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RIEMANN SURFACES WITH THE AB-MAXIMUM PRINCIPLE

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Riemann surfaces with the AB-maximum principle *

Let $\langle a_n \rangle$ be a sequence of points in $0 < |z| < 1$ with $\lim a_n = 0$, and let W be the two-sheeted Riemann surface over $0 < |z| < 1$ which has the points a_n as branch points. Then, as P. J. Myrberg [2] has observed, every bounded analytic function on W takes the same values on the two sheets of W. For the square of the difference of the values on the two sheets is a bounded analytic function of z in $0 < |z| < 1$, and hence also in $|z| < 1$. Since this function vanishes at the points a_n it must vanish identically. Somewhat similar observations had previously been made by H. L. Selberg [3].

In this example we see that each bounded analytic function on W is the composition $g \circ \tau$ of an analytic function in the disk $|z| < 1$ and the projection τ of W into the z-plane. Heins [1] has generalized this result to showing that, if W is a parabolic Riemann surface with precisely one ideal boundary component, then some end Ω of W can be mapped onto $0 < |z| < 1$ by an analytic function τ so that each bounded analytic function f on Ω is of the form $g \circ \tau$ where g is a bounded analytic function in disk $|z| < 1$.

Actually, a result of this nature holds under much weaker assumptions on W. Let (W, Γ) be a bordered Riemann surface with compact border Γ. Then W is said to satisfy the AB-maximum principle if every bounded analytic function on $W \cup \Gamma$ assumes its maximum on Γ. Then Theorem 3 asserts that there is an analytic mapping τ of $W \cup \Gamma$ into a compact subset C of some Riemann surface such that every bounded analytic function f on $W \cup \Gamma$ is the composition $g \circ \tau$ of τ with some function g defined and analytic in a neighborhood of C. Theorem 3 is slightly more general than this in that it establishes the corresponding conclusion for functions in any algebra of analytic functions on $W \cup \Gamma$ which assume their maxima on Γ.

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1. *Algebras of analytic functions on a Riemann surface.* Let W be a Riemann surface. A collection A of analytic or meromorphic functions on W is called an algebra on W if the constant functions belong to W and if $f + g$ and $f \cdot g$ belong to A whenever f and g do. If p and q are points of W, we say that A *separates* p and q if there is an element f in A with $f(p) \neq f(q)$. We say that A *weakly separates* p and q if there are elements f and g in A such that f/g has different values at p and q. Thus A separates p and q weakly iff the field of quotients of A separates p and q. Since A is an algebra, A separates p and q iff there are f and g in A such that f/g has a pole at p and is regular at q. An algebra is said to separate (weakly) on a subset of W if it separates (weakly) each pair of distinct points of the subset.

If A is an algebra on W, the Riemann surface W' with an algebra A is said to be an extension of (A, W) if $W \subset W'$ and A consists of the restrictions to W of the functions in A'. An algebra A is said to be *proper* for the Riemann surface W if A weakly separates the points of W and if (A, W) has no proper extension which separates weakly, i. e. if (A', W') is an estension of (A, W) and A' separates on W, then $W = W'$. A pair (A, W) is said to be isomorphic to the pair (A', W') if there is a one-to-one conformal map σ of W onto W' which carries A' onto A.

If we start with an algebra of analytic functions in a disk, and suppose that the algebra separates weakly on the disk, then we may use the classical method of analytic continuation to construct a maximal extension of the disk on which the algebra is defined and weakly separates points. This enables us to establish the following proposition:

Proposition 1. *Let A be an algebra of meromorphic functions on a Riemann surface W. Then there is a Riemann surface W', a proper algebra A' on W', and an analytic map τ of W into W' such that each $f \in A$ is of the form $g \circ \tau$ with $g \in A'$. The pair (A', W') is unique to within isomorphism.*

If K is a set on a Riemann surface, we say that a function f defined on K is analytic on K if f can be extended to an analytic function defined on some open set containing K. A collection of functions is said to be analytic on K if each function in the collection is analytic on K. Note that we do not suppose that there is an open set containing K on which all the functions of the collection are analytic. Whenever this latter property holds, we speak of a collection of functions uniformly analytic on K.

If (W, Γ) is a bordered Riemann surface with compact border Γ, we can also consider an algebra A of functions analytic on $W \cup \Gamma$. The preceding proposition does not apply directly, for although each f in A is defined and analytic on some Riemann surface containing $W \cup \Gamma$, there is no fixed Riemann surface containing $W \cup \Gamma$ on which all functions of A

are defined and analytic. The following proposition shows, however, that we can find a finitely generated subalgebra of A which separates as well as A does. With the help of this proposition we can establish Proposition 3, which generalizes Proposition 1 to the case of a bordered Riemann surface.

Proposition 2. *Let K be a finite union of analytic arcs on a Riemann surface W and A an algebra of meromorphic functions on K. Then there is a finitely generated subalgebra A_0 of A with the property that A_0 separates weakly each pair of points which are weakly separated by A.*

Proposition 3. *Let $W \cup \Gamma$ be a bordered Riemann surface with compact border Γ, and let A be an algebra of analytic functions on $W \cup \Gamma$. Then there is an analytic map τ of $W \cup \Gamma$ into a Riemann surface W' and an algebra A' of analytic functions on a connected compact set containing $\tau[\Gamma]$ such that a finitely generated subalgebra of A' is proper for W' and such that on Γ each $f \in A$ is of the form $g \circ \tau$ where $g \in A'$.*

2. *Some theorems from functional analysis.* Wermer [4] has proved a remarkable theorem about algebras of functions analytic on the unit circumference, and his proof can be modified to prove the following generalization:

Theorem 1. *Let A be a proper algebra for the Riemann surface W, and let K be a compact subset of W. Let K^* be the union of K and those components of $W \sim K$ whose closures are compact, and let*

$$\Delta = \{ p \in W : p \notin K^* , \exists q \in K^* , f(p) = f(q) \text{ for all } f \in A \} .$$

Then K^* is compact, Δ is an isolated set, and we have the following:

i) *The hull of K is $K^* \cup \Delta$, i.e. $K^* \cup \Delta = \{ p \in W : |f(p)| \leqq \sup_K |f| \}$.*

ii) *If π is a homomorphism of A into the complex numbers with $|\pi f| \leqq \sup_K |f|$, then there is a $p \in K^*$ with $\pi f = f(p)$.*

iii) *If ϱ is a homomorphism of A into the algebra of analytic functions on a disk D such that $\sup_D |\varrho f| \leqq \sup_K |f|$, then there is a unique analytic map ψ of D into the interior of K^* such that $\varrho f = f \circ \psi$.*

Repeated application of this theorem gives us the following theorem:

Theorem 2. *Let A_0 be a proper algebra of analytic functions on the Riemann surface W, let K be a compact connected subset of W, and let A be an algebra of analytic functions on K with $A \supset A_0$. Let K^+ be the union of K and those relatively compact components of $W \sim K$ to which each function in A_0 has an analytic extension. Then K^+ is a compact set for which the following hold:*

i) *If π is a homomorphism of A into the complex numbers with $|\pi f| \leqq \sup_K |f|$, then there is a $p \in K^+$ with $\pi f = f(p)$.*

ii) *If ϱ is a homomorphism of A into the algebra of analytic functions on a disk D so that $\sup_D |\varrho f| \leqq \sup_K |f|$, then there is a unique analytic map ψ of D into the interior of K^+ such that $\varrho f = f \circ \psi$.*

Combining Proposition 3 with Theorem 2, we obtain the following theorem, which generalizes the theorem of Heins:

Theorem 3. *Let* (W, Γ) *be a bordered Riemann surface with compact border, and* A *an algebra of bounded analytic functions on* $W \cup \Gamma$ *such that each* $f \in A$ *assumes its maximum on* Γ. *Then there is an analytic mapping* τ *of* $W \cup \Gamma$ *into a Riemann surface* W' *such that* $\tau[W \cup \Gamma]$ *has compact closure and each* $f \in A$ *is of the form* $g \circ \tau$ *where* g *is analytic in some neighborhood of the closure of* $\tau[W \cup \Gamma]$.

We say that the bordered Riemann surface (W, Γ) satisfies the AB-maximum principle if each bounded analytic function on $W \cup \Gamma$ assumes its maximum on Γ. If (W, Γ) satisfies the AB-maximum principle, we may take the algebra A in Theorem 3 to be the algebra of all bounded analytic functions. In this case each f on W of the form $g \circ \tau$ with g anlytic on the closure of $\tau[W \cup \Gamma]$ is a bounded analytic function on $W \cup \Gamma$, and so the class of bounded analytic functions consists precisely of those f which are lifted from analytic functions on $\overline{\tau[W \cup \Gamma]}$.

3. *Some examples.* Let W be the surface of Myrberg mentioned in the introduction, that is the two-sheeted covering of $0 < |z| < 1$ branched over a sequence $\langle a_n \rangle$ of points accumulating at zero. Let us take that surface W for which $|z| = 1$ is covered by two circles, i. e. for which $\{ p \in W : |a_1| < |z(p)| < 1 \}$ has two components. Then each bounded analytic function on W has the same values on both sheets. Thus if we take one (or both) of the circles over $|z| = 1$ as the border Γ of W, then the algebra A of bounded analytic functions on $W \cup \Gamma$ has the property that each $f \in A$ assumes its maximum on Γ, and so Theorem 3 applies. In this case the Riemann surface W' is the z-plane, and the mapping τ is the projection of W onto the z-plane. If we modify this example slightly by taking that part of W which is bounded by curves Γ_1 and Γ_2 lying over $|a_1| < |z| < 1$, and whose projections intersect, we obtain a surface V bordered by $\Gamma = \Gamma_1 \cup \Gamma_2$ such that the image under the mapping τ of Theorem 3 is not bounded by analytic curves (analyticity breaking down at the intersections of the images of Γ_1 and Γ_2). Thus we cannot quite assert that τ maps $W \cup \Gamma$ analytically *onto* a finite Riemann surface.

The requirement in Theorem 3 that our algebra consists of functions analytic on Γ instead of merely in W seems restrictive, but the following example shows some of the difficulties we encounter if we drop this restriction.

Let W_1 be the half plane $\operatorname{Re} z > -2$ with the circles $|z| < 1$ and $|z - 2i| < 1$ removed. Let Γ be the line $\operatorname{Re} z = -2$ with $z = \infty$ added. Then (W_1, Γ) is a bordered surface with compact border. Let W_0 be the

Myrberg surface described at the beginning of the section. Form a new bordered Riemann surface W by identifying the boundary points of W_1 on $|z| = 1$ with the corresponding points of the boundary of one sheet of W_0, and identifying the boundary points of W_1 on $|z - 2i| = 1$ with the translates of the boundary of the other sheet of W_0. Then each bounded analytic function on W is a function of z on W_1 such that for $|z| = 1$ we have $f(z + 2i) = f(z)$. Thus the function $f(z + 2i) - f(z)$ is analytic in the half plane $\operatorname{Re} z > 2$ outside the three circles $|z| < 1$, $|z - 2i| < 1$, and $|z - 4i| < 1$. Since this function vanishes identically on $|z - 2i| = 1$, it must vanish identically. Thus each bounded analytic function on W is periodic with period $2i$ as a function of z in W_1.

Conversely, each bounded periodic function in $\operatorname{Re} z > 2$ with period $2i$ defines a bounded analytic function on W. Thus every bounded analytic function on W is singular at the point $z = \infty$ on Γ, and so there are no bounded analytic functions on $W \cup \Gamma$, although there are many bounded analytic functions on W.

The function $h(z) = e^{-2z/\pi}$ is an analytic function on W, which maps W onto the circle $D = \{ z : |z| \leqq e^{4/\pi} \}$, and every bounded analytic f on W is of the form $f = g \circ h$ where g is bounded and analytic on D. Thus in this case we still have a situation similar to that of Theorem 3 but the mapping h fails to be analytic on Γ. I do not know to what extent this example represents the general situation.

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