 30, 1976, 172–199.

[7] Gehring, F., and J. Väisälä: On the geometric definition for quasiconformal mappings. - Comment. Math. Helv. 36, 1962, 19–32.

[8] Heinonen, J.: Lectures on analysis on metric spaces. - Universitext, Springer-Verlag, New York, 2001.

[9] Heinonen, J., P. Koskela, N. Shanmugalingam, and J. T. Tyson: Sobolev classes of Banach space-valued functions and quasiconformal mappings. - J. Anal. Math. 85, 2001, 87–139.

[10] Lehto, O., and K. I. Virtanen: Quasiconformal mappings in the plane. Second edition. - Grundlehren Math. Wiss. 126, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[11] Rohde, S.: Bilipschitz maps and the modulus of rings. - Ann. Acad. Sci. Fenn. Math. 22, 1997, 465–474.

[12] Tukia, P., and J. Väisälä: Quasisymmetric embeddings of metric spaces. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 97–114.

[13] Tukia, P., and J. Väisälä: Rings and bilipschitz maps in normed space. - Ann. Fenn. Math. 46, 2021, 587–591.

[14] Tyson, J. T.: Metric and geometric quasiconformality in Ahlfors regular Loewner spaces. - Conform. Geom. Dyn. 5, 2001, 21–73.

[15] Väisälä, J.: On quasiconformal mappings in space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 298, 1961, 36 pp.

[16] Väisälä, J.: Two new characterizations for quasiconformality. - Ann. Acad. Sci. Fenn. Ser. A I Math. 362, 1965, 12 pp.

[17] Väisälä, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[18] Väisälä, J.: Free quasiconformality in Banach spaces. I. - Ann. Acad. Sci. Fenn. Ser. A I Math. 15, 1990, 355–379.

[19] Väisälä, J.: Free quasiconformality in Banach spaces. II. - Ann. Acad. Sci. Fenn. Ser. A I Math. 16, 1991, 255–310.

[20] Väisälä, J.: Free quasiconformality in Banach spaces. III. - Ann. Acad. Sci. Fenn. Ser. A I Math. 17, 1992, 393–408.

[21] Väisälä, J.: Free quasiconformality in Banach spaces. IV. - Analysis and Topology, World Sci. Publ., River Edge, NJ, 1998, 697–717.

[22] Väisälä, J.: The free quasiworld. Freely quasiconformal and related maps in Banach spaces. - In: Quasiconformal geometry and dynamics (Lublin 1996), Banach Center Publications 48, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1999, 55–118.

[23] Williams, M.: Geometric and analytic quasiconformality in metric measure spaces. - Proc. Amer. Math. Soc. 140, 2012, 1251–1266.

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