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ON NORMAL MEROMORPHIC FUNCTIONS

BY

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On normal meromorphic functions

1. *Normality Criteria.* A function $f(z)$ meromorphic in $D = \{|z| < 1\}$ is called *normal* if

$$\sup_{|z|<1} (1 - |z|^2) f^{\#}(z) < \infty , \tag{1.1}$$

where

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denotes the spherical derivative. This concept was introduced by Lehto and Virtanen [7]; see also Yosida [18] and Noshiro [14].

1.1. We first prove two criteria of normality; the first of these criteria is essentially a reformulation of (1.1).

Theorem 1. *A non-constant function $f(z)$ meromorphic in D is normal if and only if there do not exist sequences $\{z_n\}$ and $\{\varrho_n\}$ with $z_n \in D$, $\varrho_n > 0$, $\varrho_n \to +0$, such that*

$$\lim_{n\to\infty} f(z_n + \varrho_n \zeta) = g(\zeta) \quad \textit{locally uniformly in } \mathbf{C} , \tag{1.2}$$

where $g(\zeta)$ is a non-constant meromorphic function in $\mathbf{C}$, the finite complex plane.

In somewhat picturesque language, we may express this theorem as follows: A function is normal if and only if its Riemann image surface does not contain asymptotically a Riemann surface of parabolic type.

Proof. (a) Suppose that $f(z)$ is not normal. By (1.1) there exists a sequence $\{z_n^*\}$ such that

$$(1 - |z_n^*|^2) f^{\#}(z_n^*) \to \infty \qquad (n \to \infty). \tag{1.3}$$

We choose $\{r_n\}$ such that $|z_n^*| < r_n < 1$ and

$$\left(1 - \frac{|z_n^*|^2}{r_n^2}\right) f^{\#}(z_n^*) \to \infty \qquad (n \to \infty). \tag{1.4}$$

Furthermore, we choose $\{z_n\}$ such that

$$(1.5) \qquad M_n \equiv \max_{|z|<r_n} \left(1 - \frac{|z|^2}{r_n^2}\right) f^{\#}(z) = \left(1 - \frac{|z_n|^2}{r_n^2}\right) f^{\#}(z_n) ;$$

the maximum exists because $f^{\#}(z)$ is continuous in $\{|z| \leq r_n\}$. Since $|z_n^*| < r_n$, it follows from (1.4) that $M_n \to \infty$.

If we set

$$(1.6) \qquad \varrho_n = \frac{1}{M_n}\left(1 - \frac{|z_n|^2}{r_n^2}\right) = \frac{1}{f^{\#}(z_n)},$$

we have

$$(1.7) \qquad \frac{\varrho_n}{r_n - |z_n|} = \frac{r_n + |z_n|}{r_n^2 M_n} \leqq \frac{2}{r_n M_n} \to 0 .$$

Therefore the functions

$$(1.8) \qquad g_n(z) = f(z_n + \varrho_n \zeta)$$

are defined for $|\zeta| < R_n$, where $R_n \to \infty$ as $n \to \infty$. It follows from (1.6) that

$$(1.9) \qquad g_n^{\#}(0) = \varrho f^{\#}(z_n) = 1.$$

We now apply Marty's criterion [2, p. 169] to show that the sequence $\{g_n(\zeta)\}$ is normal. If $|\zeta| \leqq R < R_n$, then, by (1.6)

$$g_n^{\#}(\zeta) = \varrho_n f^{\#}(z_n + \varrho_n \zeta) \leqq \frac{\varrho_n M_n}{1 - r_n^{-2}\, |z_n + \varrho_n \zeta|^2}$$

$$\leqq \frac{r_n + |z_n|}{r_n + |z_n| - \varrho_n R} \; \frac{r_n - |z_n|}{r_n - |z_n| - \varrho_n R} .$$

By (1.7) the last term of this inequality tends to 1 as $n \to \infty$, for each fixed R. Hence $\{g_n(\zeta)\}$ is a normal sequence, and we may assume that $g_n(\zeta)$ converges locally uniformly in $\mathbf{C}$. The limit function $g(\zeta)$ is meromorphic in $\mathbf{C}$, and is non-constant because, by (1.9), $g^{\#}(0) = 1 \neq 0$.

(b) Conversely, let $f(z)$ be normal in D. If (1.2) holds, then $\varrho_n/(1 - |z_n|) \to 0$. Since

$$\varrho_n f^{\#}(z_n + \varrho_n \zeta) \leqq \frac{\varrho_n}{1 - |z_n| - \varrho_n |\zeta|} \cdot (1 - |z_n + \varrho_n \zeta|^2) f^{\#}(z_n + \varrho_n \zeta) ,$$

it therefore follows from (1.1) and (1.2) that $g^{\#}(\zeta) = 0$ for all $\zeta \in \mathbf{C}$, so that $g(\zeta) \equiv$ const. This completes the proof of Theorem 1.

As an application, we can derive Schottky's theorem from Picard's theorem. Indeed, if $f(z)$ omits three values, then the same is true of the limit function $g(\zeta)$ in (1.2). Hence $g(\zeta)$ is constant, by Picard's theorem. Therefore $f(z)$ is normal, from which Schottky's theorem easily follows.

1.2. We need the following result of Ahlfors [1] in the form given in [15]; it is clear that upper semi-continuity is sufficient.

Lemma. *Let $u(z) \geq 0$ be upper semi-continuous in D. For each $z_0 \in D$, let $u(z_0) \leq 1$, or else let there exist a function $\varphi(z)$ analytic in some neighborhood of z_0 such that $|\varphi(z_0)| < 1$ and, for small $|z - z_0|$,*

$$v(z) \equiv \frac{(1 - |z|^2)\,|\varphi'(z)|}{1 - |\varphi(z)|^2} \leqq u(z)\,, \qquad v(z_0) = u(z_0)\,.$$

Then $u(z) \leq 1$ for $z \in D$.

Theorem 2. *Let E be a continuum in the extended complex plane $\hat{\mathbf{C}}$. Let $f(z)$ be meromorphic in D, and, for some $M \geq 1$, let*

$$(1.10) \qquad (1 - |z|^2) f^{\#}(z) \leqq M\,, \qquad z \in f^{-1}(E)\,.$$

Then

$$(1.11) \qquad \sup_{|z|<1} (1 - |z|^2) f^{\#}(z) \leqq K_1 M\,,$$

where K_1 is a constant depending only on E.

(In Theorem 2 and its proof, we shall denote by $K_1, K_2, \ldots$ constants that depend only on E.)

Proof. (a) We may assume that $0 \in E$ and that $E \subset \{|w| \leqq 1\}$. Let G be that component of $\hat{\mathbf{C}} \setminus E$ containing ∞. Let the univalent function

$$(1.12) \qquad g(s) = bs + b_0 + b_1 s^{-1} + \ldots$$

map $|s| > 1$ onto the simply connected domain G. We define the function $h(w)$ $(w \in G)$ by

$$(1.13) \qquad h(g(s)) = s^{-2} g(s) + 2 \int_{\infty}^{s} t^{-3} g(t)\,dt\,, \qquad |s| > 1\,.$$

Since $|g(s)| \leqq K_2$ for $1 < |s| \leqq 2$, and since $h(g(s)) = -bs^{-1} + \ldots$, we see that

$$(1.14) \qquad |h(w)| < K_3\,, \qquad w \in G\,.$$

Furthermore, (1.13) shows that $h'(g(s))\, g'(s) = s^{-2}\, g'(s)$, and therefore that $h'(g(s)) = s^{-2}$. Hence, by (1.12),

$$\frac{1}{K_4} \leqq (1 + |w|^2)\, |h'(w)| = \frac{1 + |g(s)|^2}{|s|^2} \leqq K_5 \tag{1.15}$$

for $w = g(s) \in G$.

(b) We choose $\delta = \delta(E) > 0$ so small (see (1.14)) that

$$\frac{K_5\, \delta}{1 - \delta^2 K_3^2} < \frac{1}{2}\,, \qquad \frac{\delta}{1 - \delta^2} < \frac{1}{2}\,. \tag{1.16}$$

Let $z_0 \in D$. Suppose first that $z_0 \in G^* = f^{-1}(G)$. Then the function

$$\psi(z) = \frac{\delta}{M}\, h(f(z))$$

is analytic in some neighbourhood of z_0 and satisfies $|\psi(z_0)| \leq \delta K_3/M \leq$ $\leq \delta K_3 < 1$ by (1.14) and (1.16). We see that

$$v(z) \equiv \frac{1 - |z|^2}{1 - |\psi(z)|^2}\, |\psi'(z)| = (1 - |z|^2) |f'(z)|\, \frac{\delta M^{-1} |h'(f(z))|}{1 - \delta^2\, M^{-2} |h(f(z))|^2}\,. \tag{1.17}$$

Suppose now that $z_0 \in D \setminus (G^* \cup E^*)$, where $E^* = f^{-1}(E)$. This case can occur only if E disconnects the plane. If we now consider

$$\psi(z) = \frac{\delta}{M} f(z)\,,$$

we have, instead of (1.17), that

$$v(z) = (1 - |z|^2) |f'(z)|\, \frac{\delta M^{-1}}{1 - \delta^2\, M^{-2} |f(z)|^2}\,. \tag{1.18}$$

(c) We define

$$u(z) = \begin{cases} \frac{1}{2}\, M^{-1} (1 - |z|^2)\, |f'(z)| & \text{for } z \in E^*\,, \\ v(z) & \text{for } z \notin E^*\,. \end{cases} \tag{1.19}$$

This function is continuous in $D \setminus \partial E^*$. We shall show that $u(z)$ is upper semi-continuous at each point $\zeta \in \partial E^*$. To show this, it is sufficient to consider $z \notin E^*$. If $z \in G^*$, we have

$$u(z) = v(z) \leqq (1 - |z|^2)\, |f'(z)|\, \frac{\delta\, M\, K_5}{M^2 - \delta^2\, K_3^2}$$

$$< \tfrac{1}{2}\, (1 - |z|^2)\, |f'(z)|\, M^{-1}$$

by (1.17), (1.15), (1.14), (1.16), and by the fact that $M \geq 1$. If $z \notin G^*$, we have

$$u(z) = v(z) \leq (1 - |z|^2)|f'(z)| \frac{M^{-1}\,\delta}{1 - M^{-2}\,\delta^2} < \tfrac{1}{2}(1 - |z|^2)|f'(z)|M^{-1}$$

by (1.18) and (1.16) because $\mathbf{C} \setminus (E \cup G) \subset \{|w| \leq 1\}$ and therefore $|f(z)| < 1$. In any case we must have

$$\limsup_{z \to \zeta} u(z) \leq \tfrac{1}{2}(1 - |\zeta|^2)\,|f'(\zeta)|\,M^{-1} = u(\zeta)\,.$$

It follows from (1.19) and (1.10) that $u(z) \leqq 1$ for $z \in E^*$. Hence the Lemma shows that $u(z) \leq 1$ for $z \in D$. If $z \in G^*$, we deduce from (1.17) and (1.15) that

$$(1 - |z|^2)\,f^{\#}(z) \leq \frac{M\,\delta^{-1}}{(1 + |f(z)|^2)\,|h'(f(z))|} \leq M\,\delta^{-1}\,K_4\,;$$

if $z \in D \setminus (G^* \cup E^*)$, we deduce from (1.18) that

$$(1 - |z|^2)\,f^{\#}(z) \leq M\,\delta^{-1}\,.$$

This proves (1.11) for all cases.

1.3. The condition (1.1) for normality can be written as

$$\frac{|dw|}{1 + |w|^2} \leq \text{const.} \frac{|dz|}{1 - |z|^2}\,, \qquad w = f(z)\,,$$

that is, the spherical element of length of the image is bounded in terms of the non-Euclidean element of length. We shall show that the spherical element of length can be replaced by any other, that is,

$$\varrho(w)\,|dw| \leq \text{const.} \frac{|dz|}{1 - |z|^2}\,, \qquad w = f(z)\,, \tag{1.20}$$

already implies normality.

Corollary. *Let* $\varrho(w) \not\equiv 0$ *be continuous in* $\hat{\mathbf{C}}$, *and let* $f(z)$ *be meromorphic in* D. *If*

$$\sup_{|z|<1}\,(1 - |z|^2)\,|f'(z)|\,\varrho(f(z)) < \infty\,, \tag{1.21}$$

then $f(z)$ *is normal in* D.

Proof. There exists a closed disc E such that $\varrho(w) \geq \sigma > 0$ for $w \in E$. Hence (1.21) implies that

$$(1 - |z|^2)|f'(z)| \leq \frac{1}{\sigma}$$

for $z \in f^{-1}(E)$, and the assertion follows immediately from Theorem 2.

It remains an open problem whether the hypothesis of Theorem 2 that E be a continuum can be replaced by a weaker hypothesis, for instance that E have positive capacity or analytic capacity. It might perhaps even be true that one can always find a finite set E, depending on M, such that (1.10) implies normality.

2. *Boundary Behaviour.* 2.1. Let $f(z)$ be normal in D. As Lehto and Virtanen [7] have proved, every asymptotic value is an angular limit, that is, if $f(z) \to c$ as $z \to e^{i\theta}$ along some arc ending at $e^{i\theta}$, then $f(z) \to c$ in every Stolz angle at $e^{i\theta}$. Furthermore, non-constant normal functions have no Koebe arcs [3]. This means that if there is a sequence $\{A_n\}$ of arcs in D converging to a non-degenerate boundary arc such that $f(z) \to c$ for $z \in A_n$, $n \to \infty$, then it follows that $f(z) \equiv c$. A consequence is that every analytic normal function has angular limits on a dense subset of $\partial D = \{|z| = 1\}$. The modular function shows that this subset may be countable.

Hayman ([6], [15]) has shown that for every analytic normal function

$$\log^+|f(z)| = O\left(\frac{1}{1 - |z|}\right), \qquad |z| \to 1 . \tag{2.1}$$

The modular function shows that O cannot be replaced by o in general.

Theorem 3. *Let* $f(z)$ *be analytic and normal in* D. *If*

$$\log^+|f(z)| = o\left(\frac{1}{1 - |z|}\right), \qquad |z| \to 1 , \tag{2.2}$$

then $f(z)$ *has angular limits on an uncountably dense subset of* ∂D.

This result is closely related to a theorem of Hall [5]: A function analytic in D has an uncountably dense set of asymptotic values if either (2.2) is satisfied and the function omits some finite value, or if

$$|f(z)| = o\left(\frac{1}{1 - |z|}\right) \qquad (|z| \to 1) .$$

It is not assumed in Hall's theorem that the function is normal. As a consequence, a function has an uncountably dense set of angular limits if it is a Bloch function ([8], [12], [16]), that is, if

$$\sup_{|z|<1} (1 - |z|^2)|f'(z)| < \infty .$$

Proof. (a) Let A be an open arc of the unit circle and let $\zeta_0 \in A$. We may assume that $f(z)$ is unbounded near ζ_0 because otherwise the assertion follows from Fatou's theorem. For $m = 1, 2, \ldots$, there is a component G_n of $\{z \in D : |f(z)| > m\}$ whose distance from ζ_0 is arbitrarily small. Furthermore, diam $G_n \to 0$ because otherwise $f(z)$ would have Koebe arcs for the value $c = \infty$, which is impossible for a normal function. Furthermore, $\partial G_n \cap \partial D = \emptyset$ because $f(z)$ has no poles. Therefore we can choose a value of m such that $\partial G_m \cap \partial D \subset A$.

(b) Let $\varphi(s)$ map $|s| < 1$ conformally onto the universal covering surface of $G_m \subset D$. The function $m^{-1} f(\varphi(s))$ is analytic in $|s| < 1$ and has modulus greater than 1. Hence there exists a non-decreasing function $\mu(t)$ such that

$$g(s) = \log \frac{f(\varphi(s))}{m} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + s}{e^{it} - s}\, d\mu(t) , \qquad |s| < 1 . \tag{2.3}$$

It is clear that $\mu(t)$ is not constant and that $\Re e\, g(s) > 0$ in $|s| < 1$ unless $f(z)$ is constant. We shall deduce from (2.2) that $\mu(t)$ is continuous.

Suppose that $\mu(t)$ is not continuous. Then $\mu(t)$ has a jump $\sigma > 0$, say at $t = t_0$. Since $|\varphi(s)| \leq (a + |s|)/(1 + a|s|)$ for $|s| < 1$, where $a = |\varphi(0)|$, we see that

$$1 - |\varphi(re^{it_0})| \geq (1 - a)(1 - r) , \qquad 0 < r < 1 . \tag{2.4}$$

Hence it follows from (2.3) that, for $0 < r < 1$,

$$\log |m^{-1} f(\varphi(re^{it_0}))| > \sigma \frac{1 + r}{1 - r} \geq \frac{\sigma(1 - a)}{1 - |\varphi(re^{it_0})|} ,$$

which contradicts the assumption (2.2).

Thus the function $\mu(t)$ is continuous. We consider first the case that $\mu(t)$ is absolutely continuous. Then $\mu'(t)$ exists and is positive on a set E of positive measure. Furthermore, at all points of E, $\Re e\, g(s)$ possesses angular limits. It follows from a variant of Fatou's theorem [Collingwood and Lohwater [4], p. 149] that $g(s)$ possesses angular limits at almost all points of E, that is, there exists a subset E' of E of positive measure at which $g(s)$ possesses angular limits $g(e^{it})$. Therefore we have

$$\Re e\, g(e^{it}) > 0 , \qquad t \in E' . \tag{2.5}$$

In the case that $\mu(t)$ is not absolutely continuous, it is known [see, for example, Saks [17], pp. 127—128] that $\mu'(t) = +\infty$ on a set E having the power of the continuum, and (2.3) shows that

$$\mathcal{R}e\, g(e^{it}) \to +\infty\,, \qquad r \to 1-0\,,\ t \in E\,. \tag{2.6}$$

(c) in both cases we have found a set E_0 of the power of the continuum such that $g(e^{it})$ exists and $\mathcal{R}e\, g(e^{it}) > 0$ for $t \in E_0$. Along the arc

$$\Gamma(t) = \{\varphi(re^{it}) : 0 \leqq r < 1\}\,, \qquad t \in E_0\,, \tag{2.7}$$

we have

$$f(z) = m \exp g(s) \to m \exp g(e^{it})\,.$$

Since $f(z)$ has no Koebe arcs, $\Gamma(t)$ ends at a point, so that the normal function $f(z)$ has an asymptotic value at $\zeta(t)$. Hence it has an angular limit $f(e^{it})$ there, and it satisfies $m < |f(e^{it})| \leq \infty$ by (2.5) and (2.6). The definition of G_m shows that the endpoint $\zeta(t)$ of $\Gamma(t)$ lies on ∂D. By our choice of m, we have that $\zeta(t) \in A$.

(d) Finally we show that $\zeta(t)$ attains the same value only a countable number of times. Since E_0 has the power of the continuum, this will complete the proof.

Suppose that $\zeta(t) = \zeta_1$ for $t \in E_1$. Since $f(z) \to f(\zeta_1)$ as $z \to \zeta_1$ along $\Gamma(t)$ by (2.7), a result of Lehto and Virtanen [7] shows that $f(z) \to f(\zeta_1)$ in the whole set between $\Gamma(t)$ and the radius $R_1 = [0\,,\zeta_1]$. Since $|f(\zeta_1)| > m$, it follows that, for $t \in E_1$, an arc $C(t\,,\varrho)$ of $\{|z-\zeta_1| = \varrho\}$ between $\Gamma(t)$ and R_1 lies in G_m if $0 < \varrho < 1/k$ and k is sufficiently large. Since the normal function $\varphi(s)$ has no Koebe arcs, the diameter of the component of $\varphi^{-1}(C(t\,,\varrho))$ through $[0\,,e^{it}]$ tends to 0 as $\varrho \to 0$. Hence a component of $\varphi^{-1}(R_1)$ ends at e^{it} for each $t \in E_1$. There are only countably many such components, so that E_1 is countable.

2.2. The proof of Theorem 3 gives at once the following local version:

If $f(z)$ is analytic and normal in D, and if (2.2) holds on some open arc A of ∂D, then $f(z)$ has angular limits on an uncountably dense subset of A.

We remark that we may weaken (2.2) in the following sense. The only use made of (2.2) in proof of Theorem 3 was to guarantee the continuity of the function $\mu(t)$ in (2.3); any condition which excludes the order of growth (2.1) — and guarantees the continuity of $\mu(t)$ in (2.3) — will yield the conclusion of Theorem 3. A result in this direction was proved by Mida [13] that if $f(z)$ is analytic and normal in D and if $f(z)$ has radial limits only at a set of measure zero on ∂D, then $f(z)$ has angular limits at a set of points which is dense on ∂D; indeed, if the α-points, $z_k = z_k(\alpha)$, satisfy the condition

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty, \tag{2.8}$$

then α is an angular limit of $f(z)$ at a dense subset of ∂D.

Example 1. The function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

is a Bloch function, hence normal [16]. It satisfies

$$|f(z)| = O\left(\log \frac{1}{1 - |z|}\right) \qquad (|z| \to 1),$$

and therefore certainly (2.2). It does not have any finite asymptotic values.

Example 2. The function

$$f(z) = (1 - z)^{-2} \exp \frac{1 + z}{1 - z}$$

does not satisfy (2.2). Furthermore,

$$f^{\#}(z) \leqq \left| \frac{d}{dz} \frac{1}{f(z)} \right| \leqq \left| (4 - 2z) \exp\left(- \frac{1 + z}{1 - z} \right) \right| \leqq 6,$$

and therefore

$$(1 - |z|^2) f^{\#}(z) \to 0 \qquad (|z| \to 1).$$

Hence this condition does not imply (2.2).

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