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Sharp Poincaré–Wirtinger inequalities on complete graphs

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Abstract. Let $K_n = (V, E)$ be the complete graph with $n \geq 3$ vertices (here V and E denote the set of vertices and edges of K_n respectively). We find the optimal value $\mathbf{C}_{n,p}$ such that the inequality

$$\|f - m_f\|_p \leq \mathbf{C}_{n,p} \mathrm{Var}_p f$$

holds for every $f: V \to \mathbb{R}$, where Var_p stands for the p-variation, and m_f stands for the average value of f, for all $p \in [1, 3+\delta_n^1) \cup (3+\delta_n^2, +\infty)$, for $\delta_n^1 = \frac{1}{2n^2 \log(n)} + O(1/n^3)$ and $\delta_n^2 = \frac{2}{n} + O(1/n^2)$. Moreover, we characterize all the maximizer functions in that case. The behavior of the maximizers is different in each of the intervals $(1, 2)$, $(2, 3+\delta_n^1)$ and $(3+\delta_n^2, \infty)$.

Täydellisten verkkojen tarkat Poincarén–Wirtingerin epäyhtälöt

Tiivistelmä. Olkoon $K_n = (V, E)$ täydellinen verkko, jonka solmujen ($n \geq 3$ kappaletta) ja särmien joukot ovat V ja E. Käytetään funktion f keskiarvosta ja p-heilahtelusta merkintöjä m_f ja $\mathrm{Var}_p f$. Tässä työssä löydetään paras vakio $\mathbf{C}_{n,p}$, joka toteuttaa epäyhtälön

$$\|f - m_f\|_p \leq \mathbf{C}_{n,p} \mathrm{Var}_p f$$

kaikilla funktioilla $f: V \to \mathbb{R}$ ja kaikilla arvoilla $p \in [1, 3+\delta_n^1) \cup (3+\delta_n^2, +\infty)$, missä $\delta_n^1 = \frac{1}{2n^2 \log(n)} + O(1/n^3)$ ja $\delta_n^2 = \frac{2}{n} + O(1/n^2)$. Lisäksi löydetään kaikki funktiot, joilla yhtäsuuruus saavutetaan. Nämä funktiot käyttäytyvät eri tavoin kullakin välillä $(1, 2)$, $(2, 3+\delta_n^1)$ ja $(3+\delta_n^2, \infty)$.

1. Introduction

The study of Poincaré inequalities is a central theme in analysis, in particular, they play a relevant role in partial differential equations and mathematical physics. In recent years, proving discrete analogues of classical analytic inequalities has attracted the attention of many authors, see for instance [3, 4, 9]. In particular, in [7] and [6] the optimal constant for Poincaré inequalities on the hypercube was studied. In [8] the Poincaré inequalities constant grow on graphs was studied. Poincaré inequalities also appear naturally when studying isoperimetric inequalities [2]. Poincaré inequalities on complete graphs have been studied in [5], in that paper non-sharp bounds were established [5, Proposition 10.3]. In this paper, we establish optimal Poincaré–Wirtinger type inequalities on complete graphs and we characterize the extremizers. That is the content of our main result. In the following, K_n denotes the complete graph with n vertices $a_1, a_2, \ldots, a_n$. For all $f: K_n \to \mathbb{R}$ we denote by

$$\|f\|_p = \left(\sum_{a_i \in K_n} |f(a_i)|^p \right)^{\frac{1}{p}}$$

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the ℓ^p-norm of f. Also, we denote by

$$\mathrm{Var}_p(f) = \left(\sum_{1 \le i < j \le n} |f(a_i) - f(a_j)|^p \right)^{\frac{1}{p}},$$

the p-variation of f.

Theorem 1. (Main theorem) *Let $n \in \mathbb{N}$, and let $f\colon K_n \to \mathbb{R}$ and $m = m_f := \frac{\sum_{a_i \in \mathbb{K}_n} f(a_i)}{n}$.*

(i) *If $p = 1$ and $n \ge 3$, then we have*

$$\sum_{a_i \in K_n} |f(a_i) - m| \le \frac{2}{n} \mathrm{Var}_1(f),$$

Moreover, if we assume without loss of generality that $f(a_1) \ge \cdots \ge f(a_k) \ge m \ge f(a_{k+1}) \ge \cdots \ge f(a_n)$, the equality happens iff $f(a_i) = f(a_1)$ for all $1 \le i \le k$ and $f(a_i) = f(a_n)$ for all $k+1 \le i \le n$.

(ii) *If $1 < p < 2$ and $n \ge 3$, then we have*

$$\sum_{a_i \in K_n} |f(a_i) - m|^p \le \frac{(\lfloor \frac{n}{2} \rfloor)^{p-1} + (\lceil \frac{n}{2} \rceil)^{p-1}}{n^p} \mathrm{Var}_p(f)^p.$$

In this case, the maximizers are functions $f\colon K_n \to \mathbb{R}$ with $f(K_n) = \{A, B\}$ with $A > B$ and $|f^{-1}(A)| \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$.

(iii) *If $p = 2$ and $n \ge 3$, then we have an identity*

$$\sum_{a_i \in K_n} |f(a_i) - m|^2 = \frac{1}{n} \mathrm{Var}_2(f)^2.$$

(iv) *If $2 < p \le 3$ and $n \ge 3$, then*

$$\sum_{a_i \in K_n} |f(a_i) - m|^p \le \frac{1}{2^{p-1} + n - 2} \mathrm{Var}_p(f)^p.$$

and this inequality is sharp. The equality is attained for any function $f\colon K_n \to \mathbb{R}$ assuming values $a + c, \underbrace{a, \dots, a}_{n-2}, a - c$, for $a, c \in \mathbb{R}$. Moreover, the sharp inequality can be extended to p belonging to the interval $(3, 3 + \delta_n^1)$, with the same maximizers. In this case, we take

$$\delta_n^1 = \frac{\log(\sqrt{n^2+4} + 3n) - \log(4n)}{\log(n-1)} = \frac{1}{2n^2 \log(n)} + O(1/n^3).$$

(v) *Suppose that $3 < p < 4$ such that $\frac{n-2}{n} \ge \frac{1 + (n-1)^{p-1}}{n^{p-1}}$. In particular, if*

$$p \ge 3 + \delta_n^2 := 1 + \frac{2\log(n) - \log(n^2 - 2n - 1)}{\log(n) - \log(n-1)} = 3 + \frac{2}{n} + O\left(\frac{1}{n^2}\right),$$

then

$$\sum_{a_i \in K_n} |f(a_i) - m|^p \le \frac{1 + (n-1)^{p-1}}{n^p} \mathrm{Var}_p(f)^p \tag{1.1}$$

and this inequality is sharp. The equality is attained for Dirac delta functions, i.e. functions with $f(K_n) = \{A, B\}$ with $A \ne B$ and $|f^{-1}(A)| = 1$.

(vi) *If $p \geq 4$ and $n \geq 3$, then*

$$\sum_{a_i \in K_n} |f(a_i) - m|^p \leq \frac{1 + (n-1)^{p-1}}{n^p} \mathrm{Var}_p(f)^p, \tag{1.2}$$

and this inequality is sharp. The equality is attained for Dirac delta functions.

The idea behind the proof of these results (for $p \neq 1, 2$) can be outlined as follows. First, we prove that the maximizers for the inequality

$$\|f - m_f\|_p \leq \mathbf{C}_{n,p} \mathrm{Var}_p f \tag{1.3}$$

exist.

Then, we observe that, being critical points, these maximizers should satisfy a functional equation that depends on the optimal constant $\mathbf{C}_{n,p}$. On the other hand, we prove that these functional equations are the equality case of a sharp inequality (when $\mathbf{C}_{n,p}$ is as conjectured). This step depends strongly on the range to which p belongs, and is the one that determines the different structure of the maximizers in each interval. The proof of this auxiliary inequality (Lemma 3) takes most of the efforts in this manuscript, and is based on symmetrization and concavity/convexity arguments. Therefore, by classifying the maximizers of this auxiliary inequality, we determine all the critical points of the original (1.3) inequality. From this we conclude the classification of the maximizers and we obtain the optimal constant $\mathbf{C}_{n,p}$.

We notice that some related inequalities concerning the complete graph have been studied in [1, Section 9], where a similar (but different) Poincaré–Wirtinger type inequality is proved. However, both constants and maximizers differ in a wide range of p.

2. Proofs of the main results

We start with a basic lemma.

Lemma 2. (There is an extremizer) *There is a function $f\colon K_n \to \mathbb{R}$ such that*

$$\sup_{g: K_n \to \mathbb{R}, \textit{non constant}} \frac{\sum_{a_i \in K_n} |g(a_i) - m|^p}{\mathrm{Var}_p(g)^p} = \sup_{g: K_n \to [0,1], \textit{non constant}} \frac{\sum_{a_i \in K_n} |g(a_i) - m|^p}{\mathrm{Var}_p(g)^p} = \frac{\sum_{a_i \in K_n} |f(a_i) - m|^p}{\mathrm{Var}_p(f)^p}.$$

Proof. The first equality follows from the fact that the quotient is translation and dilation invariant, more precisely, the quotient is preserved by the transformation

$$g \mapsto \frac{g - \min_i g(a_i)}{\max_{j=1,\ldots,n}[g(a_j) - \min_i g(a_i)]}.$$

Then, given $y := (y_1, \ldots, y_n) \in A := \{(y_1, y_2, \ldots, y_n) \in [0,1]^n;\ \max_{i=1,\ldots,n} y_i = 1,\ \min_{i=1,\ldots,n} y_i = 0\}$ we define $f_y\colon V \to \mathbb{R}_{\geq 0}$ by $f_y(a_i) := y_i$. We observe that $\frac{\sum_{a_i \in K_n} |f_y(a_i) - m_{f_y}|^p}{\mathrm{Var}_p(f_y)^p}$ is continuous with respect to y in the compact set A. Thus, it achieves its maximum at some point $y_0 \in A$. Then

$$\sup_{g: K_n \to \mathbb{R}, \text{non constant}} \frac{\sum_{a_i \in K_n} |g(a_i) - m|^p}{\mathrm{Var}_p(g)^p} = \frac{\sum_{a_i \in K_n} |f_{y_0}(a_i) - m|^p}{\mathrm{Var}_p(f_{y_0})^p}.$$

From where we obtain the result. □

Proof of Main theorem. Case $p = 1$. In this case, without loss of generality we assume that $f(a_1) \geq \cdots \geq f(a_k) \geq m \geq f(a_{k+1}) \geq \cdots \geq f(a_n)$. Then, we have

$$\begin{aligned}\sum_{i=1}^{n} |f(a_i) - m| &= \sum_{i=1}^{k}(f(a_i) - m) + \sum_{i=k+1}^{n} (m - f(a_i)) \\ &= \sum_{i=1}^{k} f(a_i) - \sum_{i=k+1}^{n} f(a_i) + \frac{(n-2k)}{n}\sum_{i=1}^{n} f(a_i) \\ &= \frac{2}{n}\left[(n-k)\sum_{i=1}^{k} f(a_i) - k\sum_{i=k+1}^{n} f(a_i)\right] \\ &= \frac{2}{n}\sum_{1\leq i\leq k, k+1\leq j\leq n} f(a_i) - f(a_j) \\ &\leq \frac{2}{n}\sum_{i<j} f(a_i) - f(a_j).\end{aligned}$$

In the last inequality the identity is attained iff $f(a_i) = f(a_1)$ for all $1 \leq i \leq k$ and $f(a_i) = f(a_n)$ for all $k + 1 \leq i \leq n$.

Case $p = 2$. In this case we have an identity

$$\begin{aligned}\sum_{i=1}^{n}(f(a_i) - m)^2 &= \sum_{i=1}^{n}[f(a_i)^2 + m^2 - 2mf(a_i)] \\ &= \left[\sum_{i=1}^{n} f(a_i)^2\right] + nm^2 - 2nm^2 = \left[\sum_{i=1}^{n} f(a_i)^2\right] - nm^2 \\ &= \sum_{i=1}^{n} f(a_i)^2 - \frac{1}{n}\sum_{i=1}^{n} f(a_i)^2 - \frac{2}{n}\sum_{i<j} f(a_i)f(a_j) \\ &= \frac{n-1}{n}\sum_{i=1}^{n} f(a_i)^2 - \frac{2}{n}\sum_{i<j} f(a_i)f(a_j) \\ &= \frac{1}{n}\sum_{i<j} f(a_i)^2 + f(a_j)^2 - 2f(a_i)f(a_j) \\ &= \frac{1}{n}\mathrm{Var}_2(f)^2.\end{aligned}$$

Thus, any function $f\colon K_n \to \mathbb{R}$ is an extremizer.

Now we discuss the case $2 \neq p > 1$. Let

$$C_n = C_{n,p} := \sup_{g: K_n \to \mathbb{R}} \frac{\sum_{a_i\in K_n} |g(a_i) - m|^p}{\mathrm{Var}_p(g)^p}. \tag{2.1}$$

By taking g_0 such that $g_0(a_1) = 1, g_0(a_n) = -1$ and $g_0(a_i) = 0$ for $i = 2, \ldots, n-1$ we observe that $C_n \geq \frac{1}{2^{p-1}+n-2}$. By taking g_1 such that $g_1(a_1) = 1$, and $g_1(a_i) = 0$ for $i = 2, \ldots, n$ we observe that $C_n \geq \frac{1+(n-1)^{p-1}}{n^p}$. Moreover, by taking $g_2(a_i) = 1$ for $i \leq \lfloor \frac{n}{2} \rfloor$ and $g_2(a_i) = 0$ for $i > \lfloor \frac{n}{2} \rfloor$, we also obtain that $C_n \geq \frac{\lfloor \frac{n}{2}\rfloor^{p-1}+\lceil \frac{n}{2}\rceil^{p-1}}{n^p}$. Then

$$C_n \geq \max\left\{\frac{1}{2^{p-1}+n-2}, \frac{1+(n-1)^{p-1}}{n^p}, \frac{\lfloor \frac{n}{2}\rfloor^{p-1}+\lceil \frac{n}{2}\rceil^{p-1}}{n^p}\right\}. \tag{2.2}$$

By the previous lemma, there is an extremizer function f for (2.1). We assume without loss of generality that $f(a_1) \geq f(a_2) \geq \cdots \geq f(a_n)$.

For $\epsilon > 0$ we define the auxiliary function f_ε by: $f_\varepsilon(a_i) := f(a_i)$ for $i = 2, \ldots, n-1$ and $f_\varepsilon(a_1) := f(a_1)+\varepsilon$ and $f_\varepsilon(a_n) := f(a_n)-\varepsilon$. Observe that $m_f = m_{f_\varepsilon}$. We consider the function

$$r(\varepsilon) := C_n \mathrm{Var}_p(f_\varepsilon)^p - \sum_{a_i \in K_n} |f_\varepsilon(a_i) - m|^p.$$

Observe that

$$\begin{aligned} 0 = \frac{r'(0)}{p} &= C_n \left(\sum_{a_i \in K_n} (f(a_1) - f(a_i))^{p-1} + (f(a_i) - f(a_n))^{p-1} \right) \\ &\quad - (f(a_1) - m)^{p-1} - (m - f(a_n))^{p-1}. \end{aligned} \tag{2.3}$$

Then, we observe the following:

Lemma 3. *For $p \in (1,2) \cup (2, 3+\delta_n^1] \cup [(3,4) \cap \mathcal{A}_n] \cup [4, +\infty)$ and $C^*_{n,p}$ be such that*

$$C^*_{n,p} = \begin{cases} \frac{\lfloor \frac{n}{2} \rfloor^{p-1} + \lceil \frac{n}{2} \rceil^{p-1}}{n^p}, & \text{if } p \in (1,2), \\ \frac{1+(n-1)^{p-1}}{n^p}, & \text{if } p \in ((3,4] \cap \mathcal{A}_n) \cup [4,\infty), \\ \frac{1}{2^{p-1}+n-2}, & \text{if } p \in (2, 3+\delta_n^1), \end{cases} \tag{2.4}$$

where δ_n^1 is defined in Theorem 1 and $\mathcal{A}_n := \left\{x > 0; \frac{n-2}{n} \geq \frac{1+(n-1)^{x-1}}{n^{x-1}}\right\}$. Let $f\colon K_n \to \mathbb{R}$, assuming that $f(a_1) = \max_{i=1}^n \{f(a_i)\}$ and $f(a_n) = \min_{i=1}^n \{f(a_n)\}$, and $m = m_f := \frac{1}{n}\sum_{i=1}^n f(a_i)$. Then

$$\begin{aligned} &(f(a_1) - m)^{p-1} + (m - f(a_n))^{p-1} \\ &\leq C^*_{n,p} \left(\sum_{j=1}^n (f(a_1) - f(a_j))^{p-1} + (f(a_j) - f(a_n))^{p-1} \right), \end{aligned} \tag{2.5}$$

and the constant $C^*_{n,p}$ can not be replaced by a smaller quantity. Moreover, we have that the maximizers satisfy the following property (when $f(a_1) \geq f(a_2) \geq \cdots \geq f(a_n)$):

(i) *$f(a_1) = f(a_j)$ if $j \leq \lfloor \frac{n}{2} \rfloor$ and $f(a_j) = f(a_n)$ otherwise; or $f(a_1) = f(a_j)$ if $j \leq \lceil \frac{n}{2} \rceil$ and $f(a_j) = f(a_n)$ otherwise, for $1 < p < 2$.*

(ii) *$f(a_1) > f(a_2) = f(a_3) = \cdots = f(a_{n-1}) > f(a_n)$ and $f(a_2) = \frac{f(a_1)+f(a_n)}{2}$, if $2 < p \leq 3 + \delta_n^1$.*

(iii) *$f(a_1) > f(a_2) = \cdots = f(a_n)$ or $f(a_1) = \ldots f(a_{n-1}) > f(a_n)$ if $4 \leq p$, $p \in (3,4) \cap \mathcal{A}_n$.*

Remark 4. Since $\frac{1+(n-1)^x}{n^x}$ is decreasing in x for any fixed n, we have that $\mathcal{A}_n$ is an interval for all $n \geq 3$. We observe that for $n \geq 3$ we have $4 \in \mathcal{A}_n$. And, moreover, $1 + \log_{\frac{n}{n-1}} \frac{n^2}{n^2-2n-1} \in \mathcal{A}_n$ since $\frac{1+(n-1)^{p-1}}{n^{p-1}} \leq \frac{1}{n^2} + \left(\frac{n-1}{n}\right)^{p-1}$ and $\frac{1}{n^2} + \left(\frac{n-1}{n}\right)^{p-1} = \frac{n-2}{n}$ for $p = 1 + \log_{\frac{n}{n-1}} \frac{n^2}{n^2-2n-1}$. Therefore, since $\lim_{n\to\infty} 1 + \log_{\frac{n}{n-1}} \frac{n^2}{n^2-2n-1} = 3$, we have that for any $p > 3$ there exists n_0 such that for any $n > n_0$ we have $p \in \mathcal{A}_n$.

Now we explain how Lemma 3 implies the remaining cases of Theorem 1. In fact, given that we have (2.3) for any maximizer of (2.1), by using the Lemma 3 we have

$C^*_{n,p} \geq C_{n,p}$. On the other hand, by (2.4) and (2.2),

$$C^*_{n,p} \leq \max\left\{ \frac{1}{2^{p-1}+n-2}, \frac{1+(n-1)^{p-1}}{n^p}, \frac{\lfloor \frac{n}{2} \rfloor^{p-1} + \lceil \frac{n}{2} \rceil^{p-1}}{n^p} \right\} \leq C_{n,p}.$$

Thus, we conclude that

$$C^*_{n,p} = C_{n,p}, \tag{2.6}$$

as desired. Furthermore, in Lemma 3 we characterize the maximizers for (2.5). And, by (2.1) and (2.6) the maximizers of (2.1) are the same as for (2.5).

Now, we discuss the proof of Lemma 3.

Proof of Lemma 3. Let $R_{n,p}$ be the best constant for the inequality (2.5). By considering the functions g_0, g_1 and g_2 as before, we observe that $R_{n,p} \geq \max\Big\{ \frac{1}{2^{p-1}+n-2}, \frac{1+(n-1)^{p-1}}{n^p}, \frac{\lfloor \frac{n}{2} \rfloor^{p-1} + \lceil \frac{n}{2} \rceil^{p-1}}{n^p} \Big\}$. Let f be an extremizer for (2.5) (the existence of an extremizer can be seen similarly to Lemma 2). Now, we analyze the different cases of the lemma.

Case $1 < p < 2$. We observe that, in this case, the function $x \mapsto x^{p-1}$ is concave for $x \geq 0$. Therefore, we observe by Karamata's inequality, that if $a > b > 0$, and $0 \leq \varepsilon \leq \frac{a-b}{2}$, then

$$(a-\varepsilon)^{p-1} + (b+\varepsilon)^{p-1} \geq a^{p-1} + b^{p-1}, \tag{2.7}$$

with strict inequality if $\varepsilon > 0$. If there exist $a_j \neq a_k$ with $f(a_1) > f(a_j) \geq f(a_k) > f(a_n)$, then for $\varepsilon > 0$ sufficiently small, defining $f_\varepsilon \colon K_n \to \mathbb{R}$ by $f_\varepsilon(a_j) = f(a_j) + \varepsilon$, $f_\varepsilon = f(a_k) - \varepsilon$, and $f_\varepsilon = f$ otherwise. We have that, since

$$\begin{aligned}&(f_\varepsilon(a_1) - f_\varepsilon(a_k) - \varepsilon)^{p-1} + (f_\varepsilon(a_1) - f_\varepsilon(a_j) + \varepsilon)^{p-1} \\ &> (f_\varepsilon(a_1) - f_\varepsilon(a_k))^{p-1} + (f_\varepsilon(a_1) - f_\varepsilon(a_j))^{p-1}.\end{aligned}$$

by taking $a = f_\varepsilon(a_1) - f_\varepsilon(a_k)$ and $b = f_\varepsilon(a_1) - f_\varepsilon(a_j)$ in (2.7). And, similarly

$$\begin{aligned}&(f_\varepsilon(a_j) - f_\varepsilon(a_n) - \varepsilon)^{p-1} + (f_\varepsilon(a_k) - f_\varepsilon(a_n) + \varepsilon)^{p-1} \\ &> (f_\varepsilon(a_j) - f_\varepsilon(a_n))^{p-1} + (f_\varepsilon(a_k) - f_\varepsilon(a_n))^{p-1}.\end{aligned}$$

by taking $a = f_\varepsilon(a_j) - f_\varepsilon(a_n)$ and $b = f_\varepsilon(a_k) - f_\varepsilon(a_n)$ in (2.7). Then

$$\begin{aligned}&\frac{(f(a_1) - m)^{p-1} + (m - f(a_n))^{p-1}}{\left(\sum_{j=1}^n (f(a_1) - f(a_j))^{p-1} + (f(a_j) - f(a_n))^{p-1}\right)} \\ &< \frac{(f_\varepsilon(a_1) - m)^{p-1} + (m - f_\varepsilon(a_n))^{p-1}}{\left(\sum_{j=1}^n (f_\varepsilon(a_1) - f_\varepsilon(a_j))^{p-1} + (f_\varepsilon(a_j) - f_\varepsilon(a_n))^{p-1}\right)}.\end{aligned}$$

Thus, we would have that f is not a maximizer of our inequality. This, implies that this a_j and a_k do not exist. Thus, f is of the form $f(a_1) = \cdots = f(a_j) \geq f(a_{j+1}) = x > f(a_{j+2}) = \cdots = f(a_n)$. By the invariance, we can assume that $f(a_1) = 1$ and $f(a_n) = 0$. Now, we deal with the case when $1 > x > 0$. Here, $m = \frac{j+x}{n}$, and

$$\begin{aligned}&\frac{(f(a_1) - m)^{p-1} + (m - f(a_n))^{p-1}}{\left(\sum_{j=1}^n (f(a_1) - f(a_j))^{p-1} + (f(a_j) - f(a_n))^{p-1}\right)} \\ &= \frac{1}{n^{p-1}} \frac{(n-j-x)^{p-1} + (j+x)^{p-1}}{(1-x)^{p-1} + x^{p-1} + n - 1}.\end{aligned}$$

We will observe that this is bounded from above by

$$\frac{\lfloor \frac{n}{2} \rfloor^{p-1} + \lceil \frac{n}{2} \rceil^{p-1}}{n^p}.$$

The case n even follows using that $(1-x)^{p-1}+x^{p-1} \geq 1$ (the equality occurs for $x = 1$), and since $\max\{n-j-x, j+x\} \geq \frac{n}{2} \geq \min\{n-j-x, j+x\}$, by Karamata's inequality $(n-j-x)^{p-1} + (j+x)^{p-1} \leq \lfloor \frac{n}{2} \rfloor^{p-1} + \lceil \frac{n}{2} \rceil^{p-1}$ (the equality occurs when $j + x = \frac{n}{2}$). So, the equality happens when $f(a_1) = \cdots = f(a_{n/2}) = A$ and $f(a_{n/2+1}) = \cdots = f(a_n) = B$ for some $A > B$ (by the dilation and translation invariance). For the odd case $n = 2k+1$, we observe that if $j \notin \{k, k+1\}$, then $\max\{n-j-x, j+x\} \geq k+1 \geq k \geq \min\{n-j-x, j+x\}$, therefore by the concavity of $x \mapsto x^{p-1}$ we get

$$(n-j-x)^{p-1} + (j+x)^{p-1} \leq k^{p-1} + (k+1)^{p-1},$$

from where the result follows using once again that $(1-x)^{p-1} + x^{p-1} \geq 1$.

Now, we discuss the case when $j \in \{k, k+1\}$. However, if $j = k+1$ we get that $(n-j-x, j+x) = (k-x, k+1+x)$, then by concavity $(k+1+x)^{p-1} + (k-x)^{p-1} \leq (k+1-x)^{p-1} + (k+x)^{p-1}$, that corresponds to the case when $k = j$. Therefore, we just need to prove that

$$\frac{1}{(2k+1)^{p-1}} \frac{(k+1-x)^{p-1} + (k+x)^{p-1}}{(1-x)^{p-1} + x^{p-1} + 2k} \leq \frac{k^{p-1} + (k+1)^{p-1}}{(2k+1)^p},$$

or, equivalently,

$$\begin{aligned} &(2k+1)\left((k+1-x)^{p-1} + (k+x)^{p-1}\right) \\ &\leq (k^{p-1} + (k+1)^{p-1})((1-x)^{p-1} + x^{p-1} + 2k). \end{aligned} \tag{2.8}$$

Writing $x = \frac{1}{2} + u$ for $u \in [-1/2, 1/2]$, the previous inequality reduced to prove that

$$\begin{aligned} &(2k+1)\left(k+\frac{1}{2}\right)^{p-1}\left[\left(1-\frac{u}{k+1/2}\right)^{p-1} + \left(1+\frac{u}{k+1/2}\right)^{p-1}\right] \\ &\leq (k^{p-1} + (k+1)^{p-1})\left[2k + \frac{1}{2^{p-1}}((1-2u)^{p-1} + (1+2u)^{p-1})\right]. \end{aligned}$$

Using the expansion $(1+y)^\alpha = \sum_{l=0}^{\infty} \binom{\alpha}{l} y^l$ (converging absolutely for $-1 < y < 1$), this is equivalent to

$$\begin{aligned} &2(2k+1)\left(k+\frac{1}{2}\right)^{p-1} \sum_{l=0}^{\infty} \binom{p-1}{2l} \left(\frac{u}{k+\frac{1}{2}}\right)^{2l} \\ &\leq (k^{p-1} + (k+1)^{p-1})\left(2k + \frac{2}{2^{p-1}}\left(\sum_{l=0}^{\infty} \binom{p-1}{2l} (2u)^{2l}\right)\right), \end{aligned}$$

or, equivalently,

$$\begin{aligned} &2(2k+1)\left(k+\frac{1}{2}\right)^{p-1} - (k^{p-1} + (k+1)^{p-1})\left(2k + \frac{2}{2^{p-1}}\right) \\ &\leq \sum_{l=1}^{\infty} \binom{p-1}{2l} \left(2^{2l+2-p}\left((k^{p-1} + (k+1)^{p-1}) - (2k+1)^{p-2l}\right)\right) u^{2l}. \end{aligned} \tag{2.9}$$

We observe that $\binom{p-1}{2l} < 0$ and $k^{p-1}+(k+1)^{p-1} \geq (2k+1)^{p-1}$ since $1 < p < 2$. Thus,

$$\binom{p-1}{2l}\left(2^{2l+2-p}\left((k^{p-1}+(k+1)^{p-1})-(2k+1)^{p-2l}\right)\right) < 0,$$

for any $l > 0$. Therefore, the RHS of (2.9) has only negative coefficients, and thus it is a decreasing function for u in $[0, \frac{1}{2}]$ and it is an increasing function for u in $[-\frac{1}{2}, 0]$. Then, inequality (2.9) holds if it holds for $u = \frac{1}{2}$ and $u = -\frac{1}{2}$, that follows directly by going back to the expression (2.8), in fact, we observe that for $x = 1$, and $x = 0$ we have an identity in (2.8). From where we conclude the case $1 < p < 2$.

Now, for $p > 2$ we observe the following. Let us consider the function $g\colon K_n \to \mathbb{R}$ defined by $g(a_1) := f(a_1)$, $g(a_n) := f(a_n)$ and $g(a_j) = \frac{1}{n-2}\sum_{i=2}^{n-1} f(a_i)$ for $j = 2, \ldots n-1$. Then, since $m_g = m_f$, we have

$$(g(a_1)-m_g)^{p-1} + (m_g - g(a_n))^{p-1} = (f(a_1)-m_f)^{p-1} + (m_f - f(a_n))^{p-1}.$$

On the other hand, since the function $h_1\colon [0,+\infty) \to \mathbb{R}$ defined by $h_1(x) = x^{p-1}$ is convex for $p \geq 2$, by Karamata's inequality we have

$$\begin{aligned}&\sum_{j=1}^{n}(f(a_1)-f(a_j))^{p-1} + (f(a_j)-f(a_n))^{p-1}\\ &\quad\geq \sum_{j=1}^{n}(g(a_1)-g(a_j))^{p-1} + (g(a_j)-g(a_n))^{p-1}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{|g(a_1)-m|^{p-1}+|m-g(a_n)|^{p-1}}{\sum_{j=1}^{n}(g(a_1)-g(a_j))^{p-1}+(g(a_j)-g(a_n))^{p-1}}\\ &\geq \frac{|f(a_1)-m|^{p}+|m-f(a_n)|^{p-1}}{\sum_{j=1}^{n}(f(a_1)-f(a_j))^{p-1}+(f(a_j)-f(a_n))^{p-1}}.\end{aligned} \tag{2.10}$$

Thus, we can assume without loss of generality that $f(a_1) \geq f(a_2) = f(a_3) = \cdots = f(a_{n-1}) \geq f(a_n)$. Since f is not constant then, we have three scenarios

- $f(a_1) > f(a_2) = f(a_3) = \cdots = f(a_{n-1}) > f(a_n)$,
- $f(a_1) > f(a_2) = f(a_3) = \cdots = f(a_{n-1}) = f(a_n)$,
- $f(a_1) = f(a_2) = f(a_3) = \cdots = f(a_{n-1}) > f(a_n)$.

Since the quotients we are considering are translation and dilation invariant. In the first scenario, we can assume without loss of generality that $f(a_1) = 1$, $f(a_n) = -1$ and $f(a_2) = \cdots = f(a_{n-1}) = x \geq 0$. On the other hand, in the second and third scenarios, we can assume without loss of generality that $f(a_1) = 1$ and $f(a_2) = \cdots = f(a_{n-1}) = f(a_n) = 0$ have a Dirac delta function and then conclude the analysis on those cases.

Then, we are left to prove the following inequality

$$\left(1-\frac{n-2}{n}x\right)^{p-1} + \left(1+\frac{n-2}{n}x\right)^{p-1} \leq C^*_{n,p}\left(2^p + (n-2)((1-x)^{p-1}+(1+x)^{p-1})\right).$$

Define the function $T\colon [0,1] \to \mathbb{R}$ by

$$\begin{aligned}T(x) := {}& C^*_{n,p}\left(2^p + (n-2)((1-x)^{p-1}+(1+x)^{p-1})\right)\\ &- \left(1-\frac{n-2}{n}x\right)^{p-1} - \left(1+\frac{n-2}{n}x\right)^{p-1}.\end{aligned}$$

We want to show that $T(x) \geq 0$ for all $x \in [0,1]$. We will prove that $T(0), T(1) \geq 0$ and that T is monotone in $[0,1]$, we discuss each case separately.

Case $2 < p \leq 3 + \delta_n^1$. In this case, we observe that $T(0) = 0$, and we will prove that T is increasing in $[0,1]$. For this is enough to prove that $T'(x) \geq 0$ for all $x \in [0,1]$, which is equivalent to

$$nC_{n,p}^* \geq \frac{\left(1 + \frac{n-2}{n}x\right)^{p-2} - (1 - \frac{n-2}{n}x)^{p-2}}{(1+x)^{p-2} - (1-x)^{p-2}} := G(x) \quad \text{for all} \quad x \in [0,1].$$

Subcase $2 < p \leq 3$. We observe that for a fix $x \in [0,1]$ the function $v_x\colon \left[\frac{n-2}{n}, 1\right] \to \mathbb{R}$ defined by

$$v_x(y) := \left(y + \frac{n-2}{n}x\right)^{p-2} - \left(y - \frac{n-2}{n}x\right)^{p-2}$$

is decreasing in $[\frac{n-2}{n}, 1]$ as a function in y, since its derivative is $v_x'(y) = (p-2)\left(\left(y + \frac{n-2}{n}x\right)^{p-3} - \left(y - \frac{n-2}{n}x\right)^{p-3}\right) \leq 0$. Therefore,

$$\frac{\left(1 + \frac{n-2}{n}x\right)^{p-2} - (1 - \frac{n-2}{n}x)^{p-2}}{(1+x)^{p-2} - (1-x)^{p-2}} = \left(\frac{n-2}{n}\right)^{p-2} \frac{v_x(1)}{v_x(\frac{n-2}{n})} \leq \left(\frac{n-2}{n}\right)^{p-2}. \tag{2.11}$$

Moreover, since the function $r\colon \mathbb{R} \to \mathbb{R}$ defined by $r(p) := p - 1 - 2^{p-2}$ is concave on $[2,3]$, then $r(p) \geq \min\{r(2), r(3)\} = 0$ for all $p \in [2,3]$. Then

$$2^{p-1}(n-2)^{p-2} \leq 2(p-1)(n-2)^{p-2}.$$

Moreover, by the mean value theorem $(p-1)(n-2)^{p-2} \leq \frac{n^{p-1} - (n-2)^{p-1}}{2}$. Then $2^{p-1}(n-2)^{p-2} \leq n^{p-1} - (n-2)^{p-1}$ or equivalently

$$\left(\frac{n-2}{n}\right)^{p-2} \leq \frac{n}{n-2+2^{p-1}}. \tag{2.12}$$

Combining (2.11) and (2.12) we obtain

$$\frac{\left(1 + \frac{n-2}{n}x\right)^{p-2} - (1 - \frac{n-2}{n}x)^{p-2}}{(1+x)^{p-2} - (1-x)^{p-2}} \leq \frac{n}{n-2+2^{p-1}} = nC_{n,p}^*. \tag{2.13}$$

Then, T is increasing in $[0,1]$, as claimed. Then $T(x) = 0$ only happens for $x = 0$, this corresponds to the function $1 = f(a_1) > 0 = f(a_2) = f(a_3) = \cdots = f(a_{n-1}) > f(a_n) = -1$. Therefore, the extremizers are all functions such that $f(a_2) = f(a_3) = \cdots = f(a_{n-1}) = m$, $f(a_1) = m + c$ and $f(a_n) = m - c$ for some $m \in \mathbb{R}$, $c > 0$ (by the dilation and translation invariance).

Subcase $3 < p \leq 3 + \delta_n^1$. We observe that in this range, the function $x \mapsto x^{p-3}$ is strictly concave in $[1,n]$. Therefore, we have that

$$\frac{n-2}{n-1}n^{p-3} + \frac{1}{n-1}1^{p-3} < \left[\frac{n-2}{n-1}n + \frac{1}{n-1}\right]^{p-3} = (n-1)^{p-3},$$

thus $(n-2)n^{p-3} + 1 < (n-1)^{p-2}$. We notice that this implies that

$$\lim_{x \to 0} G(x) = \frac{n-2}{n} < \frac{(n-1)^{p-2} - 1}{n^{p-2}} = G(1).$$

We also observe, that $G(1) \leq nC_{n,p}^* = \frac{n}{n-2+2^{p-1}}$. In fact, we observe that the inequality

$$(n - 2 + 4s)(n + 2 + 4s) \leq n^2,$$

is valid whenever $\frac{-\sqrt{n^2+4}-n}{4} \leq s \leq \frac{\sqrt{n^2+4}-n}{4}$. So, we have that

$$\max\left\{\frac{((n-1)^{p-3}-1)(n-1)}{4}, 2^{p-3}-1\right\} \leq \frac{\sqrt{n^2+4}-n}{4}$$

if $(n-1)^{p-3}-1 \leq \frac{\sqrt{n^2+4}-n}{4(n)}$. Here, we remark that this condition can be slightly improved, in particular, by replacing $4n$ by $n-1$ in the denominator for all $n \geq 4$, and, replacing 12 by 4 in the denominator for $n=3$. Then

$$\begin{aligned}&((n-1)^{p-2}-1)(2^{p-1}+n-2)\\ &= (((n-1)^{p-3}-1)(n-1)+(n-2))((n+2)+4(2^{p-3}-1)) \leq n^2 \leq n^{p-1},\end{aligned}$$

which implies $G(1) \leq nC^*_{n,p}$. Therefore, we need $(n-1)^{p-3}-1 \leq \frac{\sqrt{n^2+4}-n}{4(n)}$, or equivalently

$$p \leq 3 + \frac{\log(\sqrt{n^2+4}+3n)-\log(4n)}{\log(n-1)} = 3+\delta^1_n.$$

Now, we need to prove that $(1+\frac{n-2}{n}x)^{p-2}-(1-\frac{n-2}{n}x)^{p-2} \leq nC^*_{n,p}((1+x)^{p-2}-(1-x)^{p-2})$, or equivalently

$$\sum_{k=0}^{\infty}\binom{p-2}{2k+1}\left(nC^*_{n,p}-\left(\frac{n-2}{n}\right)^{2k+1}\right)x^{2k+1} \geq 0, \tag{2.14}$$

for all $x \in [0,1]$. Since $\binom{p-2}{1} > 0$ and $nC^*_{n,p} \geq G(1) > \frac{n-2}{n}$ as seen above, we have that the first coefficient is positive. On the other hand, for $k \geq 1$ we have that $\binom{p-2}{2k+1} < 0$ and $nC^*_{n,p} - \left(\frac{n-2}{n}\right)^{2k+1} \geq 0$, thus, all the other coefficients in the left hand side of (2.14) are negative. Let us consider the polynomials

$$P_N(x) := \sum_{k=0}^{N}\binom{p-2}{2k+1}\left(nC^*_{n,p}-\left(\frac{n-2}{n}\right)^{2k+1}\right)x^{2k+1}.$$

We know that (2.14) holds for $x=1$, since $nC^*_{n,p} \geq G(1)$. Then $P_N(1) > 0$ for N sufficiently large, and $P'_N(0) > 0$ (since the first coefficient in the left hand side of (2.14) is positive). Moreover, we observe that since the higher degrees coefficients are negative $\lim_{x\to\infty} P_N(x) = -\infty$. Thus, for N sufficiently large, $P_N(x)$ has a root between 1 an ∞. However, by the Descartes's rule of signs, $P_N(x)$ has only one positive root (since it's coefficients list has only one sign change), therefore $P_N(x) \geq 0$ for all $x \in [0,1]$ for N sufficiently large. Since

$$P_N(x) \to \sum_{k=0}^{\infty}\binom{p-2}{2k+1}\left(nC^*_{n,p}-\left(\frac{n-2}{n}\right)^{2k+1}\right)x^{2k+1},$$

pointwise in $0<x<1$, we conclude that (2.14) holds, and then our main theorem holds for $p \in (3, 3+\delta^1_n)$.

Cases $4 \leq p$ and $p \in (3,4) \cap A_n$. In this case we prove that T is decreasing in $[0,1]$, then $T(x) \geq T(1) = 0$ for all $x \in [0,1]$ and $R_{n,p} = \frac{1+(n-1)^{p-1}}{n^p} := C^*_{n,p}$. For this is enough to prove that

$$\frac{(1+\frac{n-2}{n}x)^{p-2}-(1-\frac{n-2}{n}x)^{p-2}}{(1+x)^{p-2}-(1-x)^{p-2}} \geq \frac{1+(n-1)^{p-1}}{n^{p-1}}, \tag{2.15}$$

since this implies that $T'(x) \leq 0$ for all $x \in [0,1]$.

We study first the case $p \in [4,\infty)$. For this, we use the following proposition.

Proposition 5. *Let $r \geq 2$ and k be a positive integer. Then, for any $y \geq 1$ we have*

$$(1+ky)^r - 1 - (ky)^r \geq (k+y)^r - k^r - y^r.$$

Proof. We observe that the statement is true for $y = 1$ (in fact, we have an identity). The result follows from the fact that

$$h(y) := (1+ky)^r + k^r + y^r - 1 - (k+y)^r - (ky)^r$$

is increasing. Since, its derivative is

$$h'(y) = kr(1+ky)^{r-1} + ry^{r-1} - r(k+y)^{r-1} - kr(ky)^{r-1},$$

and $k(1+ky)^{r-1} + y^{r-1} \geq (k+y)^{r-1} + k(ky)^{r-1}$, by Karamata's inequality, since the $(k+1)$-tuple $(1+ky, \ldots, 1+ky, y)$ majorizes $(k+y, ky, \ldots, ky)$ or $(ky, \ldots, ky, k+y)$ and $x \mapsto x^{r-1}$ is convex. □

Then, using that

$$\frac{(n-1)^{p-2} - 1}{n^{p-2}} \geq \frac{1 + (n-1)^{p-1}}{n^{p-1}},$$

for all $p \geq 4, n \geq 3$ (since $(n-1)^{p-2} \geq (n-1)^2 \geq n+1$). In order to have (2.15), we just need to prove

$$\frac{(1+\frac{n-2}{n}x)^{p-2} - (1-\frac{n-2}{n}x)^{p-2}}{(1+x)^{p-2} - (1-x)^{p-2}} \geq \frac{(n-1)^{p-2} - 1}{n^{p-2}}. \tag{2.16}$$

This can be rewritten as follows

$$\begin{aligned}&(n + (n-2)x)^{p-2} + ((n-1)(1-x))^{p-2} + (1+x)^{p-2}\\ &\geq (n - (n-2)x)^{p-2} + ((n-1)(1+x))^{p-2} + (1-x)^{p-2},\end{aligned}$$

or, equivalently

$$\begin{aligned}&\left((n-1)\frac{1+x}{1-x} + 1\right)^{p-2} + (n-1)^{p-2} + \left(\frac{1+x}{1-x}\right)^{p-2}\\ &\geq \left((n-1) + \frac{1+x}{1-x}\right)^{p-2} + \left((n-1)\frac{1+x}{1-x}\right)^{p-2} + 1,\end{aligned}$$

which follows as a consequence of our Proposition 5 by taking $r = p-2$, $k = n-1$ and $y = \frac{1+x}{1-x}$.

Now we study the case when $p \in \mathcal{A}_n \cap (3,4)$, by the definition of $\mathcal{A}_n$, similarly to (2.15) it is enough to prove that

$$\frac{(1+\frac{n-2}{n}x)^{p-2} - (1-\frac{n-2}{n}x)^{p-2}}{(1+x)^{p-2} - (1-x)^{p-2}} \geq \frac{n-2}{n}. \tag{2.17}$$

Notice that we have equality for $x = 1$. For $x \neq 1$ this equality becomes:

$$\begin{aligned}&n\left(1 + \frac{n-2}{n}x\right)^{p-2} - n\left(1 - \frac{n-2}{n}x\right)^{p-2}\\ &- (n-2)(1+x)^{p-2} + (n-2)(1-x)^{p-2} := \alpha(x) \geq 0.\end{aligned}$$

We notice that $\alpha(0)=0$. Moreover, we have that

$$\alpha'(x) = (p-2)(n-2)\Bigg[\left(1+\frac{n-2}{n}x\right)^{p-3} + \left(1-\frac{n-2}{n}x\right)^{p-3} - (1+x)^{p-3} - (1-x)^{p-3}\Bigg] \geq 0,$$

for all $x \in [0,1]$, by Karamata's inequality, since the function $x \mapsto x^{p-3}$ is concave. Then α is increasing on $[0,1]$, thus $\alpha(x) \geq 0$ for all $x \in [0,1]$ as desired. Then, in this case, the equality is only attained for $x=1$, this corresponds to the function $1 = f(a_1) = f(a_2) = f(a_3) = \cdots = f(a_{n-1}) > f(a_n) = 0$. Therefore, the extremizers are given by functions such that $f(a_1) = f(a_2) = f(a_3) = \cdots = f(a_{n-1}) = a$ and $f(a_n) = b$ for some $a \neq b$ (by the dilation and translation invariance). This concludes the proof of our main theorem. □

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