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# MEASURE PROPERTIES OF THE BRANCH SET AND ITS IMAGE OF QUASIREGULAR MAPPINGS

BY

O. MARTIO and S. RICKMAN

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SUOMALAINEN TIEDEAKATEMIA

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## 1. Introduction

Let $m_k$ denote the Lebesgue measure in the $k$-dimensional euclidean space $R^k$, $H^\alpha$ the normalized $\alpha$-dimensional Hausdorff measure in $R^n$, $\alpha \leq n$, and $C^k$ the $k$-dimensional cut measure in $R^n$, i.e. $C^k(S) = \sup m_k(S \cap P)$ over all $k$-dimensional planes $P$ in $R^n$. Note that $C^k(S) \leq H^k(S)$ and $C^n = H^n = m_n$ in $R^n$.

Given a continuous, discrete, and open mapping $f: G \to R^2$ with $G$ a domain in $R^2$, it is well known that the branch set $B_f$ of $f$ is a discrete set of points in $G$. Indeed, by Stoïlow's theorem $f$ can be represented in the form $f = g \circ h$ where $h$ is a homeomorphism and $g$ an analytic function. Hence $H^1(B_f) = H^1(fB_f) = 0$ and, if $B_f \neq \emptyset$, $H^0(B_f) > 0$ and $H^0(fB_f) > 0$.

If $f: G \to R^n$, $n \geq 3$, is continuous, discrete, and open with $B_f \neq \emptyset$, then in [7] (cf. also [9]) it was shown that $H^{n-2}(fB_f) > 0$. The argument was topological: If $y \in fB_f$, then $\complement fB_f$ has a non-trivial homotopy at $y$ [1].

A natural generalization of complex analytic functions to $R^n$ seems to be the class of quasiregular mappings. For the theory of these mappings we refer to [4—7]. If $f: G \to R^n$ is a non-constant quasiregular mapping, then $f$ is discrete and open [11]. One might conjecture that the classes of quasiregular mappings and discrete open mappings are the same from the topological point of view also for $n \geq 3$ as is the case in plane. However, in [7] it was shown that there exists in $R^n$, $n \geq 3$, a discrete and open mapping which is not topologically equivalent to any quasiregular mapping. Quasiregularity also imposes metric conditions on $B_f$ and $fB_f$. In [5] it was proved that $m_n(B_f) = m_n(fB_f) = 0$ for a non-constant quasiregular mapping $f: G \to R^n$. In [13] Rešetnjak proved that $C^{n-1}(B_f) = 0$.

In this paper we extend some of the above results. Given a discrete and open mapping $f: G \to R^3$ with $B_f \neq \emptyset$ we prove in Section 2 that $H^1(B_f) > 0$. For this we use a result of Papakyriakopoulos [8] to show that $\complement B_f$ has a non-trivial homotopy at some $x \in B_f$. The fact $H^1(B_f) > 0$ can also be derived from a result of Trohimčuk [15] but our arguments are different. In Section 3 we prove by using the method of Rešetnjak that $C^{n-1}(fB_f) = 0$ if $f: G \to R^n$ is a quasiregular mapping. In Section 4 we derive a lower bound for the Hausdorff dimension $\dim_H B_f$ of $B_f$ of a quasiregular mapping $f : G \to R^n$ with $B_f \neq \emptyset$. This lower bound depends only on $n$ and the dilatation of $f$.

Our notation is mainly that of [5].

## 2. On $\mathbf{B_f}$ and $\mathbf{fB_f}$ of discrete and open mappings

2.1. *Normal neighborhoods and covering spaces.* For $Y \subset R^n$ and $y \in Y$ we let $\pi_1(Y, y)$ be the first homotopy group of $Y$ at $y$. If $Y$ is pathwise connected, these groups for different $y$'s are all isomorphic and will be denoted also by $\pi_1(Y)$. If $\alpha : I \to Y$, $I = [0, 1]$, is a loop with base point $y \in Y$, i.e. $\alpha(0) = \alpha(1) = y$, we let $[\alpha]$ denote its homotopy class in $\pi_1(Y, y)$. The $k$-times product of a loop $\alpha$ is denoted by $\alpha^k$, the constant loop with base point $y$ is $\varepsilon_y$, and $\sim$ is the homotopy relation.

Let $f : G \to R^n$ be discrete and open where we always assume that $n \geq 2$, $G$ is a domain in $R^n$, and that $f$ is continuous. Given $x \in G$ we recall that a domain $D$ is called a normal neighborhood of $x$ if (1) $\bar{D}$ is a compact subset of $G$, (2) $f\partial D = \partial f D$, and (3) $f^{-1}(f(x)) \cap D = \{x\}$ [5, 2.1]. The property (2) means that $f|D : D \to fD$ is a closed mapping and (3) implies that $|i(x, f)| = \operatorname{card}(f^{-1}(y) \cap D)$ for every $y \in fD \setminus f(D \cap B_f)$ [5, 2.12]. Here $i(x, f)$ is the local topological index of $f$ at $x$ [5, p. 6]. By [5, 2.10] there exist arbitrarily small normal neighborhoods for every $x \in G$.

We denote by $R(f)$ the set of points $x \in G$ for which there exists a normal neighborhood $D$ of $x$ such that

$$B_f \cap D = f^{-1}(f(B_f \cap D)) \cap D \,. \tag{2.2}$$

If $x \in G \setminus B_f$, then $B_f \cap D = \emptyset$ for every normal neighborhood $D$ of $x$ [5, 2.12], hence $R(f) \supset G \setminus B_f$.

2.3. **Lemma** [1, Theorem 2.2] *The set $R(f) \cap B_f$ is dense in $B_f$. Moreover, the points $x \in B_f$ for which there exists a normal neighborhood $D$ of $x$ such that $f|B_f \cap D : B_f \cap D \to f(B_f \cap D)$ is a homeomorphism are dense in $B_f$.*

2.4. *Remark.* The condition (2.2) means that

$$f|D \setminus B_f : D \setminus B_f \to f(D \setminus B_f)$$

is a covering mapping. Note that for any domain $U$, $U \setminus B_f$ and $f(U \setminus B_f)$ are domains in $R^n$ since $\dim B_f \leq n - 2$ [16].

2.5. Suppose that $x \in R(f)$ and that $D$ is a normal neighborhood of $x$ with the property (2.2). Set $\tilde{X} = D \setminus B_f$, $X = f\tilde{X}$, and $p = f|\tilde{X}$. Then $(\tilde{X}, p)$ is a covering space of $X$ and the group $\pi_1(X, y)$ operates transitively on the right on the set $p^{-1}(y)$ for every $y \in X$. For $z \in p^{-1}(y)$ and $c \in \pi_1(X, y)$ we denote this action by $z \cdot c$, i.e. $z \cdot c \in p^{-1}(y)$ is the terminal point of the lifts of the representatives of $c$ starting at $z$ [14, p. 71]. Hence especially:

2.6. **Lemma.** *If $\gamma : I \to \tilde{X}$ is a path which joins two distinct points in $p^{-1}(y)$, then $p \circ \gamma \not\sim \varepsilon_y$ in $X$.*

We let $\tilde{R}(f)$ denote the set of points $x \in R(f)$ for which there exists a normal neighborhood $D$ with the property (2.2) and such that the projection $p$ is regular, i.e. either every lift of a loop $\alpha$ in $X$ is a loop or no lift of $\alpha$ is a loop. Note that if $D' \subset D$ is a normal neighborhood of $x$, then also $D'$ satisfies (2.2) and the covering projection $f|D' \setminus B_f : D' \setminus B_f \to f(D' \setminus B_f)$ is regular if $p$ is regular. Clearly $\tilde{R}(f) \supset G \setminus B_f$.

2.7. **Lemma.** *Suppose that $x \in R(f) \setminus \tilde{R}(f)$. Let $D$ be a normal neighborhood of $x$ such that (2.2) holds. If $U \subset D$ is a connected neighborhood of $x$, then $\pi_1(U \setminus B_f)$ is non-trivial.*

*Proof.* Let $D' \subset U$ be a normal neighborhood of $x$ such that $fD' \subset fU \setminus f\partial U$. Set $\tilde{X}' = D' \setminus B_f$, $p' = f|\tilde{X}'$, $X' = p'\tilde{X}'$. Since $x \in R(f) \setminus \tilde{R}(f)$, $p'$ is not a regular projection of the covering space $(\tilde{X}', p')$. Hence there exists a loop $a : I \to X'$ which has two lifts $\alpha_1$ and $\alpha_2$ such that $\alpha_1$ is a loop and $\alpha_2$ is not a loop in $\tilde{X}'$. If the base point of $\alpha_1$ is $z$ and if $\alpha_1 \sim \varepsilon_z$ in $\tilde{X}$, we have $\alpha = f \circ \alpha_1 \sim \varepsilon_{f(z)}$ in $X$ and hence by 2.6 $\alpha_2$ would be a loop. Thus $\alpha_1 \not\sim \varepsilon_z$ in $\tilde{X}$ and also $\alpha_1 \not\sim \varepsilon_z$ in $U \setminus B_f$. The lemma follows.

2.8. **Lemma.** *Suppose that $x \in R(f) \cap B_f$. Let $D$ be a normal neighborhood of $x$ such that (2.2) holds. If $\pi_1(U \setminus B_f)$ is trivial for some connected neighborhood $U \subset D$ of $x$, then there exists a loop $\alpha$ in $f(U \setminus B_f)$ such that $\alpha$ is not homotopic to a constant loop in $f(D \setminus B_f)$ and $\alpha^{|i(x,f)|}$ is homotopic to a constant loop in $f(U \setminus B_f)$.*

*Proof.* Let $D'$, $\tilde{X}'$, $X'$, and $p'$ be as in the proof of 2.7. Pick $y \in X'$ and let $\gamma : I \to \tilde{X}'$ be a path joining two distinct points of $p'^{-1}(y)$. By

2.6 $\alpha = f \circ \gamma \not\sim \varepsilon_y$ in $X = f(D \setminus B_f)$. Set $p'^{-1}(y) = \{x_1, \ldots, x_k\}$, $k = |i(x, f)|$. Choose a maximal set $\{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$, $l \leq k$, inductively as follows: Set $i_1 = 1$, and if $i_1, \ldots, i_j$ are chosen, pick $i_{j+1}$ so that

$$i_{j+1} \notin \{r \mid x_r = x_{i_q} \cdot [\alpha^m] \text{ for some } m = 1, 2, \ldots, 1 \leq q \leq j\}.$$

Then for each $j$, $1 \leq j \leq l$, there exists an integer $m(j)$, $1 \leq m(j) \leq k$, such that $m(j) = \inf\{r \mid x_{i_j} = x_{i_j} \cdot [\alpha^r],\ r = 1, 2, \ldots\}$. Set

$$A_j = \{x_r \mid x_r = x_{i_j} \cdot [\alpha^q],\ 1 \leq q \leq m(j)\}$$

and let $\gamma_j$ be the lift of $\alpha^{m(j)}$ starting at $x_{i_j}$. Then (i) $\bigcup A_j = p'^{-1}(y)$, (ii) $A_j'$s are disjoint, and (iii) $\gamma_j$ is a loop in $U \setminus B_f$. If $\pi_1(U \setminus B_f)$ is trivial, $\gamma_j \sim \varepsilon_{x_{i_j}}$ in $U \setminus B_f$ and hence $f \circ \gamma_j = \alpha^{m(j)} \sim \varepsilon_y$ in $f(U \setminus B_f)$. Thus $[\alpha]$ is of finite order $m$ in $\pi_1(f(U \setminus B_f), y)$. On the other hand, (i) and (ii) imply

$$\sum_{j=1}^{l} m(j) = \operatorname{card} p'^{-1}(y) = k = |i(x, f)|.$$

Because $m$ divides each $m(j)$, it also divides $|i(x, f)|$. Thus $\alpha^{|i(x,f)|} \sim \varepsilon_y$ in $f(U \setminus B_f)$.

2.9. *Local homotopy properties of* $\complement B_f$ *and* $\complement f B_f$. We recall the definition of a trivial homotopy at a point (cf. [1, Definition 6]):

2.10. **Definition.** Suppose that $C \subset R^n$ and $x \in C$. We say that $\complement C$ has a trivial homotopy at $x$ if there exist arbitrarily small neighborhoods $U$ of $x$ such that $U \setminus C$ is pathwise connected and $\pi_1(U \setminus C)$ is trivial.

2.11. **Lemma.** [1, Lemma 5.8] *Suppose that* $U \subset R^n$ *is a domain and* $C$ *closed in* $U$ *with* $\dim C \leq n - 2$. *If* $\pi_1(U \setminus C)$ *is trivial, then for any* $Z \subset C$ *closed in* $U$, $\pi_1(U \setminus Z)$ *is trivial.*

2.12. **Corollary.** *Suppose that* $V \subset R^n$ *is open,* $C$ *closed in* $V$ *with* $\dim C \leq n - 2$, *and* $x \in C$. *Then* $\complement C$ *has a trivial homotopy at* $x$ *if and only if there exist arbitrarily small simply connected neighborhoods* $U$ *of* $x$ *such that* $\pi_1(U \setminus C)$ *is trivial.*

*Proof.* The condition is clearly sufficient. The converse follows from 2.11 since if $U \subset V$ is a neighborhood of $x$ such that $U \setminus C$ is pathwise connected and $\pi_1(U \setminus C)$ is trivial, then $U$ is connected and for $Z = \emptyset$ 2.11 implies that $\pi_1(U)$ is trivial.

2.13. **Theorem.** *Suppose that* $f : G \to R^n$ *is discrete and open. If* $D$ *is a normal neighborhood of* $x \in B_f$ *with the property* (2.2), *then* $\complement f(D \cap B_f)$ *has a non-trivial homotopy at* $f(x)$. *Moreover, if* $U \subset D$ *is a connected neighborhood of* $x$, *then* $\pi_1(f(U \setminus B_f))$ *is non-trivial.*

*Proof.* Let $D$ be a normal neighborhood of $x \in B_f \cap R(f)$ such that (2.2) holds. Let $U \subset D$ be a connected neighborhood of $x$. Pick a normal neighborhood $D' \subset U$ of $x$ such that $fD' \subset fU \setminus f\partial U$. Let $y \in f(D' \setminus B_f)$. By 2.6 there exists a loop $\alpha$ in $f(D' \setminus B_f)$ with base $y$ such that $\alpha \not\sim \varepsilon_y$ in $f(D \setminus B_f)$, hence $\alpha \not\sim \varepsilon_y$ in $f(U \setminus B_f)$. Thus $\pi_1(f(U \setminus B_f), y)$ is non-trivial and the last statement is proved. Let now $V \subset fD$ be a connected neighborhood of $f(x)$. Then $U = f^{-1}V \cap D$ is connected [5, 2.6] and $f(U \setminus B_f) = V \setminus f(D \cap B_f)$, hence $\pi_1(V \setminus f(D \cap B_f))$ is non-trivial. The theorem follows.

2.14. *Remark.* In [1, Theorem 5.9] Church and Hemmingsen proved that $\complement fB_f$ has a non-trivial homotopy at $f(x)$ for every $x \in B_f$.

Next we shall study the homotopy properties of $B_f$. Lemma 2.7 gives the following result:

2.15. **Theorem.** *Suppose that* $f : G \to R^n$ *is discrete and open, and that* $x \in R(f) \setminus \tilde{R}(f)$. *Then* $\complement B_f$ *has a non-trivial homotopy at* $x$.

From Lemma 2.8 we obtain:

2.16. **Theorem.** *Suppose that* $f : G \to R^n$ *is discrete and open,* $x \in B_f \cap R(f)$, *and* $D$ *is a normal neighborhood of* $x$ *with the property* (2.2). *If there exists a connected neighborhood* $U' \subset D$ *of* $x$ *such that* $\alpha^{|i(x,f)|} \not\sim \varepsilon_y$ *in* $f(U' \setminus B_f)$ *whenever* $\alpha$ *is a loop in* $f(U' \setminus B_f)$ *with base* $y$ *and* $\alpha \not\sim \varepsilon_y$ *in* $f(U' \setminus B_f)$, *then* $\complement B_f$ *has a non-trivial homotopy at* $x$.

*Proof.* Let $U'$ be a connected neighborhood of $x$ as in the theorem and let $U \subset U'$ be a connected neighborhood of $x$. If $\pi_1(U \setminus B_f)$ is trivial, then by 2.8 there exists a loop $\alpha$ in $f(U \setminus B_f)$ such that $\alpha$ is not homotopic to a constant loop in $f(D \setminus B_f)$ and $\alpha^{|i(x,f)|}$ is homotopic to a constant loop in $f(U \setminus B_f)$. Let $y$ be the base point of $\alpha$. Hence $\alpha \not\sim \varepsilon_y$ in $f(U' \setminus B_f)$ and $\alpha^{|i(x,f)|} \sim \varepsilon_y$ in $f(U' \setminus B_f)$, a contradiction by assumption. Thus $\pi_1(U \setminus B_f)$ is non-trivial and the theorem follows.

2.17. **Theorem.** *Suppose that* $f: G \to R^3$ *is discrete and open. Then* $\complement B_f$ *has a non-trivial homotopy at every point of* $B_f \cap R(f)$.

*Proof.* Let $x \in B_f \cap R(f)$ and let $D$ be a normal neighborhood of $x$ with the property (2.2). Let $U' \subset D$ be a connected neighborhood of $x$. By 2.13 $\pi_1(f(U' \setminus B_f))$ is non-trivial and by [8, Corollary 31.8] $\pi_1(f(U' \setminus B_f))$ contains no element of finite order, hence by 2.16, $\complement B_f$ has a non-trivial homotopy at $x$.

2.18. *Homotopy and measure.* Here we study the $(n-2)$-dimensional Hausdorff measure of $B_f$ and $fB_f$.

2.19. **Lemma.** *Suppose that* $U \subset R^n$ *is open and* $C \subset U$ *is closed in* $U$. *If* $\complement C$ *has a non-trivial homotopy at* $x$ *for some* $x \in C$, *then* $H^{n-2}(C) > 0$.

*Proof.* If $\dim C \geq n-1$, $H^{n-2}(C) = \infty$ by [3, Theorem VII 2, p. 104] If $\dim C \leq n-2$, there exists by 2.12 a simply connected neighborhood $U' \subset U$ of $x$ such that $\pi_1(U' \setminus C)$ is non-trivial. The lemma now follows from [7, 3.3].

2.20. **Theorem.** *Suppose that* $f: G \to R^n$ *is discrete and open, and* $B_f \neq \emptyset$. *Then*

(a) $H^{n-2}(fB_f) > 0$ *for* $n \geq 2$.

(b) $H^{n-2}(B_f) > 0$ *for* $n = 2, 3$.

*Proof.* (a) and (b) are trivial for $n = 2$. By 2.3 there exists $x \in B_f \cap R(f)$. Let $D$ be a normal neighborhood of $x$ such that (2.2) holds. Then $C = f(D \cap B_f)$ is closed in $fD$, and by 2.13 $\complement C$ has a non-trivial homotopy at $f(x)$. Now (a) follows from Lemma 2.19. The same lemma and Theorem 2.17 imply (b) for $n = 3$.

2.21. *Remarks.* 1. (a) in 2.20 has been proved in [7, 3.4] (see also [9]). (b) also follows from the result of Trohimčuk [15]: If $f: G \to R^3$ is discrete and open and $B_f \neq \emptyset$, then $\dim B_f = 1$.

2. The properties (a) and (b) have turned out to be useful in the theory of quasiregular mappings (see [4; 7; 9]).

## 3. On the measure of $fB_f$ of a quasiregular mapping

3.1. **Theorem.** *Suppose that* $f: G \to R^n$, $n \geq 2$, *is a quasiregular mapping. Then* $C^{n-1}(fB_f) = 0$.

For the proof we shall use a modification of the method of Rešetnjak [13]. At first we present some preliminaries.

Suppose that $f: G \to R^n$ is a non-constant quasiregular mapping. Then $f$ is discrete, open, and sense-preserving [11]. Let $x_0 \in G$. For $r > 0$ we denote by $U(x_0, f, r)$ the $x_0$-component of $f^{-1}B^n(f(x_0), r)$. By [5, 2.9] there exists $\sigma_0 > 0$ such that for $0 < r \leq \sigma_0$ $U(x_0, f, r)$ is a normal neighborhood of $x_0$, $fU(x_0, f, r) = B^n(f(x_0), r)$, and

$$\operatorname{card}(f^{-1}(y) \cap U(x_0, f, r)) \leq i(x_0, f)$$

for every $y \in R^n$. As in [5, 4.1] we set for $r > 0$

$$l(r) = l(x_0, f, r) = \inf_{|x-x_0|=r} |f(x) - f(x_0)|,$$

$$L(r) = L(x_0, f, r) = \sup_{|x-x_0|=r} |f(x) - f(x_0)|,$$

and for $0 < r < d(f(x_0), \partial fG)$

$$l^*(r) = l^*(x_0, f, r) = \inf_{x \in \partial U(x_0, f, r)} |x - x_0|,$$

$$L^*(r) = L^*(x_0, f, r) = \sup_{x \in \partial U(x_0, f, r)} |x - x_0|.$$

We need the following two lemmas.

3.2. **Lemma.** *Suppose that* $f_i : G \to R^n$, $i = 1, 2, \ldots,$ *is a sequence of discrete and open mappings,* $f: G \to R^n$ *is discrete and open or a constant mapping, and* $f_i \to f$ *uniformly on compact subsets of* $G$. *If* $x_i \to x \in G$, $x_i \in B_{f_i}$, *then* $x \in B_f$.

*Proof.* We may assume that $f$ is not constant since otherwise the lemma is trivial. Then $x \in B_f$ if and only if $|i(x, f)| \geq 2$ (cf. [5, 2.12]). Set $y = f(x)$ and $y_i = f(x_i)$. Pick a domain $D$ such that $x \in D$, $\bar{D}$ is a compact subset of $G$, $y \in fD \setminus f\partial D$, and $\mu(y, f, D) = i(x, f)$ where $\mu(y, f, D)$ denotes the topological index of the triple $(y, f, D)$ ([10], [5, p. 6]). Now $\mu(y, f, D) = \mu(y, f_i, D)$ for $i > i_0$ for some $i_0$ [10, II.2.3]. Since $\bar{D}$ is compact in $G$ and $y_i \to y$, there exists $i_0'$ such that $x_i \in D$ and $y$ and $y_i$ belong to the same component of $f_iD \setminus f_i\partial D$ for $i > i_0'$. Then $\mu(y_i, f_i, D) = \mu(y, f_i, D)$ [10, Theorem 1, p. 125]. Hence for $i > \max(i_0, i_0')$

$$|i(x, f)| = |\mu(y, f, D)| = |\mu(y, f_i, D)| = |\mu(y_i, f_i, D)|$$

$$= \Big|\sum_{z \in f_i^{-1}(y_i) \cap D} i(z, f_i)\Big| \geq |i(x_i, f_i)| \geq 2$$

because each $i(z, f_i)$ has the same sign (see [5, 2.12]). The lemma follows.

3.3. **Lemma.** *Suppose that* $(f_i)$ *is a sequence of mappings such that* $f_i : G_i \to R^n$ *is either discrete and open or a constant mapping. Then* $\dim \bigcup f_i B_{f_i} \leq n - 2$.

*Proof.* Clearly $f_i B_{f_i}$ is a $F_\sigma$-set, i.e. a countable union of closed sets. Since $\dim f_i B_{f_i} \leq n - 2$ and the countable union of at most $(n-2)$-dimensional $F_\sigma$-sets is again at most $(n-2)$-dimensional [3, p. 30], the lemma follows.

*Proof of* 3.1. We may assume that $f : G \to R^n$ is a non-constant quasi-regular mapping since otherwise the theorem is trivial. Let $x_0 \in G$. Pick $\delta > 0$ so small that for $r \in (0, \delta]$ $U(x_0, f, r)$ is a normal neighborhood of $x_0$. Denote $U_0 = U(x_0, f, \delta)$, $g = f | U_0$, and $y_0 = f(x_0)$. It suffices to show that $m_{n-1}(gB_g \cap P) = 0$ for every $(n-1)$-dimensional hyperplane $P$ in $R^n$.

If $P$ is an $(n-1)$-dimensional hyperplane in $R^n$, then for $y \in P$ and $r > 0$ we let $D(y, r, P)$ denote the disk $B^n(y, r) \cap P$ and set

$$\alpha_g(y, r, P) = \sup\{\varrho/r \mid D(z, \varrho, P) \subset D(y, r, P),\ D(z, \varrho, P) \cap gB_g = \emptyset\}.$$

Note that $gB_g$ is closed in $B^n(y_0, \delta)$. Since $\dim gB_g \leq n - 2$, $gB_g$ does not contain any $(n-1)$-dimensional disk. Thus $1 \geq \alpha_g(y, r, P) > 0$ and $\alpha_g(y, r, P) = 1$ if and only if $D(y, r, P) \cap gB_g = \emptyset$.

Let $y \in B^n(y_0, \delta)$. Pick $\eta > 0$ so that $\eta < \delta - |y - y_0|$ and define

$$\beta(y, \eta) = \inf_{\substack{r, P \\ r \leq \eta}} \alpha_g(y, r, P).$$

Then $\eta \mapsto \beta(y, \eta)$ is a non-increasing function. We shall show that

$$\beta(y) = \lim_{\eta \to 0} \beta(y, \eta) > 0. \tag{3.4}$$

For $x \in g^{-1}(y)$ let $l(r) = l(x, g, r)$ and $l^*(r) = l^*(x, g, r)$. First we show that there exist $M \geq 1$ and $r > 0$ such that for every $x \in g^{-1}(y)$ and $r' \in (0, r]$ (i) $U(x, f, r')$ is a normal neighborhood of $x$ and (ii) $r'/M \leq l(l^*(r'))$. For $x \in g^{-1}(y)$ choose $\sigma_x < \eta$ as in [5, 2.9]. Then $U(x, g, r')$ is a normal neighborhood of $x$ for $r' \in (0, \sigma_x]$. If $l(l^*(r')) < r'$, let $\Gamma$ be the family of paths joining $\partial U(x, g, r')$ and $\bar{U}(x, g, l(l^*(r')))$ in $U(x, g, r')$. By [5, 3.2]

$$M(\Gamma) \leq N(g, U_0) K_O(g) M(g\Gamma) \leq i(x_0, f) K_O(f) \omega_{n-1} \left(\ln \frac{r'}{l(l^*(r'))}\right)^{1-n}. \tag{3.5}$$

Since $S^{n-1}(x, l^*(r'))$ meets both $\complement U(x, g, r')$ and $\bar{U}(x, g, l(l^*(r')))$ and $U(x, g, r') \setminus \bar{U}(x, g, l(l^*(r')))$ is a ring [5, 2.9], $M(\Gamma) \geq \delta_x > 0$ [17, 11.7] where $\delta_n$ depends only on $n$. This combined with (3.5) yields (i) and (ii) with $r = \min\{\sigma_x \mid x \in g^{-1}(y)\}$.

Let us suppose $\beta(y) = 0$. Then for $i = 1, 2, \ldots$ there exists a positive number $r_i \leq \min\{r, 1/i\}$ and a plane $P_i$ such that

$$\alpha_i = \alpha_g(y, r_i/2M, P_i) \to 0 \tag{3.6}$$

as $i \to 0$. Passing to a subsequence we may assume $P_i \to P_0$. Let $A_i : R^n \to R^n$ denote the mapping $z \mapsto (z - y)/r_i$ and for $z \in \bar{B}^n$ and $x \in g^{-1}(y)$ define $\bar{g}_i^x(z) = A_i(g(x + l^*(r_i)z))$. Set $g_i^x = \bar{g}_i^x | B^n$. Note that for $z \in \bar{B}^n$, $x + l^*(r_i)z \in \bar{B}^n(x, l^*(r_i)) \subset \bar{B}^n(x, l^*(r)) \subset \bar{U}(x, g, r) \subset U_0$. We have

$$|\bar{g}_i^x(z)| \leq \frac{L(x, g, l^*(r_i))}{r_i} = 1$$

for $z \in B^n$, hence $(g_i^x)$ is a uniformly bounded sequence of quasiregular mappings. By [6, 3.17] the mappings $g_i^x$ form a normal family and hence, by [12, p. 664], we may choose a subsequence $(g_{i_j}^x)$ which converges uniformly on compact subsets of $B^n$ to a quasiregular mapping $g_0^x : B^n \to R^n$. Let $z \in g^{-1}(y)$, $z \neq x$. Arguing as above we may choose a subsequence of $(g_{i_j}^z)$ converging uniformly on compact subsets of $B^n$ to a quasiregular mapping $g_0^z : B^n \to R^n$. Since $g^{-1}(y)$ is finite, we obtain by this method a subsequence $k_1, k_2, \ldots$ of $1, 2, \ldots$ such that $(g_{k_j}^x)$ converges uniformly on compact subsets of $B^n$ to a quasiregular mapping $g_0^x : B^n \to R^n$ for all $x \in g^{-1}(y)$. We may suppose that this subsequence is $1, 2, \ldots$

Let $Q_i = P_i - y$. Then $Q_i \to Q_0 = P_0 - y$. Set $D_i = D(0, 1/2M, Q_i)$, $i = 0, 1, \ldots$ Then for $i = 1, 2, \ldots$

$$A_i(gB_g \cap D(y, r_i/2M, P_i)) = \bigcup_{x \in g^{-1}(y)} g_i^x B_{g_i^x} \cap D_i . \tag{3.7}$$

Let $y' \in D_0$. Now (3.6) and (3.7) imply that there exists a sequence $(y_i')$ such that for each $i$, $y_i' \in g_i^x B_{g_i^x} \cap D_i$ with $x = x(i) \in g^{-1}(y)$ and $y_i' \to y'$. Passing to a subsequence, denoted again by $(y_i')$, we may assume that $y_i' \in g_i^x B_{g_i^x} \cap D_i$ for a fixed $x$ because $g^{-1}(y)$ is finite. Pick $z_i' \in B_{g_i^x}$ so that $g_i^x(z_i') = y_i'$. Then there exists $\alpha \in (0, 1)$ such that $z_i' \in \bar{B}^n(\alpha)$ for all $i$. This can be seen as follows: If not true, then

$$\liminf_{i \to \infty} d(S^{n-1}, \bar{U}(0, g_i^x, 1/2M)) = 0 . \tag{3.8}$$

Let $\Gamma_i$ be the family of paths joining $S^{n-1}$ and $\bar{U}(0, g_i^x, 1/2M)$ in $B^n$. Every path in $g_i^x \Gamma_i$ joins $\bar{B}^n(1/2M)$ and $\bar{g}_i^x S^{n-1}$. By (ii)

$$d(\bar{g}_i^x S^{n-1}, 0) \geq \frac{l(l^*(r_i))}{r_i} \geq \frac{r_i}{Mr_i} = \frac{1}{M} > \frac{1}{2M},$$

hence

$$M(g_i^x \Gamma_i) \leq \omega_{n-1} \left( \ln \frac{1/M}{1/2M} \right)^{1-n} = \omega_{n-1} (\ln 2)^{1-n}.$$

On the other hand, by [5, 3.2]

$$M(\Gamma_i) \leq K_O(g_i^x) N(g_i^x, B^n) M(g_i^x \Gamma_i) \leq K_O(f) i(x_0, f) \omega_{n-1} (\ln 2)^{1-n}. \tag{3.9}$$

Since $\bar{U}(0, g_i^x, 1/2M)$ is connected and contains 0, (3.8) implies $\limsup_{i \to \infty} M(\Gamma_i) = \infty$ which contradicts (3.9).

Since $\bar{B}^n(\alpha)$ is a compact subset of $B^n$, there exists a cluster point $z' \in \bar{B}^n(\alpha)$ of the sequence $(z_i')$ which, by Lemma 3.2, belongs to $B_{g_0^x}$. Since $g_0^x(z') = y'$, every point of $D_0$ belongs to $\bigcup_{x \in g^{-1}(y)} g_0^x B_{g_0^x}$, which is impossible by Lemma 3.3. The inequality (3.4) has been proved.

Suppose now that $m_{n-1}(gB_g \cap P) > 0$. Then $gB_g \cap P$ has a point of density $y \in B^n(y_0, \delta)$, i.e.

$$\lim_{r \to 0} \frac{m_{n-1}(gB_g \cap D(y, r, P))}{m_{n-1}(D(y, r, P))} = 1. \tag{3.10}$$

By (3.4) $\beta(y) > 0$, hence $D(y, r, P)$ contains a disk

$$D' = D(y', r(\beta(y) - \varepsilon(r)), P)$$

such that $D' \cap gB_g = \emptyset$ and $\varepsilon(r) \to 0$ as $r \to 0$. This implies

$$\frac{m_{n-1}(gB_g \cap D(y, r, P))}{m_{n-1}(D(y, r, P))} \leq 1 - (\beta(y) - \varepsilon(r))^{n-1} \to 1 - \beta(y)^{n-1} < 1$$

which contradicts (3.10). The theorem follows.

3.11. *Remark.* Theorem 3.1 and the result of Rešetnjak [13] are not true for $n \geq 3$ if $C^{n-1}$ is replaced by $H^{n-1}$. In fact for every $n \geq 3$ there exist by [2] quasiregular mappings $f$ for which $\dim_H B_f$ and $\dim_H fB_f$ are arbitrarily close to $n$.

3.12. *An application.* Theorem 3.1 or the result of Rešetnjak in [13] gives the following result on the metric structure of a quasiconformal $k$-ball, $1 \leq k \leq n-2$. We recall that a set $A \subset R^n$ is called a quasiconformal $k$-ball if there exists a domain $G \supset A$ and a quasiconformal mapping $f: G \to fG$ such that

$$fA = R^k \cap B^n = \{(x_1, \ldots, x_n) \mid x_n = x_{n-1} = \ldots = x_{k+1} = 0\} \cap B^n .$$

3.13. **Theorem.** *Suppose that* $A \subset R^n$ *is a quasiconformal* $k$*-ball,* $1 \leq k \leq n-2$. *Then* $C^{n-1}(A) = 0$.

*Proof.* Let $f: G \to fG$ be the corresponding quasiconformal mapping. Define $g_1 : R^n \to R^n$ as the winding mapping $(r, \varphi, z) \mapsto (r, 2\varphi, z)$, $z \in R^{n-2} \supset R^k$, in cylindrical coordinates of $R^n$. Set $g = f^{-1} \circ (g_1 | g_1^{-1}(fG))$. Then $g$ is quasiregular and $gB_g \supset A$. Hence, by Theorem 3.1, $C^{n-1}(A) = 0$.

## 4. A lower bound of the Hausdorff dimension of $B_f$

4.1. **Theorem.** *Suppose that* $f: G \to R^n$, $n \geq 3$, *is quasiregular and* $B_f \neq \emptyset$. *Then* $H^\alpha(B_f) > 0$ *where* $\alpha = (n-2)(2/K_I(f))^{1/(n-2)}$.

*Proof.* We may assume that $f$ is not constant. By Lemma 2.3 there exist $x_0 \in B_f$ and $r_0 > 0$ such that for $r \in (0, r_0]$ $U(r) = U(x_0, f, r)$ is a normal neighborhood of $x_0$ and if $U_0 = U(r_0)$, then $f | B_f \cap U_0$ defines a homeomorphism $B_f \cap U_0 \to f(B_f \cap U_0)$ and $i(x, f) = i(x_0, f)$ for all $x \in B_f \cap U_0$. Fix $r' > 0$ such that $B^n(x_0, 3r') \subset U_0$ and then $r_0' > 0$ so that $\bar{U}_0' = \bar{U}(r_0') \subset B^n(x_0, r')$.

Let $F = B_f \cap \bar{U}_0'$. We shall show that there exists $\delta > 0$ such that for all $x \in F$ and $r \in (0, \delta]$

$$L^*(x, f, L(x, f, r)) \leq Cr \tag{4.2}$$

where $C$ does not depend on $x$ or on $r$.

Pick $\delta > 0$ so that for all $x \in F$ and all $r \in (0, \delta]$ $L(x, f, r) < d(fF, fS^{n-1}(x_0, r'))$. Fix $x \in F$ and $r \in (0, \delta]$. Let $L = L(x, f, r)$, $L^* = L^*(x, f, L)$, $U = U(x, f, L(x, f, L^*))$, and $U' = U(x, f, L)$. Then $U' \subset U \subset U_0$ and, since $i(x, f) = i(x_0, f)$, $U$ and $U'$ are normal neighborhoods of $x$. If $L^* > r$, $E = (U, \bar{U}')$ is a normal condenser in $U_0$ [5, 5.1—6.1] and $E' = (B^n(x, L^*), \bar{B}^n(x, r))$ is a condenser in $U_0$. Since $E$ is ringlike and $\partial U$ and $\bar{U}'$ both meet $S^{n-1}(x, L^*)$ [5, 2.9, 4.3], $\operatorname{cap} E \geq \delta_n > 0$ where $\delta_n$ depends only on $n$. Hence by [5, 6.2]

$$\operatorname{cap}(fU, f\bar{U}') = \operatorname{cap} fE \geq \frac{1}{K_O(f) i(x, f)} \operatorname{cap} E \geq \frac{\delta_n}{K_O(f) i(x_0, f)} . \tag{4.3}$$

Since $f\bar{B}^n(x, r)$ is connected, contains $f(x)$, and meets $S^{n-1}(f(x), L)$, (4.3) implies

$$\text{cap}\,(fU, f\bar{B}^n(x, r)) \geq C' \tag{4.4}$$

where $C' > 0$ does not depend on $x$ or on $r$. On the other hand,

$$\begin{aligned}\text{cap}\,(fU, f\bar{B}^n(x, r)) &\leq K_I(f)\,\text{cap}\,(U, \bar{B}^n(x, r))\\ &\leq K_I(f)\,\text{cap}\,(B^n(x, L^*), \bar{B}^n(x, r))\end{aligned}$$

which together with (4.4) yields (4.2).

Suppose $H^\alpha(F) = 0$. Let $\varepsilon > 0$. There exist $x_i \in F$ and

$$r_i \in (0, \min\{\delta, r'/C\}), \quad i = 1, 2, \ldots$$

such that $\cup B^n(x_i, r_i) \supset F$ and

$$\sum_i r_i^\alpha < \varepsilon . \tag{4.5}$$

Let $L_i = L(x_i, f, r_i)$, $L_i^* = L^*(x_i, f, L_i)$, and $E_i = (U_0, \bar{U}(x_i, f, L_i))$. Then $E_i$ is a normal condenser in $G$ and by [4, 5.14 and 3.7]

$$\frac{\omega_{n-1}}{\left(\ln \dfrac{2r_0}{L_i}\right)^{n-1}} \leq \text{cap}\, fE_i \leq \frac{K_I(f)}{i(x_0, f)}\,\text{cap}\, E_i \leq \frac{K_I(f)}{2}\,\frac{\omega_{n-1}}{\left(\ln \dfrac{r'}{L_i^*}\right)^{n-1}} \tag{4.6}$$

where the last inequality is true because $i(x_0, f) \geq 2$ and $L_i^* \leq Cr_i < r'$. Now (4.6) implies

$$L_i \leq C'' L_i^{*\,(2 K_I(f))^{1/(n-1)}} \tag{4.7}$$

where $C''$ does not depend on $i$. Since $\bigcup B^n(f(x_i), L_i)$ covers $fF$ and (4.7), (4.2), and (4.5) yield

$$\sum_i L_i^{n-2} \leq C''^{n-2} \sum_i L_i^{*\alpha} \leq C^\alpha C''^{n-2} \sum_i r_i^\alpha < C^\alpha C''^{n-2} \varepsilon ,$$

$H^{n-2}(fF) = 0$. On the other hand, by Theorem 2.20,

$$H^{n-2}(fF) \geq H^{n-2}(gB_g) > 0$$

where $g = f|U_0'$. The theorem follows.

4.8. *Remark.* The number $\alpha$ in Theorem 4.1 is in the case $K_I(f) = 2$ the best possible. The mapping $f: R^n \to R^n$, $(r, \varphi, z) \mapsto (r, 2\varphi, z)$ in cylindrical coordinates of $R^n$ gives an example since $K_I(f) = 2$ and the Hausdorff dimension of $B_f$ is $\dim_H B_f = \dim_H \{(0, 0, z) \mid z \in R^{n-2}\} =$

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$n - 2$. It has been conjectured that for $n \geq 3$, $B_f \neq \emptyset$ implies $K_I(f) \geq 2$. Furthermore, it has been conjectured that $B_f \neq \emptyset$ implies $\dim B_f = n - 2$. This would imply $H^{n-2}(B_f) > 0$, which for $K_I(f) > 2$ would be a stronger result than 4.1.

University of Helsinki

University of Helsinki
University of Michigan