Annales Fennici Mathematici
Volumen 51, 2026, 223–236

Banach–Lamperti theorem for the Kurzweil–Henstock integral

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Abstract. We identify isometric isomorphisms of the space of Kurzweil–Henstock integrable functions as bi-absolutely-continuous changes of variable.

Banachin–Lampertin lause Kurzweilin–Henstockin integraalille

Tiivistelmä. Tässä työssä osoitetaan, että Kurzweilin–Henstockin mielessä integroituvien funktioiden avaruuden isometriset isomorfismit vastaavat kahteen suuntaan absoluuttisesti jatkuvia muuttujanvaihtoja.

1. Introduction

By a *cell* we mean a non-degenerate compact interval in $\mathbb{R}$. If $I \subset \mathbb{R}$ is a cell, then $KH(I)$ denotes the space of Kurzweil–Henstock integrable functions on I and $\mathbf{KH}(I)$ is the corresponding space of equivalence classes almost everywhere of members of $KH(I)$. We denote by f typical members of $KH(I)$, by $\boldsymbol{f}$ typical members of $\mathbf{KH}(I)$, and by $[f]$ the equivalence class of f. Thus, $\boldsymbol{f} = [f]$ whenever $f \in \boldsymbol{f}$. We equip $\mathbf{KH}(I)$ with Alexiewicz's norm (see [1]) defined by

$$\|\boldsymbol{f}\|_A = \max\left\{ \left| \int_{[\min I, x]} f \right| : \min I < x \leq \max I \right\}$$

whenever $f \in \boldsymbol{f} \in \mathbf{KH}(I)$. We study the isometries of this normed space. Specifically, we prove the following.

Theorem. *Let I and $\tilde{I}$ be two cells in $\mathbb{R}$ and $\mathbf{T} \colon \mathbf{KH}(\tilde{I}) \to \mathbf{KH}(I)$ a linear operator. The following are equivalent.*

(A) $\mathbf{T}$ *is a surjective isometry.*

(B) *There exist $\sigma \in \{-1, 1\}$ and an increasing homeomorphism $\phi \colon I \to \tilde{I}$ such that both ϕ and ϕ^{-1} are absolutely continuous and*

$$\mathbf{T}(\boldsymbol{f}) = [\sigma \cdot (f \circ \phi) \cdot \phi'],$$

for all $f \in \boldsymbol{f} \in \mathbf{KH}(I)$.

The proof that (B) implies (A) in section 3 boils down to establishing the proper change of variable theorem in this context (see also [9, Theorem 1] and [4, Theorem 7.3]). The proof of the reverse implication in section 4 relies on the Banach–Stone theorem at the level of indefinite integrals together with the appropriate absolute continuity property of indefinite Kurzweil–Henstock integrals. Recall that $\mathbf{KH}(I)[\|\cdot\|_A]$ is not a Banach space (though, it is barrelled but this will not be used here), therefore, a linear isometric operator may have a dense image without being surjective.

https://doi.org/10.54330/afm.180834
2020 Mathematics Subject Classification: Primary 26A39, 46B04, 26A46.
Key words: Kurzweil–Henstock integral, isometric isomorphism of normed spaces, change of variable.

This is in contrast with Banach–Lamperti's theorem (see [2, Ch. XI, §5, Espaces $L^{(p)}$, n°2, I] and [8]) asserting that the surjective linear isometries $\mathbf{T}\colon \mathbf{L}_1(\tilde{I}) \to \mathbf{L}_1(I)$ correspond to pairs (ϕ, h), where $\phi\colon I \to \tilde{I}$ is a bi-Lebesgue-measurable bijection and $h\colon I \to \mathbb{R}$ is Lebesgue-measurable such that $\mu = \phi_*\lambda \ll \lambda$ and $|h| = \frac{d\mu}{d\lambda} > 0$, by means of the formula $\mathbf{T}([f]) = [(f \circ \phi) \cdot h]$. Here, λ is the Lebesgue measure and $\frac{d\mu}{d\lambda}$ is a Radon–Nikodým derivative. Thus, the case of Kurzweil–Henstock integration corresponds, so to speak, to more regular changes of variable than the case of Lebesgue integration. Observe, however, that from a functional-analytic point of view the two problems have little in common, as $\mathbf{L}_1(I)$ is an $\mathscr{L}_1$-space whereas $\mathbf{KH}(I)$ is an $\mathscr{L}_\infty$-space (see e.g. [5, Appendix F] for the relevant definition).

Our result belongs to the general topic: If D and D' are domains in $\mathbb{R}^n$ and $F(D)$, $F(D')$ are a class of function spaces defined on D and D', respectively, then do isomorphisms $F(D) \cong F(D')$ correspond to a specific class of isomorphisms $D \cong D'$ by composition? Banach–Lamperti's theorem (regarding *isometric* isomorphisms) evoked in the previous paragraph is about the case $F(D) = L_1(D)$ and the corresponding "isomorphisms" $D \cong D'$ are the bi-Borel-measurable maps that preserve Lebesgue-null sets. The quasi-conformal, respectively bi-Lipschitz, homeomorphisms $D \cong D'$ are shown in [11] to correspond to isomorphisms of the Royden algebras $F(D) = M_p(D)$ where $p = n$, respectively $1 < p < n$ (these are isomorphisms of *algebras*). $M_p(D)$ consists of those bounded continuous functions defined on D which are absolutely continuous on almost every line parallel to the coordinate axes and whose p-Dirichlet energy is finite. Thus, more regularity of the isomorphisms $D \cong D'$ than in the Banach–Lamperti theorem is obtained from a space of more regular functions than merely Lebesgue-summable—note also the extra structure of an algebra of continuous functions borne by $M_p(D)$ which is key to inferring the existence of a map at all from D' to D from the existence of an isomorphism of algebras $M_p(D) \cong M_p(D')$. Here, we go in the opposite direction: The space of Kurzweil–Henstock integrable functions strictly contains the space of Lebesgue-summable functions on the same domain—every derivative is Kurzweil–Henstock integrable and the Kurzweil–Henstock integrability of f does not imply that of $|f|$. In fact, Kurzweil–Henstock integrable functions are to (conditionally) convergent series what Lebesgue-summable functions are to absolutely convergent series.

In an attempt to reach a large audience, we do not assume any prior knowledge of Kurzweil–Henstock integrability (the interested read may consult e.g. [3, 10, 12, 13, 14]). In particular, none of the results in the next section appear to be new but we have gathered and proved them in a fashion most appropriate for our applications. Of course, we assume familiarity with the theory of Lebesgue measure and integration in the real line.

It is a pleasure to acknowledge helpful discussions with Ph. Bouafia during the preparation of this paper.

2. Preliminaries

2.1. Vocabulary and notation. Two cells I_1 and I_2 are *non-overlapping* if $I_1 \cap I_2$ is either empty or a singleton. A *P-family* Π is a finite set of pairs (J, x), where J is a cell and $x \in J$, such that J_1 and J_2 are non-overlapping whenever $(J_1, x_1), (J_2, x_2) \in \Pi$ and $J_1 \neq J_2$. The *body* of a P-family Π is the set $\operatorname{body}(\Pi) = \cup\{J\colon (J, x) \in \Pi\}$. We say that Π is *in a cell* I if $\operatorname{body}(\Pi) \subset I$. A P-family Π is a *P-division* of a cell I if $\operatorname{body}(\Pi) = I$. The *Riemann sum* associated with a P-family

Π is

$$S(\Pi, f) = \sum_{(J,x)\in\Pi} f(x) \cdot |J|,$$

where $|J|$ is the length of J, i.e. $\max J - \min J$. A *gauge* on a set $E \subset \mathbb{R}$ is a positive function $\boldsymbol{\delta}\colon E \to \mathbb{R}^+ \setminus \{0\}$. If $\boldsymbol{\delta}$ is a gauge on a cell I and Π is a P-family in I, then we say that Π is $\boldsymbol{\delta}$*-fine* whenever $J \subset [x - \boldsymbol{\delta}(x), x + \boldsymbol{\delta}(x)]$ for all $(J, x) \in \Pi$.

2.2. *KH*-integral. Let I be a cell and $f\colon I \to \mathbb{R}$ a function. We say that f is *Kurzweil–Henstock integrable* in I if there exists a real number r satisfying the following property. For every $\varepsilon > 0$ there exists a gauge $\boldsymbol{\delta}$ on I such that $|S(\Pi, f) - r| < \varepsilon$ for all $\boldsymbol{\delta}$-fine P-divisions Π of I. In that case, r is unique and denoted $\int_I f$. The set $KH(I)$ consisting of all Kurzweil–Henstock integrable functions defined on I is a vector space and $KH(I) \to \mathbb{R}\colon f \mapsto \int_I f$ is linear. If $f \in KH(I)$ and $\hat{I} \subset I$ is a cell, then the restriction $f|_{\hat{I}} \in KH(\hat{I})$.

We now state what is known as the *Saks–Henstock* theorem (see e.g. [3, 5.4], [12, CH. 10, §8], or [14, Ch. 3, lemma 1]). If $f \in KH(I)$ and $\varepsilon > 0$, then there exists a gauge $\boldsymbol{\delta}$ on I such that

$$\sum_{(J,x)\in\Pi} \left| f(x) \cdot |J| - \int_J f|_J \right| < \varepsilon$$

for all $\boldsymbol{\delta}$-fine P-family Π in I.

2.3. Indefinite integral. If $f \in KH(I)$ we define a function $\int^{\bullet} f\colon I \to \mathbb{R}$ by means of the formula

$$\left(\int^{\bullet} f\right)(x) = \begin{cases} 0 & \text{if } x = \min I, \\ \int_{I_x} f|_{I_x} & \text{otherwise,} \end{cases}$$

where $I_x = [\min I, x]$. The function $\int^{\bullet} f$ is continuous, according to Hake's theorem (see e.g. [3, 5.6], [12, Ch. 11 §1], or [14, Ch. 3, Corollary 2]), and we call it the *indefinite integral* of f.

2.4. Additive functions. Let $\mathscr{C}(I)$ be the set of cells contained in a cell I. We say that $F\colon \mathscr{C}(I) \to \mathbb{R}$ is *additive* whenever the following holds. For all positive integer N and $J, J_1, \dots, J_N \in \mathscr{C}(I)$ if $J = \bigcup_{n=1}^N J_n$ and $J_1, \dots, J_N$ are non-overlapping, then $F(J) = \sum_{n=1}^N F(J_n)$.

If $F\colon I \to \mathbb{R}$, then $F^{\times}\colon \mathscr{C}(I) \to \mathbb{R}$ defined by $F^{\times}(J) = F(\max J) - F(\min J)$ is additive. If $F\colon \mathscr{C}(I) \to \mathbb{R}$ is additive, then we let ${}^{\times}F\colon I \to \mathbb{R}$ be defined by ${}^{\times}F(\min I) = 0$ and ${}^{\times}F(x) = F([\min I, x])$ whenever $\min I < x \leq \max I$. Note that[1] $({}^{\times}F)^{\times} = F$ and that ${}^{\times}(F^{\times}) = F$ if and only if $F(\min I) = 0$. This establishes a bijection from $\mathbb{R}^I \cap \{F\colon F(\min I) = 0\}$ to the set of additive functions $\mathscr{C}(I) \to \mathbb{R}$.

The "indefinite integral" $\left(\int^{\bullet} f\right)^{\times}$ of $f \in KH(I)$ is additive. The less pedantic choice not to distinguish between F and $F^{\times}$ leads to oddities such as "F is increasing if and only if F is positive".

2.5. Lebesgue measure. If $A \subset \mathbb{R}$ we let $|A|$ be its Lebesgue outer measure. When we say that A is *negligible* it always refer to Lebesgue measure, i.e. $|A| = 0$. The phrase "almost everywhere" always refers to Lebesgue-negligibility.

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[1]In this sentence, the first two occurrences of the symbol F and the last three refer to objects of different nature.

2.6. The condition $\boldsymbol{AC_*}$. A P-family Π in I is said to be *tagged* in a set $E \subset I$ if $x \in E$ for all $(J,x) \in \Pi$. We say that an additive function $F\colon \mathscr{C}(I) \to \mathbb{R}$ is $\boldsymbol{AC_*}$ if it satisfies the following property. For every negligible set $N \subset I$ and every $\varepsilon > 0$ there exists a gauge $\boldsymbol{\delta}$ on N such that

$$\sum_{(J,x)\in\Pi} |F(J)| < \varepsilon$$

for all $\boldsymbol{\delta}$-fine P-families in I tagged in N.

Theorem 2.7. *Let $F\colon \mathscr{C}(I) \to \mathbb{R}$ be an additive function. The following are equivalent.*

(A) *There exists $f \in KH(I)$ such that $F = \left(\int^\bullet f\right)^\times$.*
(B) *F is AC_* and ${}^\times F$ is differentiable almost everywhere.*

Furthermore, in case (B) holds, $({}^\times F)' \in KH(I)$ and ${}^\times F = \int^\bullet ({}^\times F)'$.

Proof that (A) $\Rightarrow$ (B). (i) Let $N \subset I$ be negligible and $\varepsilon > 0$. For each natural integer j define $N_j = N \cap \{j \le |f| < j+1\}$ and choose an open set U_j containing N_j and such that $\left|U_j\right| < \frac{\varepsilon}{(j+1)\cdot 2^{j+2}}$. Let $\boldsymbol{\eta}$ be a gauge on I associated to f and $\frac{\varepsilon}{2}$ by means of the Saks–Henstock theorem. Define a gauge $\boldsymbol{\delta}$ on N by setting $\boldsymbol{\delta}|_{N_j} = \min\{\frac{1}{2}\operatorname{dist}(\cdot, \mathbb{R}\setminus U_j), \boldsymbol{\eta}\}$ for all j.

Let Π be a $\boldsymbol{\delta}$-fine P-family in I tagged in N. Since Π is $\boldsymbol{\eta}$-fine, it follows from the Saks–Henstock theorem that

$$\sum_{(J,x)\in\Pi} |F(J)| < \frac{\varepsilon}{2} + \sum_{(J,x)\in\Pi} |f(x)|\cdot \left|J\right|.$$

Moreover,

$$\begin{aligned}\sum_{(J,x)\in\Pi} |f(x)|\cdot \left|J\right| &= \sum_{j=0}^{\infty} \sum_{\substack{(J,x)\in\Pi\\ x\in N_j}} |f(x)|\cdot \left|J\right| \le \sum_{j=0}^{\infty} (j+1)\cdot \sum_{\substack{(J,x)\in\Pi\\ x\in N_j}} \left|J\right| \\ &\le \sum_{j=0}^{\infty} (j+1)\cdot \left|U_j\right| < \frac{\varepsilon}{2},\end{aligned}$$

since $\operatorname{body}(\Pi_j) \subset U_j$, where $\Pi_j = \Pi \cap \{(J,x)\colon x \in N_j\}$, for all j, by the choice of $\boldsymbol{\delta}|_{N_j}$. This completes the proof that F is AC_*.

(ii) Let $G = \operatorname{int}(I) \cap \{x\colon {}^\times F \text{ is differentiable at } x \text{ and } ({}^\times F)'(x) = f(x)\}$ and $B = \operatorname{int}(I) \setminus G$. We shall show that B is negligible. To this end, we define

$$B_k = \operatorname{int}(I) \cap \left\{ x\colon \limsup_{\substack{h\to 0\\ h\neq 0}} \left| \frac{{}^\times F(x+h) - {}^\times F(x)}{h} - f(x) \right| > \frac{1}{k} \right\}$$

for each positive integer k and we notice that $B = \bigcup_k B_k$. Thus, it suffices to prove that each B_k is negligible.

Fix k. Given $\varepsilon > 0$, choose a gauge $\boldsymbol{\delta}$ on I associated with f and ε in Saks–Henstock's theorem. Define

$$(1)\quad \mathscr{V} = \mathscr{C}(I) \cap \left\{ J \colon \left| {}^{\times}F(J) - f(\min J) \cdot \lVert J \rVert \right| > \frac{\lVert J \rVert}{k} \text{ and } \lVert J \rVert \le \boldsymbol{\delta}(\min J) \text{ or } \right.$$
$$\left. \left| {}^{\times}F(J) - f(\max J) \cdot \lVert J \rVert \right| > \frac{\lVert J \rVert}{k} \text{ and } \lVert J \rVert \le \boldsymbol{\delta}(\max J) \right\}.$$

Observe that for all $x \in B_k$ there exists $J \in \mathscr{V}$ of which x is an endpoint. By the Vitali covering theorem, there are finitely many pairwise disjoint members of $\mathscr{V}$, say $J_1, \dots, J_N$, such that $\left\lVert B_k \setminus \bigcup_{n=1}^N J_n \right\rVert < \varepsilon$. For each n, let $x_n = \min J_n$ or $x_n = \max J_n$ according to whether the first or second condition in (1) witnesses membership of J_n to $\mathscr{V}$. Thus, the P-family $\{(J_n, x_n) \colon n = 1, \dots, N\}$ is $\boldsymbol{\delta}$-fine and we infer from the Saks–Henstock theorem that

$$\sum_{n=1}^N \lVert J_n \rVert \le k \cdot \sum_{n=1}^N \left| {}^{\times}F(J_n) - f(x_n) \cdot \lVert J_n \rVert \right| < k \cdot \varepsilon.$$

It ensues that $\lVert B_k \rVert \le \left\lVert B_k \setminus \bigcup_{n=1}^N J_n \right\rVert + \left\lVert \bigcup_{n=1}^N J_n \right\rVert < (1+k) \cdot \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Remark 2.8. The following comments are in order.

(1) In the above proof that (A) implies (B), we have shown, in fact, that if $f \in KH(I)$, then $\left(\int^{\bullet} f\right)'(x) = f(x)$ for almost every $x \in I$.

(2) If $f \in KH(I)$ and $\left(\int^{\bullet} f\right)(x) = 0$ for all $x \in I$, then $f(x) = 0$ for almost every $x \in I$. Indeed, under this assumption the function $x \mapsto \int_{\min I}^x f$ is differentiable everywhere and its derivative vanishes identically. The conclusion then ensues from (1).

(3) Arguing as in (i) above, one shows that if $f \colon I \to \mathbb{R}$ and $\{f \neq 0\}$ is negligible, then $f \in KH(I)$ and $\int_I f = 0$. In particular, if $f, g \colon I \to \mathbb{R}$ and $\{f \neq g\}$ is negligible, then $f \in KH(I)$ if and only if $g \in KH(I)$, in which case $\int_I f = \int_I g$. This clarifies the last sentence in the statement of 2.7 as well as the statement of our main theorem in the introduction (as $\left({}^{\times}F\right)'$, respectively ϕ', is defined merely almost everywhere).

Proof that (B) ⇒ (A). Let $N \subset I$ consist of the endpoints of I and of all points of $\operatorname{int}(I)$ at which ${}^{\times}F$ is not differentiable. Thus, N is negligible. Define $f \colon I \to \mathbb{R}$ by declaring that $f(x) = ({}^{\times}F)'(x)$ if $x \in I \setminus N$ and $f(x) = 0$ if $x \in N$. We will show that $f \in KH(I)$ and $\int_I f = F(I)$. Repeating this for all cells contained in I will complete the proof.

Fix $\varepsilon > 0$ and $x \in I \setminus N$. Since ${}^{\times}F$ is differentiable at x and its derivative is $f(x)$, one checks that there exists $\boldsymbol{\delta}_{\text{diff}}(x) > 0$ such that, for all $J \in \mathscr{C}(I)$, if[2] $x \in J$ and $\lVert J \rVert < \boldsymbol{\delta}_{\text{diff}}(x)$, then $\left| F(J) - f(x) \cdot \lVert J \rVert \right| < \varepsilon \cdot \lVert J \rVert$. Furthermore, there exists a gauge $\boldsymbol{\delta}_{\text{ac}}$ on N associated with F and ε is the definition of F being AC_*. We now

[2]The condition $x \in J$—in the definition of Kurzweil–Henstock integrability—is essential precisely here.

define a gauge $\boldsymbol{\delta}$ on I as follows:

$$\boldsymbol{\delta}(x) = \begin{cases} \boldsymbol{\delta}_{\mathrm{ac}}(x) & \text{if } x \in N, \\ \boldsymbol{\delta}_{\mathrm{diff}}(x) & \text{if } x \in I \setminus N. \end{cases}$$

If Π is any P-division of I then

$$F(I) = \sum_{(J,x)\in\Pi} F(J) = \sum_{\substack{(J,x)\in\Pi \\ x\notin N}} F(J) + \sum_{\substack{(J,x)\in\Pi \\ x\in N}} F(J),$$

$$S(\Pi, f) = \sum_{(J,x)\in\Pi} f(x)\cdot \|J\| = \sum_{\substack{(J,x)\in\Pi \\ x\notin N}} f(x)\cdot \|J\|.$$

Therefore,

$$\big|F(I) - S(\Pi, f)\big| \leq \sum_{\substack{(J,x)\in\Pi \\ x\notin N}} \big|F(J) - f(x)\cdot \|J\|\,\big| + \sum_{\substack{(J,x)\in\Pi \\ x\in N}} |F(J)|.$$

If, moreover, Π is $\boldsymbol{\delta}$-fine then

$$\big|F(I) - S(\Pi, f)\big| < \left(\sum_{\substack{(J,x)\in\Pi \\ x\notin N}} \varepsilon\cdot \|J\| \right) + \varepsilon \leq (\|I\| + 1)\cdot\varepsilon. \qquad \square$$

2.9. The condition *AC*. We say that an additive function $F\colon \mathscr{C}(I) \to \mathbb{R}$ is ***AC*** whenever the following condition is satisfied. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all positive integers N and all $J, J_1, \dots, J_N \in \mathscr{C}(I)$ if $J = \bigcup_{n=1}^N J_n$ and $J_1, \dots, J_N$ are pairwise non-overlapping and $\sum_{n=1}^N \|J_n\| < \delta$, then $\sum_{n=1}^N |F(J_n)| < \varepsilon$. This is the classical notion pertaining to measure theory.

(A) *If F is AC, then it is AC_*.*

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be as in the definition above. Let $N \subset I$ be negligible and choose an open set $U \subset \mathbb{R}$ containing N such that $\|U\| < \delta$. Define a gauge $\boldsymbol{\delta}$ on N to coincide with $\frac{1}{2}\operatorname{dist}(\cdot, \mathbb{R}\setminus U)$. If Π is a $\boldsymbol{\delta}$-fine P-family anchored in N, then $\operatorname{body}(\Pi) \subset U$. Therefore, $\sum_{(J,x)\in\Pi} \|J\| \leq \|U\| < \delta$ and, in turn, $\sum_{(J,x)\in\Pi} |F(J)| < \varepsilon$. ∎

(B) *The converse of (A) fails.*

Proof. This classical fact is witnessed for instance by $F^\times$ on $I = [0,1]$, where $F(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ if $0 < x \leq 1$ and $F(0) = 0$. ∎

The following is the 1-dimensional version of the Besicovitch covering theorem. For this and its higher dimensional analogues, see e.g. [7, Ch. I §1] or [13, 5.2.4] (the latter gives a proof leading to 14 instead of 5 subcollections but none of this matters for our subsequent application). By a *disjointed* collection of sets we mean that any two distinct members of it are disjoint.

Theorem 2.10. (A covering theorem) *Let $X \subset \mathbb{R}$ be bounded and let $\boldsymbol{\delta}$ be a bounded gauge on X. Abbreviate $J(x) = [x - \boldsymbol{\delta}(x), x + \boldsymbol{\delta}(x)]$. Then there exist five disjointed collections $\mathscr{J}_i^* \subset \{J(x)\colon x \in X\}$, $i = 1, \dots, 5$, such that $X \subset \cup_{i=1}^5 (\cup \mathscr{J}_i^*)$.*

2.11. Luzin's condition (N). We say that a function $g\colon I \to \mathbb{R}$ satisfies *Luzin's condition* (N) if $\left|g(Z)\right| = 0$ for all $Z \subset I$ such that $|Z| = 0$, i.e. g maps negligible sets to negligible sets.

The following is a special case of the Banach–Zarecki theorem (see e.g. [6, Ch. 7 §3]).

Theorem 2.12. *Let $F\colon \mathscr{C}(I) \to \mathbb{R}$ be additive and non-negative, i.e. $F(J) \geq 0$ for all $J \in \mathscr{C}(I)$. The following are equivalent.*

(A) *F is AC_*.*
(B) *${}^{\times}F$ is continuous and satisfies Luzin's condition (N).*
(C) *F is AC.*

Proof. (i) We make a preliminary comment about how we shall use the hypothesis that F be non-negative. This is equivalent to ${}^{\times}F$ being non-decreasing. Thus, for any cell $J = [a, b] \subset I$ we have ${}^{\times}F(J) = {}^{\times}F([a, b]) = \left[{}^{\times}F(a), {}^{\times}F(b)\right]$ so that

$$\left|{}^{\times}F(J)\right| = {}^{\times}F(b) - {}^{\times}F(a) = F([a, b]) = F(J). \tag{2}$$

Similarly, if $\hat{J} \subset \mathbb{R}$ is a cell, then $J := \left({}^{\times}F\right)^{-1}(\hat{J}) \subset I$ is a cell as well and, upon writing $J = [a, b]$, we have

$$F(J) = {}^{\times}F(b) - {}^{\times}F(a) \leq \left|\hat{J}\right|, \tag{3}$$

Since ${}^{\times}F(a), {}^{\times}F(b) \in \hat{J}$.

(ii) Here, we prove that (A) implies (B). The continuity of ${}^{\times}F$ readily follows from the definition of F being AC_*. Thus, given a negligible set $Z \subset I$ we ought to show that ${}^{\times}F(Z)$ is negligible as well. Given $\varepsilon > 0$, the AC_* property of F implies the existence of a gauge $\boldsymbol{\delta}$ on Z such that $\sum_{(J,x)\in\Pi} |F(J)| < \varepsilon$ whenever Π is a $\boldsymbol{\delta}$-fine P-family anchored in Z. The same holds when $\boldsymbol{\delta}$ is replaced with $\min\{\boldsymbol{\delta}, \operatorname{diam} I\}$. Accordingly, there is no restriction to assume that $\boldsymbol{\delta}$ is bounded. Since

$$Z \subset \bigcup_{x\in Z} [x - \boldsymbol{\delta}(x), x + \boldsymbol{\delta}(x)],$$

in view of theorem 2.10, we infer the existence of five disjointed collections of cells $\mathscr{J}_i^*$, $i = 1, \ldots, 5$, each member of which is of the type $J_x = I \cap [x - \boldsymbol{\delta}(x), x + \boldsymbol{\delta}(x)]$ for some $x \in Z$, such that $Z \subset \bigcup_{i=1}^5 \left(\bigcup \mathscr{J}_i^*\right)$. Given $i = 1, \ldots, 5$ and $\mathscr{F} \subset \mathscr{J}_i^*$ a finite subcollection, we define $\Pi_{\mathscr{F}} = \{(J, x(J))\colon J \in \mathscr{F}\}$, where $x(J)$ denotes the center of J. Note that $\Pi_{\mathscr{F}}$ is a $\boldsymbol{\delta}$-fine P-family anchored in Z, thus, $\sum_{J\in\mathscr{F}} |F(J)| = \sum_{(J,x)\in\Pi_{\mathscr{F}}} |F(J)| < \varepsilon$. As $\mathscr{F}$ is arbitrary, we infer that $\sum_{J\in\mathscr{J}_i^*} |F(J)| \leq \varepsilon$, $i = 1, \ldots, 5$.

Now,

$${}^{\times}F(Z) \subset {}^{\times}F\left(\bigcup_{i=1}^{5} \bigcup \mathscr{J}_i^*\right) \subset \bigcup_{i=1}^{5} \bigcup_{J\in\mathscr{J}_i^*} {}^{\times}F(J)$$

and, in view of (i) (specifically (2)), we conclude that

$$\left|{}^{\times}F(Z)\right| \leq \sum_{i=1}^{5} \sum_{J\in\mathscr{J}_i^*} \left|{}^{\times}F(J)\right| = \sum_{i=1}^{5} \sum_{J\in\mathscr{J}_i^*} |F(J)| \leq 5 \cdot \varepsilon. \tag{4}$$

Since ε is arbitrary, $\left|{}^{\times}F(Z)\right| = 0$.

(iii) We turn to showing that (B) implies (C). We briefly recall the construction of the Lebesgue–Stieltjes measure μ associated with ${}^{\times}F$. For an arbitrary $A \subset I$ we put

$$\mu^*(A) = \inf\left\{ \sum_{k=1}^{\infty} F(J_k) \colon \langle J_k \rangle_k \text{ is a sequence in } \mathscr{C}(I) \text{ and } A \subset \bigcup_{k=1}^{\infty} J_k \right\}.$$

Borel subsets of I are μ^*-measurable in the sense of Carathéodory and $\mu^*(J) = F(J)$ for all $J \in \mathscr{C}(I)$ (see e.g. [6, Theorem 3.20 conclusions 1 and 4]). In particular, the restriction μ of μ^* to the set of Borel subsets of I is a finite Borel measure on I. Notice also that $\mu(\{x\}) = 0$ for all $x \in I$, since ${}^{\times}F$ is continuous. In particular, if J_1 and J_2 are non-overlapping cells contained in I, then $\mu(J_1 \cup J_2) = \mu(J_1) + \mu(J_2)$.

We now show that μ is absolutely continuous with respect to the Lebesgue measure, i.e. if $Z \subset I$ is Borel and $|Z| = 0$, then $\mu(Z) = 0$. By hypothesis, $\left|{}^{\times}F(Z)\right| = 0$, i.e. given $\varepsilon > 0$ there exists a sequence $\langle \hat{J}_k \rangle_k$ of cells in $\mathbb{R}$ such that ${}^{\times}F(Z) \subset \bigcup_{k=1}^{\infty} \hat{J}_k$ and $\sum_{k=1}^{\infty} \left|\hat{J}_k\right| < \varepsilon$. For each k we consider the cell $J_k = \left({}^{\times}F\right)^{-1}(\hat{J}_k)$ and we notice that $Z \subset \left({}^{\times}F\right)^{-1}\left[{}^{\times}F(Z)\right] \subset \bigcup_{k=1}^{\infty} J_k$. Therefore,

$$\mu(Z) \le \sum_{k=1}^{\infty} F(J_k) \le \sum_{k=1}^{\infty} \left|\hat{J}_k\right| < \varepsilon,$$

by (i) (specifically (3)). Since ε is arbitrary, we have estbalished that $\mu(Z) = 0$.

Since μ is a *finite* Borel measure, it is now classical that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(B) < \varepsilon$ whenever $B \subset I$ is Borel and $|B| < \delta$. In particular, if $J_1, \dots, J_N$ are pairwise non-overlapping cells contained in I and $\left|\bigcup_{n=1}^{N} J_n\right| = \sum_{n=1}^{N} |J_n| < \delta$, then $\sum_{n=1}^{N} F(J_n) = \sum_{n=1}^{N} \mu(J_n) = \mu\left(\bigcup_{n=1}^{N} J_n\right) < \varepsilon$. This completes the proof that F is AC.

(iv) That (C) implies (A) is 2.9(A). □

3. Change of variable

Remark 3.1. Suppose that I and $\tilde{I}$ are cells and $\phi \colon I \to \tilde{I}$ is either increasing or decreasing and $\phi^{\times}$ is AC. Since ϕ is monotone, we infer that it is differentiable almost everywhere. Furthermore, either $\phi^{\times}$ or $-\phi^{\times}$ is non-negative. Therefore, ϕ satisfies Luzin's condition (N), by 2.12.

3.2. Let I and $\tilde{I}$ be cells and $\phi \colon I \to \tilde{I}$ a function. We say that ϕ is *bi-**AC*** if it satisfies the following properties:

(A) ϕ is a bijection;
(B) $\phi^{\times}$ is AC;
(C) $(\phi^{-1})^{\times}$ is AC.

Since ϕ is AC, it is continuous as well and, being a bijection, it is either increasing or decreasing and is a homeomorphism.

Theorem 3.3. *Let I and $\tilde{I}$ be two cells and let $\phi \colon I \to \tilde{I}$ be bi-AC. If $f \in KH(\tilde{I})$, then $(f \circ \phi) \cdot |\phi'| \in KH(I)$ and*

$$\int_I (f \circ \phi) \cdot |\phi'| = \int_{\tilde{I}} f,$$

$$I \xrightarrow{\ \phi\ } \tilde{I} \xrightarrow{\ f\ } \mathbb{R}.$$

Proof. (i) We abbreviate $G = \left({}^{\times}F \circ \phi\right)^{\times}$, where $F = \left(\int^{\bullet} f\right)^{\times}$. Thus, G is an additive function defined on $\mathscr{C}(I)$ and $G(J) = \operatorname{sign}(\phi') \cdot F(\phi(J))$ for all $J \in \mathscr{C}(I)$, where

$$\operatorname{sign}(\phi') = \begin{cases} +1 & \text{if } \phi \text{ is increasing,} \\ -1 & \text{if } \phi \text{ is decreasing.} \end{cases}$$

In the course of this proof we shall establish that G is the indefinite integral of $(f \circ \phi) \cdot \phi'$.

(ii) Here, we shall show that G is AC_*. First, we notice that ϕ is uniformly continuous on I, therefore, there exists $\boldsymbol{\eta}\colon\ ]0,\infty[\to]0,\infty[$ with the following property. For every $\delta > 0$ and every $x, y \in I$ if $|x-y| \le \boldsymbol{\eta}(\delta)$, then $|\phi(x) - \phi(y)| \le \delta$.

Let $Z \subset I$ be negligible and $\varepsilon > 0$. According to theorem 2.12, ϕ satisfies Luzin's condition (N), therefore, $\phi(Z)$ is negligible. As F is AC_*, by 2.7, there exists a gauge $\tilde{\boldsymbol{\delta}}$ on $\phi(Z)$ such that if $\tilde{\Pi}$ is a $\tilde{\boldsymbol{\delta}}$-fine P-family in $\tilde{I}$ anchored in $\phi(Z)$, then $\sum_{(\tilde{J},\tilde{x})\in\tilde{\Pi}} |F(\tilde{J})| < \varepsilon$. Define a gauge $\boldsymbol{\delta}$ on Z by letting $\boldsymbol{\delta} = \boldsymbol{\eta} \circ \tilde{\boldsymbol{\delta}} \circ \phi$. Now, let Π be a $\boldsymbol{\delta}$-fine P-family in I anchored in Z. Define $\tilde{\Pi} = \{(\phi(J), \phi(x))\colon (J,x) \in \Pi\}$ and note that this is a P-family in $\tilde{I}$, since ϕ is increasing and continuous. Moreover, $\tilde{\Pi}$ is readily anchored in $\phi(Z)$ and one happily checks that it is $\tilde{\boldsymbol{\delta}}$-fine as well, according to the definition of $\boldsymbol{\eta}$. Thus, $\sum_{(J,x)\in\Pi} |G(J)| = \sum_{(J,x)\in\Pi} |F[\phi(J)]| = \sum_{(\tilde{J},\tilde{x})\in\tilde{\Pi}} |F(\tilde{J})| < \varepsilon$. This completes the proof that G is AC_*.

(iii) Here, we shall observe that for almost every $x \in I$, ϕ is differentiable at x and ${}^{\times}F$ is differentiable at $\phi(x)$ with $\left({}^{\times}F\right)'(\phi(x)) = f(\phi(x))$, in particular, ${}^{\times}G$ is differentiable at x and $\left({}^{\times}G\right)'(x) = f(\phi(x)) \cdot \phi'(x)$. Since $\phi^{\times}$ is AC, one infers that the set $Z = I \cap \{x\colon \phi \text{ is not differentiable at } x\}$ is negligible (recall 3.1). Moreover, it follows from 2.7 (see 2.8(1)) that the set $\tilde{Z} = \tilde{I} \cap \{\tilde{x}\colon {}^{\times}F \text{ is not differentiable at } x \text{ or it is differentiable at } x \text{ but } \left({}^{\times}F\right)'(x) \ne f(x)\}$ is negligible as well. Accordingly, $\phi^{-1}(\tilde{Z})$ is negligible, since ϕ^{-1} satisfies Luzin's (N) condition, by 3.1. If $x \notin Z \cup \phi^{-1}(\tilde{Z})$, then it satisfies the claimed properties.

(iv) The conclusion of the theorem is a consequence of the conjunction of (ii), (iii), and 2.7:

$$\begin{aligned} \int_{\tilde{I}} f &= F(\tilde{I}) && \text{(by definition of } F) \\ &= F(\phi(I)) && \text{(since } \phi \text{ is a bijection)} \\ &= \operatorname{sign}(\phi') \cdot G(I) && \text{(by (i))} \\ &= \operatorname{sign}(\phi') \cdot \int_I (f \circ \phi) \cdot \phi' && \text{(by 2.7, (ii), and (iii)).} \end{aligned}$$ □

4. Isometries

Theorem 4.1. *If $\phi\colon I \to \tilde{I}$ is bi-AC, then $\{\phi' = 0\}$ is negligible.*

Proof. Of course, the set considered here is

$$A = \operatorname{int}(I) \cap \{x\colon \phi \text{ is differentiable at } x \text{ and } \phi'(x) = 0\}.$$

We observe that

$$A \subset \phi^{-1}\big(\{y\colon \phi^{-1} \text{ is not differentiable at } y\}\big). \tag{5}$$

Indeed, if $x \in A$ and $y = \phi(x)$, then the limit of

$$\frac{\phi^{-1}(\phi(x+h)) - \phi^{-1}(y)}{\phi(x+h) - y} = \frac{h}{\phi(x+h) - \phi(x)}$$

as $h \to 0$ does not exist. As $\phi(x+h) \to \phi(x)$ when $h \to 0$, this implies that ϕ^{-1} is not differentiable at y.

Since ϕ^{-1} is differentiable almost everywhere and satisfies Luzin's condition, (5) implies that $|A| = 0$. $\square$

4.2. Given a cell I we consider the vector space $KH(I)$ defined in 2.2 and its vector subspace $W(I) = KH(I) \cap \{f \colon \{f \neq 0\} \text{ is negligible}\}$. We let $\mathbf{KH}(I)$ denote the quotient space $KH(I)/W(I)$. For $\min I < x \leq \max I$ we abbreviate $I_x = [\min I, x]$. If $f \in \boldsymbol{f} \in \mathbf{KH}(I)$ then, by virtue of 2.8(3), the quantity

$$\|\boldsymbol{f}\|_A = \sup\left\{\left|\int_{I_x} f|_{I_x}\right| : \min I < x \leq \max I\right\}$$

is independent of the choice of $f \in \boldsymbol{f}$. One readily checks that $\boldsymbol{f} \mapsto \|\boldsymbol{f}\|_A$ is a seminorm on $\mathbf{KH}(I)$. In view of 2.8(2), it is, in fact, a norm. We classically call it the *Alexiewicz norm*, see [1].

The following is a proof of the implication $(B) \Rightarrow (A)$ of our main theorem.

Theorem 4.3. *Let I and $\tilde{I}$ be two cells, $\sigma \in \{-1, 1\}$, and $\phi \colon I \to \tilde{I}$ an increasing bi-AC function. Then there exists a surjective linear isometry $\mathbf{T}_\phi \colon \mathbf{KH}(\tilde{I}) \to \mathbf{KH}(I)$ such that $\mathbf{T}_\phi(\boldsymbol{f}) = [\sigma \cdot (f \circ \phi) \cdot \phi']$ for all $\boldsymbol{f} \in \mathbf{KH}(\tilde{I})$, where $f \in \boldsymbol{f}$.*

Proof. (i) First, we note that ϕ' is merely defined almost everywhere (recall 3.1) so that, what we really mean for the definition of $\mathbf{T}_\phi(\boldsymbol{f})$ is that $\mathbf{T}_\phi(\boldsymbol{f}) = [T_\phi(f)]$, where

$$T_\phi(f) \colon I \to \mathbb{R} \colon x \mapsto \begin{cases} \sigma \cdot f(\phi(x)) \cdot \phi'(x) & \text{if } \phi \text{ is differentiable at } x, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $T_\phi(f) \in KH(I)$, in view of 3.3.

(ii) Here, we show that $[T_\phi(f)]$ does not depend upon the choice of $f \in \boldsymbol{f}$. Indeed, if $\tilde{I} \cap \{f_1 \neq f_2\}$ is negligible, then so is $I \cap \{f_1 \circ \phi \neq f_2 \circ \phi\} = \phi^{-1}(\tilde{I} \cap \{f_1 \neq f_2\})$, since ϕ^{-1} satisfies Luzin's condition (N) (recall 3.1). Thus, $\mathbf{T}_\phi$ is well-defined.

(iii) One routinely checks that $\mathbf{T}_\phi$ is linear.

(iv) Here, we show that $\mathbf{T}_\phi$ is an isometry. Let $f \in \boldsymbol{f} \in \mathbf{KH}(\tilde{I})$ and $\min I < x \leq \max I$. Applying 3.3 to I_x, $\tilde{I}_{\phi(x)}$, $\phi|_{I_x}$, and $f|_{\tilde{I}_{\phi(x)}}$ (notice carefully that $\phi(I_x) = \tilde{I}_{\phi(x)}$, since ϕ is increasing) we infer that

$$\int_{\tilde{I}_{\phi(x)}} f|_{\tilde{I}_{\phi(x)}} = \int_{I_x} \left(f|_{\tilde{I}_{\phi(x)}} \circ \phi|_{I_x}\right) \cdot (\phi|_{I_x})' = \sigma \int_{I_x} T_\phi(f)|_{I_x}.$$

Accordingly,

$$\left|\int_{\tilde{I}_{\phi(x)}} f|_{\tilde{I}_{\phi(x)}}\right| = \left|\int_{I_x} T_\phi(f)|_{I_x}\right|$$

for all $\min I < x \le \max I$. Since ϕ is a bijection, we conclude that

$$\begin{aligned}\|\mathbf{T}_\phi(\boldsymbol{f})\|_A &= \sup\left\{\left|\int_{I_x} T_\phi(f)|_{I_x}\right| : \min I < x \le \max I\right\}\\ &= \sup\left\{\left|\int_{\tilde{I}_{\phi(x)}} f|_{\tilde{I}_{\phi(x)}}\right| : \min I < x \le \max I\right\}\\ &= \sup\left\{\left|\int_{\tilde{I}_{\tilde{x}}} f|_{\tilde{I}_{\tilde{x}}}\right| : \min \tilde{I} < \tilde{x} \le \max \tilde{I}\right\} = \|\boldsymbol{f}\|_A.\end{aligned}$$

(v) It remains to show that $\mathbf{T}_\phi$ is surjective. To this end, we first notice that the above construction applies to defining $T_{\phi^{-1}}\colon KH(I) \to KH(\tilde{I})$. Given $g \in \boldsymbol{g} \in \mathbf{KH}(I)$, we note the following identities almost everywhere:

$$T_{\phi^{-1}}(g) = \sigma \cdot (g \circ \phi^{-1}) \cdot (\phi^{-1})' = \sigma \cdot (g \circ \phi^{-1}) \cdot \left(\frac{1}{\phi' \circ \phi^{-1}}\right),$$

where the last identity almost everywhere is a consequence of 4.1. Thus, still up to a negligible set,

$$T_\phi[T_{\phi^{-1}}(g)] = \sigma^2 \cdot \big([(g \circ \phi^{-1})] \circ \phi\big) \cdot [(\phi')^{-1}] \circ \phi \cdot \phi' = g.$$

This completes the proof. □

4.4. Banach–Stone theorem. We let I be a cell and $C(I)$ consist of all continuous functions $F\colon I \to \mathbb{R}$. We also let $C_0(I)$ be its subspace consisting of those F such that $F(\min I) = 0$. Equipped with the maximum norm $\|\cdot\|_\infty$, the former is a Banach space and the latter its closed subspace. We recall Banach's theorem [2, Ch. XI, §4, Théorème 3 et Remarque] that if $S\colon C(\tilde{I}) \to C(I)$ is a linear surjective isometry, then there exist $\sigma \in \{-1, 1\}$ and a homeomorphism $\phi\colon I \to \tilde{I}$ such that $S(F) = \sigma \cdot (F \circ \phi)$. Note that, ibid., $\sigma\colon I \to \mathbb{R}$ is continuous and $|\sigma| = 1$, therefore, constant. Our aim, here, is to establish the following variant that will be used in the proof of 4.6.

If $T\colon C_0(\tilde{I}) \to C_0(I)$ is a linear surjective isometry, then there exist $\sigma \in \{-1, 1\}$ and a homeomorphism $\phi\colon I \to \tilde{I}$ such that

(A) $T(F) = \sigma \cdot (F \circ \phi)$ *for all* $F \in C_0(\tilde{I})$;

(B) $\phi(\min I) = \min \tilde{I}$.

Proof. (i) Either $T(F) \ge 0$ for all $F \ge 0$ or $T(F) \le 0$ for all $F \ge 0$. For if not there would exist $F \ge 0$ and $G \ge 0$ such that $T(F) > 0$ and $T(G) < 0$. For all $t > 0$, $\|T(F + t \cdot G)\|_\infty = \|F + t \cdot G\|_\infty \ge \|F\|_\infty$ and, for $t > 0$ sufficiently small, $\|T(F + t \cdot G)\|_\infty = \|T(F) + t \cdot T(G)\|_\infty < \|T(F)\|_\infty = \|F\|_\infty$. This contradiction proves the claim.

(ii) Upon observing that if the conclusions hold for $-T$ they also hold for T (replacing σ by $-\sigma$), we may henceforth assume that $T(F) \ge 0$ whenever $F \ge 0$.

(iii) $T(F)(I) = F(\tilde{I})$ for all F. To prove this, notice that $F^+, F^- \in C_0(\tilde{I})$, $T(F^+) \ge 0$, thus, $T(F^+)(I) = [0, \|T(F^+)\|_\infty] = [0, \|F^+\|_\infty]$, and $T(-F^-) \le 0$, therefore, $T(-F^+) = [-\|T(-F^-)\|_\infty, 0] = [-\|F^-\|_\infty, 0]$. We infer that[3]

$$T(F)(I) = [T(F^+) + T(-F^-)](I) = [-\|F^-\|_\infty, \|F^+\|_\infty] = F(I)$$

and the proof of the claim is complete.

[3] If $a \le s \le 0 \le t \le b$, then $a \le s + t \le b$.

(iv) We notice that $C(I) \cong C_0(I) \oplus \mathbb{R}$ via $F \mapsto (F - F(0)) \oplus F(0)$ which allows for a linear extension $S : C(\tilde{I}) \to C(I)$ of T defined by $S(F \oplus r) = T(F) \oplus r$. As T is a bijection, it easily follows that S is one, too.

(v) Referring to (iii) we observe that $S(F \oplus r)(I) = [T(F) \oplus r](I) = T(F)(I) + r = F(I) + r$. Accordingly, $\|S(F \oplus r)\|_\infty = \|F \oplus r\|_\infty$. This proves that S is a linear surjective isometry and conclusion (A) follows from Banach's theorem quoted above.

(vi) Abbreviate $\tilde{a} = \phi(\min I)$. Then $0 = T(F)(\min I) = \sigma \cdot F(\tilde{a})$ for all $F \in C_0(\tilde{I})$. This readily implies that $\tilde{a} = \min \tilde{I}$. ■

4.5. Indefinite integral, again. Recall 2.3. By virtue of 2.8(3), we may similarly define

$$\int_I^\bullet : \mathbf{KH}(I) \to C_0(I)$$

by means of the formula

$$\left(\int_I^\bullet \right) (\boldsymbol{f})(x) = \int_{I_x} f,$$

$f \in \boldsymbol{f}$. One checks that this is a linear map and an isometry, by the definition of the Alexiewciz norm, i.e.

$$\left\| \left(\int_I^\bullet \right) (\boldsymbol{f}) \right\|_\infty = \|\boldsymbol{f}\|_A .$$

We now argue that the image of $\int_I^\bullet$ is dense in $C_0(I)$. Indeed, the set of those functions $F : I \to \mathbb{R}$ of the type $F = \hat{F}|_I$ for some $\hat{F} : \mathbb{R} \to \mathbb{R}$ of class C^1 such that $\hat{F}(\min I) = 0$ is dense. For each such F, $F = \left(\int_I^\bullet\right)([\hat{F}'|_I])$, by the fundamental theorem of calculus.

The following is a proof that $(A) \Rightarrow (B)$ in our main theorem.

Theorem 4.6. *Let $\mathbf{T} : \mathbf{KH}(\tilde{I}) \to \mathbf{KH}(I)$ be a surjective linear isometry. Then there exists $\sigma \in \{-1, 1\}$ and an increasing bi-AC function $\phi : I \to \tilde{I}$ such that $\mathbf{T}(\boldsymbol{f}) = \mathbf{T}_\phi(\boldsymbol{f})$ for all $\boldsymbol{f} \in \mathbf{KH}(\tilde{I})$, where $\mathbf{T}_\phi$ is as in 4.3.*

Proof. (i) We shall define a surjective linear isometry $T^\bullet : C_0(\tilde{I}) \to C_0(I)$ in order that the following diagram be commutative.

$$\begin{array}{ccc} \mathbf{KH}(\tilde{I}) & \xrightarrow{\mathbf{T}} & \mathbf{KH}(I) \\ {\scriptstyle \int_{\tilde{I}}^\bullet}\big\downarrow & & {\scriptstyle \int_I^\bullet}\big\downarrow \\ C_0(\tilde{I}) & \xrightarrow{T^\bullet} & C_0(I) \end{array}$$

Since $\int_{\tilde{I}}^\bullet$ is injective, there is a most obvious unique way to define $T^\bullet$ on the image of $\int_{\tilde{I}}^\bullet$ so that the diagram commutes. In this way, $T^\bullet$ restricted to the image of $\int_{\tilde{I}}^\bullet$ is linear (because so are the other three arrows) and an isometry (because so are the other three arrows).

It remains to show that $T^\bullet$ is surjective. Let $G \in C_0(I)$. There is a sequence $\langle \boldsymbol{g}_n \rangle_n$ in $\mathbf{KH}(I)$ such that $\left\| G - \left(\int_I^\bullet\right)(\boldsymbol{g}_n) \right\|_\infty \to 0$ as $n \to \infty$, recall 4.5. Since $\mathbf{T}$ is surjective, there is a sequence $\langle \boldsymbol{f}_n \rangle_n$ in $\mathbf{KH}(\tilde{I})$ such that $\mathbf{T}(\boldsymbol{f}_n) = \boldsymbol{g}_n$ for all n. Abbreviate $F_n = \left(\int_{\tilde{I}}^\bullet\right)(\boldsymbol{f}_n)$. Since $\mathbf{T}$ and both indefinite integrals involved in the above commutative diagram are isometries, we infer that $\langle F_n \rangle_n$ is a Cauchy sequence in $C_0(\tilde{I})$ and, therefore, converges to some $F \in C_0(\tilde{I})$. Thus $T^\bullet(F) = \lim_n T^\bullet(F_n)$

and we are left to note that $G = \lim_n T^\bullet(F_n)$ as well:

$$\begin{aligned}\lim_n \|G - T^\bullet(F_n)\|_\infty &= \lim_n \left\| \left(\int_I^\bullet\right)(\boldsymbol{g}_n) - \left(T^\bullet \circ \int_{\tilde{I}}^\bullet\right)(\boldsymbol{f}_n)\right\|_\infty \\ &= \lim_n \left\| \left(\int_I^\bullet\right)(\boldsymbol{g}_n) - \left(\int_{\tilde{I}}^\bullet\right)(\mathbf{T}(\boldsymbol{f}_n))\right\|_\infty \\ &= \lim_n \|\boldsymbol{g}_n - \mathbf{T}(\boldsymbol{f}_n)\|_A = 0.\end{aligned}$$

(ii) It follows from (i) and the Banach–Stone theorem (recall 4.4) that there exist $\sigma \in \{-1, 1\}$ and an increasing homeomorphism $\phi\colon I \to \tilde{I}$ such that $T^\bullet(F) = \sigma\cdot(F\circ\phi)$ for all $F \in C_0(\tilde{I})$. Furthermore, if $F \in C_0(\tilde{I})$ is in the image of $\int_{\tilde{I}}^\bullet$, then $T^\bullet(F)$ is in the image of $\int_I^\bullet$, by the commutativity of our diagram. In other words, if $F^\times$ is AC_* and F is differentiable almost everywhere, then $(F \circ \phi)^\times$ is AC_* (and $F \circ \phi$ is differentiable almost everywhere, though, we shall not use this here), by 2.7. We apply this observation to $F(\tilde{x}) = \tilde{x} - \min \tilde{I}$ (which is readily the indefinite integral of the constant function equal to 1 on $\tilde{I}$). It follows that $(\phi - \min \tilde{I})^\times$ is AC_* and, hence, also $\phi^\times$ itself. As $\phi^\times$ is non-negative (because ϕ is increasing), we infer from 2.12 that $\phi^\times$ is AC.

(iii) We note that $(T^\bullet)^{-1}(G) = \sigma \cdot (G \circ \phi^{-1})$ for all $G \in C_0(I)$. Next, we apply the same reasoning as in (i) and (ii), this time to $\mathbf{T}^{-1}$ instead of $\mathbf{T}$. We infer that there exist $\tau \in \{-1, 1\}$ and an increasing homeomorphism $\psi\colon \tilde{I} \to I$ such that $\psi^\times$ is AC and such that $(T^{-1})^\bullet(G) = \tau \cdot (G \circ \psi)$ for all $G \in C_0(I)$. If $G = \left(\int_I^\bullet\right)(\boldsymbol{g})$ for some $\boldsymbol{g} \in \mathbf{KH}(I)$ then

$$(T^{-1})^\bullet(G) = (T^{-1})^\bullet \left(\int_I^\bullet\right)(\boldsymbol{g}) = \left(\int_{\tilde{I}}^\bullet\right)(\mathbf{T}^{-1}(\boldsymbol{g}))$$

and, therefore,

$$T^\bullet[(T^{-1})^\bullet(G)] = \left(T^\bullet \circ \int_{\tilde{I}}^\bullet\right)(\mathbf{T}^{-1}(\boldsymbol{g})) = \left(\int_I^\bullet \circ \mathbf{T}\right)(\mathbf{T}^{-1}(\boldsymbol{g})) = \left(\int_I^\bullet\right)(\boldsymbol{g}) = G.$$

Applying $(T^\bullet)^{-1}$ on both sides of the previous equation, we see that $(T^{-1})^\bullet = (T^\bullet)^{-1}$ on the image of $\int_I^\bullet$. Choosing $G(x) = x - \min I$, $x \in I$, we infer that $\tau \cdot (\psi(\tilde{x}) - \min I) = \sigma \cdot (\phi^{-1}(\tilde{x}) - \min I)$ for all $\tilde{x} \in \tilde{I}$. Letting $\tilde{x} = \max \tilde{I}$ we obtain that $\sigma = \tau$. Accordingly, $\psi = \phi^{-1}$, therefore, $(\phi^{-1})^\times$ is AC, which concludes the proof that ϕ is bi-AC.

(iv) Finally, we ought to show that $\mathbf{T} = \mathbf{T}_\phi$. Let $\boldsymbol{f} \in \mathbf{KH}(\tilde{I})$ and $F = \left(\int_{\tilde{I}}^\bullet\right)(\boldsymbol{f})$. Then $T^\bullet(F) = \sigma\cdot(F\circ\phi)$, by (ii), $F' = f$ almost everywhere, by 2.7 (recall 2.8(1)), and reasoning as in the proof of 3.3(iii) we have $(F \circ \phi)' = (f \circ \phi) \cdot \phi'$ almost everywhere (since ϕ is bi-AC). It then follows from 2.7 that

$$T^\bullet(F) = \left(\int_I^\bullet\right)(\sigma \cdot [(F \circ \phi)']) = \left(\int_I^\bullet\right)([\sigma \cdot (f \circ \phi) \cdot \phi']).$$

Since $T^\bullet(F) = \left(\int_I^\bullet\right)(\mathbf{T}(\boldsymbol{f}))$ as well, we conclude from the injectivity of $\int_I^\bullet$ that $\mathbf{T}(\boldsymbol{f}) = [\sigma \cdot (f \circ \phi) \cdot \phi']$. The proof is complete. □

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Received 8 May 2025 • Revision received 27 May 2025 • Accepted 2 March 2026
Published online 18 March 2026

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