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ON THE CONVERGENCE OF THE FINITE ELEMENT APPROXIMATION OF EIGENFREQUENCIES AND EIGENVECTORS TO MAXWELL'S BOUNDARY VALUE PROBLEM

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0. Introduction

This paper can be regarded as a supplementary to the work [10]. There the finite element approximation of the time-harmonic Maxwell's equations

$$(0.1)\qquad \begin{cases}\operatorname{curl} E - i\omega\mu H = J\\ \operatorname{curl}_* H + i\omega\varepsilon E = K\end{cases} \quad \text{in} \quad G \subset \boldsymbol{R}^2$$

with homogeneous boundary condition

$$(0.2)\qquad n \wedge E_{|\Gamma} = 0,$$

$\Gamma := \partial G$, $n \wedge E := n_1 E_2 - n_2 E_1$, was considered under the assumption that ω is not an eigenvalue of the system. Here G is a bounded smooth domain in the plane, n denotes the outer unit normal on the boundary Γ, E and H are vector and scalar functions and the operators curl and curl_* are in two space dimensions formally given by $\operatorname{curl} E = \partial_1 E_2 - \partial_2 E_1$, $\operatorname{curl}_* H = (\partial_2 H | -\partial_1 H)$. (The indices refer to the respective components.) Moreover, in system (0.1) ε is a function that takes positive definite bounded matrix values, μ is a strictly positive bounded real valued function and J and K are force densities. In the case where there are no nontrivial solutions of problem (0.1), (0.2) with $J=0$, $K=0$, the approximation of the electric field E is handled by Saranen in [17] for smooth and in [18] for polygonal domains of the plane. The approximation of the whole solution $(E|H)$ of problem (0.1), (0.2) for polygonal $G \subset \boldsymbol{R}^2$ has been studied by Neittaanmäki and Saranen in [12].

In this paper we consider the finite element approximation of the eigenvalue problem arising from equation (0.1), (0.2) in smooth domains. We denote

$$(0.3)\qquad MU := i(\varepsilon^{-1} \operatorname{curl}_* H | -\mu^{-1} \operatorname{curl} E),\ U := (E|H)$$

and our problem can be written

$$(0.4)\qquad (M-\omega)U = 0, \quad n \wedge U_{1|\Gamma} = 0.$$

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It is well known ([8], [9], [10], [14]) that the resolvent of M, interpreted as an operator in a certain Hilbert space, is compact so that the spectrum of M is purely discrete. We shall, in fact, first consider the approximation of eigenvalues and eigenvectors of the operator $\mathscr{A}:=M^2+1$ and reduce the situation back to the original problem (see Chapters 3 and 4).

Our method to prove error estimates for the approximation of eigenvalues is based on the use of the min—max-principle. This idea has been presented by Strang and Fix in [19] and has been developed later on to cover conforming and nonconforming schemes by many authors (see [1], [6], [7]). Our method follows partly the works of Kikuchi [7] and Ishihara [6] combined with some projection methods of Hilbert spaces presented in [10], [14] and [15]. In this form our method can be applied e.g. to the eigenvalue approximation of boundary value problems arising in [11]. For other general treatments of eigenvalue approximation we refer to [2], [3], [4], [13] and [20].

In the approximation of eigenvectors of Maxwell's equations with boundary condition of total reflection there arise some special difficulties. As the first aspect we mention that the electric field component E of the eigensolutions is solenoidal (divergence free). For the approximation of eigensolutions we use piecewise linear continuous vector fields such that the first components satisfy the condition (0.2) at boundary nodes of the triangulation, but are solenoidal only asymptotically. In order to use regularity results we need to use smooth domains. Therefore the special boundary condition makes the nonconformal method necessary. Because of this, care should be taken especially in the evaluation of the lower bounds of eigenvalues. The reduction of the bilinear form problem (connected with $\mathscr{A}=M^2+1$) back to the original question of the eigenvalue problem of M leads to difficulties, which are treated in Chapters 4.2 and 4.3.

Let us remark that the abstract methods of Chapters 3 and 4 are, with the exception of Lemma 3.1, independent of the dimension of the space. The reason why we restrict our discussions to the two dimensional case is that up to now the finite element approximations for the resolvent of M are known only in the two dimensional case.

After submitting this paper the article [21] appeared. There finite element approximation of vector fields given by curl and divergence was considered in the three dimensional case.

1. The eigenvalue problem

1.1. In order to use variational methods we formulate the eigenvalue problem in a weak sense. The familiar Sobolev spaces $H^k(G)$ $(k\in \mathbf{N},\ H^0(G)=L^2(G))$ and $H_0^1(G)$ are needed for the precise formulation. The inner product of $H^k(G)$ is denoted by $(\cdot,\ \cdot)_{H^k(G)}$ with corresponding norm $\|\cdot\|_{H^k(G)}$, $k=0, 1, 2$. The subscript will be omitted in the case $k=0$. We will write $H^k(G)$, $(\cdot, \cdot)_{H^k(G)}$ and $\|\cdot\|_{H^k(G)}$ also for the product spaces $H^k(G)\times\ldots\times H^k(G)$ without indicating the number of components which will always be clear from the context. Furthermore, we introduce in the same manner as in [9] the following special conventions

$$D_\varepsilon := \{V\in H^0(G) | \operatorname{div} \varepsilon V\in H^0(G)\},$$
$$D_{0,\varepsilon} := \{V\in D_\varepsilon | \operatorname{div} \varepsilon V = 0\},$$

and

$$R := \{V\in H^0(G) | \operatorname{curl} V\in H^0(G)\},$$
$$R_* := \{\varphi\in H^0(G) | \operatorname{curl}_* \varphi\in H^0(G)\},$$
$$R^0 := \{V\in R | (\operatorname{curl}_* \varphi, V) = (\varphi, \operatorname{curl} V) \text{ for all } \varphi\in R_*\}.$$

The derivates $\operatorname{div} \varepsilon V$, $\operatorname{curl} V$ and $\operatorname{curl}_* \varphi$ appearing above are defined distributionally in an obvious way corresponding to the formulae $\operatorname{div} \varepsilon V=\partial_i(\varepsilon_{ij}V_j)$, $\operatorname{curl} V=\partial_1 V_2-\partial_2 V_1$ and $\operatorname{curl}_* \varphi=(\partial_2\varphi|-\partial_1\varphi)$, where $\partial_i:=\partial/\partial x_i$, $i=1, 2$, and $(\cdot|\cdot)$ denotes the ordered pair.

With this terminology, we will need to make use of the Hilbert space

$$\mathscr{H}_{*,s} := (R\cap D_\varepsilon\times R_* | (\cdot, \cdot)_{*,s})$$

with the inner product

$$(U, V)_{*,s} := (U, V)_s + (MU, MV),$$

where

$$(U, V) := (U_1, \varepsilon V_1) + (U_2, \mu V_2),$$

and

$$(U, V)_s := (U, V) + s(\operatorname{div} \varepsilon U_1, \operatorname{div} \varepsilon V_1)$$

with a real parameter $s\geqq 0$. Moreover, we denote

$$\mathscr{H} := (H^0(G)\times H^0(G) | (\cdot, \cdot))$$
$$\mathscr{H}_s := (D_\varepsilon\times H^0(G) | (\cdot, \cdot)_s)$$

and

$$\mathscr{H}^0 := \{V\in\mathscr{H}_s | V_1\in D_{0,\varepsilon}\}.$$

1.2. Using the formal differential operator M, we now define the (symmetric) Maxwell operator

$$\mathscr{M}\colon D(\mathscr{M})\subset\mathscr{H}^0\to\mathscr{H}^0, \quad U\to MU,$$

where $D(\mathscr{M}) := (R^0\times R_*)\cap\mathscr{H}^0$.

We denote by $N(\mathscr{M})$ the kernel of $\mathscr{M}$ and by $W(\mathscr{M})$ the range of $\mathscr{M}$. According to [8], [10] and [14] one has the results

(i) $\mathscr{M}$ is selfadjoint in $\mathscr{H}^0$ so that the spectrum $\sigma(\mathscr{M})$ of $\mathscr{M}$ is real.

(ii) The resolvent of $\mathscr{M}$ is compact, which implies the spectrum of $\mathscr{M}$ consists of isolated points.

(iii) Let P_ω be the projection of $\mathscr{H}^0$ onto the space of eigenfunctions of $\mathscr{M}$ corresponding to ω. Then

$$P_\omega \perp P_\lambda \quad \omega \neq \lambda \tag{1.1}$$

in the sense of the scalar product $(\cdot, \cdot)$.

We shall prove an important

Theorem 1.1. *If* $\omega^2 \in \sigma(\mathscr{M}^2)$, *then* $\omega \in \sigma(\mathscr{M})$ *and* $-\omega \in \sigma(\mathscr{M})$. *Let* Q_λ *be the projection on the space of eigenfunctions of* $\mathscr{A} := \mathscr{M}^2 + 1$ *corresponding to* $\lambda := \omega^2 + 1$, $\omega \geqq 0$. *Then*

$$Q_\lambda = P_\omega + P_{-\omega} \quad \text{for} \quad \omega \neq 0$$

and

$$Q_1 = P_0.$$

Proof. A. If $\omega^2 \in \sigma(\mathscr{M}^2)$, then $\omega \in \sigma(\mathscr{M})$ or $-\omega \in \sigma(\mathscr{M})$. If the ordered pair $(U_1|U_2)$ belongs to $W(P_\omega) = N(\mathscr{M} - \omega)$, it is easily seen that $(U_1| - U_2) \in W(P_{-\omega}) = N(\mathscr{M} + \omega)$. Thus $\omega \in \sigma(\mathscr{M})$ and $-\omega \in \sigma(\mathscr{M})$.

B. In order to prove the second assertion of Theorem 1.1 let U belong to $N(\mathscr{A} - \lambda) \equiv N(\mathscr{M}^2 - \omega^2)$. Then

$$F = (\mathscr{M} + \omega) U \in N(\mathscr{M} - \omega). \tag{1.2}$$

Because $\mathscr{M}$ is selfadjoint,

$$\mathscr{H}^0 = N(\mathscr{M} + \omega) \oplus W(\mathscr{M} + \omega),$$

and it holds for all $U \in \mathscr{H}^0$ that

$$U = U_1 + U_2 \in N(\mathscr{M} + \omega) \oplus W(\mathscr{M} + \omega), \tag{1.3}$$

where the indices refer to the projection on the respective subspace. Thus, by (1.2),

$$F = (\mathscr{M} + \omega) U_2. \tag{1.4}$$

On the other hand, it holds by (1.2) that

$$\begin{aligned} (\mathscr{M} + \omega) F &= (\mathscr{M} - \omega) F + 2\omega F \\ &= 2\omega F. \end{aligned} \tag{1.5}$$

According to (1.4) and (1.5) we have for $\omega \neq 0$, $U_2 - (2\omega)^{-1} F \in N(\mathscr{M} + \omega)$. But (1.3) and (1.4) imply $U_2 - (2\omega)^{-1} F \in W(\mathscr{M} + \omega)$. Thus

$$U_2 = (2\omega)^{-1} F \in N(\mathscr{M} - \omega),$$

and therefore $Q_\lambda = P_\omega + P_{-\omega}$, according to (1.3).

The case $\omega = 0$ is trivial: If $\mathscr{M}^2 U = 0$, then $\mathscr{M} U \in N(\mathscr{M}) \cap W(\mathscr{M}) = \{0\}$. Thus $Q_1 = P_0$. □

According to Theorem 1.1 it is obviously sufficient to study the spectrum of $\mathscr{M}^2$ in order to obtain exact information as to the spectrum of $\mathscr{M}$. Without loss of generality we can move the spectrum to the right so that we can, in fact, study the properties of the spectrum of

$$\mathcal{A} := \mathcal{M}^2 + 1.$$

In what follows we are interested in the spectral points of $\mathcal{A}$ which lie between 1 and λ_0, where λ_0 is any fixed constant larger than 1.

We shall give a variational formulation to this eigenvalue problem, which is based on the following theorem:

Theorem 1.2. *Let*

$$\mathcal{K} = \min\{\|\nabla\varphi\|^2/\|\varphi\|^2 \,|\, \varphi \in H_0^1(\Omega)\},$$

$s = 2\mathcal{K}^{-1}(\lambda_0 - 1)$, *and let* $\lambda \in [1, \lambda_0]$. *Then* $U \in N(\mathcal{A} - \lambda)$ *if and only if the equation*

$$(\Phi, U)_{*,s} = \lambda(\Phi, U) \quad \text{for all} \quad \Phi \in D(\mathcal{M})$$

holds.

Proof. We note that, if $U \in N(\mathcal{A} - \lambda)$, $\operatorname{div} \varepsilon U_1 = 0$ and $\varepsilon^{-1} \operatorname{curl}_* U_2 \in R^0$. By partial integration we find that U satisfies (1.6).

Conversely, the argument follows the same lines as in the proof of Theorem 1.3 in [12] (see also [10]). □

Let us remark that we can indeed determine the "elliptization parameter" s, since $\mathcal{K}$ is the first eigenvalue of the (negative) Laplacian and its evaluation is well known in many respects (see [5], [19]).

We are now in a position to define our problem in variational form:

Problem (EP). *Find the eigenvalues* $\lambda \in [1, \lambda_0]$ *and the eigenfunctions* U *such that*

$$(\Phi, U)_{*,s} = \lambda(\Phi, U) \quad \text{for all} \quad \Phi \in D(\mathcal{M}). \tag{1.6}$$

The Problem (EP) admits a finite sequence of positive eigenvalues $\{\lambda_i\}_i$ each of finite, even multiplicity $\{v_j\}_j$ (we agree that eigenvalues are ordered and repeated according to their multiplicity) and a corresponding sequence of eigenfunctions $\{U_i\}_i$ with the normalization condition

$$(U_i, U_j) = \delta_{ij}. \tag{1.7}$$

Let $E(\lambda)$ be the spectral family of $\mathcal{A}$ and let

$$Q_\lambda := E(\lambda+) - E(\lambda-).$$

We define on $\mathcal{H}_{*,s} - \{0\}$ the Rayleigh quotient

$$\mathcal{R}(\Phi) = \frac{\|\Phi\|_{*,s}^2}{\|\Phi\|^2}. \tag{1.8}$$

The eigenvalues of the Problem (EP) can be characterized by the Rayleigh principle: The stationary points of $\mathcal{R}$ are precisely the eigenfunction of the Problem (EP) and the values of $\mathcal{R}$ at such points are the corresponding eigenvalues:

(1.9) $$\lambda_i = \min\{\mathcal{R}(\Phi)|\Phi\in W(E(\lambda_{i-1}+))^\perp,\ \Phi\neq 0\}.$$

Moreover, $\mathcal{R}(\Phi)=\lambda$ for every $\Phi\in W(Q_\lambda)$ and $\dim W(Q_{\lambda_i})=\nu_j$, $2\leqq\nu_j<\infty$ for all $\lambda_i=\lambda_k$ with $k=\sum_{\ell=1}^{j-1}\nu_\ell+1$.

By the regularity results in [10], Theorem 3.2, and by (1.7), it holds for every eigensolution U_λ of (1.6) that $U_\lambda\in H^2(G)$ and that

(1.10) $$\|U_\lambda\|_{H^2(G)}\leqq c\lambda.$$

Note that, as above, we adapt the usual convention that $c, c_1, c_2, \ldots$ always denote positive generic constants, which may vary from context to context.

2. Finite element approximation of problem (EP)

To approximate the eigenfunctions and eigenvectors of (1.6) we introduce a family of finite dimensional subspaces $\mathcal{H}^h\subset\mathcal{H}_{*,s}$ depending on a discretization parameter $0<h<1$ going to zero. The idea is to solve (1.6) on these subspaces.

For a precise formulation, let $\mathcal{T}_h$ be a family of regular triangulations of G in the usual sense ([10])

$$\bar{G} = \cup\{\bar{T}_h\ |T_h\in\mathcal{T}_h\}.$$

Let K_h be the set of all nodes of $\mathcal{T}_h$ lying on the boundary. Denoting $C(\bar{G}):=\{\varphi:\bar{G}\to C|\varphi \text{ continuous}\}$ we define, as in [10],

$$S_1^h:=\{\Phi=(\Phi_1|\Phi_2)\in C(\bar{G})\times C(\bar{G})|\ \Phi_i|_{T_h} \text{ linear},\ (n\wedge\Phi)(x)=0,\ x\in K_h\},$$

and

$$S_2^h:=\{\varphi\in C(\bar{G})|\ \varphi|_{T_h} \text{ linear}\}.$$

We define $\mathcal{H}^h:=S_1^h\times S_2^h$ as a subspace of $\mathcal{H}_{*,s}$.

The discrete analogue of Problem (EP) reads:

Problem (EP)^h. *Find the eigenvalues* $\lambda^h\in[1,\infty)$ *and the eigenfunctions* $U^h\in\mathcal{H}^h$ *such that*

(2.1) $$(\Phi, U^h)_{*,s}=\lambda^h(\Phi, U^h)\quad \textit{for all}\quad \Phi\in\mathcal{H}^h.$$

Since $\mathcal{H}^h\not\subset D(\mathcal{M})$, (2.1) is a nonconforming finite element model for solving Problem (EP).

In order to write (2.1) as an equivalent operator equation, we recall some definitions from [10]. Let P_* be the orthogonal projection on $R^0\cap D_\varepsilon\times R_*$ defined through the decomposition

$$\mathcal{H}_{*,s}=R^0\cap D_\varepsilon\times R_*\oplus N(\operatorname{div}\mu^{-1}\operatorname{curl}_*+\varepsilon)\times\{0\}.$$

We introduce the operators

$$S\colon \mathscr{H} \to P_* \mathscr{H}_{*,s}, \quad T\colon \mathscr{H} \to \mathscr{H}_{*,s}$$

$$\Phi \to \psi^{(1)} \qquad \Phi \to \psi^{(2)}$$

by equations

(2.2) $$(\eta, \psi^{(1)})_{*,s} = (\eta, \Phi) \quad \text{for all} \quad \eta \in P_* \mathscr{H}_{*,s}$$

and, respectively,

(2.3) $$(\eta, \psi^{(2)})_{*,s} = \big((1-P_*)\eta, \Phi\big) \quad \text{for all} \quad \Phi \in \mathscr{H}_{*,s}.$$

Let P^h be the orthogonal projection of $\mathscr{H}_{*,s}$ on $\mathscr{H}^h$. It is easily seen that equation (2.1) is equivalent to

(2.4) $$\big(I - \lambda^h P^h (S+T)\big) U^h = 0.$$

According to [10], Theorem 2.6, the pseudoinverse

$$\mathscr{A}_h := \big(P^h(S+T)\big)^{-1}\colon \mathscr{H}^h \to \mathscr{H}^h$$

exists and Problem $(\text{EP})^h$ corresponds to the problem

$$(\mathscr{A}_h - \lambda^h) U^h = 0.$$

We note that with respect to $\mathscr{H}$ $\mathscr{A}_h$ is a symmetric, finite dimensional operator and let $E^h(\lambda)$ be its spectral family.

Following the same lines as in [19] it can be proved:

Theorem 2.1. *Problem* $(\text{EP})^h$ *admits* $N_h := \dim \mathscr{H}^h$ *real, possibly repeated, eigenvalues* $\{\lambda_j\}_{j=1}^{N_h}$ *(arranged in increasing order* $1 \leqq \lambda_1^h \leqq \lambda_2^h \leqq \ldots \leqq \lambda_{N^h}^h < \infty$*) and corresponding eigenfunctions* $\{U_j^h\}_{j=1}^{N_h}$ *with the normalization condition*

(2.5) $$(U_i^h, U_j^h) = \delta_{ij}.$$

Moreover, the following characterizations hold:

(2.6) $$\lambda_i^h = \min \big\{\mathscr{R}(\Phi) | \Phi \in W\big(E^h(\lambda_{i-1}^h +)\big)^\perp,\ \Phi \neq 0\big\},$$

$\mathscr{R}(U_j^h) = \lambda_j^h$ *and (min—max-principle)*

(2.7) $$\lambda_i^h = \min_{\mathscr{H}_i^h \subseteq \mathscr{H}^h} \max_{\Phi \in \mathscr{H}_i^h} \mathscr{R}(\Phi),$$

where $\mathscr{H}_i^h$ *is an i-dimensional subspace of* $\mathscr{H}^h$.

Let $\tilde{Q}_{\mu^h}^h$ be the projection on the eigenspace of $\mathscr{A}_h$ relative to eigenvalue μ^h. Then

(2.8) $$E^h(\lambda) = \sum_{\mu^h \leqq \lambda} \tilde{Q}_{\mu^h}^h$$

and it holds that

(2.9) $$(\Phi, \tilde{Q}_{\mu^h}^h U)_{*,s} = \mu^h (\Phi, \tilde{Q}_{\mu^h}^h U) \quad \text{for all} \quad \Phi \in \mathscr{H}^h.$$

3. Error bounds for eigenvalue approximation

Let m be the number of the eigenvalues of $\mathscr{A}$ lying in $[1, \lambda_0]$. In this section we shall give error bounds to approximation of the eigenvalues λ_i $(i=1, \ldots, m)$ by the eigenvalues λ_i^h in $[1, \lambda_0+1]$.

At first we prove as a corollary to the abstract approximation results given in Section 3 of [10].

Lemma 3.1. *Let*

$$A_\lambda^h := P^h + P^h T\mathscr{A} \colon W(E(\lambda+)) \to W(A_\lambda^h) \subset \mathscr{H}^h.$$

Then the asymptotic estimates

$$\|(I-A_\lambda^h)U\| \leqq ch^2 \|U\|_{H^2(G)} \tag{3.1}$$

$$\|(I-A_\lambda^h)U\|_{*,s} \leqq ch \|U\|_{H^2(G)} \tag{3.2}$$

hold for $U \in W(E(\lambda+))$. *Furthermore,*

$$\dim (W(A_h^\lambda)) = \dim (W(E(\lambda+)))$$

for sufficiently small h.

Proof. A. By [10], Section 3,

$$\|(I-P^h)V\| + h\|(I-P^h)V\|_{*,s} \leqq ch^2 \|V\|_{H^2(G)} \tag{3.3}$$

holds for all $V \in P_* \mathscr{H}_{*,s} \cap H_2(G)$ and

$$\|P^h TY\| + h\|P^h TY\|_{*,s} \leqq ch^2 \|Y\| \tag{3.4}$$

for all $Y \in \mathscr{H}$.

Using the regularity (1.10) we have $U \in H^2(G)$ for every $U \in W(E(\lambda+))$. Thus the estimates (3.1) and (3.2) follow from (3.3) and (3.4).

B. To have the last assertion, taking into account that $\dim A_\lambda^h(W(E(\lambda+))) = \dim W(E(\lambda+)) - \dim N(A_\lambda^h)$, we only have to prove that $A_\lambda^h \colon W(E(\lambda+)) \to W(A_\lambda^h)$ is injective.

If $A_\lambda^h U = 0$, we have $U = (I-P^h)U - P^h T\mathscr{A}$. Estimates (3.3) and (3.4) yield

$$\|U\| \leqq ch^2 \|U\|_{H^2(G)}.$$

By the regularity argument we obtain $\|U\| \leqq ch^2 \|U\|$. Therefore $U=0$ if h is small enough. □

Using the min—max-principle and Lemma 3.1 we can prove the error estimate for the approximations of eigenvalues:

Theorem 3.2. *Let λ_i^h be the approximate eigenvalue of λ_i $(i=1, \ldots, m)$. Then, for sufficiently small h, it holds that*

$$|\lambda_i - \lambda_i^h| \leqq c\lambda_0^2 h^2. \tag{3.5}$$

Proof. A. We shall first prove

$$\lambda_i^h \leqq \lambda_i + c\lambda_0^2 h^2. \tag{3.6}$$

Using the definitions of S and T, given in (2.2) and (2.3), respectively, and the orthogonality $Q_\lambda \perp Q_\mu$, $\lambda \neq \mu$, we obtain for $F_i \in W(E(\lambda_i+))$, $F_i = \sum_{1 \leqq \lambda \leqq \lambda_i} Q_\lambda F_i$,

$$\|A_{\lambda_i}^h F_i\|_{*,s}^2 = (A_{\lambda_i}^h F_i, (I+T\mathscr{A}) F_i)_{*,s} \tag{3.7}$$

$$= (A_{\lambda_i}^h F_i, (S+T)\mathscr{A} F_i)_{*,s} = (A_{\lambda_i}^h F_i, \mathscr{A} F_i)$$

$$\leqq \|A_{\lambda_i}^h F_i\| \, \|\mathscr{A} F_i\|.$$

Inequality (3.1) implies $A_{\lambda_i}^h F_i \neq 0$ and $\|A_{\lambda_i}^h F_i\|^{-1} > 2^{-1}$, for sufficiently small h. By (3.7), (3.1), and by regularity,

$$\|A_{\lambda_i}^h F_i\|_{*,s}^2 \leqq \lambda_i \|A_{\lambda_i}^h F_i\| \left(\|A_{\lambda_i}^h F_i\| + \|(I - A_\lambda^h) F_i\|\right)$$

$$\leqq \lambda_i \|A_{\lambda_i}^h F_i\|^2 (1 + c\lambda_i h^2) \|F_i\|$$

holds. Hence

$$\mathscr{R}(A_{\lambda_i}^h F_i) \leqq \lambda_i + c\lambda_i^2 h^2 \leqq \lambda_i + c\lambda_0^2 h^2.$$

By Lemma 3.1 $\dim W(A_{\lambda_i}^h) = \sigma_k$, $\sigma_k := \sum_{j=1}^k \nu_j$, $\sum_{\ell=1}^{k-1} \nu_\ell + 1 \leqq i \leqq \sum_{\ell=1}^k \nu_\ell$ and using the min—max-principle, we get $\lambda_i^h \leqq \lambda_{\sigma_k}^h \leqq \lambda_i + c\lambda_0^2 h^2$.

B. By (3.6) we only have to prove

$$\lambda_i - c\lambda_i^2 h^2 \leqq \lambda_i^h \tag{3.8}$$

for sufficiently small h.

Let $U^h \in \mathscr{H}_\sigma^h \cap W(E(\lambda_{\sigma_{k-1}}+))^\perp$ ($H_\sigma^h \subset \mathscr{H}^h$ with $\dim(\mathscr{H}_c^h) = \sigma = \sum_{\ell=1}^{k-1} \nu_\ell + 1$; the intersection is indeed nonempty, since the condition $(U^h, U_j) = 0$, $j = 1, \ldots, \sigma_{k-1}$, contains σ_{k-1} restrictions (unknowns), σ_{k-1} and i as above). Since for all $F_i \in W(E(\lambda_{i-1}+))$

$$(SU^h, F_i)_{*,s} = \sum_{1 \leqq \lambda \leqq \lambda_{\sigma_{k-1}}} (SU^h, Q_\lambda F_i)_{*,s} = \sum_{1 \leqq \lambda \leqq \lambda_{\sigma_{k-1}}} \lambda^{-1} (U^h, Q_\lambda F_i) = 0,$$

we have $SU^h \in W(E(\lambda_{\sigma_{k-1}}))^\perp$. By (2.2) and Rayleigh's principle (1.9) we obtain

$$\lambda_i^2 \|SU^h\|^4 \leqq \|SU^h\|_{*,s}^4 = (SU^h, SU^h)_{*,s}^2$$

$$\leqq \|U^h\|^2 \|SU^h\|^2 \leqq \|U^h\|^2 \lambda_i^{-1} \|SU^h\|_{*,s}^2,$$

which implies

$$\|SU^h\|^2 \leqq \lambda_i^{-2} \|U^h\|^2 \quad \text{and} \quad \|SU^h\|_{*,s}^2 \leqq \lambda_i^{-1} \|U^h\|^2. \tag{3.9}$$

Since, according to [10], Section 3,

$$\|SU^h - P^h(S+T)U^h\| \leqq ch^2 \|U^h\|,$$

we conclude by the formulae (2.2), (2.3) and (3.9)

$$\begin{aligned}\|U^h\|^4 &= \left|(U^h, P^h(S+T)U^h)_{*,s}\right|^2 \\ &\leqq \|U^h\|^2_{*,s}\|P^h(S+T)U^h\|^2_{*,s} \\ &= \|U^h\|^2_{*,s}\left|(P^h(S+T)U^h, U^h)\right| \\ &\leqq \|U^h\|^2_{*,s}\|U^h\|\,\lambda_i^{-1}(1+c\lambda_i h^2).\end{aligned}$$

If $c\lambda_i h^2 \leqq 1$, this implies

$$\mathscr{R}(U^h) \geqq \lambda_i(1+c\lambda_i h^2)^{-1} \geqq \lambda_i - c\lambda_i^2 h^2. \tag{3.10}$$

Using the min—max-principle we obtain (3.8), since $\lambda_\sigma^h \leqq \lambda_i^h$. Estimate (3.5) follows if we combine (3.6) and (3.8). □

We now want to obtain an approximation for the corresponding eigenvalue ω of operator $\mathscr{M}$. We recall that

$$\omega = \pm\sqrt{\lambda - 1}.$$

Let

$$\omega^h := \pm\sqrt{\lambda^h - 1}. \tag{3.10}$$

We conclude by Theorem 3.2 for $\omega \neq 0$ and sufficiently small $h > 0$

$$\begin{aligned}|\omega-\omega^h| &\leqq c|\omega^2-(\omega^h)^2| \\ &= c|\lambda-\lambda^h| \leqq c\lambda_0 h^2.\end{aligned}$$

Thus we have

Theorem 3.3. *Let ω be an eigenvalue of $\mathscr{M}$, $|\omega| < \sqrt{\lambda_0 - 1}$, and let ω^h be its finite element approximation defined through equations* (2.1) *and* (3.10). *Then*

$$|\omega - \omega^h| = \mathcal{O}(h^2), \quad \textit{for} \quad h \to 0.$$

4. Error bounds for eigenfunction approximation

4.1. Let $\lambda'(\lambda'')$ be the greatest (smallest) eigenvalue with $\lambda' < \lambda$ (with $\lambda'' > \lambda$, respectively). According to Theorem 3.2 it holds for all eigenvalues λ of Problem (EP) with $1 \leqq \lambda \leqq \lambda_0$ and for the corresponding λ^h (i.e. the eigenvalue with the same index) of Problem $(\mathrm{EP})^h$ with $1 \leqq \lambda^h \leqq \lambda_0 + 1$ that

$$\lambda' < \lambda - c\lambda_0^2 h^2 \leqq \lambda^h \leqq \lambda + c\lambda_0^2 h^2 < \lambda'', \tag{4.1}$$

for sufficiently small h.

We abbreviate

$$\lambda(h) := \lambda + c\lambda_0^2 h\,|h|$$

and define

$$Q_\lambda^h := E^h(\lambda(h)) - E^h(\lambda(-h)). \tag{4.2}$$

By (4.1) it holds that

$$Q_\lambda^h = \sum_{\lambda(-h)\leqq\mu\leqq\lambda(h)} \tilde{Q}_\mu^h .$$

Using Lemma 3.1 as an essential tool we prove

Theorem 4.1. *The asymptotic error estimates*

$$\|(I-Q_\lambda^h)Q_\lambda U\| \leqq c\lambda_0^2 h^2 \|U\|, \tag{4.3}$$

$$\|(I-Q_\lambda^h)Q_\lambda U\|_{*,s} \leqq c\lambda_0^{3/2} h \|U\| \tag{4.4}$$

hold for all eigenvalues λ, $1\leqq\lambda\leqq\lambda_0$, *of operator* $\mathscr{A}$.

Proof. A. Let λ', λ and λ'' be three eigenvalues such that $1\leqq\lambda'<\lambda<\lambda''\leqq\lambda_0$, $1=\lambda<\lambda''$ or that $\lambda'<\lambda=\lambda_0$, satisfying (4.1). Using relation

$$I = E^h(\lambda'(h)) + Q_\lambda^h + (I - E^h(\lambda''(-h))), \tag{4.5}$$

Lemma 3.1 and the regularity estimate (1.10) we conclude

$$\begin{aligned} \|(I-Q_\lambda^h)Q_\lambda U\| &\leqq \|(I-A_\lambda^h)Q_\lambda U\| + \|(A_\lambda^h - Q_\lambda^h)Q_\lambda U\| \\ &\leqq c\lambda h^2\|U\| + \left\|E^h(\lambda'(h))A_\lambda^h Q_\lambda U\right\| \\ &\quad + \left\|(I-E^h(\lambda''(h)))A_\lambda^h Q_\lambda U\right\| \\ &=: c\lambda h^2\|U\| + ① + ②. \end{aligned} \tag{4.6}$$

We will prove that

$$① \leqq \frac{2\lambda}{\lambda-\lambda'}\|(I-A_\lambda^h)Q_\lambda U\| \tag{4.7}$$

and that

$$② \leqq \frac{2\lambda}{\lambda''-\lambda}\|(I-A_\lambda^h)Q_\lambda U\|. \tag{4.8}$$

If we combine (4.6), (4.7) and (4.8) with Lemma 3.1, the inequality (4.3) follows.

B. Let us first prove (4.7). By (2.9)

$$\begin{aligned} &((I-E^h(\lambda'(h)))A_\lambda^h Q_\lambda U,\ E^h(\lambda'(h))A_\lambda^h Q_\lambda U)_{*,s} \\ &= \sum_{1\leqq\mu\leqq\lambda'(h)} (A_\lambda^h Q_\lambda U,\ \tilde{Q}_\mu^h U)\mu((I-E^h(\lambda'(h)))A_\lambda^h Q_\lambda U,\ \tilde{Q}_\mu^h U) = 0. \end{aligned} \tag{4.9}$$

Using this with the identity $I=E^h(\lambda'(h))+(I-E^h(\lambda'(h)))$ we find

$$\begin{aligned} &\left\|E^h(\lambda'(h))A_\lambda^h Q_\lambda U\right\|_{*,s}^2 \\ &= (E^h(\lambda'(h))A_\lambda^h Q_\lambda U,\ A_\lambda^h Q_\lambda U)_{*,s} \\ &= \lambda(E^h(\lambda'(h))A_\lambda^h Q_\lambda U,\ Q_\lambda U). \end{aligned} \tag{4.10}$$

Moreover,

$$(E^h(\lambda'(h))A_\lambda^h Q_\lambda U,\ A_\lambda^h Q_\lambda U) = \left\|E^h(\lambda'(h))A_\lambda^h Q_\lambda U\right\|^2. \tag{4.11}$$

On the other hand, we obtain by (2.9) and by the orthogonality $\tilde{Q}^h_\lambda \perp \tilde{Q}^h_\mu$, $\lambda \neq \mu$,

(4.12)
$$\|E^h(\lambda'(h))A^h_\lambda Q_\lambda U\|^2_{*,s}$$
$$= \sum_{1 \leq \mu \leq \lambda'(h)} \mu (A^h_\lambda Q_\lambda U, \tilde{Q}^h_\mu U)^2 (\tilde{Q}^h_\mu U, \tilde{Q}^h_\mu U)$$
$$\leqq \lambda'(h) \|E^h(\lambda'(h))A^h_\lambda Q_\lambda U\|^2.$$

A combination of (4.10), (4.11) and (4.12) yields
$$(\lambda - \lambda'(h))\|E^h(\lambda'(h))A^h_\lambda Q_\lambda U\|^2$$
$$\leqq \lambda(E^h(\lambda(h))A^h_\lambda Q_\lambda U, A^h_\lambda Q_\lambda U - Q_\lambda U),$$
which implies (4.7) for sufficiently small h.

C. In order to prove (4.8) we first find by arguments similar to (4.10) that

(4.13)
$$\|(I - E^h(\lambda''(-h))A^h_\lambda Q_\lambda U\|^2_{*,s}$$
$$= \lambda((I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U, Q_\lambda U).$$

Since $I - E^h(\lambda''(-h)) = I - E^h(\lambda(h))$, we have
$$(I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U \in W(E^h(\lambda(h)))^\perp,$$
and so by (2.6)

(4.14)
$$\tilde{\lambda}^h \|(I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U\|^2$$
$$\leqq \|(I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U\|^2_{*,s},$$

where $\tilde{\lambda}^h$ is the smallest eigenvalue for which $\tilde{\lambda}^h \geqq \lambda(h)$ holds. Also $\lambda^h \geqq \lambda''(-h)$.

The orthogonality yields
$$\|(I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U\|^2$$
$$= ((I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U, A^h_\lambda Q_\lambda U).$$

Thus we obtain from (4.13) and (4.14)
$$(\tilde{\lambda}^h - \lambda)\|(I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U\|^2$$
$$\leqq \lambda((I - E^h(\lambda''(-h)))A^h_\lambda Q_\lambda U, Q_\lambda U - A^h Q_\lambda U),$$
which implies (4.8) for sufficiently small h.

D. We now prove the second assertion of Theorem 4.1. Again, by Lemma 3.1 and by regularity it holds that

(4.15)
$$\|Q_\lambda U - Q^h_\lambda Q_\lambda U\|_{*,s}$$
$$\leqq c\lambda h \|Q_\lambda U\| + \|A^h_\lambda Q_\lambda U - Q^h_\lambda Q_\lambda U\|_{*,s}.$$

Using Theorem 3.1, regularity and the fact that
$$\|Q^h_\lambda U\|^2_{*,s} - (A^h_\lambda Q_\lambda U, Q^h_\lambda Q_\lambda U)$$
$$= \sum_{\lambda(-h) \leqq \mu \leqq \lambda(h)} \mu (Q^h_\lambda Q_\lambda U - A^h_\lambda Q_\lambda U, \tilde{Q}^h_\mu Q_\lambda U),$$

we conclude

$$\|A_\lambda^h Q_\lambda U - Q_\lambda^h Q_\lambda U\|_{*,s}^2$$

$$\leqq c_1 \lambda^2 h^2 \|Q_\lambda U\| + c_2 \lambda^2 \lambda(h) \|Q_\lambda U\|^2.$$

This and inequality (4.15) yield (4.4). □

4.2. Using our result for $\mathscr{A} = \mathscr{M}^2 + 1$ we now want to obtain error estimates for the eigenfunctions of operator $\mathscr{M}$ corresponding to the eigenvalue ω. We shall first reformulate Theorem 4.1 with the help of Theorem 1.1.

According to Theorem 1.1 it holds that $Q_\lambda P_\omega U = P_\omega U$ and $Q_\lambda P_{-\omega} U = P_{-\omega}$. Let

$$Z_\omega^h := Q_\lambda^h P_\omega,\ \omega \in \sigma(\mathscr{M}),\ \lambda = \omega^2 + 1. \tag{4.16}$$

By Theorem 4.1 we have that

$$\|P_\omega U - Z_\omega^h U\| + \lambda_0^{1/2} h \|P_\omega U - Z_\omega^h U\|_{*,s} \leqq c\omega_0^2 h^2 \|P_\omega U\| \tag{4.17}$$

holds for all $-\omega_0 \leqq \omega \leqq \omega_0$, $\omega_0 := \sqrt{\lambda_0^2 - 1}$.

4.3. Theorem 4.2 gives an approximation result for the spaces of eigenfunctions of operator $\mathscr{M}$. But unfortunately Z_ω^h is not constructive, because in order to know Z_ω^h we in fact should know P_ω. On the other hand, $W(Q_\lambda^h)$ is constructive. One way to overcome this difficulty is to decompose $W(Q_\lambda^h)$ in a suitable way. To do this we must find appropriate orthogonal projectors P_ω^h and $P_{-\omega}^h$ such that

$$Q_\lambda^h = P_\omega^h + P_{-\omega}^h. \tag{4.18}$$

In order to construct P_ω^h and $P_{-\omega}^h$ we first remark that, since

$$\mathscr{M} P_{\pm\omega} U = \pm\omega P_{\pm\omega} U \tag{4.19}$$

holds, we have

$$\operatorname{Re}(P_\omega U, \mathscr{M} P_\omega U) > 0 \quad \text{and} \quad \operatorname{Re}(P_{-\omega} U, \mathscr{M} P_{-\omega} U) < 0. \tag{4.20}$$

Let

$$\hat{\mathscr{R}}(\Phi) := \frac{\operatorname{Re}(\Phi, \mathscr{M}\Phi)}{\|\Phi\|^2}$$

and let

$$\mu_i^h := \min\{\hat{\mathscr{R}}(\Phi) \mid \Phi \in W(Q_\lambda^h),\ \Phi \neq 0,\ (\Phi, \Phi_j^h) = 0,\ j = 0, 1, \ldots, i-1\}, \tag{4.21}$$

$i = 1, \ldots, \nu = \dim(W(Q_\lambda^h)) = \dim(W(Q_\lambda))$. Here $\Phi_i^h \in W(Q_\lambda^h)$, $i = 1, \ldots, \nu$, are the stationary points of $\hat{\mathscr{R}}$ with $(\Phi_i^h, \Phi_j^h) = \delta_{ij}$.

We define for $\omega > 0$

$$P_\omega^h U = \sum_{i=1}^{\nu/2} (U, \Phi_i^h) \Phi_i^h \tag{4.22}$$

and

$$P^h_{-\omega} U = \sum_{i=\nu/2+1}^{\nu} (U, \Phi_i^h)\Phi_i^h. \tag{4.23}$$

Obviously, it holds that $P^h_\omega \perp P^h_{-\omega}$. For $\omega=0$ we set $P^h_0=Z^h_0$.

It will be shown that projections P^h_ω, $\omega\in\sigma(\mathcal{M})$, are "approximations" of P_ω in a certain sense (similar to the connection between Q^h_λ and Q_λ).

We shall first give asymptotic upper and lower bounds to μ^h_i. For this purpose, let us remark that according to the min—max-principle (max—min-principle, respectively) we have

$$\begin{aligned}\mu_i^h &= \min_{\mathcal{H}_i^h \subset W(Q_\lambda^h)} \max_{\Phi\in\mathcal{H}_i^h} \hat{\mathcal{R}}(\Phi) \\ &= \max_{\mathcal{H}^h_{\nu-i+1}\subset W(Q_\lambda^h)} \min_{\Phi\in\mathcal{H}^h_{\nu-i+1}} \hat{\mathcal{R}}(\Phi).\end{aligned} \tag{4.24}$$

Using Theorem 4.2 we find

$$\|P_{\pm\omega}\Phi\| \leqq c\lambda^2 h^2 \|P_{\pm\omega}\Phi\| + \|Q^h_\lambda P_{\pm\omega}\Phi\|. \tag{4.25}$$

Thus for sufficiently small h

$$\|P_{\pm\omega}\Phi\| \leqq c\|Q^h_\lambda P_{\pm\omega}\Phi\|. \tag{4.26}$$

Let $B\colon \mathcal{H}_{*,s}\to\mathcal{H}_{*,s}$, $\Phi\to B\Phi$, be an operator defined by

$$(\psi, B\Phi)_{*,s} = (M\psi, \Phi)-(\psi, M\Phi) \quad \text{for all} \quad \psi\in\mathcal{H}_{*,s}.$$

It holds that

$$\|P^h B\Phi\| + h^{1/2}\|P^h B\Phi\|_{*,s} \leqq ch^2\|\Phi\| \tag{4.27}$$

for sufficiently small h (see [10], Section 3).

Since

$$\operatorname{Re}(P_{-\omega} U, \mathcal{M}P_{-\omega}U) = -\omega\|P_{-\omega} U\|^2,$$

we obtain, using in turn Theorem 4.2, (4.24) and inequalities (4.25)—(4.27) for $\Phi=Q^h_\lambda P_{-\omega} U$

$$\begin{aligned}&\operatorname{Re}(\Phi, \mathcal{M}\Phi) \\ &= \operatorname{Re}(P_{-\omega}U, \mathcal{M}P_{-\omega}U)+\operatorname{Re}\big(\Phi-P_{-\omega}U, \mathcal{M}(\Phi-P_{-\omega}U)\big) \\ &+\operatorname{Re}\big(P_{-\omega} U, \mathcal{M}(\Phi-P_{-\omega} U)\big)+\operatorname{Re}(\Phi-P_{-\omega} U, \mathcal{M}_{-\omega}P_{-\omega}U) \\ &= -\omega\|\Phi\|^2+\mathcal{O}(h^{3/2})\|\Phi\|^2, \quad h\to 0.\end{aligned} \tag{4.28}$$

In a similar way, it can be proved that

$$\operatorname{Re}(\Phi, \mathcal{M}\Phi) = \omega\|\Phi\|^2+\mathcal{O}(h^{3/2})\|\Phi\|^2, \quad h\to 0, \quad \text{for all} \quad \Phi = Q^h_\lambda P_\omega U. \tag{4.29}$$

By the min—max-principle and by (4.28) we obtain

$$\mu_i^h \leqq \max_{\Phi\in W(Q^h_\lambda P_{-\omega})} \hat{\mathcal{R}}(\Phi) = -\omega+\mathcal{O}(h^{3/2}), \quad h\to 0, \tag{4.30}$$

for $i=\nu/2+1, \ldots, \nu$ and, respectively, by the max—min-principle and by (4.29)

$$\mu_i^h \geqq \min_{\Phi \in W(Q_\lambda^h P_\omega)} \hat{\mathscr{R}}(\Phi) = \omega + \mathcal{O}(h^{3/2}), \quad h \to 0, \tag{4.31}$$

for $i=1, \ldots, \nu/2$.

As a corollary to Theorem 4.2 and to estimates (4.30) and (4.31) we can prove

Theorem 4.3. *The asymptotic error estimate*

$$\|P_\omega U - P_\omega^h P_\omega U\|_{*,s} \leqq c(\omega_0) h^{3/2} \|P_\omega U\| \tag{4.32}$$

holds for all $-\omega_0 \leqq \omega \leqq \omega_0$.

Proof. The case $\omega=0$ is trivial. At first let $\omega>0$. Using decomposition (4.18) and Theorem 4.2 we find

$$\|P_\omega U - P_\omega^h P_\omega U\|_{*,s} \leqq c(\omega_0) h^2 \|P_\omega U\|_{*,s} + \|P_{-\omega}^h P_\omega U\|_{*,s}. \tag{4.33}$$

Since $(\mathscr{M}+\omega)P_\omega U = 2\omega P_\omega U$, we obtain by (4.23) and by estimate (4.27)

$$\begin{aligned} &\|P_{-\omega}^h P_\omega U\|^2 \\ &\leqq \frac{1}{(2\omega)^2} \sum_{i=\nu/2+1}^{\nu} \big((\mathscr{M}+\omega)P_\omega U, \Phi_i^h\big)^2 \\ &\leqq \frac{1}{(2\omega)^2} \|P_\omega U\|^2 \Big(\sum_{i=\nu/2+1}^{\nu} \|(\mathscr{M}+\omega)\Phi_i^h\|^2 + ch^3 \|P_\omega U\|_{*,s}^2 \Big). \end{aligned} \tag{4.34}$$

Using (4.30) we conclude for $\Phi_i^h \in W(Q_\lambda^h)$, $i=\nu/2, \ldots, \nu$,

$$\begin{aligned} &\|(\mathscr{M}+\omega)\Phi_i^h\|^2 \\ &= \|\mathscr{M}\Phi_i^h\|^2 + 2\omega\mu_i^h \|\Phi_i^h\|^2 + \omega^2 \|\Phi_i^h\|^2 \\ &\leqq \|\mathscr{M}\Phi_i^h\| + \big(\mathcal{O}(h^{3/2}) - \omega^2\big) \|\Phi_i^h\|^2, \quad h \to 0. \end{aligned} \tag{4.35}$$

Since by (2.9) for all $\Phi \in \mathscr{H}$

$$\|Q_\lambda^h \Phi\|_{*,s}^2 \leqq \lambda^h \|Q_\lambda^h \Phi\|^2$$

and since $\lambda^h \leqq \omega^2 + 1 + c\lambda^2 h^2$, we obtain

$$\|\mathscr{M} Q_\lambda^h \Phi\|^2 + \|Q_\lambda \Phi\|^2 \leqq \big(\omega^2 + 1 + \mathcal{O}(h^2)\big) \|Q_\lambda^h \Phi\|^2, \quad h \to 0. \tag{4.36}$$

Moreover, recalling that $\operatorname{div} \varepsilon (Q_\lambda U)_1 = 0$, we have by Theorem 4.1

$$\|\operatorname{div}(Q_\lambda^h \Phi)_1\|^2 = \mathcal{O}(h^2) \|\Phi_i^h\|^2, \quad h \to 0. \tag{4.37}$$

Accordingly, by (4.35) and (4.36),

$$\|(\mathscr{M}+\omega)\Phi_i^h\|^2 = \mathcal{O}(h^{3/2}) \|\Phi_i^h\|^2, \quad h \to 0. \tag{4.38}$$

Hence, by (4.34)

$$\|P_{-\omega}^h P_\omega^h U\|^2 = \mathcal{O}(h^{3/2}) \|P_\omega U\|^2. \tag{4.39}$$

Using the same arguments as in the deduction for inequality (4.34) together with (4.37) we obtain

(4.40) $$\|\mathcal{M}P^h_{-\omega}P_\omega U\|^2 = \mathcal{O}(h^{3/2})\|P_\omega U\|^2\|\Phi^h_i\|^2.$$

If we combine (4.33), (4.37), (4.39) and (4.40), the assertion of Theorem 4.3 follows for $\omega>0$.

The case $\omega<0$ follows in the same way if we use (4.31) instead of (4.30). □

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