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The weakly ∞-compact approximation property and covering properties of weakly null sequences

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Abstract. We introduce the property $(\mathcal{W}_\infty)$ and the weakly ∞-compact approximation property (WICAP) of a Banach space X. We establish a characterization of the property $(\mathcal{W}_\infty)$ and relationships of the property $(\mathcal{W}_\infty)$, the approximate identities for the algebra $\mathcal{W}_\infty(X)$ and the WICAP. As a consequence, we obtain that both $\ell_p(1 < p < \infty)$ and c_0 fail property $(\mathcal{W}_\infty)$. It is also shown that the WICAP is strictly stronger than the weakly compact approximation property.

Heikosti ∞-kompakti likiarvo-ominaisuus ja heikkojen nollajonojen peiteominaisuudet

Tiivistelmä. Tässä työssä määritellään Banachin avaruuden X ominaisuus $(\mathcal{W}_\infty)$ ja heikosti ∞-kompakti likiarvo-ominaisuus (WICAP). Työssä esitellään $(\mathcal{W}_\infty)$-ominaisuuden yhtäpitävä muotoilu ja selvitetään sen suhde algebran $\mathcal{W}_\infty(X)$ yksikön likiarvoihin sekä WICAP-ominaisuuteen. Seurauksena nähdään, että mikään avaruuksista $\ell_p(1 < p < \infty)$ tai c_0 ei toteuta $(\mathcal{W}_\infty)$-ominaisuutta. Lisäksi osoitetaan, että WICAP on aidosti vahvempi kuin heikosti kompakti likiarvo-ominaisuus.

1. Introduction

A Banach space X is said to have *property* $(\mathcal{K})$ if for every compact subset (or null sequence) K of X, there exists a $T \in \mathcal{K}(X)$, the space of compact operators from X to X, such that

$$K \subset \overline{T(B_X)},$$

where B_X is the closed unit ball of X. It is an old open problem whether every Banach space has property $(\mathcal{K})$ (cf. [CJ, Problem 4(a)] and [D, Problem 4.4]). Let $\lambda \geq 1$. We say that X has the *compact approximation property* (CAP) (respectively, λ-bounded CAP) if for every compact subset K of X and every $\varepsilon > 0$, there exists an $S \in \mathcal{K}(X)$ (respectively, $S \in \mathcal{K}(X)$ with $\|S\| \leq \lambda$) such that $\sup_{x\in K} \|Sx - x\| \leq \varepsilon$. Grothendieck [G] proved that a subset K of X is relatively compact if and only if there exists $(x_n)_n \in c_0(X)$, the space of all null sequences in X, such that K is contained in the balanced and closed convex hull of $\{x_n\}_{n\in\mathbb{N}}$. Consequently, X has the CAP if (and only if) for every $(x_n)_n \in c_0(X)$ and every $\varepsilon > 0$, there exists an $S \in \mathcal{K}(X)$ such that

$$\sup_{n\in\mathbb{N}} \|Sx_n - x_n\| \leq \varepsilon.$$

Dixon [D, Theorem 4.3] proved that if X has the bounded CAP, then X has property $(\mathcal{K})$. Consequently, most classical Banach spaces have property $(\mathcal{K})$.

Sinha and Karn [SK] were motivated by Grothendieck's criterion of norm compactness to introduce a new compactness. A subset W of X is called *weakly ∞-compact* if there exists $(x_n)_n \in c_0^w(X)$, the space of all weakly null sequences in X,

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such that W is contained in the balanced and closed convex hull of $\{x_n\}_{n\in\mathbb{N}}$. According to Mazur's theorem, a weakly ∞-compact set is weakly compact. On the other hand, if every weakly compact subset W of X is weakly ∞-compact, then X has the Schur property (see [DFLORT]). An operator T from X to Y is also called weakly ∞-compact if $T(B_X)$ is a weakly ∞-compact subset of Y. We denote by $\mathcal{W}_\infty(X)$ the normed algebra equipped with the operator norm of weakly ∞-compact operators from X to X. Johnson, Lillemets and Oja [JLO] studied the relationship among $\mathcal{V}, \mathcal{W}_\infty$ and $\mathcal{W}$ and proved the identity $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. Here $\mathcal{V}$ is the ideal of completely continuous operators and $\mathcal{W}$ is the ideal of weakly compact operators. Recall that the right-hand quotient $\mathcal{A} \circ \mathcal{B}^{-1}$ of two operator ideals $\mathcal{A}$ and $\mathcal{B}$ is the operator ideal that consists of all operators $T \in \mathcal{L}(X,Y)$ such that $TS \in \mathcal{A}(X,Y)$ whenever $S \in \mathcal{B}(Z,X)$ for some Banach space Z. The result of Johnson, Lillemets and Oja plays an important role in our paper.

As in the CAP, if for every $(x_n)_n \in c_0^w(X)$ and every $\varepsilon > 0$, there exists an $S \in \mathcal{W}_\infty(X)$ (respectively, $S \in \mathcal{W}_\infty(X)$ with $\|S\| \le \lambda$) such that $\sup_{n\in\mathbb{N}} \|Sx_n - x_n\| \le \varepsilon$, then we say that X has the *weakly ∞-compact approximation property* (WICAP) (respectively, λ-bounded WICAP). We say that X has *property* $(\mathcal{W}_\infty)$ if for every $(x_n)_n \in c_0^w(X)$, there exists a $T \in \mathcal{W}_\infty(X)$ such that $\{x_n\}_{n\in\mathbb{N}} \subset \overline{T(B_X)}$. We will show that if X has the bounded WICAP, then X has property $(\mathcal{W}_\infty)$ and there exist Banach spaces failing to have property $(\mathcal{W}_\infty)$.

A normed algebra $\mathcal{A}$ is said to have the *left approximate identity* (LAI) (respectively, λ-bounded LAI) if for every $T_1, \ldots, T_m \in \mathcal{A}$ and every $\varepsilon > 0$, there exists an $S \in \mathcal{A}$ (respectively, $S \in \mathcal{A}$ with $\|S\|_\mathcal{A} \le \lambda$) such that

$$\|ST_i - T_i\|_\mathcal{A} \le \varepsilon \quad (i = 1, \ldots, m).$$

If X has the CAP, then the algebra $\mathcal{K}(X)$ has the LAI. But it is not known whether the converse is true. We also see that if X has the WICAP, then the algebra $\mathcal{W}_\infty(X)$ has the LAI (see Proposition 3.5). We will show that the converse does not hold in general.

The purpose of this paper is to initiate the study of the relationship of property $(\mathcal{W}_\infty)$, the approximate identities for the algebra $\mathcal{W}_\infty(X)$ and the weakly ∞-compact approximation properties. The established relations will be used to show that $\ell_p(1 < p < \infty)$ does not have property $(\mathcal{W}_\infty)$ and c_0 fails property $(\mathcal{W}_0)$. One may refer to [C] for various approximation properties and we refer to [DW] for various informations and results of approximate identites. Throughout this paper, Banach spaces will be denoted by X and Y over $\mathbb{R}$ or $\mathbb{C}$, with dual spaces X^* and Y^* and the closed unit ball of X will be denoted by B_X. The map $i_X\colon X \to X^{**}$ is the canonical isometry defined by $i_X(x)(x^*) = x^*(x)$ for every $x \in X$ and $x^* \in X^*$.

2. The properties $(\mathcal{W})$, $(\mathcal{W}_\infty)$ and $(\mathcal{W}_0)$

We denote by $\mathcal{W}(X)$ the Banach algebra of weakly compact operators from X to X. We say that X has *property* $(\mathcal{W})$ if for every weakly compact subset W of X, there exists a $T \in \mathcal{W}(X)$ such that $W \subset \overline{T(B_X)}$. Trivially, every reflexive Banach space has property $(\mathcal{W})$. We say that X has *property* $(\mathcal{W}_0)$ if for every $(x_n)_n \in c_0^w(X)$, there exists a $T \in \mathcal{W}(X)$ such that $\{x_n\}_{n\in\mathbb{N}} \subset \overline{T(B_X)}$. We see that if X has property $(\mathcal{W})$ or $(\mathcal{W}_\infty)$, then X has property $(\mathcal{W}_0)$.

Lemma 2.1. [D, Proposition 3.1] *There exists an operator from X onto ℓ_1 if and only if there exists a bounded biorthogonal system $(x_n, x_n^*)_{n\in\mathbb{N}}$ in $X \times X^*$ such* *that*

$$\sum_{n=1}^{\infty} |x_n^*(x)| \le \|x\|$$

for every $x \in X$.

The prototype of the following proposition for property $(\mathcal{K})$ was given in [D, Theorem 3.3].

Proposition 2.2. *If there exists an operator from X onto ℓ_1, then X has property $(\mathcal{W}_\infty)$.*

Proof. Let $(z_n)_{n\in\mathbb{N}} \in c_0^w(X)$. Let $(x_n, x_n^*)_{n\in\mathbb{N}}$ be a biorthogonal system in Lemma 2.1. Let $M := \sup_{n\in\mathbb{N}} \|x_n\|$. Let us consider the operator S from X to X defined by

$$Sx = \sum_{n=1}^{\infty} x_n^*(x)(Mz_n)$$

for every $x \in X$. Then for every $k \in \mathbb{N}$,

$$z_k = \sum_{n=1}^{\infty} x_n^*(x_k) z_n = S\left(\frac{1}{M} x_k\right) \in S(B_X).$$

Since for every $x \in B_X$, $\sum_{n=1}^{\infty} |x_n^*(x)| \le 1$,

$$S(B_X) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n (Mz_n) \colon (\alpha_n)_{n\in\mathbb{N}} \in B_{\ell_1} \right\}.$$

This means that $S \in \mathcal{W}_\infty(X)$. Hence X has property $(\mathcal{W}_\infty)$. □

Corollary 2.3. *If X has an isomorphic copy of c_0, then X^* has property $(\mathcal{W}_\infty)$.*

Proposition 2.4. *If X embeds isomorphically into c_0, then X^* has property $(\mathcal{W}_\infty)$ and $(\mathcal{W})$.*

Proof. It is well known that for every closed subspace Z of c_0, Z^* has the Schur property (cf. [JLO, Remark 2.3]). It was shown in [CJ, Corollary 3.6] that for every closed subspace Z of c_0, Z^* has property $(\mathcal{K})$. Hence the conclusions follow. □

As an immediate consequence, ℓ_1 has property $(\mathcal{W}_\infty)$ and $(\mathcal{W})$. It is obvious from reflexivity that $\ell_p(1 < p < \infty)$ has property $(\mathcal{W})$. However, it is not clear whether $\ell_p(1 < p < \infty)$ has property $(\mathcal{W}_\infty)$. We will show that $\ell_p(1 < p < \infty)$ does not have property $(\mathcal{W}_\infty)$. In the next section, we show that c_0 does not have property $(\mathcal{W}_0)$ and hence fails property $(\mathcal{W}_\infty)$. We denote by $\mathcal{L}_{w^*}(Y^*, X)$ the space of *weak** to *weak* continuous operators from Y^* to X. The following lemma is well known. For the completeness of our presentation, we give a proof.

Lemma 2.5. *Let $(x_n)_n \in c_0^w(X)$. If the operator $T\colon \ell_1 \to X$ is defined by $Te_n = x_n$ for all n, where $(e_n)_n$ is the unit vector basis of ℓ_1, then $T \in \mathcal{L}_{w^*}(\ell_1, X)$.*

Proof. It is well known that an operator $U\colon Y^* \to X$ is *weak** to *weak* continuous if (and only if) for every net $(y_\alpha^*)_{\alpha\in I}$ in B_{Y^*} with $\lim_{\alpha\in I} y_\alpha^* = 0$ with respect to the *weak** topology, $\lim_{\alpha\in I} Uy_\alpha^* = 0$ with respect to the *weak* topology on X (cf. [K, the version of *weak** to *weak* continuity of Proposition 3.1]).

Now, let $((t_n^\alpha)_n)_{\alpha\in I}$ be a net in B_{ℓ_1} with $\lim_{\alpha\in I}(t_n^\alpha)_n = 0$ with respect to the *weak** topology. To show that $\lim_{\alpha\in I} T(t_n^\alpha)_n = 0$ with respect to the *weak* topology on X. Let $x^* \in X^*$ and let $\varepsilon > 0$ be given. Since $(x_n)_n \in c_0^w(X)$, there exists an $N \in \mathbb{N}$ such that $n > N$ implies

$$|x^*(x_n)| \leq \frac{\varepsilon}{2}.$$

Since $\lim_{\alpha \in I}(t_n^\alpha)_n = 0$ with respect to the *weak** topology, there exists an $\alpha_0 \in I$ such that $\alpha \succeq \alpha_0$ implies

$$\sum_{n=1}^{N} |t_n^\alpha x^*(x_n)| \leq \frac{\varepsilon}{2}.$$

Hence if $\alpha \succeq \alpha_0$, then

$$|x^*(T(t_n^\alpha)_n)| \leq \sum_{n=1}^{\infty} |t_n^\alpha x^*(x_n)| \leq \sum_{n=1}^{N} |t_n^\alpha x^*(x_n)| + \frac{\varepsilon}{2} \sum_{n>N} |t_n^\alpha| \leq \varepsilon.$$

Hence $\lim_{\alpha \in I} T(t_n^\alpha)_n = 0$ with respect to the *weak* topology on X. □

Remark 2.6. In view of Lemma 2.5, we see that for every Banach space Z and every $(z_n)_n \in c_0^w(Z)$, there exists a $T \in \mathcal{W}_\infty(\ell_1, Z)$ such that $\{z_n\}_{n \in \mathbb{N}} \subset T(B_{\ell_1})$.

Proposition 2.7. *For every Banach space Z, $Z \oplus \ell_1$ has the property $(\mathcal{W}_\infty)$.*

Proof. Let $(y_n)_n \in c_0^w(Z \oplus \ell_1)$. Then by Remark 2.6, there exists a $T \in \mathcal{W}_\infty(\ell_1, Z \oplus \ell_1)$ such that $\{y_n\}_{n \in \mathbb{N}} \subset T(B_{\ell_1})$. Let $P \colon Z \oplus \ell_1 \to \ell_1$ be the canonical propjection. Then $TP \in \mathcal{W}_\infty(Z \oplus \ell_1, Z \oplus \ell_1)$ and

$$\{y_n\}_{n \in \mathbb{N}} \subset T(B_{\ell_1}) = TP(B_{Z \oplus \ell_1}).$$

Hence $Z \oplus \ell_1$ has the property $(\mathcal{W}_\infty)$. □

Now, we establish some characterizations of property $(\mathcal{W}_\infty)$ and $(\mathcal{W}_0)$. The prototype of the following theorem for property $(\mathcal{K})$ was given in [CJ, Proposition 3.2].

Theorem 2.8. *The following statements are equivalent.*

(a) *X has property $(\mathcal{W}_\infty)$.*

(b) *For every $T \in \mathcal{L}_{w^*}(\ell_1, X)$, there exist an $R \in \mathcal{W}_\infty(X)$ and a sequence $(S_n)_n$ in $\mathcal{L}(\ell_1, X)$ with $\|S_n\| \leq 1$ such that*

$$\lim_{n \to \infty} \|RS_n - T\| = 0.$$

(c) *For every $T \in \mathcal{L}_{w^*}(\ell_1, X)$, there exist an $R \in \mathcal{W}_\infty(X)$ and an $S \in \mathcal{L}(\ell_1, X^{**})$ such that*

$$i_X T = R^{**} S.$$

Proof. (a) ⇒ (b): Let $T \in \mathcal{L}_{w^*}(\ell_1, X)$. Since $\lim_{k \to \infty} e_k = 0$ with respect to the *weak** topology, $(Te_k)_k \in c_0^w(X)$. Then by (a), there exists an $R \in \mathcal{W}_\infty(X)$ such that

$$\{Te_k\}_k \subset \overline{R(B_X)}.$$

For every $k \in \mathbb{N}$, choose an $x_k^n \in B_X$ so that

$$\|Te_k - Rx_k^n\| \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$.

Now, for each $n \in \mathbb{N}$, let us define an operator $S_n \colon \ell_1 \to X$ by

$$S_n\Big(\sum_{k=1}^{\infty} \alpha_k e_k\Big) = \sum_{k=1}^{\infty} \alpha_k x_k^n.$$

Then S_n is well defined operator and $\|S_n\| \le 1$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|RS_n - T\| = \lim_{n\to\infty} \sup_{(\alpha_k)_k \in B_{\ell_1}} \left\| \sum_{k=1}^{\infty} \alpha_k (Te_k - Rx_k^n) \right\| \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

(b) ⇒ (c): Let $T \in \mathcal{L}_{w^*}(\ell_1, X)$. Let R and S_n be the operators in (b) for all $n \in \mathbb{N}$. By using *weak** compacness, we can find a subnet $(S_\alpha)_{\alpha\in I}$ of the sequence (S_n) and an operator $S\colon \ell_1 \to X^{**}$ such that for every $(t_n)_n \in \ell_1$,

$$S(t_n)_n = \lim_{\alpha\in I} i_X S_\alpha (t_n)_n$$

with respect to the *weak** topology on X^{**}.

To show that $i_X T = R^{**}S$, let $(t_n)_n \in \ell_1$. Then for every $x^* \in X^*$,

$$\begin{aligned} &\lim_{\alpha\in I} |(R^{**} i_X S_\alpha (t_n)_n)(x^*) - (R^{**} S(t_n)_n)(x^*)| \\ &= \lim_{\alpha\in I} |i_X S_\alpha (t_n)_n (R^* x^*) - S(t_n)_n (R^* x^*)| = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{\alpha\in I} \|(R^{**} i_X S_\alpha (t_n)_n - i_X T(t_n)_n\| &\le \lim_{\alpha\in I} \|R^{**} i_X S_\alpha - i_X T\| \|(t_n)_n\|_{\ell_1} \\ &= \|(t_n)_n\|_{\ell_1} \lim_{\alpha\in I} \|RS_\alpha - T\| = 0. \end{aligned}$$

Hence $i_X T(t_n)_n = R^{**} S(t_n)_n$.

(c) ⇒ (a): Let $(x_n)_n \in c_0^w(X)$. Let us consider the operator $T\colon \ell_1 \to X$ defined by $Te_n = x_n$ for all n. Then by Lemma 2.5, $T \in \mathcal{L}_{w^*}(\ell_1, X)$. By (c), there exist an $R \in \mathcal{W}_\infty(X)$ and an $S \in \mathcal{L}(\ell_1, X^{**})$ such that

$$i_X T = R^{**} S.$$

Then $\{i_X x_n\}_n \subset i_X T(B_{\ell_1}) = R^{**} S(B_{\ell_1}) \subset \|S\| \overline{R^{**}(B_{X^{**}})}$. Since $R \in \mathcal{W}(X)$, we see that $i_X(\overline{R(B_X)}) = R^{**}(B_{X^{**}})$. Hence $\{x_n\}_n \subset \overline{\|S\| R(B_X)}$. □

We say that an operator T is *completely continuous* from X to Y if for every $(x_n)_n \in c_0^w(X)$, one has that $(Tx_n)_n \in c_0(Y)$. This notion is equivalent to that T takes weakly compact sets into norm compact sets. We denote by $\mathcal{V}(X,Y)$ the space of completely continuous operators from X to Y.

Lemma 2.9. [JLO, Proposition 3.1] *For every Banach spaces X and Y,*

$$\mathcal{K}(X,Y) \subset \mathcal{W}_\infty(X,Y) \subset \mathcal{V}(X,Y).$$

Corollary 2.10. *For every reflexive Banach space X and every Banach space Y,*

$$\mathcal{W}_\infty(X,Y) = \mathcal{K}(X,Y).$$

From Theorem 2.8 and Corollary 2.10, we have

Corollary 2.11. *If X is reflexive, then X has property $(\mathcal{W}_\infty)$ if and only if for every $T \in \mathcal{L}_{w^*}(\ell_1, X)$, there exist an $R \in \mathcal{K}(X)$ and an $S \in \mathcal{L}(\ell_1, X)$ such that*

$$T = RS.$$

Corollary 2.12. *$\ell_p (1 < p < \infty)$ does not have property $(\mathcal{W}_\infty)$.*

Proof. Suppose that $\ell_p(1 < p < \infty)$ would have property $(\mathcal{W}_\infty)$. Let us consider the inclusion map $i\colon \ell_1 \to \ell_p$. It is easy to see that $i \in \mathcal{L}_{w^*}(\ell_1, \ell_p)$. According to Corollary 2.11, $i\colon \ell_1 \to \ell_p$ is compact. This is a contradiction. □

The proof of the theorem below is similar to the proof of Theorem 2.8 and we omit it.

Theorem 2.13. *The following statements are equivalent.*

(a) *X has property $(\mathcal{W}_0)$.*
(b) *For every $T \in \mathcal{L}_{w^*}(\ell_1, X)$, there exist an $R \in \mathcal{W}(X)$ and a sequence $(S_n)_n$ in $\mathcal{L}(\ell_1, X)$ with $\|S_n\| \le 1$ such that*
$$\lim_{n\to\infty} \|RS_n - T\| = 0.$$
(c) *For every $T \in \mathcal{L}_{w^*}(\ell_1, X)$, there exist an $R \in \mathcal{W}(X)$ and an $S \in \mathcal{L}(\ell_1, X^{**})$ such that*
$$i_X T = R^{**} S.$$

3. The weakly ∞-compact approximation property

In this section, we discover some relationships between the weakly ∞-compact approximation properties, the left approximation identities for the normed algebra $\mathcal{W}_\infty(X)$ and property $(\mathcal{W}_\infty)$.

Proposition 3.1. *If X has the WICAP, then X has the Schur property.*

Proof. Let $(x_n)_n \in c_0^w(X)$. To show that $(x_n)_n \in c_0(X)$, let $\varepsilon > 0$ be given. Since X has the WICAP, there exists an $S \in \mathcal{W}_\infty(X)$ such that
$$\sup_{n\in\mathbb{N}} \|Sx_n - x_n\| \le \frac{\varepsilon}{4}.$$
Also, by Lemma 2.9, there exists an $N \in \mathbb{N}$ such that if $n \ge N$, then
$$\|Sx_n\| \le \frac{\varepsilon}{4}.$$
Hence, if $n, m \ge N$, then we have
$$\|x_n - x_m\| \le \|x_n - Sx_n\| + \|Sx_n - Sx_m\| + \|Sx_m - x_m\| \le \varepsilon.$$
Consequently, $(x_n)_n \in c_0(X)$. □

Let $\lambda \ge 1$. We say that X has the *weakly compact approximation property* (WCAP) (respectively, λ-bounded WCAP) if for every weakly compact subset W of X and every $\varepsilon > 0$, there exists an $S \in \mathcal{W}(X)$ (respectively, $S \in \mathcal{W}(X)$ with $\|S\| \le \lambda$) such that
$$\sup_{x\in W} \|Sx - x\| \le \varepsilon.$$
In [AT], [OT], [ST], [T1] and [T2], this notion was introduced and studied for related topics.

Corollary 3.2. *The following statements are equivalent.*

(a) *For every weakly compact subset W of X and every $\varepsilon > 0$, there exists an $S \in \mathcal{K}(X)$ such that*
$$\sup_{x\in W} \|Sx - x\| \le \varepsilon.$$
(b) *X has the WICAP.*
(c) *X has the Schur property and the WCAP.*

Proof. (a) $\Rightarrow$ (b) is clear. (b) $\Rightarrow$ (c) follows from Grothendieck's criterion of norm compactness and Proposition 3.1. (c) $\Rightarrow$ (a) follows from the Schur property. □

By a similar argument, we have

Corollary 3.3. *Let $\lambda \geq 1$. Then the following statements are equivalent.*

(a) *For every weakly compact subset W of X and every $\varepsilon > 0$, there exists an $S \in \mathcal{K}(X)$ with $\|S\| \leq \lambda$ such that*

$$\sup_{x\in W} \|Sx - x\| \leq \varepsilon.$$

(b) *X has the λ-bounded WICAP.*

(c) *X has the Schur property and the λ-bounded WCAP.*

We remark that there exist non-reflexive Banach spaces having the WCAP but failing to have the WICAP in [OT, ST].

The following lemma is immediate.

Lemma 3.4. *Let $\lambda \geq 1$. Then X has the WICAP (respectively, λ-bounded WICAP) if and only if for every finite $(x_n^1)_n, \ldots, (x_n^m)_n \in c_0^w(X)$ and $\varepsilon > 0$, there exists an $S \in \mathcal{W}_\infty(X)$ (respectively, $S \in \mathcal{W}_\infty(X)$ with $\|S\| \leq \lambda$) such that $\sup_{n\in\mathbb{N}} \|Sx_n^i - x_n^i\| \leq \varepsilon$ for all $i = 1, \ldots, m$.*

Proposition 3.5. *Let $\lambda \geq 1$. Then the following statements hold.*

(a) *If X has the WICAP, then $\mathcal{W}_\infty(X)$ has the LAI.*

(b) *If X has the λ-bounded WICAP, then $\mathcal{W}_\infty(X)$ has the λ-bouned LAI.*

Proof. We only show (a) using Lemma 3.4. Let $T_1, \ldots, T_m \in \mathcal{W}_\infty(X)$ and let $\varepsilon > 0$ be given. Then there exist $(x_n^1)_n, \ldots, (x_n^m)_n \in c_0^w(X)$ such that $T_i(B_X)$ is contanined in the balanced and closed convex hull of $\{x_n^i\}_n$ for $i = 1, \ldots, m$. Since X has the WICAP, there exists an $S \in \mathcal{W}_\infty(X)$ such that

$$\sup_{n\in\mathbb{N}} \|Sx_n^i - x_n^i\| \leq \varepsilon$$

for all $i = 1, \ldots, m$. Then for all $i = 1, \ldots, m$,

$$\|ST_i - T_i\| = \sup_{x\in T_i(B_X)} \|Sx - x\| \leq \sup_{n\in\mathbb{N}} \|Sx_n^i - x_n^i\| \leq \varepsilon.$$

Hence $\mathcal{W}_\infty(X)$ has the LAI. □

The normed algebra $\mathcal{A}$ is said to have the *left approximate unit* (LAU) (respectively, λ-*bounded LAU*) if for every $T \in \mathcal{A}$ and every $\varepsilon > 0$, there exists an $S \in \mathcal{A}$ (respectively, $S \in \mathcal{A}$ with $\|S\|_{\mathcal{A}} \leq \lambda$) such that

$$\|ST - T\|_{\mathcal{A}} \leq \varepsilon.$$

Corollary 3.6. *Assume that X has property $(\mathcal{W}_\infty)$. Then the following statements are equivalent.*

(a) *X has the WICAP.*

(b) *$\mathcal{W}_\infty(X)$ has the LAI.*

(c) *$\mathcal{W}_\infty(X)$ has the LAU.*

Proof. (b) $\Rightarrow$ (c) is trivial and (a) $\Rightarrow$ (b) follows from Proposition 3.5.

(c) $\Rightarrow$ (a): Let $(x_n)_n \in c_0^w(X)$ and let $\varepsilon > 0$ be given. Since X has the property $(\mathcal{W}_\infty)$, there exists a $T \in \mathcal{W}_\infty(X)$ such that $\{x_n\}_n \subset \overline{T(B_X)}$. Then by (c), there exists an $S \in \mathcal{W}_\infty(X)$ such that

$$\varepsilon \geq \|ST - T\| = \sup_{x\in T(B_X)} \|Sx - x\| \geq \sup_{n\in\mathbb{N}} \|Sx_n - x_n\|.$$

Hence X has the WICAP. □

Corollary 3.7. *Let $\lambda \geq 1$. Assume that X has the property $(\mathcal{W}_\infty)$. Then the following statements are equivalent.*

(a) *X has the λ-bounded WICAP.*
(b) *$\mathcal{W}_\infty(X)$ has the λ-bounded LAI.*
(c) *$\mathcal{W}_\infty(X)$ has the λ-bounded LAU.*

Corollary 3.8. *$\mathcal{W}_\infty(c_0)$ has the bounded LAI but c_0 does not have the WICAP. Moreover, c_0 does not have property $(\mathcal{W}_0)$.*

Proof. Recall that for every Banach space Z,

$$\mathcal{K}(Z) \subset \mathcal{W}_\infty(Z) \subset \mathcal{W}(Z).$$

Now, let $T \in \mathcal{W}(c_0)$. Then $T^* \in \mathcal{W}(\ell_1) = \mathcal{K}(\ell_1)$. Thus $T \in \mathcal{K}(c_0)$. Consequently, $\mathcal{K}(c_0) = \mathcal{W}_\infty(c_0) = \mathcal{W}(c_0)$. According to [D, Theorem 2.6], X has the λ-bounded CAP if and only if $\mathcal{K}(X)$ has the λ-bounded LAI. Since c_0 has the bounded CAP, $\mathcal{W}_\infty(c_0)$ has the bounded LAI. By Corollary 3.2, c_0 does not have the WICAP. On the other hand, suppose that c_0 would have property $(\mathcal{W}_0)$. Then c_0 has property $(\mathcal{W}_\infty)$. As a consequence of Corollary 3.6, this is a contradiction. □

Remark 3.9. We can directly show that c_0 does not have property $(\mathcal{W}_0)$. Indeed, suppose that c_0 would have property $(\mathcal{W}_0)$. Since the sequence $(e_n)_n$ of standard unit vectors in c_0 is a weakly null sequence, there exists a $T \in \mathcal{W}(c_0) = \mathcal{K}(c_0)$ such that $\{e_n\}_{n\in\mathbb{N}} \subset \overline{T(B_{c_0})}$. Thus $\{e_n\}_{n\in\mathbb{N}}$ is a relatively norm compact subset of c_0. This is a contradiction.

Let $\mathcal{A}$ be a normed algebra. We say that a Banach space X is a *left Banach $\mathcal{A}$-module* if there exists an $M \geq 1$ such that for every $A, B \in \mathcal{A}$ and $x, y \in X$, scalar t,

(i) $Ax \in X$;
(ii) $(A + B)x = Ax + Bx$ and $A(x + y) = Ax + Ay$;
(iii) $(tA)x = t(Ax) = A(tx)$;
(iv) $(AB)x = A(Bx)$;
(v) $\|Ax\| \leq M\|A\|\|x\|$.

For instance, X is a left Banach $\mathcal{K}(X)$, $\mathcal{W}(X)$ and $\mathcal{W}_\infty(X)$-module. We see that a normed algebra $\mathcal{A}$ has the LAI (respectively, bounded LAI) if and only if there exists a net (respectively, bounded net) $(A_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ such that for every $A \in \mathcal{A}$,

$$\lim_{\alpha\in I} \|A_\alpha A - A\| = 0$$

(cf. [DW, Proposition 1.2]). We call the net a *left approximate identity* (LAI) (respectively, *bounded LAI*) for $\mathcal{A}$. The following lemma is in [DW, Theorem 17.1].

Lemma 3.10. *Let $\mathcal{A}$ be a Banach algebra with a bounded LAI $(A_\alpha)_{\alpha\in I}$ and let X be a left Banach $\mathcal{A}$-module. If B is a bounded subset of X such that*

$$\lim_{\alpha\in I} \sup_{x\in B} \|A_\alpha x - x\| = 0,$$

then there exists an $A \in \mathcal{A}$ and a bounded subset C of X such that

$$B = \{Ax \colon x \in C\}.$$

Proposition 3.11. *If X has the bounded WCAP, then X has property $(\mathcal{W})$.*

Proof. By our hypothesis, there exists a bounded net $(S_\alpha)_{\alpha\in I}$ in $\mathcal{W}(X)$ such that for every weakly compact subset W of X,

$$\lim_{\alpha\in I}\sup_{x\in W}\|S_\alpha x - x\| = 0.$$

Then for every $T\in\mathcal{W}(X)$,

$$\lim_{\alpha\in I}\|S_\alpha T - T\| = \lim_{\alpha\in I}\sup_{x\in B_X}\|S_\alpha Tx - Tx\| = 0.$$

Consequently, $(S_\alpha)_{\alpha\in I}$ is a bounded LAI for $\mathcal{W}(X)$.

Now, let W be a weakly compact subset of X. Then by Lemma 3.10, there exists an $R\in\mathcal{W}(X)$ and a bounded subset C of X such that

$$W = \{Rx\colon x\in C\}\subset tR(B_X)$$

for some $t>0$. Since $tR\in\mathcal{W}(X)$, X has property $(\mathcal{W})$. □

Proposition 3.12. *If X has the bounded WICAP, then X has property $(\mathcal{W}_\infty)$ and $(\mathcal{W})$.*

Proof. The second part follows from Proposition 3.11. By Proposition 3.1, if X has the bounded WICAP, then X has the Schur property. Consequently, $\mathcal{K}(X) = \mathcal{W}_\infty(X) = \mathcal{W}(X)$. Hence X has property $(\mathcal{W}_\infty)$. □

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