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LIPSCHITZ AND QUASICONFORMAL TUBULAR NEIGHBOURHOODS OF SPHERES IN CODIMENSION TWO

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In this paper it is shown that if X is a codimension 2 sphere in S^n, $n \neq 4, 5, 6$, then X has either a Lipschitz or a quasiconformal tubular neighbourhood if X is either locally Lipschitz flat or locally quasiconformally flat.

The notation of this paper is the same as that established in [GV]. In particular C denotes either of the categories LIP or QC. Theorem 3.3 of [GV] tells us that if, X is a locally C-flat codimension 2 sphere in S^n, $n \neq 4, 6$, and if X is homotopically unknotted in S^n, then (S^n, X) is C-homeomorphic to (S^n, S^{n-2}). In this paper we consider the case where X might be knotted, obtaining the following result.

Theorem 1. *Let* $X \subset S^n$ *be a locally* C*-flat* TOP $(n-2)$*-sphere in* S^n. *If* $n \neq 4, 5$ *or* 6 *then there is a neighbourhood* N *of* S^{n-2} *in* S^n *and a* C*-embedding* $(N, S^{n-2}) \to (S^n, X)$.

Analogously with Theorem 3.4 of [GV], we have the following result.

Theorem 2. *Let* g: $S^{n-2} \to S^n$ *be a locally* C*-flat embedding. If* $n \neq 4$ *or* 5 *then* g *extends to a* C*-embedding of a neighbourhood of* S^{n-2} *in* S^n.

Proof of Theorem 1. Encasing as in the proof of Theorem 3.3 of [GV], since $n \neq 6 \Rightarrow n-2 \neq 4$ we may assume that only two C-encasings are necessary to exhibit the local C-flatness of X.

Now transfer everything to $\bar{\boldsymbol{R}}^n$. Using the C-Schoenflies theorem we may extend one of the encasings to a C-homeomorphism of $\bar{\boldsymbol{R}}^n$. If we replace X by its inverse image under this homeomorphism, we see that it may be assumed that one of the two C-encasings is the inclusion. By reflection, we may assume that we have the following situation: $X \cap [\bar{\boldsymbol{R}}^n \setminus B^n(a)] = \bar{\boldsymbol{R}}^{n-2} \setminus B^{n-2}(a)$ for some $a < 1$, and there is a C-embedding h: $B^n \to \boldsymbol{R}^n$ with $h^{-1}X = B^{n-2}$ and $X \cap \bar{B}^n \subset hB^{n-2}$. Thus the knotted part of X is trapped inside B^n where it is encased by a single C-encasing. Assume that the norm on $\boldsymbol{R}^n$ is $|(x_i)| = \max\{|x_i|\}$ rather than the pythagorean norm so that B^n is a cubic ball rather than a round ball, thus allowing PL methods.

Choose α: $(V, \bar{\boldsymbol{R}}^{n-2}) \to (\bar{\boldsymbol{R}}^n, X)$, a topological embedding where V is a neighbourhood of $\bar{\boldsymbol{R}}^{n-2}$ in $\bar{\boldsymbol{R}}^n$. The existence of α follows from the topological local flatness of X: by [KS_2], X admits a normal disc bundle in $\bar{\boldsymbol{R}}^n$ since $n \neq 4$; as noted in [K],

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the 2-disc bundles are classified by $H^2(X; \mathbf{Z})$ which, when $n \neq 4$, is the trivial group. Thus X has a trivial normal disc bundle in $\bar{\mathbf{R}}^n$ thereby providing us with the embedding α. Since αV is a neighbourhood of $\bar{\mathbf{R}}^{n-2} \setminus B^{n-2} = X \cap [\bar{\mathbf{R}}^n \setminus B^n]$, we may assume that $\bar{\mathbf{R}}^n \setminus B^n \subset \alpha V$ so, using the relative TOP-Schoenflies theorem, [B] and [GV], we may extend $\alpha | \bar{\mathbf{R}}^n \setminus B^n(b)$ for some $b \in (a, 1)$ to a homeomorphism β of $\bar{\mathbf{R}}^n$ so that $\beta \bar{\mathbf{R}}^{n-2} = \bar{\mathbf{R}}^{n-2}$. Choose $r > 0$ sufficiently small so that $\bar{B}^{n-2} \times B^2(r) \subset \beta^{-1} V$ and $\alpha\beta[\bar{B}^{n-2} \times B^2(r)] \subset hB^n$. Let $\gamma = \alpha\beta | \bar{B}^{n-2} \times B^2(r)$. Then the embedding γ satisfies the following properties: $\operatorname{im} \gamma \subset \bar{B}^n \cap hB^n$; γ is the identity on a neighbourhood of $S^{n-3} \times B^2(r)$; $\gamma[B^{n-2} \times 0] = X \cap B^n$.

Suppose we can construct a C-embedding

$$\delta \colon \bar{B}^{n-2} \times \bar{B}^2(r/2) \to \bar{B}^n$$

which is the identity on a neighbourhood of $S^{n-3} \times \bar{B}^2(r/2)$ and satisfies $X \cap \bar{B}^n \subset \delta[\bar{B}^{n-2} \times 0]$. Let

$$N = [\bar{\mathbf{R}}^n \setminus \bar{B}^n] \cup [\bar{B}^{n-2} \times B^2(r/2)],$$

and extend δ over N by the identity. Then N is a neighbourhood of $\bar{\mathbf{R}}^{n-2}$ in $\bar{\mathbf{R}}^n$ and δ is a C-embedding. Moreover $\delta \bar{\mathbf{R}}^{n-2} = [\bar{\mathbf{R}}^{n-2} \setminus \bar{B}^{n-2}] \cup \delta[\bar{B}^{n-2} \times 0] = X$. Thus, apart from the change of scenery from S^n to $\bar{\mathbf{R}}^n$, δ is the required C-embedding. Thus it is sufficient to construct the C-embedding δ as above.

Consider the TOP handle γ: this is PL straight on $\partial \bar{B}^{n-2} \times B^2(r) = S^{n-3} \times B^2(r)$, being the inclusion there. Since $n \neq 4$ or 5, either $n \leqq 3$ or $n-2 \neq 3$ and $n \geqq 5$. Using [M] in the former case and [KS$_1$] in the latter case, we may straighten γ. More precisely, there is an isotopy $\gamma_t \colon \bar{B}^{n-2} \times B^2(r) \to \bar{B}^n$ $(0 \leqq t \leqq 1)$ with $\gamma_0 = \gamma$, $\gamma_1 | \bar{B}^{n-2} \times \bar{B}^2(r/2)$ PL and $\gamma_t = \gamma$ on a neighbourhood of

$$[S^{n-3} \times B^2(r)] \cup [\bar{B}^{n-2} \times (B^2(r) \setminus \bar{B}^2(s))]$$

for some $s < r$. Let

$$Y = (\bar{\mathbf{R}}^{n-2} \setminus h^{-1}\gamma[\bar{B}^{n-2} \times 0]) \cup h^{-1}\gamma_1[\bar{B}^{n-2} \times 0].$$

It is claimed that Y is a locally C-flat TOP $(n-2)$-sphere in $\bar{\mathbf{R}}^n$ with $\bar{\mathbf{R}}^n \setminus Y$ homotopy equivalent to S^1.

(i) Y is a TOP $(n-2)$-sphere: this follows from the fact that $\gamma[\bar{B}^{n-2} \times 0]$ is homeomorphic to $\gamma_1[\bar{B}^{n-2} \times 0]$ by a homeomorphism which is the identity on the boundary. Note that $\gamma_1[B^{n-2} \times 0] \cap X \setminus B^n = \emptyset$, since $\gamma_1[B^{n-2} \times 0] \subset \gamma_1[B^{n-2} \times B^2(r)] = \gamma[B^{n-2} \times B^2(r)] \subset B^n$, so that $h^{-1}\gamma_1[B^{n-2} \times 0] \cap (\bar{\mathbf{R}}^{n-2} \setminus h^{-1}\gamma[B^{n-2} \times 0]) = \emptyset$.

(ii) Y is locally C-flat: at points of $\bar{\mathbf{R}}^{n-2} \setminus h^{-1}\gamma[\bar{B}^{n-2} \times 0]$ this is immediate; at points of $h^{-1}\gamma_1[B^{n-2} \times 0]$ this follows from the fact that $h^{-1}\gamma_1 | B^{n-2} \times B^2(r/2)$ is a C-embedding; at the remaining points of Y, viz $h^{-1}[S^{n-3} \times 0]$, it follows from

the fact that γ and γ_t are the inclusion on a neighbourhood of $S^{n-3}\times B^2(r)$, so that in a neighbourhood of $h^{-1}[S^{n-3}\times 0]$, Y is still $\bar{\boldsymbol{R}}^{n-2}$.
(iii) $\bar{\boldsymbol{R}}^n\setminus Y$ is homotopy equivalent to S^1: in fact γ_t provides an isotopy of $\bar{\boldsymbol{R}}^n$ throwing $\bar{\boldsymbol{R}}^{n-2}$ onto Y, so Y is even topologically unknotted.

Now apply Theorem 3.3 of [GV] to $(\bar{\boldsymbol{R}}^n, Y)$. Since $n\neq 4$ or 6, there is a C-homeomorphism f: $(\bar{\boldsymbol{R}}^n, \bar{\boldsymbol{R}}^{n-2})\to(\bar{\boldsymbol{R}}^n, Y)$. Moreover, because of the way the C-homeomorphism was constructed in [GV], we may assume that f is the identity on a neighbourhood of $\bar{\boldsymbol{R}}^{n-2}\setminus h^{-1}\gamma[B^{n-2}\times 0]$. Let $\delta=hf^{-1}h^{-1}\gamma_1|\bar{B}^{n-2}\times\bar{B}^2(r/2)$. We check the required properties of δ.
(a) δ is a C-embedding:

$$\gamma_1[\bar{B}^{n-2}\times B^2(r/2)]\subset\gamma_1[\bar{B}^{n-2}\times B^2(r)]=\gamma[\bar{B}^{n-2}\times B^2(r)]\subset hB^n,$$

so $h^{-1}\gamma_1|\bar{B}^{n-2}\times B^2(r/2)$ is a C-embedding as, therefore, is $f^{-1}h^{-1}\gamma_1|\bar{B}^{n-2}\times B^2(r/2)$ Making r smaller if necessary, we can be sure that

$$f^{-1}h^{-1}\gamma_1[\bar{B}^{n-2}\times B^2(r/2)]\subset B^n,$$

so that δ is a well-defined C-embedding.
(b) δ is the identity on a neighbourhood of $S^{n-3}\times\bar{B}^2(r/2)$: this follows from the facts that γ_1 is the inclusion on such a set and f is the identity on a neighbourhood of $\bar{\boldsymbol{R}}^{n-2}\setminus h^{-1}\gamma[B^{n-2}\times 0]$ hence on a neighbourhood of

$$h^{-1}\gamma[S^{n-3}\times B^2(r/2)]=h^{-1}\gamma_1[S^{n-3}\times B^2(r/2)]$$

provided r is small enough.

(c) $X\cap\bar{B}^n\subset\delta[\bar{B}^{n-2}\times 0]$: in fact,

$$f^{-1}h^{-1}\gamma_1[\bar{B}^{n-2}\times 0]=h^{-1}\gamma[\bar{B}^{n-2}\times 0],$$

$$\text{so}\quad \delta[\bar{B}^{n-2}\times 0]=\gamma[\bar{B}^{n-2}\times 0]=X\cap\bar{B}^n.$$

This completes the construction of δ and hence completes the proof of Theorem 1. □

Proof of Theorem 2. The proof of Theorem 2 is much the same as that of Theorem 1 but one uses instead (C, g)-encasing and [GV, Theorem 3.4] neither of which requires the restriction $n\neq 6$. □

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