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ON THE COEFFICIENT PROBLEM FOR FUNCTIONS OF BOUNDED BOUNDARY ROTATION

BY

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1. Introduction

For $k \geq 2$ let V_k denote the class of locally univalent analytic functions

(1.1) $$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

that map $|z| < 1$ conformally onto a domain whose boundary rotation is at most $k\pi$. (See [9] for the definition and basic properties of the class V_k.)

The function

(1.2) $$f_k(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}} - 1\right] = \sum_{n=1}^{\infty} A_n z^n$$

belongs to V_k and the 'coefficient conjecture' for the class V_k is that for a function (1.1) in V_k,

(1.3) $$|a_n| \leq A_n \qquad (n \geq 1).$$

This conjecture was proved for $n = 2$ by Pick (see [9]), for $n = 3$ by Lehto [9] and for $n = 4$ in [17], [10], [2] and [4]. In support of the conjecture Noonan [12] has shown that for a given function (1.1) in V_k, $\lim_{n\to\infty} \frac{|a_n|}{A_n}$ exists and is less than 1 unless $f(z) = e^{-i\varphi} f_k(e^{i\varphi} z)$ for some real φ.

In this paper we prove the conjecture (1.3) for $k \geq 4$. If $2 \leq k \leq 4$ we prove the conjecture for all $n \leq 14$; for all n if $f(z)$ has real coefficients; and, for all n, that $|a_n| \leq 1.05\, A_n$. Each of these results holds in fact for the larger class of close-to-convex functions of order $\beta = k/2 - 1 > 0$. The definition is given in § 2 of this class of functions.

2. Sharp coefficient bounds

A function $f(z) = z + a_2 z^2 \ldots$ analytic in $|z| < 1$, is said to be close-to-convex of order β $(\beta > 0)$ if there exists a starlike function $s(z) = z + b_2 z^2 + \ldots$ $(|z| < 1)$ and a constant a such that

$$\left| \arg \frac{zf'(z)}{as(z)} \right| < \beta \frac{\pi}{2} \qquad (|z| < 1) . \tag{2.1}$$

The class of such functions $f(z)$ will be denoted by $K(\beta)$. This class of functions was introduced by Pommerenke [14] who was primarily concerned with β in the range $0 \leq \beta \leq 1$. The case $\beta > 1$ was studied by Goodman [6]. When $\beta = 1$, $K(\beta)$ reduced to the well known class of close-to-convex functions introduced by Biernacki [1] and Kaplan [8].

We will make reference to one additional class of functions, namely the class $\mathcal{P}$ of functions $p(z)$ that are analytic and satisfy $Re p(z) > 0$ in $|z| < 1$.

The condition (2.1) is equivalent to the statement that there exists a $p(z)$ in $\mathcal{P}$ such that

$$zf'(z) = a[p(z)]^{\beta} s(z) \qquad (|z| < 1) . \tag{2.2}$$

Since $f'(0) = 1$, it is clear from (2.2) that if $p(0) = b$ then $ab^{\beta} = 1$, and we may assume (as we always will) that $|a| = |b| = 1$.

Our first result is a simple observation relating the classes V_k and $K(\beta)$.

Theorem 2.1. *For all* $k > 2$, V_k *is properly contained in* $K(\beta)$, *where* $\beta = k/2 - 1$.

Proof. The geometrical interpretation of (2.1) is that for $0 < r < 1$, the tangent to $C_r = \{\omega : \omega = f(re^{i\theta}),\ 0 \leq \theta \leq 2\pi\}$ does not turn back on itself by more than $\beta\pi$ as θ increases from 0 to 2π (see [14]). If $f(z) \in V_k$ then the total variation of the argument of the tangent to C_r is at most $k\pi$. Hence the tangent to C_r cannot turn back on itself by more than $(k/2 - 1)\pi$ as θ increases from 0 to 2π and thus $f(x) \in K(k/2 - 1)$. It is clear from the geometrical interpretation of (2.1) that V_k is properly contained in $K(k/2 - 1)$.

It follows from Theorem 2.1 that if $f(z) \in V_k$,

$$zf'(z) = a[p(z)]^{\frac{k}{2} - 1} s(z) ,$$

where $p(z) \in \mathcal{P}$ and $s(z)$ is starlike. Sharp estimates for the coefficients of $s(z)$ are of course known, so we turn our attention to estimating the size of the coefficients of functions of the form $[p(z)]^{\alpha}$ where $p(z) \in \mathcal{P}$. This information is contained in the next theorem which is an extension of the classical Herglotz formula.

Before stating the theorem we recall some notation and facts from the theory of linear topological spaces that will be needed for the proof. If E is a subset of such a space, then a point in E is an extreme point of E if it cannot be non-trivially expressed as the convex combination of two points in E. We denote by $\operatorname{ext} E$ the set of extreme points of E and by coE and $\overline{co}E$ the convex hull of E and the closed convex hull of E respectively; i.e., the smallest convex set and the smallest closed convex set containing E. The Krein-Milman Theorem [5, p. 440] asserts that if E is a compact convex subset of a locally convex linear space, then $E = \overline{co}(\operatorname{ext} E)$.

Theorem 2.2. *Let c be a constant of modulus 1 at most and let E denote the set of the functions subordinate to*

$$T(z) = \frac{1 + cz}{1 - z} .$$

If $\alpha \geq 1$ and $h(z) \in E$, then there exists an increasing function $\mu(t)$ on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$ such that

$$[h(z)]^\alpha = \int_0^{2\pi} \left[\frac{1 + ce^{it}z}{1 - e^{it}z}\right]^\alpha d\mu(t) .$$

Proof. By (a trivial modification of) the Herglotz formula [15, p. 30], if $h(z) \in E$ then there exists a $\mu(t)$ satisfying the conditions stated in the theorem such that

$$h(z) = \int_0^{2\pi} \frac{1 + ce^{it}z}{1 - e^{it}z}\, d\mu(t) .$$

It follows (15, p. 30] that $\operatorname{ext} E$ consists of the functions

$$\frac{1 + ce^{i\varphi}z}{1 - e^{i\varphi}z} \qquad (0 \leq \varphi \leq 2\pi) .$$

Let $E_\alpha = \{[h(z)]^\alpha : h(z) \in E\}$. By Montel's Theorem, E_α and $\overline{co}E_\alpha$ are compact in the linear space A of functions analytic in $|z| < 1$ when the topology of A is the topology of local uniform convergence in $|z| < 1$. It is known that this topology for A is locally convex [19, p. 150] and hence by Milman's Theorem [5, p. 440] if $g(z) \in \operatorname{ext}[\overline{co}E_\alpha]$, $g(z) \in E_\alpha$. Suppose then that $g(z) = [h(z)]^\alpha$ belongs to E_α and $h(z) \notin \operatorname{ext} E$. Then there exists distinct functions $h_1(z)$ and $h_2(z)$ in E and a t, $0 < t < 1$, such that

$$\begin{aligned} g(z) &= [h(z)]^{\alpha-1} h(z) \\ &= [h(z)]^{\alpha-1} \{th_1(z) + (1 - t) h_2(z)\} \\ &= t[h(z)]^{\alpha-1} h_1(z) + (1 - t) [h(z)]^{\alpha-1} h_2(z) . \end{aligned}$$

Since $[h(z)]^{\alpha-1} h_1(z)$ and $[h(z)]^{\alpha-1} h_2(z)$ are distinct functions in E_α, $g(z)$ is not an extreme point of E_α. Thus an extreme point of E_α must be of the form

(2.3) $$\left[\frac{1 + ce^{i\varphi}z}{1 - e^{i\varphi}z}\right]^{\alpha} \qquad (0 \leq \varphi \leq 2\pi) .$$

By the Krein-Milman Theorem, $\overline{co}E_{\alpha}$ is the closed convex hull of the set of functions (2.3). Hence if $h(z) \in \overline{co}E_{\alpha}$ there is a sequence of functions $(h_n(z))$,

$$h_n(z) = \sum_{j=1}^{\sigma(n)} t_{j,n} \left[\frac{1 + ce^{i\varphi_j}z}{1 - e^{i\varphi_j}z}\right]^{\alpha} .$$

$t_{j,n} > 0$ and $\sum_{j=1}^{\sigma(n)} t_{j,n} = 1$ such that $\lim_{n\to\infty} h_n(z) = h(z)$ locally uniformly in $|z| < 1$.
Now,

$$h_n(z) = \int_0^{2\pi} \left[\frac{1 + ce^{it}z}{1 - e^{it}z}\right]^{\alpha} d\mu_n(t) ,$$

where $\mu_n(t)$ is an increasing step function on $[0 , 2\pi]$ with jump $t_{j,n}$ at φ_j . By the Helly Selection Theorem, there is an increasing function $\mu(t)$ on $[0 , 2\pi]$ which is the pointwise limit on $[0 , 2\pi]$ of an infinite subsequence of $(\mu_n(t))$ and hence

(2.4) $$h(z) = \lim h_n(z) = \int_0^{2\pi} \left[\frac{1 + ce^{it}z}{1 - e^{it}z}\right]^{\alpha} d\mu(t) ,$$

where the limit is considered as $n \to \infty$ through the appropriate subsequence. (2.4) holds in particular if $h(z) \in E_{\alpha}$, and the proof is complete.

We note for later reference that, if $\alpha > 0$, $\beta > 0$ and $0 \leq \theta_1 , \theta_2 , \leq 2\pi$, then, by Theorem 2.2, there exists an increasing function $\mu(t)$ on $[0 , 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$ such that

(2.5) $$\frac{1}{(1 - e^{i\theta_1}z)^{\alpha}} \frac{1}{(1 - e^{i\theta_2}z)^{\beta}} = \int_0^{2\pi} \frac{1}{(1 - e^{it}z)^{\alpha+\beta}} d\mu(t) \qquad (|z| < 1) .$$

Indeed, if $\alpha + \beta \geq 1$, then (2.5) is an immediate consequence of Theorem 2.2 since

(2.5)′ $$\left(\frac{1}{1 - e^{i\theta_1}z}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{1}{1 - e^{i\theta_2}z}\right)^{\frac{\beta}{\alpha+\beta}}$$

is subordinate to $1/(1 - z)$ in $|z| < 1$.

If $0 < \alpha + \beta < 1$, then the linear operator defined by

$$L(f)(z) = \frac{1}{\alpha + \beta} zf'(z) + f(z)$$

transforms functions of the form (2.5)′ into function of the form

$$\frac{\alpha}{\alpha+\beta}\frac{1}{(1-e^{i\theta_1}z)^{\alpha+1}(1-e^{i\theta_2})^{\beta}}+\frac{\beta}{\alpha+\beta}\frac{1}{(1-e^{i\theta_1}z)^{\alpha}(1-e^{i\theta_2}z)^{\beta+1}}\,.$$

It follows from (2.5) as established above that if $\alpha+\beta>0$ and $f(z)$ is of the form (2.5)′, then

$$L(f)\,(z)=\int_0^{2\pi}\frac{1}{(1-e^{it}z)^{\alpha+\beta+1}}\,d\mu(t)\qquad(|z|<1)\,.$$

By an argument similar to the one used in [3, Theorem 6] it can be shown that the extreme points of the set of functions of the form

$$\int_0^{2\pi}\frac{1}{(1-e^{it}z)^{\alpha+\beta+1}}\,d\mu(t)$$

are precisely the set of functions of the form

$$\frac{1}{(1-e^{i\varphi}z)^{\alpha+\beta+1}}\qquad(0\le\varphi\le2\pi)\,.$$

Since L is linear and $1-1$, the extreme points of the set of functions on the left of (2.5) are the functions $(1-e^{i\varphi}z)^{-\alpha-\beta}$ $(0\le\varphi\le2\pi)$. Hence (2.5) is now established using the Helly Selection Theorem as in the proof of Theorem 2.2.

We easily obtain a generalization of [3, Theorem 5] from (2.5). Using the notation of [3], let I_α denote the set of functions

$$h(z)=\int_0^{2\pi}\frac{1}{(1-e^{it}z)^{\alpha}}\,d\mu(t)\,,$$

where $\mu(t)$ ranges over the set of increasing functions on $[0\,,\,2\pi]$ with $\mu(2\pi)-\mu(0)=1$. If $\alpha\ge0$, $\beta\ge0$ and we show that

$$I_\alpha\cdot I_\beta\subset I_{\alpha+\beta}\,,\tag{2.6}$$

i.e. if $h_\alpha(z)\in I_\alpha$ and $h_\beta(z)\in I_\beta$ there exists a function $h_{\alpha+\beta}(z)$ in $I_{\alpha+\beta}$ such that $h_\alpha(z)\,.\,h_\beta(z)=h_{\alpha+\beta}(z)$. Indeed suppose that

$$h_\alpha(z)=\int_0^{2\pi}\frac{1}{(1-e^{it}z)^{\alpha}}\,d\mu(t)$$

and

$$h_\beta(z) = \int_0^{2\pi} \frac{1}{(1 - e^{it}z)^\beta}\, d\nu(t) \,;$$

then

$$h_\alpha(z) \cdot h_\beta(z) = \int_0^{2\pi}\int_0^{2\pi} \frac{1}{(1 - e^{it}z)^\alpha} \cdot \frac{1}{(1 - e^{i\tau}z)^\beta}\, d\mu(t) d\nu(\tau) \,.$$

By (2.5)

$$\frac{1}{(1 - e^{it}z)^\alpha} \cdot \frac{1}{(1 - e^{i\tau}z)^\beta} \in I_{\alpha+\beta}$$

for each t and τ in $[0, 2\pi]$ and since $I_{\alpha+\beta}$ is closed and convex we have that $h_\alpha(z)h_\beta(z) \in I_{\alpha+\beta}$. This establishes (2.6). (2.6) was proved in [3] (by a different method) only for positive integral values of α and β.

Let

$$f(z) = \sum_{j=0}^{\infty} b_j z^j \text{ and } F(z) = \sum_{j=0}^{\infty} B_j z^j$$

be analytic in $|z| < 1$. It will be convenient to denote the condition

$$|b_j| \leq |B_j| \qquad (0 \leq j \leq n)$$

by $f(z) \underset{n}{\ll} F(z)$. If this relation holds for all n, we write $f(z) \ll F(z)$. The following result is an immediate consequence of Theorem 2.2.

Corollary 2.1. *Let $h(z)$ be subordinate to* $\dfrac{1 + cz}{1 - z}$ *in* $|z| < 1$ *and let* $\alpha \geq 1$. *Then*

$$[h(z)]^\alpha \ll \left[\frac{1 + cz}{1 - z}\right]^\alpha .$$

Note. Theorem 2.2 no longer holds if $0 < \alpha < 1$. Indeed let $n \geq 2$. Then

$$\left[\frac{1 + z}{1 - z}\right]^\alpha = 1 + 2\alpha z + \sum_{j=2}^{\infty} a_j z^j \qquad (|z| < 1)$$

and

$$\left[\frac{1 + z^n}{1 - z^n}\right]^\alpha = 1 + 2\alpha z^n + \dots \qquad (|z| < 1) .$$

By a theorem due to Rogosinski [16], $|a_j| \leq 2\alpha$ for $j \geq 2$ which shows that Corollary 2.1 (and hence Theorem 2.2) fails when $0 < \alpha < 1$.

We note that when $0 < \alpha < 1$, every function of the form

$$\left[\frac{1+e^{it}z^n}{1-e^{it}z^n}\right]^\alpha$$

is an extreme point of E_α. It is not clear whether or not there are other extreme points in E_α.

We now prove a result that will circumvent the difficulty arising from the fact that Corollary 2.1 fails when $0<\alpha<1$.

Lemma 2.1. *Let* $f(z) \in K(\beta)$ $(\beta > 0)$ *with*

$$zf'(z) = a[p(z)]^\beta s(z) \qquad (p(0)=b). \tag{2.7}$$

Then

$$f'(z) = \sum_{j=0}^{\infty}\left\{\int_0^{2\pi} a\cdot\frac{1+e^{it}z}{1-e^{it}z}\cdot[p(z)]^\beta e^{2jit}d\mu(t)\right\} z^{2j}. \tag{2.8}$$

where $\mu(t)$ *is an increasing function on* $[0, 2\pi]$ *with* $\mu(2\pi)-\mu(0)=1$.

Proof. Let $c(z) = z + a_2z^2 + \ldots$ $(|z|<1)$ be a convex univalent function in $|z|<1$. By a result of Strohhäcker [18], $Re\, c(z)/z > \frac{1}{2}$ $(|z|<1)$, and hence by the Herglotz formula

$$c(z) = \int_0^{2\pi} \frac{z}{1-e^{it}z}\, d\mu(t),$$

where $\mu(t)$ is increasing on $[0, 2\pi]$ with $\mu(2\pi)-\mu(0)=1$. As was observed in [3], this implies that if $s(z)$ is a normalized starlike function in $|z|<1$ (since $s(z) = zc'(z)$ for some convex function $c(z)$) then

$$s(z) = \int_0^{2\pi} \frac{z}{(1-e^{it}z)^2}\, d\mu(t). \tag{2.9}$$

It follows that if $f(z) \in K(\beta)$ and is given by (2.7) then

$$f'(z) = a[p(z)]^\beta \int_0^{2\pi} \frac{z}{(1-e^{it}z)^2}\, d\mu(t)$$

$$= a[p(z)]^\beta \int_0^{2\pi} \frac{1+e^{it}z}{1-e^{it}z}\cdot\frac{1}{1-e^{2it}z^2}\, d\mu(t).$$

and the result follows by expanding $(1-e^{2it}z^2)^{-1}$ in powers of z^2.

Since for each $t (0 \le t \le 2\pi)$

$$q(z, t) = a\,\frac{1+e^{it}z}{1-e^{it}z}\cdot[p(z)]^\beta \tag{2.10}$$

is subordinate to $\left[\frac{1+cz}{1-z}\right]^{\beta+1}$ (with $c = b^{\frac{2\beta}{\beta+1}}$), it follows from Corollary 2.1 that

$$\int_0^{2\pi} q(z\,,\,t)e^{2jit}d\mu(t) \ll \left[\frac{1+cz}{1-z}\right]^{\beta+1} \quad (j = 0\,,1\,,2\,,\ldots)\,. \tag{2.11}$$

Combining (2.8), (2.10) and (2.11), we easily obtain a simple proof of the coefficient conjecture for $K(\beta)$ for functions with real coefficients. More generally we have

Theorem 2.3. *Let* $f(z) \in K(\beta)$ *where* $\beta = k/2 - 1 > 0$ *and suppose that in* (2.2)

$$zf'(z) = a[p(z)]^\beta s(z)$$

with $p(0) = 1$. *Then* $f(z) \ll f_k(z)$ *where* $f_k(z)$ *is defined by* (1.2).

Proof. If $1 = p(0) = b$, then in (2.11) $c = 1$. Since for any $\gamma > 0$, $\left[\frac{1+z}{1-z}\right]^\gamma$ has positive coefficients it follows from (2.8), (2.10) and (2.11) that

$$\begin{aligned} f'(z) &\ll \sum_{k=0}^{\infty} \left[\frac{1+z}{1-z}\right]^{\beta+1} z^{2k} \\ &= \left[\frac{1+z}{1-z}\right]^{\beta+1} \cdot \frac{1}{1-z^2} \\ &= f_k'(z)\,. \end{aligned}$$

We note that if $f(z) \in K(\beta)$ and has real coefficients then it follows that the $p(z)$ in (2.7) can be chosen so that $p(0) = 1$. The following simple proof of this fact was suggested by Ch. Pommerenke. If $f(z)$ has real coefficients

$$zf'(z) = a[p(z)]^\beta s(z)$$

and

$$zf'(z) = \bar{a}[\overline{p(\bar{z})}]^\beta \overline{s(\bar{z})}\,.$$

Thus

$$zf'(z) = [(p(z)\overline{p(\bar{z})})^{1/2}]^\beta \cdot [s(z) \cdot \overline{s(\bar{z})}]^{1/2}$$

and $(p(z)\overline{p(\bar{z})})^{1/2} \in \gamma$ (and has real coefficients) and $[s(z)\overline{s(\bar{z})}]^{1/2}$ is a univalent starlike mapping of $|z| < 1$.

The proof of the coefficient conjecture for $K(\beta)$ would follow from (2.8), (2.10) and (2.11) in precisely the same way if one could show that for $|\varkappa| = 1$ and $\alpha \geq 1$,

$$\left[\frac{1+\varkappa z}{1-z}\right]^{\alpha} \ll \left[\frac{1+z}{1-z}\right]^{\alpha} . \tag{2.12}$$

(2.12) is trivially true if α is an integer and hence gives a proof of the coefficient conjecture for V_k when k is an even integer (see [11] for a different proof of this fact). We suspect that (2.12) is true for arbitrary $\alpha \geq 1$ but have been unable to find a proof. On the other hand our next result shows that for $\alpha \geq 1$, $\left[\frac{1+\varkappa z}{1-z}\right]^{\alpha} \underset{13}{\ll} \left[\frac{1+z}{1-z}\right]^{\alpha}$, which proves the coefficient conjecture for $K(\beta)$ for $n \leq 14$.

Theorem 2.4. *Let* $f(z) \in K(\beta)$ $(\beta = k/2 - 1 > 0)$. *Then*

$$f(z) \underset{14}{\ll} f_k(z) .$$

Proof. As noted above the result follows from (2.8), (2.10) and (2.11) once we show that

$$\left[\frac{1+\varkappa z}{1-z}\right]^{\alpha} \underset{13}{\ll} \left[\frac{1+z}{1-z}\right]^{\alpha} \qquad (\alpha \geq 1) .$$

As observed above (2.12) is true when α is a positive integer. Hence if we restrict in the first instance α to satisfy $1 < \alpha < 2$ the general result will follow by considering $\left[\frac{1+\varkappa z}{1-z}\right]^{\alpha+n}$ with $1 < \alpha < 2$ and n a positive integer. This depends on the fact that $\left(\frac{1+z}{1-z}\right)^{\gamma}$ has positive coefficients for any $\gamma > 0$. We assume in what follows, therefore, that $1 < \alpha < 2$.

Let

$$\left[\frac{1+\varkappa z}{1-z}\right]^{\alpha} = \sum_{m=0}^{\infty} A_m(\varkappa) z^m .$$

Then for $m \geq 1$,

$$\begin{aligned} A_m(\varkappa) \equiv A_m(\varkappa, \alpha) &= \sum_{\nu=0}^{m} \frac{\alpha(\alpha+1)\ldots(\alpha+m-\nu-1)}{(m-\nu)!} \binom{\alpha}{\nu} \varkappa^{\nu} \\ &= a_0(m) + a_1(m)\varkappa + a_2(m)\varkappa^2 + \sum_{\nu=3}^{m} (-1)^{\nu} a_{\nu}(m)\varkappa^{\nu} , \end{aligned} \tag{2.13}$$

where each $a_\nu(m) > 0$. Since

$$\left[\frac{1+z}{1-z}\right]^{\alpha} = \sum_{m=0}^{\infty} A_m(1) z^m$$

we must show $|A_m(\varkappa)| \leq A_m(1)$ for $m \leq 13$. This is easily seen to be the case for $m \leq 4$. However to prove the result for $m \leq 13$, it is more convenient to prove a stronger statement, namely,

(2.14) $$|A_m(\varkappa)|^2 + |A_m(-\bar{\varkappa})|^2 \leq A_m^2(1) + A_m^2(-1) = A_m^2(1) .$$

If $\varkappa = e^{i\varphi}(0 \leq \varphi \leq 2\pi)$, then

(2.15) $$\begin{aligned} &|A_m(e^{i\varphi})|^2 + |A_m(-e^{-i\varphi})|^2 \\ &= 2(a_0^2 + \ldots + a_m^2) + 4\sum_{j=1}^{M}(a_0a_{2j} - a_1a_{2j+1} + \sum_{s=2}^{m-2j} a_s a_{2j+s}) \cos 2j\varphi , \end{aligned}$$

where $M = m/2$ if m is even and $(m-1)/2$ if m is odd. (2.14) is certainly valid if for each j the coefficient of $\cos 2j\varphi$ in (2.15) is positive. This is clearly the case if

(2.16) $$a_0a_{2j} - a_1a_{2j+1} \geq 0 \qquad (1 \leq j \leq M)$$

i.e.

$$\frac{a_0}{a_1} \geq \frac{a_{2j+1}}{a_{2j}} \qquad (1 \leq j \leq M) .$$

Now

$$\frac{a_0}{a_1} = \frac{\alpha + m - 1}{\alpha m}$$

and

$$\frac{a_{2j+1}}{a_{2j}} = \frac{m - 2j}{\alpha + m - 2j - 1} \cdot \frac{2j - \alpha}{2j + 1} .$$

Thus (2.16) is equivalent to

$$\frac{\alpha + m - 1}{\alpha m} \geq \frac{m - 2j}{\alpha + m - 2j - 1} \cdot \frac{2j - \alpha}{2j + 1} \quad (j \leq M,\ 1 < \alpha < 2,\ m \geq 2) .$$

When $m \leq 12$, or $m = 13$ and $j = 1, 2, 3$, and 6 this inequality is easily verified by lengthy but elementary calculation. If $m = 13$ and $j = 4$ or 5 we show that the coefficient of $\cos 8\varphi$ and $\cos 10\varphi$ are positive by showing that

$$a_0a_8 - a_1a_9 + a_2a_{10} > 0 \qquad (j = 4)$$

and

$$a_0a_{10} - a_1a_{11} + a_2a_{12} > 0 \qquad (j = 5) .$$

Again the required calculation is elementary and will be left to the reader. This completes the proof of (2.14) and hence the theorem.

It seems clear that by using additional terms from the coefficients of $\cos 2j\varphi$ in (2.15) that (2.14) and hence the theorem can be extended to larger values of m. What is not clear is whether a proof for all m can be found by this method.

3. Uniform coefficient estimates

Theorem 3.1. *Let* $f(z) \in K(\beta)$ $(\beta = k/2 - 1 > 0)$. *Then*

$$f(z) \ll 1.05\, f_k(z)\,.$$

The proof of this theorem follows immediately from (2.8), (2.10) and (2.11) provided we show that (with the notation of § 2)

$$|A_m(e^{i\varphi})| < 1.05\, A_m(1) \qquad (m \geq 1)\,. \tag{3.1}$$

It is only necessary to prove (3.1) for α in the range $1 < \alpha < 2$. The observation which enabled the proof of Theorem 2.4 to be restricted to the range $1 < \alpha < 2$ also applies in the present situation. The proof of (3.1) for $1 < \alpha < 2$ is established with the aid of two lemmas. First we introduce some notation. Let

$$B_m(\varkappa) = a_1\varkappa - \sum_{j=1}^{M} a_{2j+1}\varkappa^{2j+1}$$

and

$$C_m(\varkappa) = \sum_{j=1}^{M} a_{2j}\,\varkappa^{2j}\,,$$

where the a_k are defined by (2.13), and M is defined in (2.15). Then

$$B_m(\varkappa) = \tfrac{1}{2}\{A_m(\varkappa) - A_m(-\varkappa)\}$$
$$C_m(\varkappa) = \tfrac{1}{2}\{A_m(\varkappa) + A_m(-\varkappa)\}\,.$$

and

$$\sum_{m=1}^{\infty} B_m(\varkappa)z^m = \frac{(1+\varkappa z)^\alpha - (1-\varkappa z)^\alpha}{2(1-z)^\alpha}$$
$$\sum_{m=0}^{\infty} C_m(\varkappa)z^m = \frac{(1+\varkappa z)^\alpha + (1-\varkappa z)^\alpha}{2(1-z)^\alpha}\,. \tag{3.2}$$

It follows immediately from (3.2) that $B_m(1) > 0\,(m \geq 1)$.

Lemma 3.1. *Let* $1 \leq m \leq N$. *There exists a polynomial* $\gamma_{N,m}(\varkappa)$ *with non-negative coefficients and a positive number* $\lambda_{N,m}$ *such that*

$$B_m(\varkappa) = \gamma_{N,m}(\varkappa) + \lambda_{N,m} B_N(\varkappa) .$$

Proof. For a given m it follows from the definition of the a_k in (2.13) that

$$B_m(\varkappa) - \frac{m-1}{\alpha + m - 2} B_{m+1}(\varkappa) = \gamma_{m+1}(\varkappa) \tag{3.3}$$

is a polynomial with non-negative coefficients. The lemma then follows by repeated application of (3.3).

Lemma 3.2.. *Let* $\varkappa$ *be given with* $|\varkappa| = 1$. *If* N *is sufficiently large then*

$$|B_j(\varkappa)| < 1.10\, B_j(1) \qquad (j \geq N) .$$

Proof. Since

$$\sum_{m=1}^{\infty} B_m(\varkappa) z^m = \frac{(1+\varkappa z)^\alpha - (1-\varkappa z)^\alpha}{2(1-z)^\alpha} ,$$

a standard application of the 'major-minor arc' method as it appears in [7, p. 108] shows that $B_m(\varkappa)$ is asymptotic to the m^{th} coefficient of $\dfrac{(1+\varkappa)^\alpha - (1-\varkappa)^\alpha}{2(1-z)^\alpha}$ as $m \to \infty$, i.e.

$$B_m(\varkappa) \sim \frac{(1+\chi)^\alpha - (1-\chi)^\alpha}{2} (-1)^m \binom{-\alpha}{m} \qquad (m \to \infty) .$$

Let $\varkappa = e^{2i\varphi} \left(-\dfrac{\pi}{2} \leq \varphi \leq \dfrac{\pi}{2} \right)$. Then

$$\begin{aligned} g_\alpha(\varkappa) &= |(1+\varkappa)^\alpha - (1-\varkappa)^\alpha| \\ &= 2^\alpha \left[\cos^{2\alpha}\varphi + \sin^{2\alpha}\varphi - 2 \cos \alpha \frac{\pi}{2} \cdot \sin^\alpha\varphi \cdot \cos^\alpha\varphi \right]^{1/2} . \end{aligned}$$

If $1 < \alpha < 2$ is fixed, $g_\alpha(\varkappa)$ attains its maximum at $\varphi = \pi/4$ and

$$2^{-\alpha} g_\alpha(i) = 2^{1-\alpha/2} \sin \frac{\alpha\pi}{4} .$$

Further $2^{-\alpha} g_\alpha(i)$ for varying α assumes its maximum when

$$\begin{aligned} \alpha &= \frac{4}{\pi} \arctan \left(\frac{\pi}{2 \log 2} \right) \\ &= 1.47 \ldots . \end{aligned}$$

Thus $g_\alpha(\varkappa) < 1.10\; 2^\alpha\; (1 < \alpha < 2)$, and so for given χ , if N is sufficiently large (depending upon χ) ,

$$|B_m(\varkappa)| < 1.10\ 2^{\alpha-1}(-1)^m \binom{-\alpha}{m} .$$

Since

$$B_m(1) \sim 2^{\alpha-1}(-1)^m \binom{-\alpha}{m} \qquad (m \to \infty) ,$$

the proof is complete.

We now complete the proof of (3.1). Let $\varkappa$, $|\varkappa| = 1$, be fixed and let $m \geq 2$ be a positive integer. ((3.1) is trivially true if $m = 1$.) By Lemma 3.2 we can choose an $N > m$ sufficiently large that

$$|B_N(\varkappa)| < 1.10\, B_N(1) .$$

By Lemma 3.1,

$$\begin{aligned} |B_m(\varkappa)| &= |\gamma_{N,m}(\varkappa) + \lambda_{N,m} B_N(\varkappa)| \\ &\leq \gamma_{N,m}(1) + \lambda_{N,m}|B_N(\varkappa)| \\ &\leq \gamma_{N,m}(1) + 1.10\, \lambda_{N,m} B_N(1) \\ &< 1.10\, B_m(1) . \end{aligned}$$

Now

$$A_m(\varkappa) = B_m(\varkappa) + C_m(\varkappa)$$

and since the coefficients of $C_m(\chi)$ are non-negative,

$$\begin{aligned} |A_m(\varkappa)| &\leq |B_m(\varkappa)| + |C_m(\varkappa)| \\ &< 1.10\, B_m(1) + C_m(1) . \end{aligned}$$

It follows from (3.2) that $B_m(1) = C_m(1)$ $(m \geq 1)$ and hence

$$\begin{aligned} |A_m(\varkappa)| &< 1.10\, B_m(1) + C_m(1) \\ &= 2.10\, B_m(1) \\ &= 1.05\, A_m(1) . \end{aligned}$$

This establishes (3.1) and completes the proof of Theorem 3.1.

Furthermore we have

Theorem 3.2. *Let* $f(z) \in K(\beta)$ $(\beta = k/2 - 1 > 1)$. *Then*

$$f(z) \ll f_k(z) .$$

Proof. Using the notation of the above, it is clear that to prove our result it is sufficient to prove that

$$|A_m(e^{i\varphi})| \leq A_m(1) \qquad (m \geq 1) \tag{3.4}$$

for $2 \leq \lambda \leq 3$. Reworking the proof of Theorem 3.1 and making the necessary changes we can verify (3.4) without much trouble.

4. Extreme points of $K(\beta)$ and V_k.

In [3] the authors determined the set of extreme points of various classes of univalent functions. Combining Theorem 2.2 with a method from [3] we obtain a generalization of [3, Theorem 6]. The following notation will simplify the statement of our theorem.

For $\beta > 0$, $0 \leq t < 2\pi$ and $0 \leq \tau \leq 2\pi$ set

$$\begin{aligned} g(z\,;e^{it}, e^{i\tau}) &= \frac{1}{\beta+1}\cdot\frac{1}{e^{it}-e^{i\tau}}\left[\left(\frac{1+e^{it}z}{1+e^{i\tau}z}\right)^{\beta+1}-1\right] \qquad (t \neq \tau) \\ g(z\,;e^{it}, e^{it}) &= \lim_{\tau\to t} g(z\,;e^{it}, e^{i\tau}) = \frac{z}{1+e^{it}z}\,, \end{aligned} \tag{4.1}$$

such that and let $\mathcal{K}_\beta$ denote the set of functions

$$F(z) = \int_0^{2\pi}\!\!\int_0^{2\pi} g(z\,;e^{it}, e^{i\tau})\, d\nu(t\,,\tau)\,,$$

where $\nu(t\,,\tau)$ ranges over the positive Borel measures on $[0\,,2\pi]\times[0\,,2\pi]$

$$\int_0^{2\pi}\!\!\int_0^{2\pi} d\nu(t\,,\tau) = 1\,.$$

Theorem 4.1. *For* $\beta = \frac{1}{2}k - 1 \geq 1$,

$$\mathcal{K}_\beta = \overline{co}\,K(\beta) = \overline{co}\,V_k\,.$$

Further, the extreme points of $\overline{co}\,K(\beta)$ $(=\overline{co}\,V_k)$ *are precisely the functions* (4.1) *with* $t \neq \tau$.

Proof. Let $F(z) \in K(\beta)$. Then by (2.2), (2.9) and the Herglotz formula

$$F'(z) = \left[\int_0^{2\pi}\frac{1+ce^{it}z}{1-e^{it}z}\,d\mu(t)\right]^{\beta}\cdot\int_0^{2\pi}\frac{1}{(1-e^{i\tau}z)^2}\,d\nu(\tau),$$

where $\mu(t)$ and $\nu(\tau)$ are increasing functions on $[0\,,2\pi]$ with

$$\mu(2\pi)-\mu(0) = 1 = \nu(2\pi)-\nu(0)\,.$$

Since $\beta \geq 1$ it follows from Theorem 2.2 that there exists an increasing function $\gamma(t)$ on $[0\,,2\pi]$ with $\gamma(2\pi)-\gamma(0) = 1$ such that

$$F'(z) = \int_0^{2\pi} \left[\frac{1 + ce^{it}z}{1 - e^{it}z}\right]^\beta d\gamma(t) \cdot \int_0^{2\pi} \frac{1}{(1 - e^{i\tau}z)^2} d\nu(\tau)$$

(4.2)

$$= \int_0^{2\pi}\int_0^{2\pi} \left[\frac{1 + ce^{it}z}{1 - e^{it}z}\right]^\beta \frac{1}{(1 - e^{i\tau}z)^2} d\gamma(t)d\nu(\tau) .$$

By (2.5), for each t and τ in $[0 , 2\pi]$,

$$\left[\frac{1 + ce^{it}z}{1 - e^{it}z}\right]^\beta \frac{1}{(1 - e^{i\tau}z)^2} = \int_0^{2\pi} \frac{(1 + ce^{it}z)^\beta}{(1 - e^{i\varphi}z)^{\beta+2}} d\sigma(\varphi) , \tag{4.3}$$

where $\sigma(\varphi)$ is increasing on $[0 , 2\pi]$ with $\sigma(2\pi) - \sigma(0) = 1$. It follows that

$$\left[\frac{1 + ce^{it}z}{1 - e^{it}z}\right]^\beta \frac{1}{(1 - e^{i\tau}z)^2}$$

belongs to $\mathcal{K}'_\beta$, the set of derived functions $g'(z)$ of functions $g(z) \in \mathcal{K}_\beta$. Since $\mathcal{K}'_\beta$ is closed and convex, (4.2) and (4.3) imply that $F'(z) \in \mathcal{K}'_\beta$ and hence $\overline{co}\, K(\beta) \subset \mathcal{K}'_\beta$.

The reverse inclusion follows with only a slight modification of the argument in [3, Theorem 6].

As is easily seen, the functions (4.1) belong to $V_k \subset K(\beta)$ and the remainder of the proof follows using the arguments in [3, Theorem 6].

Our result leaves open the question of whether or not the functions (4.1) $(t \neq \tau)$ are the extreme points for $K(\beta)$ when $0 < \beta < 1$. We note in passing that using (2.5) one can easily show that the set of functions $z\,(1 - e^{i\tau}z)^{-\alpha}$ $(0 \leq \tau \leq 2\pi)$ are precisely the set of extreme points for the functions starlike of order α , $0 \leq \alpha \leq 1$.

If I is a continuous linear functional on A (the space of functions analytic in $|z| < 1$) then the maximum of $Re\, I$ on $\overline{co}\, K(\beta)$ must be attained at some extreme point. Consequently, there is a function of the form (4.1) that solves the coefficient conjecture for V_k if $k \geq 4$. This observation greatly simplifies what was previously known about the nature of the extremal functions for this problem (see e.g. [4]).

Addendum. (2.12) has now been established by D. Aharonov and S. Friedland and their proof will be contained in a forthcoming paper.

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