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ON THE LENGTH OF ASYMPTOTIC PATHS OF ENTIRE FUNCTIONS OF ORDER ZERO

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Suppose that f is an entire function of finite order and that P is a locally rectifiable path on which $f(z)\to\infty$. Let $l(r)$ be the length of P in $|z|<r$. We shall consider the following question of Erdős (Hayman [1, Problem 2.41]): If f has zero order, or more generally finite order, can a path P be found for which $l(r)=O(r)$ $(r\to\infty)$? Such a path P exists if

$$\log M(r,f) = O\big((\log r)^2\big). \tag{A}$$

In fact, Hayman [2] has proved that if f satisfies (A) we may choose a ray through the origin for P. We shall show that (A) is the best possible growth condition under which there exists a path P satisfying $l(r)=O(r)$.

Theorem. *Given any increasing function $\varphi(r)$ such that $\varphi(r)\to\infty$ as $r\to\infty$, here exists an entire function f such that*

$$\log M(r,f) = O\big(\varphi(r)(\log r)^2\big), \tag{1}$$

and if P is any locally rectifiable curve on which $f(z)\to\infty$ then

$$\limsup_{r\to\infty}\frac{l(r)}{r}=\infty. \tag{2}$$

Proof. Let Γ_n be the path

$$|z| = \exp\left\{\frac{\arg z}{4\pi n}\right\},$$

$0\leqq\arg z<\infty$, and let f_n be an entire function such that $f_n(z)\to 0$ on Γ_n. We set $g_n(z)=z^{-m}f_n(z)$, where m is chosen such that $g_n(0)\neq 0$ (if $f_n(0)\neq 0$ then $m=0$) and

$$h_n(z)=\frac{g_n(\varrho_n z)}{g_n(0)},$$

where $\varrho_n>0$ is chosen such that $|h_n(z)|<1/8$ on the path

$$\gamma_n:\ |z| = \exp\left\{\frac{\arg z}{4\pi n}\right\},$$

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$0 \leqq \arg z \leqq 4\pi n$. Since h_n is an entire function we see by means of the Taylor-series that there exists a polynomial P_n such that $P_n(0)=1$ and $|P_n(z)|<1/4$ on γ_n. Let

$$P_n(z) = \prod_{k=1}^{t_n} \left(1 - \frac{z}{a_{n,k}}\right).$$

We get the desired function writing

$$f(z) = \prod_{n=1}^{\infty} (P_n(z/r_n))^{s_n},$$

where $r_1=e$, $s_1=1$, $r_{n+1}>r_n^4$, and for $n \geqq 2$,

$$\text{(i)} \qquad \varphi(\sqrt{r_n}) > 8t_n \sum_{k=1}^{n-1} s_k t_k,$$

and

$$\text{(ii)} \qquad \frac{s_n}{\log r_n} = 2 \sum_{k=1}^{n-1} s_k t_k < \frac{\log r_n}{t_n}.$$

We denote $b_n = \min \{|a_{n,k}|: k=1, 2, \ldots, t_n\}$. Assume that $b_n \sqrt{r_n} > 2$. Then we get for $|z| \leqq \sqrt{r_n}$

$$\left|\log |P_n(z/r_n)^{s_n}|\right| \leqq \left|s_n t_n \log \left(1 - \frac{\sqrt{r_n}}{r_n b_n}\right)\right| \leqq s_n t_n \frac{2}{b_n \sqrt{r_n}}.$$

It follows from (ii) that

$$\left|\log |P_n(z/r_n)^{s_n}|\right| \leqq \frac{2(\log r_n)^2}{b_n \sqrt{r_n}} \to 0$$

as $r_n \to \infty$. Therefore we may assume that $r_n \to \infty$ as $n \to \infty$ so rapidly that

$$\text{(iii)} \qquad 1/2 < \left| \prod_{k=n+1}^{\infty} (P_k(z/r_k))^{s_k} \right| < 2$$

on $|z| \leqq \sqrt{r_{n+1}}$.

Let $\sqrt{r_n} \leqq |z| \leqq \sqrt{r_{n+1}}$. Then it follows from (iii) that

$$\log |f(z)| = \sum_{k=1}^{n} \log |(P_k(z/r_k))^{s_k}| + \log \left| \prod_{k=n+1}^{\infty} (P_k(z/r_k))^{s_k} \right| \leqq \sum_{k=1}^{n} s_k t_k \log |z| + \log 2.$$

If $\sqrt{r_n} \leqq r \leqq \sqrt{r_{n+1}}$ we see now from (ii) that

$$\log M(r,f) \leqq 2 s_n t_n \log r \leqq 4t_n \left(\sum_{k=1}^{n-1} s_k t_k\right) (\log r_n) \log r.$$

Since $r_n \leqq r^2$ we get

$$\log M(r,f) \leqq 8t_n \left(\sum_{k=1}^{n-1} s_k t_k\right) (\log r)^2$$

and it follows from (i) that

$$\text{(iv)} \qquad \log M(r,f) \leqq \varphi(r)(\log r)^2$$

if $\sqrt{r_n} \leqq r \leqq \sqrt{r_{n+1}}$. Therefore f satisfies the condition (1).

We denote by β_n the path

$$|z| = r_n \exp\left\{\frac{\arg z}{4\pi n}\right\},$$

$0 \leqq \arg z \leqq 4\pi n$. Let $z \in \beta_n$. Then

$$\log |f(z)| \leqq \sum_{k=1}^{n-1} s_k t_k \log |z| + s_n \log (1/4),$$

and because $|z| \leqq e r_n$, we see from (ii) that

$$\log |f(z)| \leqq 2\left(\sum_{k=1}^{n-1} s_k t_k\right) \log r_n - s_n = 0.$$

This implies that

(v) $$|f(z)| \leqq 1$$

on β_n.

Let P be any rectifiable path on which $f(z) \to \infty$. It follows from (v) that P does not intersect the path β_n if n is large enough. Therefore $l(er_n) > 2n\pi r_n$ for all large values of n and we get

$$\limsup_{r \to \infty} \frac{l(r)}{r} = \infty.$$

The theorem is proved.

Remark. After this paper had been written, I was told that the same result was proved by A. A. Goldberg and A. E. Eremenko: On the asymptotic paths of entire functions of finite order (in Russian). Mat. Sb. 109 (151) No 4, 1979, 555—581.

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